Lower Bound of Decay Rate for Higher Order Derivatives of Solution to the Compressible Quantum Magnetohydrodynamic Model

Jincheng Gao Zeyu Lyu Zheng-an Yao

School of Mathematics, Sun Yat-Sen University,
510275, Guangzhou, P.R. China

Abstract

The lower bound decay rate of global solution to the compressible viscous quantum magnetohydrodynamic model in three-dimensional whole space under the $H^5 \times H^4 \times H^4$ framework is investigated in this paper. We firstly show that the lower bound of decay rate for the density, velocity and magnetic field converging to the equilibrium state (1,0,0) in L^2-norm is $(1+t)^{-\frac{3}{4}}$ when the initial data satisfies some low frequency assumption. Moreover, we prove that the lower bound of decay rate of $k(k \in [1,3])$ order spatial derivative for the density, velocity and magnetic field converging to the equilibrium state (1,0,0) in L^2-norm is $(1+t)^{-\frac{3+2k}{4}}$. Then we show that the lower bound of decay rate for the time derivatives of density and velocity converging to zero in L^2-norm is $(1+t)^{-\frac{5}{4}}$, but the lower bound of decay rate for the time derivative of magnetic field converging to zero in L^2-norm is $(1+t)^{-\frac{7}{4}}$.

Keywords: Compressible quantum magnetohydrodynamic model, lower bound decay rate.

2010 Mathematics Subject Classification: 35Q35,35B40,35Q40.

1 Introduction

The purpose of this paper is to consider the lower bounds of decay rate for the global solution to the following compressible viscous quantum magnetohydrodynamic(in short, vQMHD) model in three-dimensional whole space:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\mu+\lambda)\nabla \mathrm{div} u + \nabla P(\rho) - \dfrac{h^2}{2}\rho \nabla\Big(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big) = (\nabla \times B) \times B, \\ \partial_t B - \nabla \times (u \times B) = -\nabla \times (\nu \nabla \times B), \ \mathrm{div} B = 0, \end{cases} \tag{1.1}$$

where $t \geq 0$ is time, $x \in \mathbb{R}^3$ is spatial coordinate, the unknown functions $\rho = \rho(x,t)$, $u = (u_1,u_2,u_3)(x,t)$ and $B = (B_1,B_2,B_3)(x,t)$ represent density, velocity and magnetic field respectively. The function $P(\rho)$ which denotes pressure is smooth in a neighborhood of 1 with $P'(1) > 0$.

Email: *gaojc1998@163.com(J.C.Gao), lvzy3@mail2.sysu.edu.cn(Z.Y.Lyu), mcsyao@mail.sysu.edu.cn(Z.A.Yao).*

The constant viscosity coefficients μ and λ satisfy the following physical conditions:

$$\mu > 0,\ 3\lambda + 2\mu \geq 0.$$

The constant $\nu > 0$ acting as a magnetic diffusion coefficient of the magnetic field is the magnetic diffusivity. The constant h denotes Planck constant and it satisfies $h > 0$. The symbol $\otimes$ denotes the Kronecker tensor product. As usual, we refer to the first equation of the system (1.1) as the continuity equation, the second equation of the system (1.1) as the momentum balance equation, and the third equation of the system (1.1) as the magnetic equation. The expression $\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ can be interpreted as a quantum potential, the so-called Bohm potential, which satisfies

$$2\rho\nabla\Big(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big) = \mathrm{div}(\rho\nabla^2\rho) = \nabla\Delta\rho + \frac{|\nabla\rho|^2\nabla\rho}{\rho^2} - \frac{\nabla\rho\Delta\rho}{\rho} - \frac{\nabla\rho\cdot\nabla^2\rho}{\rho} = \nabla\Delta\rho - 4\mathrm{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}).$$

In order to complete the system (1.1), this system is supplemented with initial data

$$(\rho, u, B)(x,t)|_{t=0} = (\rho_0(x), u_0(x), B_0(x)). \tag{1.2}$$

Furthermore, we assume that as the space variable tends to infinity, the initial disturbances satisfy

$$\lim_{|x|\to\infty}(\rho_0 - 1, u_0, B_0)(x) = 0. \tag{1.3}$$

The quantum fluid model can provide many pieces of information for the particles in the semiconductor simulation and it could be used to describe quantum semiconductors [5], weakly interacting Bose gases [10], and quantum trajectories of Bohmian mechanics [32]. Madelung [21] found a hydrodynamic form of the singlestate Schrödinger equation. Later, the quantum hydrodynamic(QHD) model, which may be viewed as a quantum corrected version of the classical hydrodynamic equations, was derived by Ferry and Zhou [5] from the Wigner equation. The quantum hydrodynamic model for plasmas was introduced in [22]. The quantum magnetohydrodynamic(QMHD) model that plays an important role in modeling and simulating electron transport was extended by Hass [14] later from a Wigner-Maxwell system and this model could be used to describe the global properties of quantum plasmas. It should be noted that system (1.1) will reduce to the compressible MHD equations without quantum effects.

There are huge literatures on the well-posedness of solutions to the quantum fluid model. The one-dimensional problems have been studied extensively, refer to [6, 17] and the references therein. The existence and uniqueness of local and global solutions to one-dimensional isentropic quantum Euler-Poisson system under a subsonic condition was proved by Jüngel and Li in [17]. Gamba et al.[6] showed the global existence of weak solution to the viscous quantum hydrodynamic equations. For the multi-dimensional case, Jüngel [18] used Faedo-Galerkin method and weak compactness technique to prove the global existence of weak solution to the viscous quantum Euler model in $\mathbb{R}^3$ and showed the global existence of weak solution to the barotropic compressible quantum Navier-Stokes equations in a three-dimensional torus under the condition that the viscosity constant is smaller then the scaled Planck constant. Later, Yang and Ju [35] applied the same method as in [18] to obtain the global existence of weak solutions to the viscous quantum magnetohydrodynamic equations with large data in a three-dimensional torus. Guo and Wang [11] established the local existence of the smooth solutions to the quantum hydrodynamic models in $\mathbb{R}^d$ with $d \geq 1$. Recently, under the condition that the initial perturbation around a constant state is small enough, Pu and Li [26] showed the existence of global smooth solutions to the initial boundary value problem for compressible quantum hydrodynamic model with damping and heat diffusion in a bounded domain in $\mathbb{R}^3$. We also remark that Chen and Dreher [1] proved the local existence of solution to the viscous model of quantum hydrodynamics in $\mathbb{R}^1$ and they showed the local existence of solution in higher dimensions provided boundary is periodical. For more results about the well-posedness of solutions to the quantum fluid model, readers can refer to [3, 4, 12, 15, 19, 20, 24] and the references therein.

The study for decay rates of solutions to the quantum magnetohydrodynamic model have attracted much attention of mathematicians. Pu and Guo [25] established the optimal decay rates of classical solutions near constant states by virtue of spectral method in $\mathbb{R}^3$ provided the initial data belong to L^1. Pu and Xu [28] obtained the decay rate of classical solutions to the viscous quantum magnetohydrodynamic model when the initial perturbation belong to L^1, more precisely, the time decay rates are as follows

$$\|\nabla^k(\rho-1)(t)\|_{H^{5-k}} + \|\nabla^k u(t)\|_{H^{4-k}} + \|\nabla^k B(t)\|_{H^{4-k}} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}},\ k=0,1.$$

Pu and Xu [27] used the pure energy method as in [13] to obtain the optimal decay rates of higher-order spatial derivatives of solutions to the full hydrodynamic equations with quantum effects under the condition that the initial perturbation belongs to $(H^{N+2}\bigcap \dot{H}^{-s})\times(H^{N+1}\bigcap \dot{H}^{-s})\times(H^{N}\bigcap \dot{H}^{-s})$ for $N \geq 3$ and $s \in [0, \frac{3}{2})$. Recently, by using Fourier splitting method, Xi et al.[33] established the optimal time decay rates for the higher-order spatial derivatives of solutions to the viscous quantum magnetohydrodynamic model when the initial perturbation belongs to L^1, which improved the work in [28], more precisely, they got

$$\|\nabla^k(\rho-1)(t)\|_{H^{5-k}} + \|\nabla^k u(t)\|_{H^{4-k}} + \|\nabla^k B(t)\|_{H^{4-k}} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}},\ k=0,1,2,3. \tag{1.4}$$

For more results about the large time behavior of solutions to the quantum fluid model, interested readers may refer to [25, 34] and the references therein.

It should be noted that the time decay rate (1.4) is called optimal in the sense that this rate of solution for the nonlinear part is coincide with the decay rate of linearized one. *To the best knowledge of the authors', there has been no result of lower bound(coincide with upper rate) of decay rate of solution to the compressible viscous magnetohydrodynamic model in* $\mathbb{R}^3$. *Thus, the purpose of this paper is to solve this problem. In other words, the aim of this work is to show that the decay rate* (1.4) *obtained in [33] is really optimal.*

Notation: Throughout this paper, we use ∇^k with an integer $k \geq 0$ to represent the usual any spatial derivatives of order k. The Fourier transform of the function f is denoted by $\mathcal{F}(f) := \hat{f}$. The pseudo-differential operator Λ^s is defined by $\Lambda^s f = \mathcal{F}^{-1}(|\xi|^s \hat{f}(\xi))$. We denote $H^s(\mathbb{R}^3)$ by the s^{th} order Sobolev space with $s \geq 0$. Particularly, when $s=0$, $H^0(\mathbb{R}^3) = L^2(\mathbb{R}^3)$. For the sake of simplicity, we write $\int f dx := \int_{\mathbb{R}^3} f dx$ and $\|(A,B)\|_X := \|A\|_X + \|B\|_X$.

First of all, we recall the main results obtained in [28], [33] in the following.

Theorem 1.1 ([28] & [33]). *Suppose that the initial data* $(\rho_0 - 1, u_0, B_0) \in H^5 \times H^4 \times H^4$*, there exists a small constant* $\delta > 0$ *such that if*

$$\|\rho_0 - 1\|_{H^5} + \|u_0\|_{H^4} + \|B_0\|_{H^4} \leq \delta, \tag{1.5}$$

then the solution (ρ, u, B) *of* (1.1)-(1.3) *satisfy for all* $t \geq 0$

$$\begin{aligned}
&\|(\rho-1, u, B)(t)\|_{H^4}^2 + \|h\nabla\rho(t)\|_{H^4}^2 + \int_0^t \|\nabla(u, B, h\rho)(s)\|_{H^4}^2 ds \\
\leq & C(\|\rho_0 - 1\|_{H^5}^2 + \|u_0\|_{H^4}^2 + \|B_0\|_{H^4}^2).
\end{aligned} \tag{1.6}$$

Moreover, provided that $\|(\rho_0 - 1, u_0, B_0)\|_{L^1}$ *is finite additionally, then the solution* (ρ, u, B) *satisfy*

$$\|\nabla^k(\rho-1)(t)\|_{H^{5-k}} + \|\nabla^k u(t)\|_{H^{4-k}} + \|\nabla^k B(t)\|_{H^{4-k}} \leq C(1+t)^{-\frac{3+2k}{4}}, \tag{1.7}$$

with $k = 0, 1, 2, 3$*, here the positive constant* C *is independent of time.*

For the sake of simplicity, we only establish the lower bound of time decay rates for global solution under the $H^5 \times H^4 \times H^4$ framework. Our main results are stated in the following theorems:

Theorem 1.2. *Let* $\varrho_0 := \rho_0 - 1$ *and* $m_0 := \rho_0 u_0$*. Suppose that the Fourier transform of the functions* (ϱ_0, m_0, B_0) *satisfy*

$$|\widehat{\varrho}_0| \geq c_0, \ \widehat{m}_0 = 0, |\widehat{B}_0| \geq c_0, \ 0 \leq |\xi| \ll 1, \tag{1.8}$$

with c_0 *a positive constant. Then, the global solution* (ρ, u, B) *obtained in Theorem 1.1 has the decay rates for all* $t \geq t_*$

$$c_1(1+t)^{-\frac{3+2k}{4}} \leq \|\nabla^k(\rho-1)(t)\|_{L^2} \leq C_1(1+t)^{-\frac{3+2k}{4}}, k = 0, 1, 2, 3; \tag{1.9}$$

$$c_1(1+t)^{-\frac{3+2k}{4}} \leq \|\nabla^k u(t)\|_{L^2} \leq C_1(1+t)^{-\frac{3+2k}{4}}, k = 0, 1, 2, 3; \tag{1.10}$$

$$c_1(1+t)^{-\frac{3+2k}{4}} \leq \|\nabla^k B(t)\|_{L^2} \leq C_1(1+t)^{-\frac{3+2k}{4}}, k = 0, 1, 2, 3. \tag{1.11}$$

Here t_* *is a positive large time, two positive constants* c_1 *and* C_1 *are independent of time.*

Remark 1.1. *To our best knowledge, there was no the result about the lower bounds of decay rates* (1.9)-(1.11) *for the derivatives of density, velocity and magnetic field to the compressible viscous quantum magnetohydrodynamic model* (1.1) *before. Thus in this paper, this result was obtained for the first time.*

Remark 1.2. *Even though we only establish the time decay rates under the* $H^5 \times H^4 \times H^4$ *framework in Theorem 1.2, the method we used here can actually be applied to the* $H^{N+2} \times H^{N+1} \times H^{N+1} (N \geq 3)$ *framework. If the condition* (1.8) *holds, the global classical solution* (ρ, u, B) *of the system* (1.1) *has the decay rates for all* $t \geq t_*$

$$\begin{gathered}
c_1(1+t)^{-\frac{3+2k}{4}} \leq \|\nabla^k(\rho-1)(t)\|_{L^2} \leq C_1(1+t)^{-\frac{3+2k}{4}}, k \in [0, N]; \\
c_1(1+t)^{-\frac{3+2k}{4}} \leq \|\nabla^k u(t)\|_{L^2} \leq C_1(1+t)^{-\frac{3+2k}{4}}, k \in [0, N]; \\
c_1(1+t)^{-\frac{3+2k}{4}} \leq \|\nabla^k B(t)\|_{L^2} \leq C_1(1+t)^{-\frac{3+2k}{4}}, k \in [0, N].
\end{gathered}$$

Here t_* *is a positive large time, two positive constants* c_1 *and* C_1 *are independent of time.*

Next, we will establish the lower bound of decay rate for the time derivatives of solution to the compressible viscous quantum magnetohydrodynamic model (1.1).

Theorem 1.3. *If the condition* (1.8) *holds, the global classical solution* (ρ, u, B) *obtained in Theorem 1.1 satisfy for all* $t \geq t_*$

$$\begin{aligned} c_2(1+t)^{-\frac{5}{4}} \leq \|\partial_t u(t)\|_{L^2} \leq C_2(1+t)^{-\frac{5}{4}}; \\ c_2(1+t)^{-\frac{7}{4}} \leq \|\partial_t B(t)\|_{L^2} \leq C_2(1+t)^{-\frac{7}{4}}. \end{aligned} \tag{1.12}$$

Moreover, if the velocity u *satisfies* $\|u_0\|_{L^1} \leq \delta_1$ *with* δ_1 *a small constant, it holds on for all* $t \geq t_*$

$$c_2(1+t)^{-\frac{5}{4}} \leq \|\partial_t \rho(t)\|_{L^2} \leq C_2(1+t)^{-\frac{5}{4}}. \tag{1.13}$$

Here t_* *is a positive large time,* c_2 *and* C_2 *are two positive constants independent of time.*

Remark 1.3. *To the best knowledge of the authors', there has been no result of lower bounds of decay rates for the time derivatives of density, velocity and magnetic field for the compressible viscous quantum magnetohydrodynamic model in the* L^2 *norm. That is to say, this result was obtained for the first time.*

Finally, we state two inequalities which play an important role in energy estimates. The first inequality here is called Sobolev interpolation of the Gagliardo-Nirenberg inequality, see [23].

Lemma 1.4. *Let* $0 \leq m, \alpha \leq l$ *and the function* $f \in C_0^\infty(\mathbb{R}^n)$*, then we have*

$$\|\nabla^\alpha f\|_{L^p} \leq C\|\nabla^m f\|_{L^2}^{1-\theta}\|\nabla^l f\|_{L^2}^{\theta}, \tag{1.14}$$

where θ *satisfies*

$$0 \leq \theta \leq 1,$$

and α, m, l *satisfy*

$$\frac{1}{p} - \frac{\alpha}{n} = (\frac{1}{2} - \frac{m}{n})(1-\theta) + (\frac{1}{2} - \frac{l}{n})\theta.$$

The following inequality has been shown in [31].

Lemma 1.5. *Suppose that* $\|\varrho\|_{L^\infty} \leq 1$. *Let* $f(\varrho)$ *be a smooth function of* ϱ *and its derivatives of any order are bounded, then for any integer* $l \geq 1$*, it holds on*

$$\left\|\nabla^l f(\varrho)\right\|_{L^\infty} \leq C\left\|\nabla^l \varrho\right\|_{L^\infty}. \tag{1.15}$$

Now we make comments on the analysis in this paper. We firstly give the lower bound of decay rate for the higher order spatial derivative of solution to the compressible quantum magnetohydrodynamic model (1.1). In order to solve this problem, we consider the difference between the lower bound of decay rate for the solution of the linearized part and the upper bound of decay rate for the difference between the solution of nonlinear and linearized problem. It is easy to obtain the lower bound of decay rate for the linearized part by applying the spectral analysis to the semigroup for the linearized quantum magnetohydrodynamic model (2.4), see Proposition 2.1 in Section 2. Therefore, it is significant to obtain upper bound of decay rate of the k^{th} order derivative of the difference between the solution of nonlinear and linearized problem with $k \geq 0$. To achieve this goal, we establish the energy estimate first of all, see (2.9) and (2.10) in Lemma 2.2. We notice that under the $H^5 \times H^4 \times H^4$ framework, Xi et al.[33] deduced the following inequality

$$\frac{d}{dt}\mathcal{F}_l^m(t) + C(\|\nabla^{l+1}\varrho\|^2_{H^{m+1-l}} + \|\nabla^{l+1}u\|^2_{H^{m-l}} + \|\nabla^{l+1}B\|^2_{H^{m-l}}) \leq 0, \tag{1.16}$$

for $0 \leq l \leq m$, $m \leq 4$, where

$$\mathcal{F}_l^m(t) := \|\nabla^l\varrho\|^2_{H^{m+1-l}} + \|\nabla^l u\|^2_{H^{m-l}} + \|\nabla^l B\|^2_{H^{m-l}} + \delta\sum_{k=l}^{m}\int \nabla^k u \cdot \nabla^{k+1}\varrho dx,$$

and δ a small positive constant. By combining Duhamel principle formula, the upper bound decay estimate (1.7) and the Fourier Splitting method developed by Schonbek [30], we could complete our proof.

Next, we use the second and third equation of (2.31) and the lower bound of first order spatial derivative to obtain the upper and lower bounds of decay rate for the time derivative of velocity and magnetic field. By combining transport equation and the assumption that the L^1 norm of initial velocity is small enough(i.e.,$\|u_0\|_{L^1} < \delta$ with δ small enough), the upper and lower bounds of decay rates for the time derivative of density are obtained.

The rest of this paper is organized as follows. Section 2 is devoted to establishing the lower bound of decay rate for the solution itself and its derivatives, then we establish the lower bound of decay rate for the time derivative of solution. Section 3 is devoted to proving technical estimates used in Section 2.

2 Lower Bounds of Decay for Spatial Derivative

This section is concerned with the lower bound of decay rate for the solution itself and its derivative. In order to achieve this goal, we establish the upper decay rate for the difference between the nonlinear and linearized parts. Then, we address the upper decay rate of solution for the higher order spatial derivative.

2.1. Lower Bounds of Decay for Spatial Derivative

In this subsection, we will establish optimal time decay rates of solution for the compressible quantum magnetohydrodynamic model (1.1)-(1.3). Without loss of generality, we assume that $P'(1) = 1$. Let us denote $\varrho := \rho - 1, m := \rho u$, then we rewrite (1.1) in the perturbation form as

$$\begin{cases} \partial_t\varrho + \mathrm{div}m = 0, \\ \partial_t m - \mu\Delta m - (\mu+\lambda)\nabla\mathrm{div}m + \nabla\varrho - \dfrac{h^2}{4}\nabla\Delta\varrho = -\mathrm{div}S_1, \\ \partial_t B - \nu\Delta B = \nabla\times S_2,\ \mathrm{div}B = 0, \end{cases} \tag{2.1}$$

where the functions $S_1 = S_1(\varrho, u, B)$ and $S_2 = S_2(u, B)$ are defined as

$$\begin{cases} S_1 =(\varrho+1)u\otimes u + \mu\nabla(\varrho u) + (\mu+\lambda)\mathrm{div}(\varrho u)\mathbb{I}_{3\times3} \\ \qquad + (P(1+\varrho) - P(1) - \varrho)\mathbb{I}_{3\times3} + \dfrac{1}{2}|B|^2 I_{3\times3} - B\otimes B \\ \qquad + h^2(\nabla\sqrt{1+\varrho}\otimes\nabla\sqrt{1+\varrho})\ ; \\ S_2 =u\times B. \end{cases} \tag{2.2}$$

The initial data is given as

$$(\varrho, m, B)(x,t)|_{t=0} = (\varrho_0, m_0, B_0)(x) \to (0,0,0) \quad \text{as} \quad |x| \to \infty. \tag{2.3}$$

In order to obtain the lower decay estimate, it is sufficient to analysis the linearized part:

$$\begin{cases} \partial_t \varrho_l + \mathrm{div} m_l = 0, \\ \partial_t m_l - \mu \Delta m_l - (\mu+\lambda)\nabla \mathrm{div} m_l + \nabla \varrho_l - \dfrac{h^2}{4}\nabla\Delta\varrho_l = 0, \\ \partial_t B_l - \nu \Delta B_l = 0, \ \mathrm{div} B_l = 0, \end{cases} \tag{2.4}$$

with the initial data

$$(\varrho_l, m_l, B_l)(x,t)|_{t=0} = (\varrho_0, m_0, B_0)(x) \to (0,0,0) \quad \text{as} \quad |x| \to \infty. \tag{2.5}$$

Notice that the first and second equation of linearized part (2.4) are similar to the linearized system of compressible fluid models of Korteweg type in [8]. We still use the method in [8] to obtain the lower bound of decay rate of solution to the system (2.4)-(2.5), which is stated as follows. For the sake of simplicity, we omit the proof here.

Proposition 2.1. *Let $\varrho_0 \in H^5(\mathbb{R}^3)\bigcap L^1(\mathbb{R}^3)$, $m_0 \in H^4(\mathbb{R}^3)\bigcap L^1(\mathbb{R}^3)$ and $B_0 \in H^4(\mathbb{R}^3)\bigcap L^1(\mathbb{R}^3)$. Assume that the Fourier transform $\mathcal{F}(\varrho_0, m_0, B_0) = (\widehat{\varrho}_0, \widehat{m}_0, \widehat{B}_0)$ satisfies $|\widehat{\varrho}_0| \ge c_0,\ |\widehat{m}_0| = 0, |\widehat{B}_0| \ge c_0,\ 0 \le |\xi| \ll 1$ with c_0 a positive constant, then we have for $k = 0,1,2,3$, it holds on*

$$\begin{aligned} \|\nabla^k \varrho_l(t)\|_{L^2} &\ge c(1+t)^{-(\frac{3}{4}+\frac{k}{2})}; \\ \|\nabla^k m_l(t)\|_{L^2} &\ge c(1+t)^{-(\frac{3}{4}+\frac{k}{2})}; \\ \|\nabla^k B_l(t)\|_{L^2} &\ge c(1+t)^{-(\frac{3}{4}+\frac{k}{2})}, \end{aligned} \tag{2.6}$$

here c is a positive constant that independent of time t.

In the sequel, we give the upper decay rate for the difference between the nonlinear and linearized part so that we can obtain the lower bound for the solution of the compressible quantum magnetohydrodynamic model (2.1). Hence, we denote

$$\varrho_\delta := \varrho - \varrho_l, m_\delta := m - m_l, B_\delta := B - B_l,$$

then the functions $(\varrho_\delta, m_\delta, B_\delta)$ satisfy the following system

$$\begin{cases} \partial_t \varrho_\delta + \mathrm{div} m_\delta = 0, \\ \partial_t m_\delta - \mu \Delta m_\delta - (\mu+\lambda)\nabla \mathrm{div} m_\delta + \nabla \varrho_\delta - \dfrac{h^2}{4}\nabla\Delta\varrho_\delta = -\mathrm{div} S_1, \\ \partial_t B_\delta - \nu \Delta B_\delta = \nabla \times S_2, \ \mathrm{div} B_\delta = 0, \end{cases} \tag{2.7}$$

and the initial data satisfy

$$(\varrho_\delta, m_\delta, B_\delta)(x,t)|_{t=0} = (0,0,0). \tag{2.8}$$

Here the functions S_1 and S_2 are defined in (2.2). Now we will establish the energy estimate of solution $(\varrho_\delta, m_\delta, B_\delta)$ of equation (2.7) in the following.

Lemma 2.2. *For any smooth solution* $(\varrho_\delta, m_\delta)$ *of the equation* (2.7)*, it holds on*

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla^l(\varrho_\delta, m_\delta)\|_{H^{4-l}}^2 + \frac{h^2}{4}\|\nabla^{l+1}\varrho_\delta\|_{H^{4-l}}^2) + \mu\|\nabla^{l+1}m_\delta\|_{H^{4-l}}^2 + (\mu+\lambda)\|\nabla^l \mathrm{div} m_\delta\|_{H^{4-l}}^2 \\
\leq& C\|\nabla^l(\varrho, u, B)\|_{H^{4-l}}^2(\|\nabla\varrho\|_{H^2}^2 + \|\nabla(u,B)\|_{H^1}^2) + C\|\nabla(\varrho, u)\|_{H^1}^2(\|\nabla^5\varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2),
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\sum_{k=l}^{4}\frac{d}{dt}\int \nabla^l m_\delta\cdot\nabla^{l+1}\varrho_\delta dx + \|\nabla^{l+1}\varrho_\delta\|_{H^{4-l}}^2 + \frac{h^2}{4}\|\nabla^{l+2}\varrho_\delta\|_{H^{4-l}}^2 \\
\leq& C\|\nabla^{l+1}m_\delta\|_{H^{4-l}}^2 + C\|\nabla^l(\varrho, u, B)\|_{H^{4-l}}^2(\|\nabla\varrho\|_{H^2}^2 + \|\nabla(u,B)\|_{H^1}^2) \\
&+ C\|\nabla(\varrho, u)\|_{H^1}^2(\|\nabla^5\varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2),
\end{aligned} \tag{2.10}$$

where $l = 0, 1, 2, 3$*,* C *is a positive constant that independent of time.*

The above inequalities (2.9) and (2.10) in Lemma 2.2 will be proved later in Section 3. By multiplying inequality (2.10) by a small constant δ and adding with (2.9), for all $t \geq 0$, it holds on

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_l^4(t) + \delta\|\nabla^{l+1}\varrho_\delta\|_{H^{4-l}}^2 + \frac{h^2}{4}\delta\|\nabla^{l+2}\varrho_\delta\|_{H^{4-l}}^2 + \frac{\mu}{2}\|\nabla^{l+1}m_\delta\|_{H^{4-l}}^2 \\
\leq& C(1+t)^{-(4+l)} + C(1+t)^{-\frac{5}{2}}(\|\nabla^5\varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2),
\end{aligned} \tag{2.11}$$

where $l = 0, 1, 2, 3$. Here the energy $\mathcal{E}_l^4(t)$ is defined by

$$\mathcal{E}_l^4(t) := \|\nabla^l(\varrho_\delta, m_\delta)\|_{H^{4-l}}^2 + \frac{h^2}{4}\|\nabla^{l+1}\varrho_\delta\|_{H^{4-l}}^2 + \delta\sum_{k=l}^{4}\int \nabla^k m_\delta\cdot\nabla^{k+1}\varrho_\delta dx. \tag{2.12}$$

Due to the smallness of δ, there are two constants C_* and C^*(independent of time) such that

$$C_*(\|\nabla^l\varrho_\delta(t)\|_{H^{5-l}}^2 + \|\nabla^l m_\delta(t)\|_{H^{4-l}}^2) \leq \mathcal{E}_l^4(t) \leq C^*(\|\nabla^l\varrho_\delta(t)\|_{H^{5-l}}^2 + \|\nabla^l m_\delta(t)\|_{H^{4-l}}^2). \tag{2.13}$$

Now we establish the upper bound decay rate of solution $(\varrho_\delta, m_\delta)$ for the equation (2.7).

Lemma 2.3. *Suppose that the conditions in Theorem 1.1 hold on, then the smooth solution* $(\varrho_\delta, m_\delta)$ *of equation* (2.7) *satisfy*

$$\|\nabla^l\varrho_\delta(t)\|_{H^{5-l}} + \|\nabla^l m_\delta(t)\|_{H^{4-l}} \leq C(1+t)^{-\frac{5+2l}{4}}, \tag{2.14}$$

where $l = 0, 1, 2, 3$.

Proof. We will take the strategy of induction to prove the estimate (2.14) in the following. Taking $l = 0$ in (2.11), then we have

$$\frac{d}{dt}\mathcal{E}_0^4(t) + C(\|\nabla\varrho_\delta\|_{H^5}^2 + \|\nabla m_\delta\|_{H^4}^2) \leq C(1+t)^{-4} + C(1+t)^{-\frac{5}{2}}(\|\nabla^5\varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2).$$

We notice that the term $\mathcal{E}_0^4(t)$ is equivalent to $\|\varrho_\delta\|_{H^5}^2 + \|m_\delta\|_{H^4}^2$. Due to the fact that the dissipation term $\|\nabla\varrho_\delta\|_{H^5}^2 + \|\nabla m_\delta\|_{H^4}^2$ could not control the energy term $\mathcal{E}_0^4(t)$ in above inequality, we add both sides of the above inequality with term $\|(\varrho_\delta, m_\delta)\|_{L^2}^2$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_0^4(t) + C(\|\varrho_\delta\|_{H^5}^2 + \|m_\delta\|_{H^4}^2) \\
\leq& C\|(\varrho_\delta, m_\delta)\|_{L^2}^2 + C(1+t)^{-4} + C(1+t)^{-\frac{5}{2}}(\|\nabla^5\varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2).
\end{aligned} \tag{2.15}$$

In order to estimate the term $\|(\varrho_\delta, m_\delta)(t)\|_{L^2}$, we utilize the Duhamel principle formula and estimate (1.7) to arrive at

$$\begin{aligned}
\|(\varrho_\delta, m_\delta)(t)\|_{L^2} &\le \int_0^t (1+t-\tau)^{-\frac{5}{4}}(\||\xi|^{-1}\mathcal{F}(\mathrm{div}S_1)\|_{L^\infty} + \|\mathrm{div}S_1\|_{L^2})d\tau \\
&\le \int_0^t (1+t-\tau)^{-\frac{5}{4}}(\|S_1\|_{L^1} + \|\mathrm{div}S_1\|_{L^2})d\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{2}}d\tau \\
&\le C(1+t)^{-\frac{5}{4}},
\end{aligned} \tag{2.16}$$

where we apply Hölder inequality, use decay rate (1.7) and take $k=1$ in (3.4) to obtain

$$\begin{aligned}
&\|S_1\|_{L^1} + \|\mathrm{div}S_1\|_{L^2} \\
\le& C\|(1+\varrho)u\otimes u\|_{L^1} + C\|\nabla(\varrho u)\|_{L^1} + C\|P(1+\varrho)-P(1)-\varrho\|_{L^1} + C\|B\cdot B\|_{L^1} \\
&+ C\|\frac{|\nabla\varrho|^2}{1+\varrho}\|_{L^1} + C\|\nabla S_1\|_{L^2} \\
\le& C\|1+\varrho\|_{L^\infty}\|u\|_{L^2}^2 + C\|\nabla\varrho\|_{L^2}\|u\|_{L^2} + C\|\varrho\|_{L^2}\|\nabla u\|_{L^2} + C\|\varrho\|_{L^2}^2 + C\|B\|_{L^2}^2 \\
&+ C\|\frac{1}{1+\varrho}\|_{L^\infty}\|\nabla\varrho\|_{L^2}^2 + C\|\nabla S_1\|_{L^2} \\
\le& C\|(\varrho,u)\|_{L^2}\|\nabla(\varrho,u)\|_{L^2} + C\|\varrho\|_{H^2}^2 + C\|(u,B)\|_{L^2}^2 + C\|\nabla\varrho\|_{H^1}\|\nabla^3\varrho\|_{L^2} \\
&+ C\|\nabla(\varrho,u,B)\|_{H^1}(\|\nabla\varrho\|_{H^2} + \|\nabla(u,B)\|_{H^1}) \\
\le& C(1+t)^{-\frac{3}{2}}.
\end{aligned}$$

Using (2.15), (2.16) and equivalent relation (2.13), one arrives at

$$\frac{d}{dt}\mathcal{E}_0^4(t) + \frac{C}{C^*}\mathcal{E}_0^4(t) \le C(1+t)^{-\frac{5}{2}} + C(1+t)^{-4} + C(1+t)^{-\frac{5}{2}}(\|\nabla^5\varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2),$$

then we can get

$$\mathcal{E}_0^4(t) \le C\int_0^t e^{-\frac{C}{C^*}(t-\tau)}(1+\tau)^{-\frac{5}{2}}d\tau + C\int_0^t e^{-\frac{C}{C^*}(t-\tau)}(1+\tau)^{-\frac{5}{2}}(\|\nabla^5\varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2)d\tau.$$

It is easy to obtain that

$$\int_0^t e^{-\frac{C}{C^*}(t-\tau)}(1+\tau)^{-\frac{5}{2}}d\tau \le C(1+t)^{-\frac{5}{2}},$$

one can refer to [8] for the detail. In the sequel, we only need to deal with the term

$$\int_0^t e^{-\frac{C}{C^*}(t-\tau)}(1+\tau)^{-\frac{5}{2}}(\|\nabla^5\varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2)d\tau.$$

We claim the following estimate(which will be proved in Section 3),

$$\int_0^t e^{-\frac{C}{C^*}(t-\tau)}(1+\tau)^{-\frac{5}{2}}(\|\nabla^5\varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2)d\tau \le C(1+t)^{-\frac{5}{2}}. \tag{2.17}$$

Then we can easily obtain that

$$\|\varrho_\delta(t)\|_{H^5}^2 + \|m_\delta(t)\|_{H^4}^2 \le C(1+t)^{-\frac{5}{2}}.$$

We now assume that the decay rate (2.14) holds on for the case $k = l$, i.e.,

$$\|\nabla^l \varrho_\delta(t)\|_{H^{5-l}} + \|\nabla^l m_\delta(t)\|_{H^{4-l}} \le C(1+t)^{-\frac{5+2l}{4}}, \tag{2.18}$$

for $l = 1, 2$. Then, we should verify that the estimate (2.14) holds on for the case $k = l+1$. Indeed, we replace l by $l+1$ in (2.11) to obtain that

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}^4_{l+1}(t) + C(\|\nabla^{l+2}\varrho_\delta\|^2_{H^{4-l}} + \|\nabla^{l+2} m_\delta\|^2_{H^{3-l}})\\ \le& C(1+t)^{-(5+l)} + C(1+t)^{-\frac{5}{2}}(\|\nabla^5\varrho\|^2_{H^1} + \|\nabla^5 u\|^2_{L^2}).\end{aligned} \tag{2.19}$$

For some constant R that will be defined below, let us denote the time sphere(see [30])

$$S_0 := \left\{\xi \in \mathbb{R}^3 \big| \, |\xi| \le \left(\frac{R}{1+t}\right)^{\frac{1}{2}}\right\},$$

it follows immediately

$$\begin{aligned}&\|\nabla^{l+2}\varrho_\delta\|^2_{H^{4-l}} \ge \frac{R}{1+t}\|\nabla^{l+1}\varrho_\delta\|^2_{H^{4-l}} - \frac{R^2}{(1+t)^2}\|\nabla^l\varrho_\delta\|^2_{H^{4-l}};\\ &\|\nabla^{l+2} m_\delta\|^2_{H^{3-l}} \ge \frac{R}{1+t}\|\nabla^{l+1} m_\delta\|^2_{H^{3-l}} - \frac{R^2}{(1+t)^2}\|\nabla^l m_\delta\|^2_{H^{3-l}}.\end{aligned} \tag{2.20}$$

By substituting (2.20) into (2.19), we can easily get

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}^4_{l+1}(t) + \frac{CR}{1+t}(\|\nabla^{l+1}\varrho_\delta\|^2_{H^{4-l}} + \|\nabla^{l+1} m_\delta\|^2_{H^{3-l}})\\ \le& \frac{CR^2}{(1+t)^2}(\|\nabla^l\varrho_\delta\|^2_{H^{4-l}} + \|\nabla^l m_\delta\|^2_{H^{3-l}}) + C(1+t)^{-(5+l)} + C(1+t)^{-\frac{5}{2}}(\|\nabla^5\varrho\|^2_{H^1} + \|\nabla^5 u\|^2_{L^2})\\ \le& CR^2(1+t)^{-\frac{9+2l}{2}} + C(1+t)^{-(5+l)} + C(1+t)^{-\frac{5}{2}}(\|\nabla^5\varrho\|^2_{H^1} + \|\nabla^5 u\|^2_{L^2}),\end{aligned}$$

where we have used the assumption (2.18). Notice that the term $\mathcal{E}^4_{l+1}(t)$ is equivalent to the norm $\|\nabla^{l+1}\varrho_\delta\|^2_{H^{4-l}} + \|\nabla^{l+1} m_\delta\|^2_{H^{3-l}}$, hence, we obtain

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}^4_{l+1}(t) + \frac{CR}{C^*(1+t)}\mathcal{E}^4_{l+1}(t)\\ \le& CR^2(1+t)^{-\frac{9+2l}{2}} + C(1+t)^{-(5+l)} + C(1+t)^{-\frac{5}{2}}(\|\nabla^5\varrho\|^2_{H^1} + \|\nabla^5 u\|^2_{L^2}).\end{aligned}$$

We choose $R = C^*(l+4)/C$ and then multiply the above inequality by $(1+t)^{l+4}$ to obtain that

$$\frac{d}{dt}[(1+t)^{l+4}\mathcal{E}^4_{l+1}(t)] \le C(1+t)^{-\frac{1}{2}} + C(1+t)^{l+\frac{3}{2}}(\|\nabla^5\varrho\|^2_{H^1} + \|\nabla^5 u\|^2_{L^2}). \tag{2.21}$$

We claim that the following estimate holds on(which will be proved in Section 3),

$$\int_0^t (1+\tau)^{l+\frac{3}{2}}(\|\nabla^5\varrho\|^2_{H^1} + \|\nabla^5 u\|^2_{L^2})d\tau \le C, \tag{2.22}$$

which together with (2.21) yield directly

$$\mathcal{E}^4_{l+1}(t) \le C(1+t)^{-\frac{7+2l}{2}}.$$

Then, due to the fact that the term $\mathcal{E}^4_{l+1}(t)$ is equivalent to the norm $\|\nabla^{l+1}\varrho_\delta\|^2_{H^{4-l}}+\|\nabla^{l+1}m_\delta\|^2_{H^{3-l}}$, by integrating the inequality (2.21) about time over $[0,t]$, we obtain

$$\|\nabla^{l+1}\varrho_\delta\|^2_{H^{4-l}}+\|\nabla^{l+1}m_\delta\|^2_{H^{3-l}}\le C(1+t)^{-\frac{7+2l}{2}}.$$

Thus, by the general step of induction, we have given the proof for (2.14). □

Next, we give the energy estimate for the the magnetic field, we claim the following inequality holds on(which will be proved in Section 3).

Lemma 2.4. *For any smooth solution* $(\varrho_\delta, m_\delta, B_\delta)$ *of the equation* (2.7)*, it holds on*

$$\frac{d}{dt}\int|\nabla^l B_\delta|^2dx+\nu\int|\nabla^{l+1}B_\delta|^2dx\le C\|(u,B)\|^2_{L^\infty}\|\nabla^l(u,B)\|^2_{L^2}, \tag{2.23}$$

for $l=0,1,2,3$*.*

Now, we will establish the upper decay rate for the difference of magnetic field between the nonlinear and linearized parts in what follows.

Lemma 2.5. *Under the conditions of Theorem 1.1, the smooth solution* B_δ *of equation* (2.7) *satisfies*

$$\|\nabla^l B_\delta(t)\|_{L^2}\le C(1+t)^{-\frac{5+2l}{4}}, \tag{2.24}$$

where $l=0,1,2,3$*.*

Proof. We will take the strategy of induction to prove the estimate (2.24) in the following. Similar with the estimate for the term $\|B_\delta(t)\|_{L^2}$ obtained in [7], with the help of the Duhamel principle formula and estimate (1.7), we have

$$\begin{aligned}\|B_\delta(t)\|_{L^2}&\le\int_0^t(1+t-\tau)^{-\frac54}(\||\xi|^{-1}\mathcal{F}(\nabla\times S_2)\|_{L^\infty}+\|\nabla S_2\|_{L^2})d\tau\\&\le\int_0^t(1+t-\tau)^{-\frac54}(\|S_2\|_{L^1}+\|\nabla S_2\|_{L^2})d\tau\\&\le C\int_0^t(1+t-\tau)^{-\frac54}(1+t)^{-\frac32}d\tau\\&\le C(1+t)^{-\frac54},\end{aligned}\tag{2.25}$$

where we have used the basic fact that

$$\begin{aligned}&\|S_2\|_{L^1}+\|\nabla S_2\|_{L^2}\\\le&C\|u\times B\|_{L^1}+C\|\nabla(u\times B)\|_{L^2}\\\le&C\|u\|_{L^2}\|B\|_{L^2}+C\|\nabla u\cdot B\|_{L^2}+C\|u\cdot\nabla B\|_{L^2}\\\le&C\|u\|_{L^2}\|B\|_{L^2}+C\|\nabla u\|_{L^3}\|B\|_{L^6}+C\|u\|_{L^6}\|\nabla B\|_{L^3}\\\le&C\|u\|_{H^2}\|B\|_{H^2}\\\le&C(1+t)^{-\frac32}.\end{aligned}$$

We now assume that the decay rate (2.24) holds on for the case $k=l$, i.e.,

$$\|\nabla^l B_\delta(t)\|_{L^2}\le C(1+t)^{-\frac{5+2l}{4}}, \tag{2.26}$$

for $l=1,2$. Then, we verify that the estimate (2.24) holds on for the case $k=l+1$. Replacing l by $l+1$ in (2.23), we can easily obtain that

$$\frac{d}{dt}\int|\nabla^{l+1}B_\delta|^2dx+\nu\int|\nabla^{l+2}B_\delta|^2dx\leq C\|(u,B)\|_{L^\infty}^2\|\nabla^{l+1}(u,B)\|_{L^2}^2\leq C(1+t)^{-(5+l)}, \tag{2.27}$$

where we used the decay estimate (1.7). For some constant R that will be defined below, let us denote the time sphere(see [30])

$$S_0:=\left\{\xi\in\mathbb{R}^3\big|\,|\xi|\leq\left(\frac{R}{1+t}\right)^{\frac{1}{2}}\right\},$$

it follows immediately

$$\|\nabla^{l+2}B_\delta\|_{L^2}^2\geq\frac{R}{1+t}\|\nabla^{l+1}B_\delta\|_{L^2}^2-\frac{R^2}{(1+t)^2}\|\nabla^{l}B_\delta\|_{L^2}^2. \tag{2.28}$$

Substituting (2.28) into inequality (2.27), which together with (2.26), we can get

$$\begin{aligned}&\frac{d}{dt}\|\nabla^{l+1}B_\delta\|_{L^2}^2+\frac{\nu R}{1+t}\|\nabla^{l+1}B_\delta\|_{L^2}^2\\ \leq&\frac{CR^2}{(1+t)^2}\|\nabla^{l}B_\delta\|_{L^2}^2+C(1+t)^{-(5+l)}\\ \leq&C(1+t)^{-(\frac{9}{2}+l)},\end{aligned} \tag{2.29}$$

where we have used the assumption (2.26). By choosing $R=(l+4)/\nu$ and multiplying the above inequality by $(1+t)^{l+4}$, one arrives at

$$\frac{d}{dt}[(1+t)^{l+4}\|\nabla^{l+1}B_\delta\|_{L^2}^2]\leq C(1+t)^{-\frac{1}{2}},$$

then by integrating the above inequality about time over $[0,t]$, we obtain

$$\|\nabla^{l+1}B_\delta\|_{L^2}^2\leq C(1+t)^{-(\frac{7}{2}+l)}.$$

Hence, we have verified that (2.24) holds on for the case $k=l+1$. By the general step of induction, we complete the proof of the lemma. □

Finally, we establish the lower bound estimates.

Lemma 2.6. *Under the conditions of Theorem 1.2, the solution* (ρ,u,B) *of equation* (1.1) *have the following estimates*

$$\begin{aligned}&\|\nabla^k(\rho-1)(t)\|_{L^2}\geq C(1+t)^{-\frac{3+2k}{4}},\ k=0,1,2,3;\\ &\|\nabla^k u(t)\|_{L^2}\geq C(1+t)^{-\frac{3+2k}{4}},\ k=0,1,2,3;\\ &\|\nabla^k B(t)\|_{L^2}\geq C(1+t)^{-\frac{3+2k}{4}},\ k=0,1,2,3,\end{aligned} \tag{2.30}$$

for all $t\geq T_*$, *here* T_* *is a positive constant.*

Proof. Remember the definition

$$\varrho_\delta := \varrho - \varrho_l,\ m_\delta := m - m_l,\ B_\delta := B - B_l.$$

By virtue of triangle inequality, we have

$$\|\varrho_l\|_{L^2} = \|\varrho - \varrho_\delta\|_{L^2} \le \|\varrho\|_{L^2} + \|\varrho_\delta\|_{L^2},$$

which, together with estimates (2.6) and (2.14), yield directly

$$\begin{aligned}\|\varrho(t)\|_{L^2} &\ge \|\varrho_l(t)\|_{L^2} - \|\varrho_\delta(t)\|_{L^2}\\ &\ge C_1(1+t)^{-\frac{3}{4}} - C_2(1+t)^{-\frac{5}{4}}\\ &\ge C(1+t)^{-\frac{3}{4}},\end{aligned}$$

here C_1, C_2 and C are positive constants independent of time. Similarly, using estimates (2.6) and (2.14), together with triangle inequality, we also have

$$\begin{aligned}&\|\nabla^k\varrho(t)\|_{L^2} \ge C(1+t)^{-\frac{3+2k}{4}}, k=1,2,3;\\ &\|\nabla^k m(t)\|_{L^2} \ge C(1+t)^{-\frac{3+2k}{4}}, k=0,1,2,3;\\ &\|\nabla^k B(t)\|_{L^2} \ge C(1+t)^{-\frac{3+2k}{4}}, k=0,1,2,3.\end{aligned}$$

Finally, we establish the lower bound decay rate for the velocity u. We use the decay rate (1.7) and apply Sobolev's inequality to get

$$\begin{aligned}\|\nabla^k m\|_{L^2} &\le \|\nabla^k u\|_{L^2} + \|\nabla^k(\varrho u)\|_{L^2}\\ &\le \|\nabla^k u\|_{L^2} + \|\varrho\|_{L^\infty}\|\nabla^k u\|_{L^2} + \|u\|_{L^\infty}\|\nabla^k\varrho\|_{L^2}\\ &\le \|\nabla^k u\|_{L^2} + C(1+t)^{-\frac{9+2k}{4}},\end{aligned}$$

which, together with (2.6), yield directly

$$\|\nabla^k u\|_{L^2} \ge \|\nabla^k m\|_{L^2} - C(1+t)^{-\frac{9+2k}{4}} \ge C(1+t)^{-\frac{3+2k}{4}} - C(1+t)^{-\frac{9+2k}{4}} \ge C(1+t)^{-\frac{3+2k}{4}}.$$

Therefore, we complete the proof of lemma. □

2.2. Upper and Lower Bounds of Decay for Time Derivative

This subsection is devoted to establishing the lower bound for the time derivative of density, velocity and magnetic field. Denoting $\varrho := \rho - 1$, Xi et al.[33] have rewritten (1.1) in the perturbation form as

$$\begin{cases}\partial_t\varrho + \mathrm{div}u = G_1,\\ \partial_t u - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}u + \nabla\varrho - \dfrac{h^2}{4}\nabla\Delta\varrho = G_2,\\ \partial_t B - \nu\Delta B = G_3,\end{cases} \tag{2.31}$$

where the function $G_i (i = 1, 2, 3)$ is defined as

$$\begin{cases}G_1 = -\varrho\mathrm{div}u - u\cdot\nabla\varrho,\\ G_2 = -u\cdot\nabla u - \dfrac{\varrho}{\varrho+1}(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u) - (\dfrac{P'(\varrho+1)}{\varrho+1} - 1)\nabla\varrho - \dfrac{h^2}{4}\dfrac{\varrho}{\varrho+1}\nabla\Delta\varrho\\ \qquad + \dfrac{h^2}{4}(\dfrac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3} - \dfrac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2} - \dfrac{\nabla\varrho\cdot\nabla^2\varrho}{(1+\varrho)^2}) + \dfrac{1}{1+\varrho}((\nabla\times B)\times B),\\ G_3 = \nabla\times(u\times B).\end{cases}$$

The initial data are given as

$$(\varrho, u, B)(x,t)|_{t=0} = (\varrho_0, u_0, B_0)(x) \to (0,0,0) \quad \text{as} \quad |x| \to \infty. \tag{2.32}$$

Now, we establish the upper and lower bound decay rates for the time derivative of solution (ρ, u, B) in the L^2 norm. The lower bound decay rate for the time derivative of density, velocity and magnetic field can be obtained by using the method in [7, 8]. Nevertheless, we still give the estimate in detail due to the appearance of the quantum potential and magnetic field simultaneously.

Lemma 2.7. *Under all the assumptions of Theorem 1.2, the global solution (ϱ, u, B) of equation* (2.31) *has the following estimate*

$$\begin{aligned} C(1+t)^{-\frac{5}{4}} &\le \|\partial_t \varrho(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}};\\ C(1+t)^{-\frac{5}{4}} &\le \|\partial_t u(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}};\\ C(1+t)^{-\frac{7}{4}} &\le \|\partial_t B(t)\|_{L^2} \le C(1+t)^{-\frac{7}{4}}, \end{aligned} \tag{2.33}$$

for all $t \ge T_*$. Here C is a positive constant independent of time.

Proof. At first, we establish upper bound of time decay rate for $\partial_t \varrho$, $\partial_t u$ and $\partial_t B$ in the L^2 norm. With the help of the equation (2.31), we can easily obtain

$$\|\partial_t \varrho\|_{L^2} \le C\|\mathrm{div} u\|_{L^2} + \|G_1\|_{L^2},$$

$$\|\partial_t u\|_{L^2} \le C\|\Delta u\|_{L^2} + C\|\nabla \mathrm{div} u\|_{L^2} + C\|\nabla \varrho\|_{L^2} + C\|\nabla\Delta\varrho\|_{L^2} + \|G_2\|_{L^2},$$

and

$$\|\partial_t B\|_{L^2} \le C\|\Delta B\|_{L^2} + \|G_3\|_{L^2}.$$

By virtue of Sobolev's inequality and time decay rate (1.7), we have

$$\begin{aligned} \|G_1\|_{L^2} &\le C\|\varrho \cdot \mathrm{div} u\|_{L^2} + C\|u \cdot \nabla \varrho\|_{L^2}\\ &\le C\|\varrho\|_{L^\infty}\|\mathrm{div} u\|_{L^2} + C\|u\|_{L^\infty}\|\nabla\varrho\|_{L^2}\\ &\le C\|(\varrho, u)\|_{L^\infty}\|\nabla(\varrho, u)\|_{L^2}\\ &\le C(1+t)^{-\frac{5}{2}}, \end{aligned}$$

$$\begin{aligned} \|G_2\|_{L^2} \le& C\|u\cdot\nabla u\|_{L^2} + C\|\frac{\varrho}{1+\varrho}\nabla^2 u\|_{L^2} + C\|(\frac{P'(1+\varrho)}{1+\varrho} - 1)\nabla\varrho\|_{L^2} + C\|\frac{\varrho}{1+\varrho}\nabla\Delta\varrho\|_{L^2}\\ &+ C\|\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3}\|_{L^2} + C\|\frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2}\|_{L^2} + C\|\frac{\nabla\varrho\nabla^2\varrho}{(1+\varrho)^2}\|_{L^2} + C\|\frac{1}{1+\varrho}(\nabla\times B)\times B\|_{L^2}\\ \le& C\|(\varrho,u)\|_{L^\infty}\|\nabla(\varrho,u)\|_{L^2} + C\|\varrho\|_{L^\infty}\|\nabla^2 u\|_{L^2} + C\|\varrho\|_{L^\infty}\|\nabla^3\varrho\|_{L^2} + C\|\nabla\varrho\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}\\ &+ C\|B\|_{L^\infty}\|\nabla B\|_{L^2}\\ \le& C(1+t)^{-\frac{5}{2}}, \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} \|G_3\|_{L^2} \le& C\|\nabla\times(u\times B)\|_{L^2}\\ \le& C\|\nabla u\cdot B\|_{L^2} + C\|B\cdot\nabla u\|_{L^2}\\ \le& C\|(u,B)\|_{L^\infty}\|\nabla(u,B)\|_{L^2}\\ \le& C(1+t)^{-\frac{5}{2}}. \end{aligned}$$

Then, we can easily derive that

$$\|\partial_t \varrho\|_{L^2} \le C(1+t)^{-\frac{5}{4}},$$

$$\|\partial_t u\|_{L^2} \le C(1+t)^{-\frac{5}{4}},$$

and

$$\|\partial_t B\|_{L^2} \le C(1+t)^{-\frac{7}{4}}.$$

Next, we establish lower bound time decay rate for $\partial_t u$ and $\partial_t B$ in the L^2 norm. Using the second equation in (2.31), we have

$$\|\nabla \varrho\|_{L^2} \le \|\partial_t u\|_{L^2} + C\|\nabla^2 u\|_{L^2} + C\|\nabla \Delta \varrho\|_{L^2} + \|G_2\|_{L^2}.$$

And hence, we get

$$\begin{aligned}\|\partial_t u\|_{L^2} &\ge \|\nabla \varrho\|_{L^2} - C\|\nabla^2 u\|_{L^2} - C\|\nabla \Delta \varrho\|_{L^2} - \|G_2\|_{L^2} \\ &\ge C(1+t)^{-\frac{5}{4}} - C(1+t)^{-\frac{7}{4}} - C(1+t)^{-\frac{9}{4}} - C(1+t)^{-\frac{5}{2}} \\ &\ge C(1+t)^{-\frac{5}{4}},\end{aligned} \tag{2.35}$$

for some large time t. Using the third equation in (2.31), we have

$$\|\Delta B\|_{L^2} \le C\|\partial_t B\|_{L^2} + C\|G_3\|_{L^2}.$$

Then we have for some large time t,

$$\begin{aligned}\|\partial_t B\|_{L^2} &\ge C\|\Delta B\|_{L^2} - C\|G_3\|_{L^2} \\ &\ge C(1+t)^{-\frac{7}{4}} - C(1+t)^{-\frac{5}{2}} \\ &\ge C(1+t)^{-\frac{5}{2}}.\end{aligned} \tag{2.36}$$

Finally, we establish lower bound time decay rate for $\partial_t \varrho$ in the L^2 norm. In order to achieve this target, we use the transport equation in equation (2.31) to obtain

$$\|\text{div} u\|_{L^2} \le \|\partial_t \varrho\|_{L^2} + \|G_1\|_{L^2}.$$

Hence, we obtain

$$\|\partial_t \varrho\|_{L^2} \ge \|\text{div} u\|_{L^2} - C(1+t)^{-\frac{11}{4}}. \tag{2.37}$$

Now, we need to establish the lower bound decay rate for $\|\text{div} u\|_{L^2}$. We can get

$$\|\text{div} u\|_{L^2} \ge C\|\nabla u\|_{L^2} - C\|\nabla \times u\|_{L^2} \ge C(1+t)^{-\frac{5}{4}} - C\|\nabla \times u\|_{L^2}, \tag{2.38}$$

due to the differential relation $\Delta = \nabla \text{div} - \nabla \times \nabla \times$. It is sufficient to establish upper bound decay rate for $\|\nabla \times u\|_{L^2}$. To achieve this target, we apply the operator $\nabla\times$ to the second equation in (2.31) to get

$$\partial_t(\nabla \times u) - \mu\Delta(\nabla \times u) = \nabla \times G_2.$$

Using Sobolev's inequality, uniform bound (1.6) and decay rate (1.7), we have

$$
\begin{aligned}
\|G_2\|_{L^1}+\|G_2\|_{L^2}\leq & C\|u\cdot\nabla u\|_{L^1}+C\|\frac{\varrho}{1+\varrho}\nabla^2u\|_{L^1}+C\|(\frac{P'(1+\varrho)}{1+\varrho}-1)\nabla\varrho\|_{L^1}+C\|\frac{\varrho}{1+\varrho}\nabla\Delta\varrho\|_{L^1}\\
&+C\|\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3}\|_{L^1}+C\|\frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2}\|_{L^1}+C\|\frac{\nabla\varrho\nabla^2\varrho}{(1+\varrho)^2}\|_{L^1}+C\|\frac{1}{1+\varrho}(\nabla\times B)\times B\|_{L^1}\\
&+\|G_2\|_{L^2}\\
\leq & C\|(\varrho,u)\|_{L^2}(\|\nabla(\varrho,u)\|_{L^2}+\|\nabla^2u\|_{L^2}+\|\nabla^3\varrho\|_{L^2})+C\|\nabla\varrho\|_{L^2}\|\nabla\varrho\|_{H^1}\\
&+C\|B\|_{L^2}\|\nabla B\|_{L^2}+C\|(\varrho,u)\|_{L^\infty}\|\nabla(\varrho,u)\|_{L^2}+C\|\varrho\|_{L^\infty}\|\nabla^2u\|_{L^2}\\
&+C\|\varrho\|_{L^\infty}\|\nabla^3\varrho\|_{L^2}+C\|\nabla\varrho\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}+C\|B\|_{L^\infty}\|\nabla B\|_{L^2}\\
\leq & C\delta(1+t)^{-\frac{5}{2}}.
\end{aligned}
\tag{2.39}
$$

By virtue of the Duhamel principle formula and (2.39), we get

$$
\begin{aligned}
\|\nabla\times u\|_{L^2}\leq & C(1+t)^{-\frac{5}{4}}(\|\Lambda^{-1}\mathcal{F}(\nabla\times u_0)\|_{L^\infty}+\|\Lambda^{-1}\mathcal{F}(\nabla\times u_0)\|_{L^2})\\
&+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}(\|\Lambda^{-1}\mathcal{F}(\nabla\times G_2)\|_{L^\infty}+\|\Lambda^{-1}\mathcal{F}(\nabla\times G_2)\|_{L^2})d\tau\\
\leq & C(1+t)^{-\frac{5}{4}}(\|u_0\|_{L^1}+\|u_0\|_{L^2})+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}(\|G_2\|_{L^1}+\|G_2\|_{L^2})d\tau\\
\leq & C(\delta+\delta_1)(1+t)^{-\frac{5}{4}}+C\delta\int_0^t(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{5}{4}}d\tau\\
\leq & C(\delta+\delta_1)(1+t)^{-\frac{5}{4}},
\end{aligned}
\tag{2.40}
$$

which, together with estimates (2.37) and (2.38), yields directly

$$
\|\partial_t\varrho\|_{L^2}\geq C(1+t)^{-\frac{5}{4}}-C(\delta+\delta_1)(1+t)^{-\frac{5}{4}}-C(1+t)^{-\frac{5}{2}}.
$$

Then, by virtue of the smallness of δ and δ_1, we have

$$
\|\partial_t\varrho\|_{L^2}\geq C(1+t)^{-\frac{5}{4}},
$$

for some large time t. Therefore, we complete the proof of this lemma. □

3 Proof of Some Technical Estimates

In this section, we will prove the estimates which have been claimed in Section 2 . In other words, we will establish the claim estimates (2.9), (2.10), (2.17), (2.22) and (2.23) in the sequel. Even though the proof process of several claim estimates below are similar to the claim estimates in [7, 8], we still give proof in detail for completeness.

Proof of inequality (2.9): By multiplying the first and second equation of (2.7) by ϱ_δ and m_δ respectively, we can obtain that

$$
\frac{1}{2}\frac{d}{dt}\int(|\varrho_\delta|^2+|m_\delta|^2)dx+\mu\int|\nabla m_\delta|^2dx+(\mu+\lambda)\int|\mathrm{div}m_\delta|^2dx-\frac{h^2}{4}\int\nabla\Delta\varrho_\delta\cdot m_\delta dx=\int S_1\cdot\nabla m_\delta dx.
$$

By virtue of the Taylor expression formula, it holds on

$$P(1+\varrho)-P(1)-\varrho\sim\varrho^2,$$

which, together with Sobolev's inequality, yield directly

$$\|S_1\|_{L^2}\le C\|(\varrho,u,B)\|_{H^2}(\|\nabla\varrho\|_{H^1}+\|\nabla(u,B)\|_{L^2}),$$

here we have used the fact that $P'(1)=1$ and the symbol $\sim$ represents the equivalent relation. By using integration by parts, together with the transport equation, it is easy to arrive at

$$\int\nabla\Delta\varrho_\delta\cdot m_\delta dx=-\frac{1}{2}\frac{d}{dt}\int|\nabla\varrho_\delta|^2dx.$$

Then, we obtain that

$$\begin{aligned}&\frac{d}{dt}(\|(\varrho_\delta,m_\delta)\|_{L^2}^2+\frac{h^2}{4}\int|\nabla\varrho_\delta|^2dx)+\mu\|\nabla m_\delta\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}m_\delta\|_{L^2}^2\\&\le C\|(\varrho,u,B)\|_{H^2}^2(\|\nabla\varrho\|_{H^1}^2+\|\nabla(u,B)\|_{L^2}^2).\end{aligned}\tag{3.1}$$

Applying the operator ∇^k to the first and second equation (2.7), then multiplying both sides by $\nabla^k\varrho_\delta$ and $\nabla^k m_\delta$ respectively, it is easy to obtain for $k=1,2,3,4$,

$$\begin{aligned}&\frac{d}{dt}\|\nabla^k(\varrho_\delta,m_\delta)\|_{L^2}^2+\mu\|\nabla^{k+1}m_\delta\|_{L^2}^2+(\mu+\lambda)\|\nabla^k\mathrm{div}m_\delta\|_{L^2}^2-\frac{h^2}{4}\int\nabla^{k+1}\Delta\varrho_\delta\cdot\nabla^k m_\delta dx\\&\le\|\nabla^kS_1\|_{L^2}\|\nabla^{k+1}m_\delta\|_{L^2}.\end{aligned}\tag{3.2}$$

At first, we estimate the term $\int\nabla^{k+1}\Delta\varrho_\delta\cdot\nabla^k m_\delta dx$ for $k=1,2,3,4$. Using integration by parts and the first equation in (2.7), we can obtain

$$\int\nabla^{k+1}\Delta\varrho_\delta\cdot\nabla^k m_\delta dx=-\frac{1}{2}\frac{d}{dt}\int|\nabla^{k+1}\varrho_\delta|^2dx.\tag{3.3}$$

Now we give the estimates for $\|\nabla^kS_1\|_{L^2}^2, k=1,2,3,4$. Indeed, we can easily apply Sobolev's inequality to obtain

$$\begin{aligned}\|\nabla^k((1+\varrho)u\otimes u)\|_{L^2}&\le C\|1+\varrho\|_{L^\infty}\|u\|_{L^\infty}\|\nabla^k u\|_{L^2}+\|u\|_{L^\infty}^2\|\nabla^k\varrho\|_{L^2}\\&\le C(1+\|\nabla u\|_{H^1})\|\nabla u\|_{H^1}\|\nabla^k(\varrho,u)\|_{L^2}\\&\le C\|\nabla u\|_{H^1}\|\nabla^k(\varrho,u)\|_{L^2},\end{aligned}$$

Similarly, we also have for $k=1,2,3,4$,

$$\|\nabla^{k+1}(\varrho u)\|_{L^2}+\|\nabla^k\mathrm{div}(\varrho u)\|_{L^2}\le C\|\nabla(\varrho,u)\|_{H^1}\|\nabla^{k+1}(\varrho,u)\|_{L^2}.$$

Due to the fact that $P'(1)=1$, by virtue of the Taylor expression formula, we get

$$\begin{aligned}&\nabla(P(1+\varrho)-P(1)-\varrho)\sim\varrho\nabla\varrho;\\&\nabla^2(P(1+\varrho)-P(1)-\varrho)\sim\nabla\varrho\nabla\varrho+\varrho\nabla^2\varrho;\\&\nabla^3(P(1+\varrho)-P(1)-\varrho)\sim\nabla\varrho\nabla\varrho\nabla\varrho+\nabla\varrho\nabla^2\varrho+\varrho\nabla^3\varrho;\\&\nabla^4(P(1+\varrho)-P(1)-\varrho)\sim|\nabla\varrho|^4+|\nabla\varrho|^2\nabla^2\varrho+\nabla\varrho\nabla^3\varrho+|\nabla^2\varrho|^2+\varrho\nabla^4\varrho.\end{aligned}$$

Then, we use Sobolev's inequality to obtain for $k = 1, 2, 3, 4$,

$$\|\nabla^k (P(1+\varrho) - P(1) - \varrho)\|_{L^2} \le C\|\nabla \varrho\|_{H^1}\|\nabla^k \varrho\|_{L^2}.$$

Next, we only have to estimate the terms $\|\nabla^k(\nabla\sqrt{1+\varrho}\otimes\nabla\sqrt{1+\varrho})\|_{L^2}$ for $k = 1, 2, 3, 4$. Applying Newton-Leibniz inequality, Sobolev's inequality and Cauchy inequality, it holds on

$$\begin{aligned}
&\|\nabla^k(\nabla\sqrt{1+\varrho}\otimes\nabla\sqrt{1+\varrho})\|_{L^2}\\
\le& C\|\nabla^k(\frac{|\nabla\varrho|^2}{1+\varrho})\|_{L^2}\\
\le& C\sum_{l=0}^{k}\sum_{m=0}^{k-l}\|\nabla^l(\frac{1}{1+\varrho})\nabla^{m+1}\varrho\nabla^{k-l-m+1}\varrho\|_{L^2}.
\end{aligned}$$

For the case $l = k$, with the aid of (1.14), (1.15) and Cauchy inequality, one arrives at

$$\begin{aligned}
&\|\nabla^k(\frac{1}{1+\varrho})|\nabla\varrho|^2\|_{L^2}\\
\le& C\|\nabla^k\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^2}\\
\le& C\|\nabla^{k+1}\varrho\|_{L^2}^{\frac{1}{2}}\|\nabla^{k+2}\varrho\|_{L^2}^{\frac{1}{2}}\|\nabla^2\varrho\|_{L^2}^{\frac{1}{2}}\|\nabla^3\varrho\|_{L^2}^{\frac{1}{2}}\|\nabla\varrho\|_{L^2}\\
\le& C\|\nabla\varrho\|_{L^2}\|\nabla^{k+2}\varrho\|_{L^2} + C\|\nabla^2\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}.
\end{aligned}$$

For the case $0 \le l \le k-1$, applying Newton-Leibniz formula, it is easy to see that

$$\begin{aligned}
&\|\nabla^{k-1}(\frac{|\nabla\varrho|^2}{1+\varrho})\|_{L^2}\\
\le& C\sum_{l=0}^{k-1}\|\nabla^l(\frac{1}{1+\varrho})\nabla\varrho\nabla^{k-l+1}\varrho\|_{L^2} + C\sum_{l=0}^{k-1}\sum_{m=1}^{k-l-1}\|\nabla^l(\frac{1}{1+\varrho})\nabla^{m+1}\varrho\nabla^{k-l-m+1}\varrho\|_{L^2}\\
=&: I_1 + I_2.
\end{aligned}$$

We deal with the term I_1 first of all. For the case $l = 0$, we use (1.15), Hölder inequality and Sobolev's inequality to obtain

$$\begin{aligned}
&\|\frac{1}{1+\varrho}\cdot\nabla\varrho\nabla^{k+1}\varrho\|_{L^2}\\
\le& C\|\frac{1}{1+\varrho}\|_{L^\infty}\|\nabla\varrho\|_{L^3}\|\nabla^{k+1}\varrho\|_{L^6}\\
\le& C\|\nabla\varrho\|_{H^1}\|\nabla^{k+2}\varrho\|_{L^2}.
\end{aligned}$$

For the case $1 \le l \le k-1$, by combining the estimate obtained in [9] and Cauchy inequality, one arrives at

$$\|\nabla^l(\frac{1}{1+\varrho})\nabla\varrho\nabla^{k-l+1}\varrho\|_{L^2} \le C\|\nabla\varrho\|_{L^2}\|\nabla^{k+2}\varrho\|_{L^2} + C\|\nabla^2\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}.$$

By combining above two inequalities, we get

$$I_1 \le C\|\nabla\varrho\|_{H^1}\|\nabla^{k+2}\varrho\|_{L^2} + C\|\nabla^2\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}.$$

Next, we estimate the term I_2. For the case $l=0$, by virtue of the estimate obtained in [9], it is easy to attain the estimate

$$\begin{aligned}
&\sum_{l=1}^{k-1}\|\frac{1}{1+\varrho}\cdot\nabla^{m+1}\varrho\nabla^{k-m+1}\varrho\|_{L^2}\\
\le& C\sum_{l=1}^{k-1}\|\nabla\varrho\|_{L^2}^{\frac12}\|\nabla^2\varrho\|_{L^2}^{\frac12}\|\nabla^{k+2}\varrho\|_{L^2}\\
\le& C\|\nabla\varrho\|_{H^1}\|\nabla^{k+2}\varrho\|_{L^2},
\end{aligned}$$

where we have used the interpolation inequality (1.14) as follows,

$$\|\nabla^{\frac32}\varrho\|_{L^2}\le C\|\nabla\varrho\|_{L^2}^{\frac12}\|\nabla^2\varrho\|_{L^2}^{\frac12}.$$

For the case $1\le l\le k-1$, we use the estimate in [9] once again and Cauchy inequality to obtain

$$\begin{aligned}
&\sum_{l=1}^{k-1}\sum_{m=1}^{k-l-1}\|\nabla^l\varrho\|_{L^\infty}\|\nabla^{m+1}\varrho\|_{L^3}\|\nabla^{k-l-m+1}\varrho\|_{L^6}\\
\le& C\|\nabla\varrho\|_{L^2}\|\nabla^{k+2}\varrho\|_{L^2}+C\|\nabla^2\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}.
\end{aligned}$$

By combining above two inequalities, it holds on

$$I_2\le C\|\nabla\varrho\|_{H^1}\|\nabla^{k+2}\varrho\|_{L^2}+C\|\nabla^2\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}.$$

Then, we deduce that

$$\|\nabla^k(\nabla\sqrt{1+\varrho}\otimes\nabla\sqrt{1+\varrho})\|_{L^2}\le C\|\nabla\varrho\|_{H^1}\|\nabla^{k+2}\varrho\|_{L^2}+C\|\nabla^2\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}.$$

Thus, it holds on for $k=1,2,3,4$,

$$\begin{aligned}
\|\nabla^k S_1\|_{L^2}\le& C\|\nabla^k(\varrho,u,B)\|_{H^1}(\|\nabla\varrho\|_{H^2}+\|\nabla(u,B)\|_{H^1})+C\|\nabla\varrho\|_{H^1}\|\nabla^{k+2}\varrho\|_{L^2}\\
&+C\|\nabla^2\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}.
\end{aligned}\tag{3.4}$$

Which together with (3.2), (3.3) and Cauchy inequality yields directly

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla^k(\varrho_\delta,m_\delta)\|_{L^2}^2+\frac{h^2}{4}\|\nabla^{k+1}\varrho_\delta\|_{L^2}^2)+\mu\|\nabla^{k+1}m_\delta\|_{L^2}^2+(\mu+\lambda)\|\nabla^k\mathrm{div}m_\delta\|_{L^2}^2\\
\le& C\|\nabla^k(\varrho,u,B)\|_{H^1}^2(\|\nabla\varrho\|_{H^2}^2+\|\nabla(u,B)\|_{H^1}^2)+C\|\nabla\varrho\|_{H^1}^2\|\nabla^{k+2}\varrho\|_{L^2}^2+C\|\nabla^2\varrho\|_{L^2}^2\|\nabla^{k+1}\varrho\|_{L^2}^2.
\end{aligned}$$

Therefore, we complete the proof of claim estimate (2.9).

Proof of inequality (2.10)**:** By taking $k(k=0,1,2,3,4)-$th spatial derivatives to the second equation of (2.7) and then multiplying both side of the equation by $\nabla^{k+1}\varrho_\delta$, it is easily to derive that

$$\begin{aligned}
&\int\partial_t\nabla^k m_\delta\cdot\nabla^{k+1}\varrho_\delta dx+\int|\nabla^{k+1}\varrho_\delta|^2dx-\frac{h^2}{4}\int\nabla^{k+1}\Delta\varrho_\delta\cdot\nabla^{k+1}\varrho_\delta dx\\
=&\int(\mu\nabla^k\Delta m_\delta+(\mu+\lambda)\nabla^{k+1}\mathrm{div}m_\delta)\cdot\nabla^{k+1}\varrho_\delta dx-\int\nabla^k\mathrm{div}S_1\cdot\nabla^{k+1}\varrho_\delta dx.
\end{aligned}$$

Using the first equation of (2.7), by virtue of integration by parts, it holds on

$$\begin{aligned}\int \partial_t \nabla^k m_\delta \cdot \nabla^{k+1} \varrho_\delta dx &= \frac{d}{dt}\int \nabla^k m_\delta \cdot \nabla^{k+1}\varrho_\delta dx - \int \nabla^k m_\delta \cdot \nabla^{k+1}\partial_t \varrho_\delta dx \\ &= \frac{d}{dt}\int \nabla^k m_\delta \cdot \nabla^{k+1}\varrho_\delta dx + \int \nabla^k m_\delta \cdot \nabla^{k+1}\mathrm{div} m_\delta dx \\ &= \frac{d}{dt}\int \nabla^k m_\delta \cdot \nabla^{k+1}\varrho_\delta dx - \int |\nabla^k \mathrm{div} m_\delta|^2 dx.\end{aligned}$$

After integration by parts, it holds on

$$\int \nabla^{k+1}\Delta \varrho_\delta \cdot \nabla^{k+1}\varrho_\delta dx = -\int |\nabla^{k+2}\varrho_\delta|^2 dx.$$

Thus, by combining the above three equalities, we arrive at

$$\begin{aligned}&\frac{d}{dt}\int \nabla^k m_\delta \cdot \nabla^{k+1}\varrho_\delta dx + \int |\nabla^{k+1}\varrho_\delta|^2 dx + \frac{h^2}{4}\int |\nabla^{k+2}\varrho_\delta|^2 dx \\ =&\int |\nabla^k \mathrm{div} m_\delta|^2 dx - \int \nabla^k \mathrm{div} S_1 \cdot \nabla^{k+1}\varrho_\delta dx + \int (\mu \nabla^k \Delta m_\delta + (\mu+\lambda)\nabla^{k+1}\mathrm{div} m_\delta)\cdot \nabla^{k+1}\varrho_\delta dx,\end{aligned}$$

which, together with integration by parts and Cauchy inequality, yield directly

$$\frac{d}{dt}\int \nabla^k m_\delta \cdot \nabla^{k+1}\varrho_\delta dx + \int |\nabla^{k+1}\varrho_\delta|^2 dx + \frac{h^2}{4}\int |\nabla^{k+2}\varrho_\delta|^2 dx \le C(\|\nabla^{k+1} m_\delta\|_{L^2}^2 + \|\nabla^k S_1\|_{L^2}^2).$$

This and the estimate (3.4) implies (2.10). Therefore, we complete proof of claim estimate (2.10).

Proof of inequality (2.17): It is obvious to obtain that

$$\int_0^t e^{-\frac{C}{C^*}(t-\tau)}(1+\tau)^{-\frac{5}{2}}(\|\nabla^5 \varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2)d\tau \le \int_0^t e^{-\frac{C}{C^*}(t-\tau)}(\|\nabla^5 \varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2)d\tau.$$

By multiplying the inequality (1.16) with $l = 4$ and $m = 4$ by $e^{\frac{C}{C^*}t}$, it holds on

$$\frac{d}{dt}[e^{\frac{C}{C^*}t}\mathcal{F}_4^4(t)] + Ce^{\frac{C}{C^*}t}(\|\nabla^5 \varrho\|_{H^1}^2 + \|\nabla^5 u\|_{L^2}^2) \le Ce^{\frac{C}{C^*}t}\mathcal{F}_4^4(t),$$

then by integrating about time over $[0, t]$, due to the fact that the equivalence of the term $\mathcal{F}_4^4(t)$ is $\|\nabla^5 \varrho(\tau)\|_{H^1}^2 + \|\nabla^5 u(\tau)\|_{L^2}^2$, one arrives at

$$\begin{aligned}&\mathcal{F}_4^4(t) + C\int_0^t e^{-\frac{C}{C^*}(t-\tau)}(\|\nabla^5 \varrho(\tau)\|_{H^1}^2 + \|\nabla^5 u(\tau)\|_{L^2}^2)d\tau \\ \le& Ce^{-\frac{C}{C^*}t}\mathcal{F}_4^4(0) + C\int_0^t e^{-\frac{C}{C^*}(t-\tau)}\mathcal{F}_4^4(\tau)d\tau \\ \le& Ce^{-\frac{C}{C^*}t} + C\int_0^t e^{-\frac{C}{C^*}(t-\tau)}(1+\tau)^{-\frac{9}{2}}d\tau \\ \le& Ce^{-\frac{C}{C^*}t} + C(1+t)^{-\frac{9}{2}} \\ \le& C(1+t)^{-\frac{9}{2}},\end{aligned}$$

here we utilize decay estimate (1.7), and we have used the fact that $e^{-\frac{C}{C^*}t} \le C(1+t)^{-\frac{9}{2}}$. Therefore, we complete proof of claim estimate (2.17).

Proof of inequality (2.22): Replacing l by $l+1$ in (1.16) with $m=4$, and then multiplying both sides by $(1+t)^{l+\frac{3}{2}}$, we arrive at

$$\frac{d}{dt}[(1+t)^{l+\frac{3}{2}}\mathcal{F}_{l+1}^4(t)]+C(1+t)^{l+\frac{3}{2}}(\|\nabla^{l+2}\varrho\|_{H^{4-l}}^2+\|\nabla^{l+2}u\|_{H^{3-l}}^2)\le C(1+t)^{l+\frac{1}{2}}\mathcal{F}_{l+1}^4(t).$$

The integration of above inequality about time over $[0,t]$ implies that

$$\begin{aligned}&\int_0^t(1+\tau)^{l+\frac{3}{2}}(\|\nabla^{l+2}\varrho(\tau)\|_{H^{3-l}}^2+\|\nabla^{l+2}u(\tau)\|_{H^{2-l}}^2)d\tau\\ \le& C\mathcal{F}_{l+1}^4(0)+C\int_0^t(1+\tau)^{l+\frac{1}{2}}\mathcal{F}_{l+1}^4(\tau)d\tau\\ \le& C+C\int_0^t(1+\tau)^{l+\frac{1}{2}}(1+\tau)^{-\frac{5+2l}{2}}d\tau\\ \le& C,\end{aligned}$$

where we used the estimate $\mathcal{F}_l^4(t)\le C(1+t)^{-\frac{3+2l}{2}}$ for $l=0,1,2,3$. Therefore, we complete proof of claim estimate (2.22).

Proof of inequality (2.23): Multiplying both sides of the third equation of (2.7) by $\nabla^l B$ and then integrating, we obtain for $l=0,1,2,3$,

$$\frac{1}{2}\frac{d}{dt}\int|\nabla^l B_\delta|^2dx+\nu\int|\nabla^{l+1}B_\delta|^2dx\le\|\nabla^l S_2\|_{L^2}\|\nabla^{l+1}B_\delta\|_{L^2}. \tag{3.5}$$

Using Sobolev's inequality, we find

$$\|\nabla^l(u\times B)\|_{L^2}\le C\|(u,B)\|_{L^\infty}\|\nabla^l(u,B)\|_{L^2}. \tag{3.6}$$

Then using Cauchy inequality, one arrives at

$$\frac{d}{dt}\int|\nabla^l B_\delta|^2dx+\nu\int|\nabla^{l+1}B_\delta|^2dx\le C\|(u,B)\|_{L^\infty}^2\|\nabla^l(u,B)\|_{L^2}^2. \tag{3.7}$$

Therefore, we complete the proof of claim estimate (2.23).

Acknowledgements

This work was partially supported by NSFC(Grants No.11801586 and No.11971496).

References

[1] L.Chen,M.Dreher, The viscous model of quantum hydrodynamics in several dimensions, Math. Models Methods Appl. Sci. 17(7) (2007) 1065-1093.

[2] L.Chen,M.Dreher, Viscous quantum hydrodynamics and parameter-elliptic systems, Math. Methods Appl. Sci. 34(5) (2011) 520-531.

[3] L.Chen,Q.C.Ju, Existence of weak solution and semiclassical limit for quantum drift-diffusion model, Z. Angew. Math. Phys. 58(1) (2007) 1-15.

[4] X.Q.Chen,L.Chen,H.Y.Jian, The existence and long-time behavior of weak solution to bipolar quantum drift-diffusion model, Chin. Ann. Math. Ser. B 28(6) (2007) 651-664.

[5] D.Ferry,J.R.Zhou, Form of the quantum potential for use in hydrodynamic equations for semiconductor device modeling, Phys.Rev.B 48 (1993) 7944-7950.

[6] I.M.Gamba,A.Jüngel,A.Vasseur, Global existence of solutions to one-dimensional viscous quantum hydrodynamic equations, J. Differential Equations 247(11) (2009) 3117-3135.

[7] J.C.Gao,Z.Y.Lyu,Z.A.Yao, Lower bound and space-time decay rates of higher order derivatives of solution for the compressible Navier-Stokes and Hall-MHD equations, arXiv:1909.13269.

[8] J.C.Gao,Z.Y.Lyu,Z.A.Yao, Lower Bound of Decay Rate for Higher Order Derivatives of Solution to the Compressible Fluid Models of Korteweg Type, preprint.

[9] J.C.Gao,Q.Tao,Z.A.Yao, Long-time behavior of solution for the compressible nematic liquid crystal flows in $\mathbb{R}^3$, J. Differential Equations 261(4) (2016) 2334-2383.

[10] J.Grant, Pressure and stress tensor expressions in the fluid mechanical formulation of the Bose condensate equations, J.Phys.A 6 (1973) L151-L153.

[11] B.L.Guo,G.W.Wang, Blow-up of the smooth solution to quantum hydrodynamic models in $\mathbb{R}^d$, J. Differential Equations 261(7) (2016) 3815-3842.

[12] B.L.Guo,B,Q.Xie, Global existence of weak solutions to the three-dimensional full compressible quantum equations, Ann. Appl. Math. 34(1) (2018) 1-31.

[13] Y.Guo,Y.J.Wang, Decay of dissipative equations and negative Sobolev spaces, Comm. Partial Differential Equations 37(12) (2012) 2165-2208.

[14] F.Haas, A magnetohydrodynamic model for quantum plasmas, Phys. Plasmas 12 (2005) 062117-062126.

[15] A.Jüngel, A steady-state quantum Euler-Poisson system for potential flows, Comm. Math. Phys. 194(2) (1998) 463-479.

[16] A.Jüngel,H.L.Li, Quantum Euler-Poisson systems: existence of stationary states, Arch. Math. (Brno) 40(4) (2004) 435-456.

[17] A.Jüngel,H.L.Li, Quantum Euler-Poisson systems: global existence and exponential decay, Quart. Appl. Math. 62(3) (2004) 569-600.

[18] A.Jüngel, Global weak solutions to compressible Navier-Stokes equations for quantum fluids, SIAM J. Math. Anal. 42(3) (2010) 1025-1045.

[19] H.Y.Li,M.Cheng,W.P.Yan, Global existence and large time behavior of solutions for compressible quantum magnetohydrodynamics flows in T^3, J. Math. Anal. Appl. 452(2) (2017) 1209-1228.

[20] H.L.Li,P.Marcati, Existence and asymptotic behavior of multi-dimensional quantum hydrodynamic model for semiconductors, Comm. Math. Phys. 245(2) (2004) 215-247.

[21] E.Madelung, Quantentheorie in hydrodynamischer form, Z. Physik 40 (1927) 322-326.

[22] G.Manfredi,F.Haas, Self-consistent fluid model for a quantum electron gas, Phys. Rev. B64 075316 (2001).

[23] L.Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm. Super. Pisa Cl. Sci.(3) 13 (1959) 115-162.

[24] X.K.Pu,B.L.Guo, Global existence and semiclassical limit for quantum hydrodynamic equations with viscosity and heat conduction, Kinet. Relat. Models 9(1) (2015) 165-191.

[25] X.K.Pu,B.L.Guo, Optimal decay rate of the compressible quantum Navier-Stokes equations, Ann. Appl. Math. 32(3) (2016) 275-287.

[26] X.K.Pu,M.Li, Global solutions for the compressible quantum hydrodynamic model in a bounded domain, Nonlinear Anal. 184 (2019) 148-171.

[27] X.K.Pu,X.L.Xu, Asymptotic behaviors of the full quantum hydrodynamic equations, J. Math. Anal. Appl. 454(1) (2017) 219-245.

[28] X.K.Pu,X.L.Xu, Decay rates of the magnetohydrodynamic model for quantum plasmas, Z. Angew. Math. Phys. 68(1) (2017) Art. 18, 17 pp.

[29] M.E.Schonbek, Decay of solutions to parabolic conservation laws, Comm.Partial Differential Equations 5(5) (1980) 449-473.

[30] M.E.Schonbek, L^2 decay for weak solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal. 88 (1985) 209-222.

[31] Y.J.Wang, Decay of the Navier-Stokes-Poisson equations, J. Differential Equations 253 (2012) 273-297.

[32] R.Wyatt, Quantum Dynamics with Trajectories, Springer, New York, 2005.

[33] X.Y.Xi,X.K.Pu,B.L.Guo, Long-time behavior of solutions for the compressible quantum magnetohydrodynamic model in $\mathbb{R}^3$, Z. Angew. Math. Phys. 70(1) (2019) Art. 7, 16 pp.

[34] B.Q.Xie,X.Y.Xi,B.L.Guo, Long-time behavior of solutions for full compressible quantum model in $\mathbb{R}^3$, Appl. Math. Lett. 80 (2018) 54-58.

[35] J.W.Yang,Q.C.Ju, Global existence of the three-dimensional viscous quantum magnetohydrodynamic model, J. Math. Phys. 55(8) (2014) 081501, 12 pp.