Received <day> <Month>, <year>; | Revised <day> <Month>, <year>; | Accepted <day> <Month>, <year>

DOI: xxx/xxxx

RESEARCH ARTICLE

Some Liouville-type theorems for the stationary 3D magneto-micropolar fluids

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Abstract

In this paper we prove some Liouville-type theorems for the stationary magneto-micropolar fluids under suitable conditions in three space dimensions. We first prove that the solutions are trivial under the assumption of certain growth conditions for the mean oscillations of the potentials. And then we show similar results assuming that the the solutions are contained in $L^p(\mathbb{R}^3)$ with $p \in [2, 9/2)$. Finally we show the same result for lower values of $p \in [1, 9/4)$ with the further assumption that the solutions vanish at infinity.

KEYWORDS:
stationary magneto-micropolar equations, Liouville-type theorem

1 | INTRODUCTION

In the present paper, we consider the stationary magneto-micropolar fluid equations in $\mathbb{R}^3$, which consists of the following partial differential equations:

$$\begin{cases} -\Delta u + (u\cdot\nabla)u + \nabla\Pi = \chi\nabla\times w + (b\cdot\nabla)b, \\ -\gamma\Delta w + (u\cdot\nabla)w = \nabla(\nabla\cdot w) + \chi\nabla\times u - 2\chi w, \\ -\nu\Delta b + (u\cdot\nabla)b = (b\cdot\nabla)u, \\ \nabla\cdot u = \nabla\cdot b = 0, \end{cases} \tag{1}$$

where $u = (u_1, u_2, u_3)$, $w = (w_1, w_2, w_3)$, $b = (b_1, b_2, b_3)$ and Π denote the fluid velocity, the angular velocity of the rotation of the fluid particles, the magnetic fields and pressure respectively. The positive constant γ in (1) correspond to the angular viscosity, ν is the inverse of the magnetic Reynolds number and χ is the micro-rotational viscosity. In this paper, we assume that $\gamma=\nu=\chi=1$ for simplicity. Equation $(1)_1$ is similar to the classical Navier–Stokes equations, but here it is coupled with equations $(1)_2$ for w and $(1)_3$ for b. Equation $(1)_2$ describes the motion in the macro-volumes as they go through micro-rotational effects, represented by the micro-rotational velocity vector w. If the fluids have no micro structure, w vanishes and the system (1) becomes a magneto-hydrodynamics system. Equation $(1)_3$ is the Maxwell system for the electric field. This model was first introduced by Ahmadi and Shahinpoor[1]. After that, Rojas-Medar[2] proved the local-in-time existence and uniqueness of strong solutions in a bounded domain based on the spectral Galerkin method. Furthermore, Rojas-Medar and Boldrini[3] established the existence of weak solutions to the model (1) in a bounded domain and in particular, the uniqueness was also proved for a two-dimensional domain. The existence of global-in-time strong solutions was addressed by Ortega-Torres and Rojas-Medar[4].

After Galdi's work in[5], Liouville-type problems for the stationary fluid equations has been extensively studied and there are a large number of works on the Liouville type-problems even to these dates (see e.g.[6,7,8,9,10] and a review paper[11]). Here, we shall study some Liouville-type results under the assumptions with regard to the potential functions. We say that $\Phi \in L^1_{loc}(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ is the potential functions for the vector fields $u \in L^1_{loc}(\mathbb{R}^3)$, if $\operatorname{div}\Phi = u$. In[12], Seregin obtained Liouville-type theorems for the steady-state Navier-Stokes equations under the assumptions that the potential $\Phi \in BMO(\mathbb{R}^3)$ and $u \in L^6(\mathbb{R}^3)$, and in[13] the integrability condition for the velocity was dropped. After that, very recently, Chae and Wolf[14] showed Liouville-type theorem for the stationary Navier-Stokes equations under the assumption

$$\left(\frac{1}{|B_r|} \int_{B_r} |\Phi - \Phi_{B_r}|^s \, \mathrm{d}x \right)^{\frac{1}{s}} \lesssim r^{\frac{1}{3}-\frac{1}{s}} \qquad \forall 1 < r < +\infty$$

for some $3 < s < +\infty$, and similar results were proved for MHD equations in[15]. The first theorem of the present paper is the extension of the result of[15]. Here however, we shall adopt a different approach to control the pressure term by introducing an auxiliary function and utilizing it as a test function. In specific, we first aim in this paper to prove the following Liouville-type result.

Theorem 1. Let (u, b, w, Π) be a smooth solution to the equations (1). Assume that there exist potentials $\Phi, \Psi, \Upsilon \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times3})$ such that $\nabla \cdot \Phi = u$, $\nabla \cdot \Psi = b$, $\nabla \cdot \Upsilon = w$ and

$$\left(\frac{1}{|B_r|} \int_{B_r} \left|\Phi - \Phi_{B_r}\right|^s \mathrm{d}x \right)^{\frac{1}{s}} + \left(\frac{1}{|B_r|} \int_{B_r} \left|\Psi - \Psi_{B(r)}\right|^s \mathrm{d}x \right)^{\frac{1}{s}} + \left(\frac{1}{|B_r|} \int_{B_r} \left|\Upsilon - \Upsilon_{B_r}\right|^s \mathrm{d}x \right)^{\frac{1}{s}} \le C r^{\frac{1}{3}-\frac{1}{s}}, \qquad r > 1 \tag{2}$$

for some $3 < s \le 6$. Then $u \equiv b \equiv w \equiv 0$.

Remark 1. In the case of $w \equiv 0$ Theorem 1 reduces to[15, Theorem 1.1].

Later, Zhang et. al.[16] proved that if smooth solutions of the stationary MHD equations are bounded in $L^{\frac{9}{2}}(\mathbb{R}^3)$ and have finite Dirichlet integral, then they are also identically zero. After that, Schulz[17] obtained the Liouville theorem for this equations provided that the smooth solution (u, b) are contained in $L^p(\mathbb{R}^3) \cap BMO^{-1}(\mathbb{R}^3)$ with $p \in (2, 6]$. Recently, Yuan and Xiao[18] proved that if smooth solution $(u, b) \in L^p(\mathbb{R}^3)$ with $2 \le p \le \frac{9}{2}$, then $u = b = 0$. In this direction, the second objective of this paper is as follows.

Theorem 2. Let $p \in [2, \frac{9}{2})$. Assume that (u, b, w, Π) is a smooth solution to the equations (1) with $u, b, w \in L^p(\mathbb{R}^3)$. Then $u \equiv b \equiv w \equiv 0$.

Furthermore, parallel to the result of Liu and Liu[19], we shall also prove the following theorem.

Theorem 3. Let $p \in [1, \frac{9}{4})$. Assume that (u, b, w, Π) is a smooth solution to the equations (1) with $u, b, w \in L^p(\mathbb{R}^3)$ satisfying $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} b(x) = \lim_{|x|\to\infty} w(x) = 0$. Then $u \equiv b \equiv w \equiv 0$.

Remark 2. Even if we consider the model with the variable density, that is, density dependent models, Theorem 1, 2 and 3 still hold under suitable additional assumptions (see, for example,[19]).

Remark 3. In the light of the work of Liu and Liu[19], through the similar approach, we can also obtain the Liouville-type results in Lorentz spaces (see e.g.[20,21]).

2 | PRELIMINARIES

In this section, we introduce some notations and auxiliary results which will be used throughout the paper. We denote the ball with center x_0 and radius R by $B_R(x_0)$. If $x_0 = 0$, we simply write $B_R = B_R(0)$. Throughout the paper, the notation $P \lesssim Q$ implies that there exists some constant $C > 0$ such that $P \le CQ$. Also, C denotes a generic positive constant, which may change at each appearance.

Let us also define a family of cut-off functions. For $0 < r < r'$, we let $\xi = \xi_{r,r'} \in C_c^\infty(B_{r'})$ be a radially non-increasing scalar function such that

$$\xi_{r,r'}(x) = \begin{cases} 1, & x \in B_r, \\ 0, & x \in B_{r'}^c, \end{cases} \tag{3}$$

with the properties $|\nabla \xi_{r,r'}| < C_1/(r' - r)$, and $|\nabla^2 \xi_{r,r'}| < C_2/(r' - r)^2$ for some constant $C_1, C_2 > 0$.

Next, for a bounded domain $\Omega \subset \mathbb{R}^3$, we consider the following problem: for given $f \in L^p(\Omega)$ with

$$\int_{\Omega} f(x)\,\mathrm{d}x = 0, \tag{4}$$

find a vector-valued function $v \in W_0^{1,p}(\Omega)^d$ satisfying

$$\begin{aligned} \nabla \cdot v &= f, \\ \|\nabla v\|_p &\le C\|f\|_p \end{aligned} \tag{5}$$

for some constant $C = C(d, p, \Omega)$. For this matter, we have the following theorem which is quoted from[5].

Theorem 4. [5, Theorem III.3.1] Assume that Ω satisfies the cone condition. Then for given $f \in L^p(\Omega)$ with $1 < p < \infty$ satisfying (4), there exists at least one solution for the problem (5).

We will use this result to construct an auxiliary function to control the pressure term in the next section. We will also use the following iteration lemma frequently in our analysis.

Lemma 1. [22, Lemma 3.1] Let $f(r)$ be a non-negative bounded function on $[r_0, r_1] \subset \mathbb{R}_{\ge 0}$. Suppose that there exist non-negative constants A, B, D, E and positive numbers $d < b < a$ and a parameter $\theta \in (0, 1)$ such that for any $r_0 \le s < t \le r_1$,

$$f(s) \le \theta f(t) + \frac{A}{(t-s)^a} + \frac{B}{(t-s)^b} + \frac{D}{(t-s)^d} + E.$$

Then we have

$$f(s) \le C(a, b, d, \theta)\left[\frac{A}{(t-s)^a} + \frac{B}{(t-s)^b} + \frac{D}{(t-s)^d} + E\right].$$

We shall also use the following lemma for the proof of Theorem 1.

Lemma 2. [23, Lemma 2.1 and Lemma 2.2] Suppose that $R > 1$ and $f \in W^{1,2}(B_R; \mathbb{R}^3)$. For $0 < \rho < R$, we let $\psi \in C_c^\infty(B_R)$ such that $0 \le \psi \le 1$ and $|\nabla \psi| \le C/(R-\rho)$ for some constant $C > 0$. Assume further that there exists the potential $F \in W^{2,2}(B_R; \mathbb{R}^{3\times 3})$ with $\nabla \cdot F = f$ and the growth condition

$$\left(\frac{1}{|B_r|}\int_{B_r} \left|F - F_{B_r}\right|^s \mathrm{d}x\right)^{\frac{1}{s}} \lesssim r^{\frac{1}{3}-\frac{1}{s}}, \qquad r > 1$$

for some $3 < s \le 6$. Then there holds

$$\|\psi^2 f\|^2_{L^2(B_R)} \lesssim R^{\frac{11}{6}-\frac{1}{s}}\|\psi\nabla f\|_{L^2(B_R)} + R^{\frac{11}{3}-\frac{2}{s}}(R-\rho)^{-2} \tag{6}$$

and

$$\|\psi^3 f\|^3_{L^3(B_R)} \lesssim R\|\psi\nabla f\|^{\frac{18}{s+6}}_{L^2(B_R)} + R^{4-\frac{3}{s}}(R-\rho)^{-3} + R\left((R-\rho)^{-1}\|\psi^2 f\|_{L^2(B_R)}\right)^{\frac{18}{s+6}}. \tag{7}$$

3 | PROOF OF THEOREM 1

Let φ_R be a cut-off function in $C_c^\infty(\mathbb{R}^3)$ given by $\varphi_R = \xi_{\rho,\tau}$ for $1 < \frac{R}{2} < \rho < \frac{2}{3}R < R < \tau < 2R$. We begin with some estimates for the terms related to w. By using the Hölder and Young's inequality, we note that

$$\begin{aligned} -\int_{\mathbb{R}^3} \nabla(\nabla\cdot w)(w\varphi_R^2)\,\mathrm{d}x &= \int_{\mathbb{R}^3} (\nabla\cdot w)\nabla\cdot(w\varphi_R^2)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} |\nabla\cdot w|^2\varphi_R^2\,\mathrm{d}x + 2\int_{\mathbb{R}^3} (\nabla\cdot w)w\varphi_R\cdot\varphi_R\,\mathrm{d}x \\ &\ge \int_{\mathbb{R}^3} |\nabla\cdot w|^2\varphi_R^2\,\mathrm{d}x - 2\left|\int_{\mathbb{R}^3} (\nabla\cdot w)w\varphi_R\cdot\varphi_R\,\mathrm{d}x\right| \\ &\ge \int_{\mathbb{R}^3} |\nabla\cdot w|^2\varphi_R^2\,\mathrm{d}x - \varepsilon\int_{\mathbb{R}^3} |\nabla\cdot w|^2\varphi_R^2\,\mathrm{d}x - C(\varepsilon)\int_{\mathbb{R}^3} |w|^2|\nabla\varphi_R|^2\,\mathrm{d}x. \end{aligned}$$

Next, using the vector identity $\nabla\times(u\varphi_R^2)=\varphi_R^2\nabla\times u+\nabla\varphi_R^2\times u$ and Young's inequality, we also note that

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla\times w\cdot u\varphi_R^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\times u\cdot w\varphi_R^2\,\mathrm{d}x&=\int_{\mathbb{R}^3}w\cdot\nabla\times(u\varphi_R^2)\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\times u\cdot w\varphi_R^2\,\mathrm{d}x\\&=2\int_{\mathbb{R}^3}w\varphi_R^2\cdot\nabla\times u\,\mathrm{d}x+2\int_{\mathbb{R}^3}w\cdot\varphi_R\nabla\varphi\times u\,\mathrm{d}x\\&\le\frac{2}{3}\int_{\mathbb{R}^3}|\nabla u|^2\varphi_R^2\,\mathrm{d}x+\frac{3}{2}\int_{\mathbb{R}^3}|w|^2\varphi_R^2\,\mathrm{d}x\\&\quad+\varepsilon\int_{\mathbb{R}^3}|w|^2\varphi_R^2\,\mathrm{d}x+C(\varepsilon)\int_{\mathbb{R}^3}|u|^2|\nabla\varphi_R|^2\,\mathrm{d}x.\end{aligned}$$

Before proceeding more, we introduce an auxiliary function which is needed to handle the pressure term. We set $\Omega=B_\tau\setminus B_\rho$ and $f=\varphi_R u$. As we know by Green's Theorem that

$$\int_{B_\tau\setminus B_\rho}\nabla\cdot(\varphi_R u)\,\mathrm{d}x=\int_{\partial B_\tau}\varphi u\cdot\nu\,\mathrm{d}x=0,$$

we can apply Theorem 4 to show the existence of vector-valued function $W_R\in W_0^{1,p}(B_\tau\setminus B_\rho)$ satisfying

$$\nabla\cdot W_R=\nabla\cdot(\varphi_R u)\quad\text{in } B_\tau\setminus B_\rho,$$

with

$$\|\nabla W_R\|_{L^p(B_\tau\setminus B_\rho)}\lesssim\|\nabla\varphi_R\cdot u\|_{L^p(B_\tau\setminus B_\rho)}. \tag{8}$$

Now, we multiply the equations $(1)_1$, $(1)_2$ and $(1)_3$ by $u\varphi_R^2-W_R$, $b\varphi_R^2$ and $w\varphi_R^2$, respectively and integrate over $\mathbb{R}^3$. Then integration by parts with the use of divergence-free conditions yields that

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(|\nabla u|^2+|\nabla b|^2+|\nabla w|^2\right)\varphi_R^2\,\mathrm{d}x+\int_{\mathbb{R}^3}|\nabla\cdot w|^2\varphi_R^2\,\mathrm{d}x+\int_{\mathbb{R}^3}|w|^2\varphi_R^2\,\mathrm{d}x\\&\lesssim\int_{\mathbb{R}^3}\left(|u|^2+|b|^2+|w|^2\right)|\nabla\varphi_R|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\left(|u|^2+|b|^2+|w|^2\right)u\cdot\varphi_R\nabla\varphi_R\,\mathrm{d}x\\&\quad+\int_{\mathbb{R}^3}\nabla u\cdot\nabla W_R\,\mathrm{d}x-\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot W_R\,\mathrm{d}x-\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot W_R\,\mathrm{d}x\\&\quad+\int_{\mathbb{R}^3}\nabla\times w\cdot W_R\,\mathrm{d}x+\int_{\mathbb{R}^3}(u\cdot B)(B\cdot\nabla)\varphi_R^2\,\mathrm{d}x\end{aligned} \tag{9}$$

where we have used the above estimates. We shall estimate the terms on the right-hand side of (9). First, by (8), Hölder's inequality and Young's inequalities we have

$$\int_{\mathbb{R}^3}\nabla u\cdot\nabla W_R\,\mathrm{d}x\le\|\nabla u\|_{L^2(B_\tau)}\|\nabla W_R\|_{L^2(B_\tau)}\le\frac{1}{4}\|\nabla u\|_{L^2(B_\tau)}^2+C(\tau-\rho)^{-2}\|u\|_{L^2(B_\tau\setminus B_\rho)}^2.$$

Next, by (8) and Hölder's inequality, we obtain

$$\int_{\mathbb{R}^3} (u\cdot\nabla)u\cdot W_R\,\mathrm{d}x = \sum_{i,j}\int_{B_\tau\setminus B_\rho} u_i\partial_i u_j (W_R)_j\,\mathrm{d}x = -\sum_{i,j}\int_{B_\tau\setminus B_\rho} u_i u_j \partial_i (W_R)_j\,\mathrm{d}x$$
$$\lesssim \left(\int_{B_\tau\setminus B_\rho} |u|^3\,\mathrm{d}x\right)^{2/3}\left(\int_{B_\tau\setminus B_R} |\nabla W_R|^3\,\mathrm{d}x\right)^{1/3}$$
$$\lesssim (\tau-\rho)^{-1}\left(\int_{B_\tau\setminus B_\rho} |u|^3\,\mathrm{d}x\right)^{2/3}\left(\int_{B_\tau\setminus B_R} |u|^3\,\mathrm{d}x\right)^{1/3}$$
$$\lesssim (\tau-\rho)^{-1}\int_{B_\tau\setminus B_\rho} |u|^3\,\mathrm{d}x,$$

and similarly, we get

$$\int_{\mathbb{R}^3} (b\cdot\nabla)b\cdot W_R\,\mathrm{d}x \lesssim (\tau-\rho)^{-1}\int_{B_\tau\setminus B_\rho} |b|^3\,\mathrm{d}x.$$

Furthermore, note that

$$\int_{\mathbb{R}^3} \nabla\times w\cdot W_R \le \|w\|_{L^2(B_\tau)}\|\nabla W_R\|_{L^2(B_\tau)} \le \frac{1}{4}\|w\|^2_{L^2(B_\tau)} + C(\tau-\rho)^{-2}\|u\|^2_{L^2(B_\tau\setminus B_\rho)}.$$

Finally, by Hölder's inequality and Young's inequality, we note that

$$\int_{\mathbb{R}^3} (u\cdot b)(b\cdot\nabla)\varphi_R^2\,\mathrm{d}x \lesssim \int_{B_\tau\setminus B_\rho} |u||b|^2|\nabla\varphi|\,\mathrm{d}x \lesssim (\tau-\rho)^{-1}\int_{B_\tau\setminus B_\rho} \left(|u|^3+|b|^3\right)\,\mathrm{d}x.$$

Altogether, we obtain from (9) that

$$\int_{B_\rho} \left(|\nabla u|^2+|\nabla b|^2+|\nabla w|^2\right)\,\mathrm{d}x + \int_{B_\rho} |w|^2\,\mathrm{d}x$$
$$\lesssim (\tau-\rho)^{-2}\int_{B_\tau\setminus B_\rho} \left(|u|^2+|b|^2+|w|^2\right)\,\mathrm{d}x + (\tau-\rho)^{-1}\int_{B_\tau\setminus B_\rho} \left(|u|^3+|b|^3+|w|^3\right)\,\mathrm{d}x$$
$$+\frac{1}{4}\int_{B_\tau} \left(|\nabla u|^2+|\nabla b|^2+|\nabla w|^2\right)\,\mathrm{d}x + \frac{1}{4}\int_{B_\tau} |w|^2\,\mathrm{d}x.$$

Then by Lemma 1, we conclude that

$$\int_{B_\rho} \left(|\nabla u|^2+|\nabla b|^2+|\nabla w|^2\right)\,\mathrm{d}x + \int_{B_\rho} |w|^2\,\mathrm{d}x$$
$$\lesssim (\tau-\rho)^{-2}\int_{B_\tau\setminus B_\rho} \left(|u|^2+|b|^2+|w|^2\right)\,\mathrm{d}x + (\tau-\rho)^{-1}\int_{B_\tau\setminus B_\rho} \left(|u|^3+|b|^3+|w|^3\right)\,\mathrm{d}x \tag{10}$$

Before proceeding further, let us briefly describe the strategy of the proof. We set $\tau = 2\rho$ for convenience and we shall firstly show

$$\rho^{-1}\int_{B_{2\rho}\setminus B_\rho} (|u|^3+|b|^3+|w|^3)\,\mathrm{d}x \to 0 \qquad \text{as } \rho\to\infty. \tag{11}$$

For the remaining part in (10), by the Hölder's inequality, we note that

$$\rho^{-2}\int_{B_{2\rho}\setminus B_\rho} (|u|^2+|b|^2+|w|^2)\,\mathrm{d}x \lesssim \rho^{-\frac{1}{3}}\left(\rho^{-1}\int_{B_{2\rho}\setminus B_\rho} (|u|^3+|b|^3+|w|^3)\,\mathrm{d}x\right)^{\frac{2}{3}}.$$

Hence, if we show

$$\rho^{-1}\int\limits_{B_{2\rho}\setminus B_\rho}(|u|^3+|b|^3+|w|^3)\,\mathrm{d}x<C, \tag{12}$$

for some constant $C>0$, we can get

$$\rho^{-2}\int\limits_{B_{2\rho}\setminus B_\rho}(|u|^2+|b|^2+|w|^2)\,\mathrm{d}x\to 0 \qquad \text{as } \rho\to\infty. \tag{13}$$

Due to (11) and (13), we have from (10) that

$$\int\limits_{B_\rho}(|\nabla u|^2+|\nabla b|^2+|\nabla w|^2)\,\mathrm{d}x\to 0 \qquad \text{as} \qquad \rho\to\infty,$$

which implies that u, b and w must be constants. Thanks to (11), we finally conclude that $u\equiv b\equiv w\equiv 0$.

As described above, we first aim to prove (11). Recall that $R>\rho>R/4>1$ and set $\psi=\xi_{\rho,R}-\xi_{\rho/4,R/4}$. By Lemma 2, we have from Young's inequality that

$$\begin{aligned}\int\limits_{B_R}|\psi^3 u|^3\,\mathrm{d}x &\lesssim R\|\psi\nabla u\|_{L^2(B_R)}^{\frac{18}{s+6}}+R^{4-\frac{3}{s}}(R-\rho)^{-3}+R\left((R-\rho)^{-2}R^{\frac{11}{6}-\frac{1}{s}}\|\psi\nabla u\|_{L^2(B_R)}+R^{\frac{11}{3}-\frac{2}{s}}(R-\rho)^{-4}\right)^{\frac{9}{s+6}}\\ &\lesssim R\|\psi\nabla u\|_{L^2(B_R)}^{\frac{18}{s+6}}+R^{4-\frac{3}{s}}(R-\rho)^{-3}+R\big(\|\psi\nabla u\|_{L^2(B_R)}^2+\rho^{\frac{11}{3}-\frac{2}{s}}(R-\rho)^{-4}\big)^{\frac{9}{s+6}}.\end{aligned}$$

By taking $\rho=2r$ and $R=4r$ for $r>1$, we deduce that

$$r^{-1}\int\limits_{B_{2r}\setminus B_r}|u|^3\,\mathrm{d}x\lesssim\|\nabla u\|_{L^2(B_{4r}\setminus B_{r/2})}^{\frac{18}{s+6}}+r^{-\frac{3}{s}}. \tag{14}$$

Similarly, we can also obtain

$$r^{-1}\int\limits_{B_{2r}\setminus B_r}\left(|b|^3+|w|^3\right)\,\mathrm{d}x\lesssim\left(\|\nabla b\|_{L^2(B_{4r}\setminus B_{r/2})}^{\frac{18}{s+6}}+\|\nabla w\|_{L^2(B_{4r}\setminus B_{r/2})}^{\frac{18}{s+6}}\right)+r^{-\frac{3}{s}}. \tag{15}$$

Next, we shall show that

$$\int\limits_{\mathbb{R}^3}\left(|\nabla u|^2+|\nabla b|^2+|\nabla w|^2\right)\,\mathrm{d}x\le C, \tag{16}$$

for some constant $C>0$. We set $R>\rho>1$ and $\overline{R}=(R+\rho)/2$. If we take $\varphi=\xi_{\rho,\overline{R}}$ as a cut-off function and proceed with the same argument used to derive (10), we obtain that

$$\int\limits_{B_\rho}\left(|\nabla u|^2+|\nabla b|^2+|\nabla w|^2\right)\,\mathrm{d}x\lesssim(R-\rho)^{-2}\int\limits_{B_{\overline{R}}\setminus B_\rho}\left(|u|^2+|b|^2+|w|^2\right)\,\mathrm{d}x+(R-\rho)^{-1}\int\limits_{B_{\overline{R}}\setminus B_\rho}\left(|u|^3+|b|^3+|w|^3\right)\,\mathrm{d}x.$$

And then we set $\psi=\xi_{\overline{R},R}$. Then as $\psi=1$ on $B(\overline{R})$, we note that

$$\int\limits_{B_\rho}(|\nabla u|^2+|\nabla b|^2+|\nabla w|^2)\,\mathrm{d}x\le C(\mathrm{I}_1+\mathrm{I}_2),$$

where

$$\mathrm{I}_1:=(R-\rho)^{-2}\int\limits_{B_R}\left(|\psi^2 u|^2+|\psi^2 b|^2+|\psi^2 w|^2\right)\,\mathrm{d}x,$$

and

$$\mathrm{I}_2:=(R-\rho)^{-1}\int\limits_{B_R}\left(|\psi^3 u|^3+|\psi^3 b|^3+|\psi^3 w|^3\right)\,\mathrm{d}x.$$

Then by Lemma 2 together with the assumption (2), and Young's inequality, we have that

$$\begin{aligned}(R-\rho)^{-1}\int\limits_{B_R}|\psi^3 u|^3\,\mathrm{d}x &\le CR(R-\rho)^{-1}\|\psi\nabla u\|_{L^2(B_R)}^{\frac{18}{s+6}}+CR^{4-\frac{3}{s}}(R-\rho)^{-4}+CR(R-\rho)^{-1}\big((R-\rho)^{-1}\|\psi^2 u\|_{L^2(B_R)}\big)^{\frac{18}{s+6}}\\ &\le\frac{1}{4}\|\psi\nabla u\|_{L^2(B(R))}^2+CR^{\frac{s+6}{s-3}}(R-\rho)^{-\frac{s+6}{s-3}}+\mathrm{I}_1.\end{aligned}$$

In the same way, if we proceed with the above argument for b and w, we obtain

$$\mathrm{I}_2 \le \frac{1}{4}\int_{B_R} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right)\,\mathrm{d}x + CR^{\frac{s+6}{s-3}}(R-\rho)^{-\frac{s+6}{s-3}} + \mathrm{I}_1.$$

Next, for I_1, using (6), we have by Young's inequality that

$$\begin{aligned}(R-\rho)^{-2}\int_{B_R} |\psi^2 u|^2\,\mathrm{d}x &\le CR^{\frac{11}{6}-\frac{1}{s}}(R-\rho)^{-2}\|\psi\nabla u\|_{L^2(B_R)} + CR^{\frac{11}{3}-\frac{2}{s}}(R-\rho)^{-4}\\ &\le CR^2(R-\rho)^{-2}\|\psi\nabla u\|_{L^2(B_R)} + R^4(R-\rho)^{-4}\\ &\le \frac{1}{4}\|\psi\nabla u\|^2_{L^2(B_R)} + CR^4(R-\rho)^{-4}.\end{aligned}$$

If we use the same method with b and w, it follows that

$$\mathrm{I}_1 \le \frac{1}{4}\int_{B_R} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right)\,\mathrm{d}x + CR^4(R-\rho)^{-4}.$$

Collecting the estimates for I_1 and I_2 yields

$$\int_{B_\rho} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right)\,\mathrm{d}x \le \frac{1}{2}\int_{B_R} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right)\,\mathrm{d}x + CR^{\frac{s+6}{s-3}}(R-\rho)^{-\frac{s+6}{s-3}},$$

where we have used the facts $R(R-\rho)^{-1} > 1$ and $3 < s \le 6$. Then applying Lemma 1 gives us the estimate

$$\int_{B_\rho} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right)\,\mathrm{d}x \le CR^{\frac{s+6}{s-3}}(R-\rho)^{-\frac{s+6}{s-3}}.$$

If we take $R = 2\rho$ and let $\rho \to \infty$, we obtain (16), and hence from (14) and (15), we conclude that (11) holds.

It remains to show (12). By direct computation we observe that

$$\begin{aligned}r^{-1}\int_{B_r} \left(|u|^3 + |b|^3 + |w|^3\right)\,\mathrm{d}x &= \sum_{j=1}^{\infty} 2^{-j}(2^{-j}r)^{-1}\int_{B_{2^{-(j-1)}r}\setminus B_{2^{-j}r}} \left(|u|^3 + |b|^3 + |w|^3\right)\,\mathrm{d}x\\ &\le \sup_{1/2\le\rho\le r/2} \rho^{-1}\int_{B_{2\rho}\setminus B_\rho} \left(|u|^3 + |b|^3 + |w|^3\right)\,\mathrm{d}x\\ &\quad + \int_{B_1} \left(|u|^3 + |b|^3 + |w|^3\right)\,\mathrm{d}x.\end{aligned}$$

Therefore from (11) we have (12), and consequently, we deduce that the convergence (13) holds.

Now we are ready to conclude $u \equiv b \equiv w \equiv 0$. From (10) together with (11) and (13), we have

$$\int_{B_\rho} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right)\,\mathrm{d}x \to 0 \qquad \text{as } \rho \to \infty,$$

which means that u, b and w are constants. By (11), we finally obtain that $u \equiv b \equiv w \equiv 0$.

4 | PROOF OF THEOREM 2

From (9) in the proof in Theorem 1, we know that

$$\begin{aligned}&\int_{B_\rho} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right)\,\mathrm{d}x\\ &\lesssim (\tau-\rho)^{-2}\int_{B_\tau\setminus B_\rho} \left(|u|^2 + |b|^2 + |w|^2\right)\,\mathrm{d}x + (\tau-\rho)^{-1}\int_{B_\tau\setminus B_\rho} \left(|u|^3 + |u||b|^2 + |u||w|^2\right)\,\mathrm{d}x \end{aligned}\tag{17}$$

In order to deal with the right-hand side of (17), let us consider two cases as follows:

(Case 1) $p \in [3, \frac{9}{2})$: Then we have by Hölder's inequality,

$$\int\limits_{B_\rho} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right) \, \mathrm{d}x \lesssim (\tau-\rho)^{-2}\|(u,b,w)\|^2_{L^2(B_{2R})} + (\tau-\rho)^{-1}\|(u,b,w)\|^3_{L^3(B_{2R})}$$
$$\lesssim \rho^{1-\frac{6}{p}}\|(u,b,w)\|^2_{L^p(B_{2\rho})} + \rho^{2-\frac{9}{p}}\|(u,b,w)\|^3_{L^p(B_{2\rho})},$$

where we have chosen $\tau = 2\rho$. Hence if we let $\rho \to \infty$, we have

$$\int\limits_{\mathbb{R}^3} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right) \, \mathrm{d}x = 0. \tag{18}$$

(Case 2) $p \in [2, \frac{18}{5})$: For this case, let us consider a non-negative cut-off function $\theta(x) = \xi_{\tau,2R}(x) - \xi_{\frac{R}{2},\rho}(x)$ with $\|\nabla\theta\|_{L^\infty} \lesssim \max\{\frac{1}{\rho-\frac{R}{2}}, \frac{1}{2R-\tau}\}$. Note that by the interpolation inequality,

$$\|w\|^2_{L^4(B_\tau\setminus B_\rho)} \lesssim \|w\theta\|^2_{L^4(\mathbb{R}^3)} \lesssim \|w\theta\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}\|\nabla(w\theta)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)}$$
$$\lesssim \|w\theta\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}\left(\|(\nabla w)\theta\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} + \|w(\nabla\theta)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)}\right)$$
$$\lesssim \|w\|^{\frac{1}{2}}_{L^2(B_{2R}\setminus B_{R/2})}\left(\|\nabla w\|^{\frac{3}{2}}_{L^2(B_{2R}\setminus B_{R/2})} + \frac{1}{R^{\frac{3}{2}}}\|w\|^{\frac{3}{2}}_{L^2(B_{2R}\setminus B_{R/2})}\right)$$
$$\lesssim \|w\|^{\frac{1}{2}}_{L^2(B_{2R}\setminus B_{R/2})}\|\nabla w\|^{\frac{3}{2}}_{L^2(B_{2R}\setminus B_{R/2})} + \frac{1}{R^{\frac{3}{2}}}\|w\|^2_{L^2(B_{2R}\setminus B_{R/2})}.$$

And thus, by Hölder's inequality and Young's inequality,

$$(\tau-\rho)^{-1}\int\limits_{B_\tau\setminus B_\rho} |u||w|^2 \, \mathrm{d}x \lesssim (\tau-\rho)^{-1}\|u\|_{L^2(B_\tau\setminus B_\rho)}\|w\|^2_{L^4(B_\tau\setminus B_\rho)}$$
$$\lesssim (\tau-\rho)^{-1}\|u\|_{L^2(B_{2R})}\left(\|w\|^{\frac{1}{2}}_{L^2(B_{2R})}\|\nabla w\|^{\frac{3}{2}}_{L^2(B_{2R})} + \frac{1}{R^{3/2}}\|w\|^2_{L^2(B_{2R})}\right)$$
$$\lesssim (\tau-\rho)^{-4}\|(u,w)\|^6_{L^2(B_{2R})} + \frac{1}{4}\|\nabla w\|^2_{L^2(B_{2R})} + \frac{(\tau-\rho)^{-1}}{R^{3/2}}\|(u,w)\|^3_{L^2(B_{2R})}.$$

Therefore, we have

$$\int\limits_{B_\rho} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right) \, \mathrm{d}x \le \frac{1}{4}\int\limits_{B_{2R}} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right) \, \mathrm{d}x + \frac{C}{(\tau-\rho)^2}\|(u,b,w)\|^2_{L^2(B_{2R})}$$
$$+ \frac{C}{(\tau-\rho)^4}\|(u,b,w)\|^6_{L^2(B_{2R})} + \frac{C}{(\tau-\rho)R^{\frac{3}{2}}}\|(u,b,w)\|^3_{L^2(B_{2R})}$$
$$\le \frac{1}{4}\int\limits_{B_{2R}} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right) \, \mathrm{d}x + CR^{1-\frac{6}{p}}\|(u,b,w)\|^2_{L^p(B_{2R})}$$
$$+ CR^{5-\frac{18}{p}}\|(u,b,w)\|^6_{L^p(B_{2R})} + CR^{2-\frac{9}{p}}\|(u,b,w)\|^3_{L^p(B_{2R})}$$

If we apply Lemma 1 with $f(r) := \int_{B_r}(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2)\,\mathrm{d}x$, we get

$$\int\limits_{B_{R/2}} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right) \, \mathrm{d}x \lesssim R^{1-\frac{6}{p}}\|(u,b,w)\|^2_{L^p(B_{2R})} + R^{5-\frac{18}{p}}\|(u,b,w)\|^6_{L^p(B_{2R})} + R^{2-\frac{9}{p}}\|(u,b,w)\|^3_{L^p(B_{2R})}$$

If we let $R \to \infty$ we can immediately find that

$$\int\limits_{\mathbb{R}^3} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right) \, \mathrm{d}x = 0,$$

which implies that u, b and w are a constant vectors in R^3. Since $u, b, w \in L^p(\mathbb{R}^3)$ for $p \in [2, \frac{9}{2})$, we conclude that $u \equiv b \equiv w \equiv 0$ in $\mathbb{R}^3$.

5 | PROOF OF THEOREM 3

We first recall the fact that any continuous functions vanishing at infinity must be bounded; thus, from the assumption of the theorem, we have $u, b, w \in L^\infty(\mathbb{R}^3)$. From (10) in the proof in Theorem 1, we know that

$$\int_{B_\rho} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right) \,\mathrm{d}x \lesssim (\tau-\rho)^{-2} \int_{B_\tau \setminus B_\rho} \left(|u|^2 + |b|^2 + |w|^2\right) \,\mathrm{d}x + (\tau-\rho)^{-1} \int_{B_\tau \setminus B_\rho} \left(|u|^3 + |b|^3 + |w|^3\right) \,\mathrm{d}x \tag{19}$$

In order to control the right-hand side in (19), let us consider it in two cases as follows:

(Case 1) $p \in [1, 3/2)$: We note that with the choice $\tau = 2R$ and $\rho = R$,

$$(\tau-\rho)^{-2} \int_{B_\tau \setminus B_\rho} |u|^2 \,\mathrm{d}x \lesssim (\tau-\rho)^{-2} \|u\|_{L^\infty(\mathbb{R}^3)} \int_{B_\tau \setminus B_\rho} |u| \,\mathrm{d}x \lesssim R^{1-\frac{3}{p}} \|u\|_{L^p(B_{2R})}.$$

And thus, with the same arguments for b and w, we get

$$(\tau-\rho)^{-2} \int_{B_\tau \setminus B_\rho} \left(|u|^2 + |b|^2 + |w|^2\right) \,\mathrm{d}x \lesssim R^{1-\frac{3}{p}} \|(u, b, w)\|_{L^p(B_{2R})}.$$

For the second term on the right-hand side of (19),

$$(\tau-\rho)^{-1} \int_{B_\tau \setminus B_\rho} |u|^3 \,\mathrm{d}x \lesssim (\tau-\rho)^{-1} \|u\|^2_{L^\infty(\mathbb{R}^3)} \int_{B_\tau \setminus B_\rho} |u| \,\mathrm{d}x \lesssim R^{2-\frac{3}{p}} \|u\|_{L^p(B_{2R})},$$

and similarly for b and w, we have

$$(\tau-\rho)^{-1} \int_{B_\tau \setminus B_\rho} \left(|u|^3 + |b|^3 + |w|^3\right) \,\mathrm{d}x \lesssim R^{2-\frac{3}{p}} \|(u, b, w)\|_{L^p(B_{2R})}. \tag{20}$$

(Case 2) $p \in [\frac{3}{2}, \frac{9}{4})$: In this case, we note that

$$(\tau-\rho)^{-2} \int_{B_\tau \setminus B_\rho} |u|^2 \,\mathrm{d}x \lesssim (\tau-\rho)^{-2} \|w\|^{1/2}_{L^\infty(\mathbb{R}^3)} \int_{B_\tau \setminus B_\rho} |w|^{\frac{3}{2}} \,\mathrm{d}x \lesssim R^{1-\frac{9}{2p}} \|w\|^{\frac{3}{2}}_{L^p(B_{2R})},$$

and in the same way for b and w, we have

$$(\tau-\rho)^{-2} \int_{B_\tau \setminus B_\rho} \left(|u|^2 + |b|^2 + |w|^2\right) \,\mathrm{d}x \lesssim R^{1-\frac{9}{2p}} \|(u, b, w)\|^{\frac{3}{2}}_{L^p(B_{2R})}.$$

For the second term, in a similar way, we also get,

$$(\tau-\rho)^{-1} \int_{B_\tau \setminus B_\rho} |u|^3 \,\mathrm{d}x \lesssim (\tau-\rho)^{-1} \|u\|^{\frac{3}{2}}_{L^\infty(\mathbb{R}^3)} \int_{B_\tau \setminus B_\rho} |u|^{\frac{3}{2}} \,\mathrm{d}x \lesssim R^{1-\frac{3}{p}} \|u\|_{L^p(B_{2R})},$$

which directly implies that

$$(\tau-\rho)^{-1} \int_{B_\tau \setminus B_\rho} \left(|u|^3 + |b|^3 + |w|^3\right) \,\mathrm{d}x \lesssim R^{1-\frac{3}{p}} \|(u, b, w)\|_{L^p(B_{2R})}. \tag{21}$$

Collecting (20) and (21), we have

$$\int_{B_R} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right)\, \mathrm{d}x \lesssim \begin{cases} (R^{1-\frac{3}{p}} + R^{2-\frac{3}{p}})\|(u,b,w)\|_{L^p(B_{2R})}, & p \in [1, \frac{3}{2}), \\ (R^{1-\frac{9}{2p}} + R^{2-\frac{9}{2p}})\|(u,b,w)\|^{3/2}_{L^p(B_{2R})}, & p \in [\frac{3}{2}, \frac{9}{4}). \end{cases}$$

If we let $R \to +\infty$ in the above estimate, we can immediately find that

$$\int_{\mathbb{R}^3} \left(|\nabla u|^2 + |\nabla b|^2 + |\nabla w|^2\right) dx = 0,$$

which implies that u, b and w are a constant vectors in $\mathbb{R}^3$. As we know $u, b, w \in L^p(\mathbb{R}^3)$ for $p \in [1, \frac{9}{4})$, we finally conclude that $u \equiv b \equiv w \equiv 0$ in $\mathbb{R}^3$.

ACKNOWLEDGMENTS

Jae-Myoung Kim was supported by National Research Foundation of Korea Grant funded by the Korean Government (NRF-2020R1C1C1A01006521).

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