Study on the transfer probability density function of a class of stochastic differential dynamical systems *

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Abstract: In this paper, the solution and validity of the transfer probability density function for a stochastic dynamical system excited by white Gaussian noise are discussed. Based on the exponential polynomial closure method, not only the numerical solution of FPK equation is accepted, but also the validity of the method is shown from different views. On the one hand, the exact solution expression of the stationary transition probability density of some kind of system is received and its error compared with the numerical solution is analyzed. On the other hand, by establishing a kind of potential function to observe the stable region of the state variables in probabilistic sense, it is found that the stable region of the state variables determined by the potential function is highly consistent with the stable region determined by the stationary transition probability density function after long-term observation.

Keywords: $FPK(Fokker-Planck-Kolmogorov)$ *equation; Steady state probability density; Smooth potential; Generalized stationary; Stochastic final boundedness;*

1. Introduction

When studying stochastic differential dynamical system, studying FPK equation is an important method to explore the response of nonlinear stochastic dynamical system.By solving FPK equation, the transition probability density function of state variables can be understood, which is helpful for effective qualitative and quantitative analysis of state variables.However, the exact solution of the FPK equation of most systems cannot be directly obtained. Under the joint efforts of many scholars, many approximate solutions have been developed, such as finite element method [18], path integration method [3−5], finite difference method [6−10], Gaussian closure method [11−17], etc. The study on the accuracy of these numerical solutions is worthy of our further work.This paper for a class of nonlinear stochastic dynamic system, we first establish the corresponding FPK equation, the approximate solution of the second assumption FPK equation form as the index of polynomial, then we used the method of undetermined coefficients for solving the coefficient, and we can get a numerical solution of FPK equations .Finally, we verify the accuracy of this method from different views.On the one hand, we study the error between the exact solution of FPK equation and the numerical solution. At the same time, we are using some sort of potential function analysis the stability of the steady state variables

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in probability sense area.After a long time observation, we found that the potential function to determine the stability of the state variables of the smooth area and transition probability density function is to determine the stability of the area with high consistency, which shows that the method is simple, high precision, and it is an effective method for solving FPK equations.

2. The main content

2.1 Approximate method

For a general nonlinear dynamical system:

$$\frac{dX_i}{dt} = f_i(X) + \sum_{l=1}^{m} g_{il}(X)W_l(t), i = 1, \ldots, n \tag{1}$$

Where $X(t) = [x_1, x_2, \ldots, x_n]^\top$ and $W_l(t)$ are Gaussian white noise, and their correlation function is

$$E[W_l(t)W_s(t+\tau)] = 2\pi K_{ls}\delta(\tau), s = 1, \ldots, m \tag{2}$$

If the system has stationary transition probability density $p(X)$, the following simplified FPK equation can be determined:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} G_i = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[a_i(x)p - \frac{1}{2}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}[b_{ij}p]] = 0 \tag{3}$$

Where G_i is the probability stream in the i direction. a_i, b_{ij} is the moment of first and second derivatives respectively, which can be derived from equation (1) :

$$a_i(x) = f_i(x) + \pi \sum_{k=1}^{n} \sum_{l,s=1}^{m} K_{ls} g_{ks}(x) \frac{\partial}{\partial x_k} g_{il}(x)$$

$$b_{ij}(x) = 2\pi \sum_{l,s=1}^{m} K_{ls} g_{il}(x) g_{js}(x) \tag{4}$$

In some special cases, we can obtain the exact solution of the system under the generalized stationary condition. For system (1), we add a set of sufficient conditions for (3), as follows:

$$G_i = a_i(x)p - \frac{1}{2}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}[b_{ij}p] = 0 \tag{5}$$

In this case, system (1) belongs to the stationary potential class, and we can express the stationary transition probability density as $p(x) = C\exp[-\varphi(x)]$. C is the normalized constant, and $\varphi(x)$ is called the probability potential function.

However, it is usually difficult to solve general FPK equation. Therefore, based on previous studies by scholars, we assume that the stationary transition probability density function of the system is:

$$p_n(x) = C\exp[a_{11}x_1 + a_{12}x_2 + a_{21}x_1^2 + a_{22}x_1x_2 + a_{23}x_2^2 + \ldots + a_{ij}x_1^{i+j-1}x_2^{j-1} + \ldots + a_{nn+1}x_2^n]$$

C is the normalized constant, $a_{11}, a_{12}, \ldots, a_{nn+1} (n \geq 2)$ is the constant. The approximate solution is substituted into FPK equation, and the undetermined coefficients are used to solve the coefficients on both sides of the equation. We apply this method to two stochastic dynamical systems and verify its validity from different angles.

2.2

Consider the following two degree of freedom nonlinear systems that are both excited by random parameters and externally excited:

$$\ddot{x} + 2\alpha[1 + W_1(t)]\dot{x} + \omega^2[1 + W_2(t)]x + \beta_1(x^2 + \frac{\dot{x}^2}{\omega^2})\dot{x} = W_3(t) \tag{6}$$

Where $W_1(t), W_2(t), W_3(t)$ is a process of zero mean Gaussian white noise in the sense of independent *Stratonovich*, and its spectral density constants are respectively k_1, k_2, k_3. α, ω, β_1 is a constant.

Let$x_1 = x, x_2 = \dot{x}$, and the corresponding stochastic differential equation of *Stratonovich* is:

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = [-2\alpha x_2 - \omega^2 x_1 - \beta_1(x_1^2 + \frac{x_2^2}{\omega^2})x_2]dt - 2\alpha x_2\sqrt{2\pi k_1} \circ dB_1(t) - \omega^2 x_1\sqrt{2\pi k_2} \circ dB_2(t) \\ \quad +\sqrt{2\pi k_3} \circ dB_3(t) \end{cases} \tag{7}$$

By adding the correction term of $Wong - Zakai$, the corresponding stochastic differential equation of *Itó* can be converted into:

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = [-2\alpha x_2 - \omega^2 x_1 - \beta_1(x_1^2 + \frac{x_2^2}{\omega^2})x_2 + 4\alpha^2 x_2\pi k_1]dt - 2\alpha x_2\sqrt{2\pi k_1}dB_1(t) - \omega^2 x_1\sqrt{2\pi k_2}dB_2(t) \\ \quad +\sqrt{2\pi k_3}dB_3(t) \end{cases} \tag{8}$$

Then the moment of the first and second derivatives is obtained as follows:

$$a_1 = x_2; a_2 = -2\alpha x_2 - \omega^2 x_1 - \beta_1(x_1^2 + \frac{x_2^2}{\omega^2})x_2 + 4\alpha^2 x_2\pi k_1$$

$$b_{11} = b_{12} = b_{21} = 0; b_{22} = 8\alpha^2 x_2^2\pi k_1 + 2\omega^4 x_1^2\pi k_2 + 2\pi k_3 \tag{9}$$

The simplified FPK equation is obtained as follows:

$$\frac{\partial x_2 p}{\partial x_1} + \frac{\partial[-2\alpha x_2 - \omega^2 x_1 - \beta_1(x_1^2 + \frac{x_2^2}{\omega^2})x_2 + 4\alpha^2 x_2\pi k_1]p}{\partial x_2} - \frac{1}{2}\frac{\partial^2[8\alpha^2 x_2^2\pi k_1 + 2\omega^4 x_1^2\pi k_2 + 2\pi k_3]p}{\partial x_2^2} = 0 \tag{10}$$

By dividing the moment of first derivative into reversible and irreversible components, the exact solvable class can be extended from stationary potential to detailed equilibrium. Similarly, we not only separate the moment of the first derivative, but also the moment of the second derivative, so as to further expand the exact solvable class to obtain the exact solution of the FPK equation of the system. The last item on the left end of equation (10) can be written as:

$$\frac{\partial^2[8\alpha^2 x_2^2\pi k_1 + 2\omega^4 x_1^2\pi k_2 + 2\pi k_3]p}{\partial x_2^2} = \frac{\partial[8\alpha^2 x_2 2\pi k_1 p]}{\partial x_2} + \frac{\partial[(8\alpha^2 x_2^2\pi k_1 + 2\omega^4 x_1^2\pi k_2 + 2\pi k_3)\frac{\partial p}{\partial x_2}]}{\partial x_2} \tag{11}$$

By substituting the above equation into FPK equation (10), we can get:

$$\frac{\partial x_2 p}{\partial x_1}+\frac{\partial[-2\alpha x_2-\omega^2 x_1-\beta_1(x_1^2+\frac{x_2^2}{\omega^2})x_2-4\alpha^2 x_2\pi k_1]p}{\partial x_2}$$
$$-\frac{1}{2}\frac{\partial[(8\alpha^2 x_2^2\pi k_1+2\omega^4 x_1^2\pi k_2+2\pi k_3)\frac{\partial p}{\partial x_2}]}{\partial x_2}=0 \tag{12}$$

That is:

$$\frac{\partial x_2 p}{\partial x_1}-\omega^2 x_1\frac{\partial p}{\partial x_1}+\frac{\partial[-2\alpha x_2-\beta_1(x_1^2+\frac{x_2^2}{\omega^2})x_2-4\alpha^2 x_2\pi k_1]p}{\partial x_2}$$
$$-\frac{1}{2}\frac{\partial[(8\alpha^2 x_2^2\pi k_1+2\omega^4 x_1^2\pi k_2+2\pi k_3)\frac{\partial p}{\partial x_2}]}{\partial x_2}=0 \tag{13}$$

If the following sufficient conditions are satisfied:

$$\begin{cases}\frac{\partial x_2 p}{\partial x_1}-\omega^2 x_1\frac{\partial p}{\partial x_2}=0\\ (-2\alpha x_2-\beta_1(x_1^2+\frac{x_2^2}{\omega^2})x_2-4\alpha^2 x_2\pi k_1)p-\frac{1}{2}(8\alpha^2 x_2^2\pi k_1+2\omega^4 x_1^2\pi k_2+2\pi k_3)\frac{\partial p}{\partial x_2}=0\end{cases} \tag{14}$$

Then, equation (13) is satisfied.Using the idea of the probability potential function, let's say $p(x)=C\exp[-\varphi(x)]$,where $\varphi(x_1,x_2)=\varphi(\lambda)$, $\lambda=\frac{1}{2}x_2^2+\frac{1}{2}\omega^2x_1^2$ and C are normalized constants. Then equation (14) is equivalent to:

$$\begin{cases}x_2\frac{\partial\varphi}{\partial x_1}-\omega^2 x_1\frac{\partial\varphi}{\partial x_2}=0\\ (2\alpha x_2+\beta_1(x_1^2+\frac{x_2^2}{\omega^2})x_2+4\alpha^2 x_2\pi k_1)+\frac{1}{2}[(8\alpha^2 x_2^2\pi k_1+2\omega^4 x_1^2\pi k_2+2\pi k_3)\frac{\partial\varphi}{\partial x_2}]=0\end{cases} \tag{15}$$

The following simplification can be obtained:

$$\frac{d\varphi}{d\lambda}=\frac{2\alpha+\beta_1(x_1^2+\frac{x_2^2}{\omega^2})+4\alpha^2\pi k_1}{(4\alpha^2 x_2^2\pi k_1+\omega^4 x_1^2\pi k_2+\pi k_3)} \tag{16}$$

If $4\alpha^2 k_1=k_2\omega^2$,then

$$\frac{d\varphi}{d\lambda}=\frac{2\alpha+\beta_1(x_1^2+\frac{x_2^2}{\omega^2})+4\alpha^2\pi k_1}{(\pi k_2\omega^2 x_2^2+\omega^4 x_1^2\pi k_2+\pi k_3)} \tag{17}$$

If $\beta_1=\frac{(2\alpha+4\alpha^2\pi k_1)\omega^4 k_2}{k_3}$,Then

$$\frac{d\varphi}{d\lambda}=\frac{(2\alpha+4\alpha^2\pi k_1)(1+\frac{\omega^4 k_2}{k_3}(x_1^2+\frac{x_2^2}{\omega^2}))}{\pi k_3(1+\frac{\omega^4 k_2}{k_3}(x_1^2+\frac{x_2^2}{\omega^2}))}=\frac{2\alpha+4\alpha^2\pi k_1}{\pi k_3} \tag{18}$$

Therefore, under the generalized stationary condition, the exact solution of FPK equation of the system can be obtained as follows:

$$p_e(X)=C\exp[-\frac{2\alpha+4\alpha^2\pi k_1}{\pi k_3}(\frac{1}{2}x_2^2+\frac{1}{2}\omega^2x_1^2)] \tag{19}$$

The numerical solution of the system is solved based on the closed exponential polynomial method. Assume that the stationary transition probability density function of the system is:

$$p_n(x) = C\exp[a_{11}x_1 + a_{12}x_2 + a_{21}x_1^2 + a_{22}x_1x_2 + a_{23}x_2^2 + \ldots + a_{ij}x_1^{i+j-1}x_2^{j-1} + \ldots + a_{nn+1}x_2^n] \quad (20)$$

Substitute equation (20) into FPK equation (10) above, fix n first, and we have

$$\frac{\partial x_2 p_n}{\partial x_1} + \frac{\partial[-2\alpha x_2 - \omega^2 x_1 - \beta_1(x_1^2 + \frac{x_2^2}{\omega^2})x_2 + 4\alpha^2 x_2 \pi k_1]p_n}{\partial x_2} - \frac{1}{2}\frac{\partial^2[8\alpha^2 x_2^2 \pi k_1 + 2\omega^4 x_1^2 \pi k_2 + 2\pi k_3]p_n}{\partial x_2^2} = 0 \quad (21)$$

The undetermined coefficient method is used to determine the coefficient of p_n.Thus the numerical solution of the FPK equation of the system can be obtained.When $\omega = 1, k_1 = k_2 = k_3 = 1$,$\alpha = 0.5, \beta 1 = 0.28$.

When we solve for $n = 2$, we get $a_{11} = a_{12} = a_{22} = 0, a_{21} = -0.579577, a_{23} = -0.579577$.

So $p_2(X) = -0.579577x_1^2 - 0.579577x_2^2$.

When we solve for $n = 6$, we get$a_{11} = a_{12} = a_{22} = a_{31} = a_{32} = a_{33} = a_{34} = a_{42} = a_{44} = 0$.

$a_{51} = a_{52} = a_{53} = a_{54} = a_{55} = a_{56} = a_{62} = a_{64} = a_{66} = 0$.

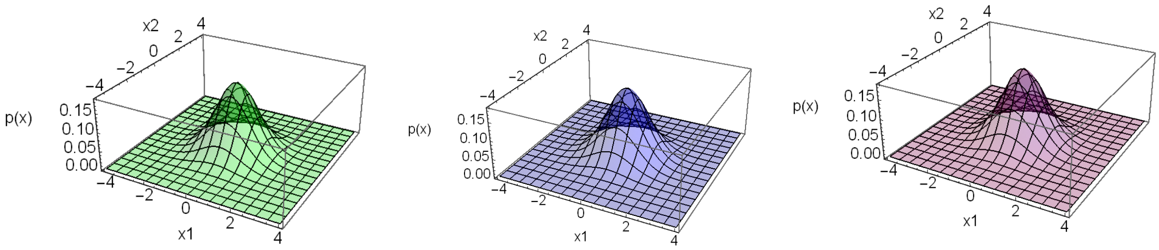
$a_{21} = -0.579577, a_{23} = -0.579577, a_{41} = 0.000126739, a_{43} = 0.000253479, a_{45} = 0.000126739$

$a_{61} = -0.0000844929, a_{63} = -0.000253479, a_{65} = -0.000253479, a_{66} = -0.0000844929$.

So

$p_6(X) = -0.579577x_1^2 + 0.000126739x_1^4 - 0.0000844929x_1^6 - 0.579577x_2^2 + 0.000253479x_1^2x_2^2 - 0.000253479x_1^4x_2^2 + 0.000126739x_2^4 - 0.000253479x_1^2x_2^4 - 0.0000844929x_2^6$

Simulation results are as follows:



(a) Graph of function $p_2(X)$. (b) Graph of function $p_6(X)$. (c) Graph of function $p_e(X)$.

Figure 1: Schematic diagram of the steady state transition probability density function.

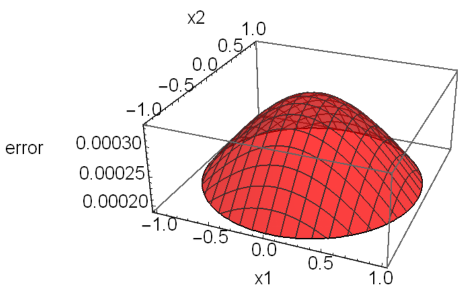


Figure 2: In the disk of $x_1^2 + x_2^2 \leq 1$, error diagram of $p_6(X)$ and $p_e(X)$

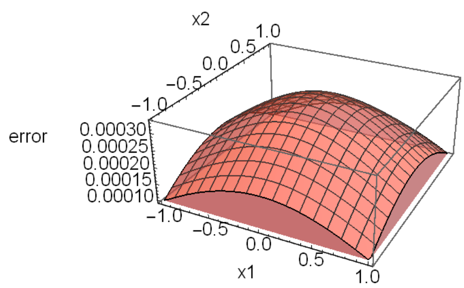


Figure 3: In the $|x_1| \leq 1; |x_2| \leq 1$ region, error diagram of $p_6(X)$ and $p_e(X)$

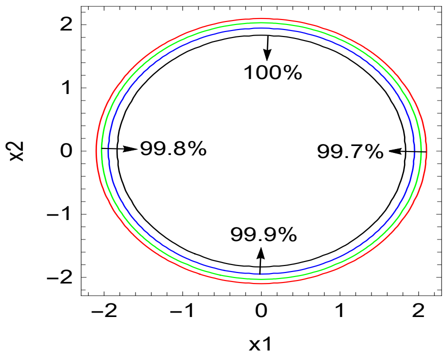


Fig 4: Stability zone accuracy diagram.

Table 1: List of joint stationary probability densities of numerical and exact solutions in different regions near the equilibrium state

$p(x)$	$x_1^2+x_2^2 \le 1$	$x_1^2+x_2^2 \le 0.25$	$\lvert x_1\rvert \le 1; \lvert x_2\rvert \le 1$	$\lvert x_1\rvert \le 0.5; \lvert x_2\rvert \le 0.5$
$n=2$	0.439865	0.134886	0.516040	0.167808
$n=6$	0.440668	0.135130	0.516982	0.168112
Exact solution	0.439865	0.134886	0.516040	0.167808

We can find that in the diagram (1) the probability of system state variables to around 0.5 concentrated near the equilibrium state, when time is infinite, basically stable in 2 for the center with the origin as the radius of a circle domain, and we can see from figure (4) the precision of this method is the highest in the region,which shows that the numerical results highly close to the precise value. Figure (2) and (3) give the error graph of the accurate solution and the numerical solution. Table (1) and (2) give the error value of the joint steady-state probability density corresponding to the numerical solution and the accurate solution in the corresponding region. We find that the numerical solution calculated from (3) is highly consistent with the accurate solution obtained from a group of sufficient conditions (5).

2.2

Considering the following system

$$\ddot{X} + [a + (bX + c\dot{X})^2]\dot{X} + X = (dX + e\dot{X})W_1(t) + W_2(t) \tag{22}$$

Table 2: List of joint stationary probability density errors of numerical and accurate solutions in different regions near the equilibrium state

Compare $p_6(X)$ to $p_e(X)$	$x_1^2+x_2^2 \le 1$	$x_1^2+x_2^2 \le 0.25$	$\lvert x_1\rvert \le 1; \lvert x_2\rvert \le 1$	$\lvert x_1\rvert \le 0.5; \lvert x_2\rvert \le 0.5$
Error	0.000803	0.000244	0.000942	0.000304
Relative error	0.1825%	0.1808%	0.1825%	0.1811%

Where a, b, c, d, e is a constant, $W_1(t), W_2(t)$ is an independent Gaussian white noise whose power spectral density constant is k_1, k_2 respectively.

Assuming $X_1 = X, X_2 = \dot{X}$, the stochastic differential equation of *Stratonovich* of the system can be obtained as follows:

$$\begin{cases} dX_1 = X_2 dt \\ dX_2 = \{-[a + (bX_1 + cX_2)^2]X_2 - X_1\}dt + (dX_1 + eX_2)\sqrt{2\pi k_1} \circ dB_1(t) + \sqrt{2\pi k_2} \circ dB_2(t) \end{cases} \tag{23}$$

Where $B_1(t), B_2(t)$is the unit Wiener process.The *Itó* stochastic differential equation of the system is:

$$\begin{cases} dX_1 = X_2 dt \\ dX_2 = \{-[a + (bX_1 + cX_2)^2]X_2 - X_1 + \pi k_1 e(dX_1 + eX_2)\}dt + (dX_1 + eX_2)\sqrt{2\pi k_1} dB_1(t) \\ \quad +\sqrt{2\pi k_2} dB_2(t) \end{cases} \tag{24}$$

Where $\pi k_1 e(dX_1 + eX_2)$ is the correction term for $Wong - Zakai$.The corresponding steady state FPK equation is:

$$\frac{\partial[x_2 p]}{\partial x_1} + \frac{\partial\{-[a + (bX_1 + cX_2)^2]X_2 - X_1 + \pi k_1 e(dX_1 + eX_2)\}p}{\partial x_2} - \frac{\partial^2[(dx_1 + ex_2)^2\pi k_1 + \pi k_2]p}{\partial x_2^2} = 0 \tag{25}$$

We assume that the stationary transition probability density function of system (22) is:

$$p_n(x) = C\exp[a_{11}x_1 + a_{12}x_2 + a_{21}x_1^2 + a_{22}x_1x_2 + a_{23}x_2^2 + \ldots + a_{ij}x_1^{i+j-1}x_2^{j-1} + \ldots + a_{nn+1}x_2^n]$$

C is the normalized constant. Substitute the approximate solution into the FPK equation:

$$\frac{\partial[x_2 p_n]}{\partial x_1} + \frac{\partial\{-[a + (bX_1 + cX_2)^2]X_2 - X_1 + \pi k_1 e(dX_1 + eX_2)\}p_n}{\partial x_2} - \frac{\partial^2[(dx_1 + ex_2)^2\pi k_1 + \pi k_2]p_n}{\partial x_2^2} = 0 \tag{26}$$

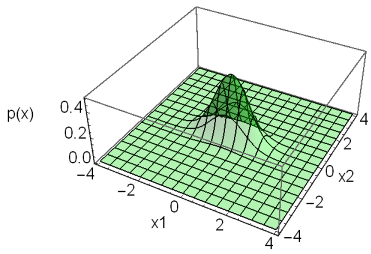
Let $a = 4, b = \frac{5}{4}, c = -\frac{15}{4}, d = \frac{1}{2}, e = -\frac{3}{2}, k_1 = k_2 = \frac{1}{\pi}$.Firstly,the coefficient of stationary transition probability density function is solved by undetermined coefficient method, and the numerical solution of stationary FPK equation of the system is obtained:

$$p_n(X) = \frac{25}{16\pi}\exp[-\frac{25}{32}(x_1^2 + 4x_2^2)] \tag{27}$$

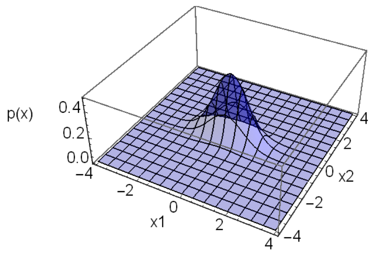
The simulation results are shown in the following figures:

Then $Milsteins$ discretization method was adopted to consider the discretization system corresponding to model (22):

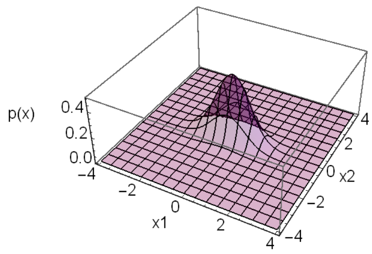
$$\begin{cases} X_1(t_{j+1}) = X_1(t_j) + X_2(t_j)\Delta t \\ X_2(t_{j+1}) = X_2(t_j) + \{-[a + (bX_1(t_j) + cX_2(t_j))^2]X_2(t_j) - X_1(t_j) + \pi k_1 e(dX_1(t_j) + eX_2(t_j))\}\Delta t \\ \qquad +(dX_1(t_j) + eX_2(t_j))\sqrt{2\pi k_1}\Delta W_1(t) + \sqrt{2\pi k_2}\Delta W_2(t) \\ \qquad +\frac{1}{2}\{[d(dX_1(t_j) + eX_2(t_j))2\pi k_1](\Delta W_1(t)^2 - \Delta t) + [e(dX_1(t_j) + eX_2(t_j))2\pi k_1](\Delta W_2(t)^2 - \Delta t)\} \end{cases} \tag{28}$$



(a) Graph of function $p_2(X)$.



(b) Graph of function $p_4(X)$.



(c) Graph of function $p_6(X)$.

Figure 5: Schematic diagram of the steady state transition probability density function.

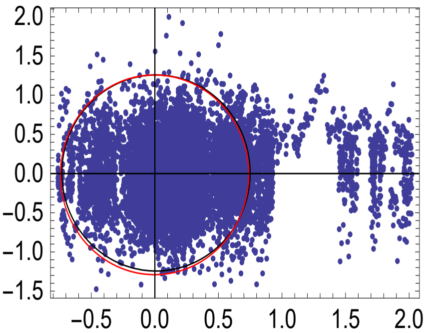


Figure 6: Phase diagram of system (22)

$\Delta W_1(t), \Delta W_2(t)$ is a Gaussian random variable that obeys $N(0, \Delta t)$. The motion track of the state variable is simulated by *Mathematica* as shown in the following figure: From figure $(5-6)$, we can see that the numerical solution result of the stationary transition probability density function is consistent with the simulation result of discretization of system state variables. When we observe for a long time, the state variables are concentrated near the origin with high probability.

Next, we observe the stable region of state variables in probabilistic sense by establishing a kind of potential function of the system.For model $d(X) = f(X)dt + g(X)dB(t), X = (x_1, x_2, \ldots, x_n)$, $B(t)$ is a Unit Wiener process, and the solution of the model is called random and finally bounded. If there is a positive number for α, such that for $\forall \varepsilon \in (0,1)$,the solution of the model satisfies:

$$\lim_{t\to\infty} supP\{|X(t)| > \alpha\} < \varepsilon$$

Lemma 1. When $2\pi k_1 d^2 + 3\pi k_1 de \le -1, 1 - 2a + 4\pi k_1 e^2 + 3\pi k_1 de \le 0$,positive $H = 4\pi k_2$ exists, independent of the initial value $X_0 = (x_{1,0}, x_{2,0}) \in \mathbf{R^2}$, so that the solution of the model satisfies:

$$\lim_{t\to\infty} supE\|(x_1(t), x_2(t))\|^2 \le H$$

.

Proof:Defining

$$V(x_1, x_2) = x_1^2 + x_2^2, (x_1, x_2) \in \mathbf{R^2}$$

From the *Itó* formula, we can get:

$$dV(x_1, x_2) = LV(x_1, x_2)dt + 2x_2(dx_1 + ex_2)\sqrt{2\pi k_1}dB_1(t) + 2x_2\sqrt{2\pi k_2}dB_2(t) \tag{29}$$

Among them:

$$LV = 2x_1x_2 - 2x_2^2[a + (bx_1 + cx_2)^2] - 2x_1x_2 + 2\pi k_1 ex_2(dx_1 + ex_2) + 2\pi k1(dx_1 + ex_2)^2 + 2\pi k_2$$

$$\leq 2x_1x_2 - 2x_2^2 a - 2x_1x_2 + 2\pi k_1 ex_2(dx_1 + ex_2) + 2\pi k1(dx_1 + ex_2)^2 + 2\pi k_2 + x_1^2 + x_2^2 - V(x_1, x_2)$$

$$\leq x_1^2(2\pi k_1 d^2 + 1) + x_2^2(-2a + 4\pi k_1 e^2 + 1) + x_1x_2(2\pi k_1 de + 4\pi k_1 de) + 2\pi k_2 - V(x_1, x_2)$$

$$\leq x_1^2(2\pi k_1 d^2 + 1) + x_2^2(-2a + 4\pi k_1 e^2 + 1) + 3\pi k_1 de(x_1^2 + x_2^2) + 2\pi k_2 - V(x_1, x_2)$$

$$\leq x_1^2(2\pi k_1 d^2 + 1 + 3\pi k_1 de) + x_2^2(-2a + 4\pi k_1 e^2 + 1 + 3\pi k_1 de) + 2\pi k_2 - V(x_1, x_2) \tag{30}$$

Thus, it can be concluded that:

$$dV(x_1, x_2) \leq [2\pi k_2 - V(x_1, x_2)]dt + 2x_2(dx_1 + ex_2)\sqrt{2\pi k_1}dB_1(t) + 2x_2\sqrt{2\pi k_2}dB_2(t) \tag{31}$$

Using the *Itó* formula again:

$$d(e^t V(x_1, x_2)) = e^t[V(x_1, x_2)dt + dV(x_1, x_2)] \leq e^t[2\pi k_2 dt + 2x_2(dx_1 + ex_2)\sqrt{2\pi k_1}dB_1(t) + 2x_2\sqrt{2\pi k_2}dB_2(t)] \tag{32}$$

By integrating both sides of the above equation and calculating the mean value, we can get:

$$e^t EV(x_1, x_2) \leq V(x_{1,2}, x_{2,0}) + (e^t - 1)2\pi k_2 \tag{33}$$

Thus,

$$\lim_{t\to\infty} supEV(x_1(t), x_2(t)) \leq 2\pi k_2 \tag{34}$$

On the other hand,

$$\|(x_1(t), x_2(t))\|^2 = x_1^2 + x_2^2 \leq 2max\{x_1^2, x_2^2\} \leq 2V(x_1, x_2) \tag{35}$$

From this we can conclude:

$$\lim_{t\to\infty} supE\|(x_1(t), x_2(t))\|^2 \leq 2 \lim_{t\to\infty} supEV(x_1(t), x_2(t)) \leq 4\pi k_2 \tag{36}$$

.

That's true with respect to $H = 4\pi k_2$.

It is proved that the model is stochastic and finally bounded.According to *Chebyshev* inequality, for $\forall \varepsilon > 0$, let $\theta = \sqrt{\frac{4\pi k_2}{\varepsilon}}$, then:

$$\lim_{t\to\infty} supP\{|X(t)| \geq \theta\} \leq \varepsilon \tag{37}$$

That is, the model is stochastic and ultimately bounded.

When we take $\varepsilon = 0.01$,$\theta = 20$. In this case, the state variable is mainly stable in the region whose modulus is less than θ near the equilibrium state, which is highly consistent with the stable region determined by the stationary transition probability density function (27) obtained by system *FPK* equation, indicating that this method is an effective algorithm for calculating the *FPK* equation.

3.The conclusion

In this paper, we solve the problem of transfer probability density function and the validity of the solution for a class of nonlinear stochastic differential dynamical systems. For the more general and complex stochastic differential dynamical system, we can make a submodule of the system conform to the idea of this paper by simplifying, and then we can use the undetermined coefficient method to find the coefficient by assuming that the stationary transition probability density function of the system is of exponential polynomial form. In this paper, the method is applied to the two systems, we not only find out in such a system under the condition of generalized steady transition probability density function of the exact solutions, and through the establishment of such a system of a kind of potential function, through long time observation, we obtained the stable state variables in probability sense the stability of the region, the results of the above analysis, respectively, compared with numerical solution of transition probability density function, we found that both have high consistency, thanks in large part proved the effectiveness of the method and feasibility.

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