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RESEARCH ARTICLE

Partial practical stability for fractional-order nonlinear systems

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Summary

In this paper, using the Lyapunov-like functions, the practical stability with respect to part of the variables of fractional-order nonlinear systems depending on a small parameter is studied.

KEYWORDS:
Fractional differential equations, stability, Lyapunov function.

1 | INTRODUCTION

In the last few decades, the field of fractional calculus has obtained significant popularity and influence due mainly to its practical applications in many areas of engineering and sciences. On the work of[12,15,19] the reader can refer to important sources of various applications and examples in aerodynamics, chemistry, physics and thermo-elasticity. In recent years, there has been a rapid evolution in the theoretical features, for example asymptotic behavior, periodicity, controllability, observability, and many others.
Analogical to the qualitative particular of the non-integer and fractional differential equations, the generation of the Lyapunov theory, fixed point theory and the Mittag-Leffler function allows to construct many and remarkable results in the stability, the exponential stability, and the Mittag-Leffler stability[2,4,5,6,7,8,9,10,13,14,16,20,21,24,25,26,27,28]. Indeed, Authors in[4] have studied a converse Lyapunov theorem for the notion of uniform practical exponential stability of parametric differential equations in presence of small perturbation. In[5], Abdellatif ben Makhlouf has described the stability with respect to part of the variables of nonlinear Caputo fractional differential equations. On the other hand, the authors in[6] have introduced a practical Mittag-Leffler stability for fractional-order nonlinear systems depending on a parameter. In addition, authors in[13] has introduced a smooth solutions to the mixed-order fractional differential systems with applications to stability analysis. Furthermore, Boulbaba Ghanmi in[14] has studied the practical exponential stability result for impulsive dynamic systems depending on a parameter.
To the best of our knowledge, there is no works in literature treats the same subject on the fractional-order nonlinear systems.
In the present paper, the notion of partial practical asymptotic stability of nonlinear fractional-order systems depending on a parameter is introduced and described. Such stability ensures the convergence of a part of the solutions towards a ball containing the origin of the state space as the radius of the ball can be made arbitrarily small. The main objective of the work is to investigate the partial practical stability of nonlinear fractional-order systems depending on a parameter by using the Lyapunov techniques. Precisely, sufficient conditions are given to ensures the partial practical stability of such systems.
The present paper is organized as follows. In Section 2, some necessary definitions and Lemmas are presented. In

section 3, some sufficient conditions on practical stability of nonlinear fractional differential equations are given. In section 4, some classes of triangular fractional systems are studied in point of view of stability. Three examples are provided in Section 5. Finally, in Section 6 some conclusions are given.

2 | PRELIMINARIES

In this section, some notations and preliminaries results are introduced.

Definition 1. [11] Given an interval $[a,b]$ of R, the Riemann-Liouville fractional integral of a function $x \in L^1([a,b])$ of order $\alpha > 0$ is defined by

$$I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \quad t \in [a,b],$$

where Γ is the Gamma function.
For $\alpha = 0$, $I_a^0 := I$, the identity operator.
Definition 2. [11] Given an interval $[a,b]$ of R, the Caputo fractional derivative of a function x of order $\alpha > 0$ is defined by

$${}^C D_{a,t}^\alpha x(t) = I_a^{m-\alpha} x^{(m)}(t), \quad t \in [a,b],$$

where $0 < m-1 < \alpha \leq m$.

When $0 < \alpha < 1$, then the Caputo fractional derivative of order α of an absolutely continuous function x on $[a,b]$ reduces to

$${}^C D_{t_0,t}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-\tau)^{-\alpha} x'(\tau) d\tau, \quad t \in [a,b]. \tag{1}$$

Lemma 1. [1] Let $\alpha \in (0,1)$ and $P \in R^{n\times n}$ a constant, square, symmetric and positive definite matrix. Then the following relationship holds

$$\frac{1}{2}\, {}^C D_{t_0,t}^\alpha (x^T(t) P x(t)) \leq x^T(t) P\, {}^C D_{t_0,t}^\alpha x(t), \ t \geq t_0.$$

Definition 3. [17] The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha > 0$, $\beta > 0$, $z \in C$.
When $\beta = 1$, we have $E_\alpha(z) = E_{\alpha,1}(z)$.

We consider the nonhomogeneous linear fractional differential equation with Caputo fractional derivative

$$\begin{aligned} {}^C D_{t_0,t}^\alpha x(t) &= \lambda x + h(t), \ t \geq t_0 \\ x(t_0) &= x_0. \end{aligned} \tag{2}$$

The problem (2) has been studied by Kilbas et al.[17] (see pp. 295, (5.2.83)), and its solution has the form

$$x(t;t_0,x_0) = x_0 E_\alpha(\lambda(t-t_0)^\alpha) + \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) h(s) ds. \tag{3}$$

Lemma 2. [8] If one sets $h(t) = d$ in (2) with a constant d, then the solution reduces to

$$x(t;t_0,x_0) = x_0 E_\alpha(\lambda(t-t_0)^\alpha) + d(t-t_0)^\alpha E_{\alpha,\alpha+1}(\lambda(t-t_0)^\alpha). \tag{4}$$

Lemma 3. [23] For $0 < \alpha < 1$, we have $E_\alpha(-t)$ is nonincreasing in t.
Lemma 4. [8] Let $0 < \alpha < 1$ and $|arg(\lambda)| > \frac{\pi\alpha}{2}$. Then, one has

$$t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha) = -\frac{1}{\lambda} - \frac{1}{\Gamma(1-\alpha)\lambda^2 t^\alpha} + O\big(\frac{1}{\lambda^3 t^{2\alpha}}\big) \ as \ t \longrightarrow \infty.$$

Definition 4. A continuous function $\psi : R_+ \longrightarrow R_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\psi(0) = 0$. It is to belong to class $\mathcal{K}_\infty$ if in addition $\lim\limits_{s \longrightarrow +\infty} \psi(s) = +\infty$.

Lemma 5. [27] If $\psi \in \mathcal{K}$, then for all $a_1, a_2 \in R_+$, we have

$$\psi(a_1 + a_2) \leq \psi(2a_1) + \psi(2a_2).$$

Lemma 6. [18] For all $p \geq 1$ and $a, b \geq 0$, we have $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ and $(a+b)^{\frac{1}{p}} \leq (a^{\frac{1}{p}} + b^{\frac{1}{p}})$.

3 | PARTIAL PRACTICAL STABILITY

Consider a parameterized family of fractional differential equations with a Caputo derivative for $0 < \alpha < 1$ having the following form:

$$\begin{gathered} {}^{C}D^{\alpha}_{t_0,t} x(t) \;=\; f(t,x,\varepsilon),\ t \geq t_0, \\ x \in R^n,\ x = (y,z)\quad,\quad y \in R^m,\ z \in R^p,\ m > 0, \end{gathered} \tag{5}$$

with initial condition $x(t_0) = x_0 = (y_0, z_0)$, where $\alpha \in (0,1)$, $t_0 \in R_+$, $\varepsilon \in R_+^*$ and $f(.,.,\varepsilon) : R_+ \times R^n \longrightarrow R^n$. Suppose that $f(.,.,\varepsilon)$ is smooth enough to ensure the existence and uniqueness of global solutions for each initial condition (t_0, x_0). Some sufficient conditions for the existence and uniqueness of global solutions $x_\varepsilon(t; t_0, x_0) \in C[t_0, +\infty) \cap C^1(t_0, +\infty)$ are given in [3,9,22,13].

Definition 5. The fractional-order system (5) is said to be

(i) ε^*-practically uniformly stable with respect to y, if for every $c_2 > 0$ there exist $c_1 > 0$ and $\hat{\varepsilon} \in]0, \varepsilon^*]$ such that for all $t_0 \in R_+$, for all $x_0 \in R^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in]0, \hat{\varepsilon}]$, $\|y_\varepsilon(t; t_0, x_0)\| < c_2$ for all $t \geq t_0$.

(ii) ε^*-practically uniformly bounded with respect to y, if for every $c_1 > 0$, there exist $c_2 > 0$ and $\hat{\varepsilon} \in]0, \varepsilon^*]$ such that for all $t_0 \in R_+$, for all $x_0 \in R^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in]0, \hat{\varepsilon}]$, $\|y_\varepsilon(t; t_0, x_0)\| < c_2$ for all $t \geq t_0$.

(iii) ε^*-globally uniformly practically attractive with respect to y, if for every $c_1 > 0$, $c_2 > 0$ there exists $T > 0$ and $\hat{\varepsilon} \in]0, \varepsilon^*]$ such that for all $t_0 \in R_+$, for all $x_0 \in R^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in]0, \hat{\varepsilon}]$, $\|y_\varepsilon(t; t_0, x_0)\| < c_2$ for all $t \geq t_0 + T$.

(iv) ε^*-practically globally uniformly asymptotically stable with respect to y, if it is ε^*-practically uniformly stable with respect to y, ε^*-practically uniformly bounded with respect to y and ε^*-globally uniformly practically attractive with respect to y.

(v) ε^*-practically uniformly Mittag Leffler stable with respect to y, if for all $0 < \varepsilon \leq \varepsilon^*$ there exist positive scalars $K(\varepsilon)$, $\lambda(\varepsilon)$ and $\rho(\varepsilon)$ such that:

$$\|y_\varepsilon(t; t_0, x_0)\| \leq \Big[K(\varepsilon) m\big(x_0\big) E_\alpha\big(-\lambda(\varepsilon)(t-t_0)^\alpha\big)\Big]^b + \rho(\varepsilon),\ \forall t \geq t_0 \geq 0, \tag{6}$$

with $b > 0$, $\rho(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$ and there exist $K,\ \lambda_1,\ \lambda_2 > 0$ such that $\lambda_1 \leq \lambda(\varepsilon) \leq \lambda_2$, $0 < K(\varepsilon) \leq K$ for all $\varepsilon \in]0, \varepsilon^*]$, $m(0) = 0$, $m(x) \geq 0$ and m is locally Lipschitz.

Theorem 1. Let $\varepsilon^* > 0$. Assume that for all $0 < \varepsilon \leq \varepsilon^*$, there exist a continuously differentiable function $V_\varepsilon : R_+ \times R^n \longrightarrow R$, a continuous function $\mu : R_+ \longrightarrow R_+$, class $\mathcal{K}$ functions α_i, $(i = 1, 2)$ and positive constants scalar $a_1(\varepsilon)$, $a_2(\varepsilon)$, $r_1(\varepsilon)$ and $r_2(\varepsilon)$ such that

1. $$a_1(\varepsilon)\alpha_1(\|y\|) \leq V_\varepsilon(t,x) \leq a_2(\varepsilon)\alpha_2(\|w\|) + r_1(\varepsilon),\ \forall t \geq 0,\ x \in R^n. \tag{7}$$

2. $${}^{C}D^{\alpha}_{t_0,t} V_\varepsilon(t, x_\varepsilon(t; t_0, x_0)) \leq \mu(t) r_2(\varepsilon),\ \forall t \geq t_0, \tag{8}$$

   where $w = (x_1, x_2, ..., x_k)$, $k \in \{m, m+1, ...n\}$.

with

- $\forall \varepsilon \in]0, \varepsilon^*]$, $0 < \frac{a_2(\varepsilon)}{a_1(\varepsilon)} \leq K$, where $K > 0$.
- $t \mapsto \int_0^t (t-s)^{\alpha-1}\mu(s)ds$ is a bounded function.
- $d(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$ where
$$d(\varepsilon) = \frac{\big(r_1(\varepsilon) + M r_2(\varepsilon)\big)}{a_1(\varepsilon)},$$
with
$$M = \sup_{t \geq 0} \int_0^t (t-s)^{\alpha-1}\mu(s)ds.$$

Then, the system (5) is ε^*-practically uniformly stable with respect to y.
Moreover, if $\alpha_i \in \mathcal{K}_\infty$, $(i = 1, 2)$, then, the system (5) is ε^*-practically uniformly bounded with respect to y.

Proof. Let $c_2 > 0$.
We consider $c_1 > 0$ and $\hat{\varepsilon} \in]0, \varepsilon^*]$ such that
$$\alpha_2(c_1) < \frac{\alpha_1(c_2)}{2K}$$
and
$$\frac{r_1(\varepsilon) + r_2(\varepsilon)M}{a_1(\varepsilon)} < \frac{\alpha_1(c_2)}{2}, \ \forall \varepsilon \in]0, \hat{\varepsilon}].$$
It follows from (8) and (7) that
$$\begin{aligned} V_\varepsilon(t, x_\varepsilon(t; t_0, x_0)) &\leq V_\varepsilon(t_0, x_0) + r_2(\varepsilon)\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\mu(s)ds \\ &\leq a_2(\varepsilon)\alpha_2(\|x_0\|) + r_1(\varepsilon) + r_2(\varepsilon)M. \end{aligned} \tag{9}$$
Then,
$$\begin{aligned} \alpha_1(\|y_\varepsilon(t; t_0, x_0)\|) &\leq \frac{a_2(\varepsilon)}{a_1(\varepsilon)}\alpha_2(\|x_0\|) + \frac{r_1(\varepsilon) + r_2(\varepsilon)M}{a_1(\varepsilon)} \\ &\leq K\alpha_2(c_1) + \frac{r_1(\varepsilon) + r_2(\varepsilon)M}{a_1(\varepsilon)} \\ &< \alpha_1(c_2), \ \forall t \geq t_0. \end{aligned} \tag{10}$$
Thus,
$$\|y_\varepsilon(t; t_0, x_0)\| < c_2, \ \forall t \geq t_0.$$
Therefore, the system (5) is ε^*-practically uniformly stable with respect to y.
Considering now the case where $\alpha_i \in \mathcal{K}_\infty$, $(i = 1, 2)$.
Let $d_1 > 0$.
We consider $\hat{\varepsilon}_1 \in]0, \varepsilon^*]$ such that
$$\frac{r_1(\varepsilon) + r_2(\varepsilon)M}{a_1(\varepsilon)} < 1, \ \forall \varepsilon \in]0, \hat{\varepsilon}_1].$$
In this case (10) becames :
$$\alpha_1((\|y_\varepsilon(t; t_0, x_0)\|) < K\alpha_2(d_1) + 1.$$
Then,
$$\|y_\varepsilon(t; t_0, x_0)\| < \alpha_1^{-1}\big(K\alpha_2(d_1) + 1\big), \ \forall t \geq t_0.$$
Hence, the system (5) is ε^*-practically uniformly bounded with respect to y.

$\square$

Remark 1. For the ε^*-practical uniform boundedness of system (5), it suffices to take the condition $d(\varepsilon) \longrightarrow 0$ is bounded on $(0, \varepsilon^*]$ instead of the condition $\lim_{\varepsilon \to 0} d(\varepsilon) = 0$.

Theorem 2. Let $\varepsilon^* > 0$. Assume that for all $0 < \varepsilon \leq \varepsilon^*$, there exist a continuously differentiable function $V_\varepsilon : R_+ \times R^n \longrightarrow R$, a continuous function $\mu : R_+ \longrightarrow R_+$, class $\mathcal{K}_\infty$ functions α_i, $(i = 1, 2)$ and positive constants scalar $a_1(\varepsilon)$, $a_2(\varepsilon)$, $a_3(\varepsilon)$, $r_1(\varepsilon)$ and $r_2(\varepsilon)$ such that

1.
$$a_1(\varepsilon)\alpha_1(\|y\|) \leq V_\varepsilon(t, x) \leq a_2(\varepsilon)\alpha_2(\|w\|) + r_1(\varepsilon), \ \forall t \geq 0, \ x \in R^n. \tag{11}$$

2.
$${}^C D^\alpha_{t_0,t} V_\varepsilon(t, x_\varepsilon(t)) \leq -a_3(\varepsilon)\alpha_2(\|x_\varepsilon(t)\|) + \mu(t) r_2(\varepsilon), \ \forall t \geq t_0, \tag{12}$$
where $w = (x_1, x_2, ..., x_k)$, $k \in \{m, m+1, ...n\}$.

with

- $\forall \varepsilon \in]0, \varepsilon^*]$, $\frac{a_3(\varepsilon)}{a_2(\varepsilon)} \geq c$ and $0 < \frac{a_2(\varepsilon)}{a_1(\varepsilon)} \leq K$, with c, $K > 0$.
- $t \mapsto \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\Big(-c(t-s)^\alpha\Big)\mu(s)ds$ is a bounded function.
- $d(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$ where
$$d(\varepsilon) = r_1(\varepsilon)\frac{(a_2(\varepsilon) + M a_3(\varepsilon))}{a_1(\varepsilon)a_2(\varepsilon)} + r_2(\varepsilon)\frac{M}{a_1(\varepsilon)},$$
with, $M = M_1 + M_2$, where
$$M_1 = \sup_{s \geq 0}\Big(s^\alpha E_{\alpha,\alpha+1}\big(-cs^\alpha\big)\Big)$$
and
$$M_2 = \sup_{t \geq 0} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\Big(-c(t-s)^\alpha\Big)\mu(s)ds.$$

Then, the system (5) is ε^*-practically globally uniformly asymptotically stable with respect to y.

Proof. From equations (11) and (12) we give :
$$\begin{aligned} {}^C D^\alpha_{t_0,t} V_\varepsilon(t, x_\varepsilon(t; t_0, x_0)) &\leq -cV_\varepsilon(t, x_\varepsilon(t; t_0, x_0)) + \frac{a_3(\varepsilon)}{a_2(\varepsilon)} r_1(\varepsilon) + \mu(t) r_2(\varepsilon) \\ &\leq -cV_\varepsilon(t, x_\varepsilon(t; t_0, x_0)) + \rho(t) l(\varepsilon), \ \forall t \geq t_0, \end{aligned} \tag{13}$$
where $\rho(t) = \big(1 + \mu(t)\big)$ and $l(\varepsilon) = r_1(\varepsilon)\frac{a_3(\varepsilon)}{a_2(\varepsilon)} + r_2(\varepsilon)$.
Let
$$M(t) = {}^C D^\alpha_{t_0,t} V_\varepsilon(t, x_\varepsilon(t; t_0, x_0)) + cV_\varepsilon(t, x_\varepsilon(t; t_0, x_0)).$$
Then,
$$\begin{aligned} V_\varepsilon(t, x_\varepsilon(t; t_0, x_0)) &= E_\alpha\big(-c(t-t_0)^\alpha\big)V_\varepsilon(t_0, x_0) + \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\Big(-c(t-s)^\alpha\Big)M(s)ds \\ &\leq E_\alpha\big(-c(t-t_0)^\alpha\big)V_\varepsilon(t_0, x_0) + l(\varepsilon)\int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\Big(-c(t-s)^\alpha\Big)\rho(s)ds. \end{aligned} \tag{14}$$
Thus,
$$V_\varepsilon(t, x_\varepsilon(t; t_0, x_0)) \leq E_\alpha\big(-c(t-t_0)^\alpha\big)V_\varepsilon(t_0, x_0) + Ml(\varepsilon), \ \forall \ t \geq t_0.$$

By (11), we get

$$\alpha_1\Big(\|y_\varepsilon(t;t_0,x_0)\|\Big) \le \frac{1}{a_1(\varepsilon)}\Big(E_\alpha\big(-c(t-t_0)^\alpha\big)\big(a_2(\varepsilon)\alpha_2(\|x_0\|)+r_1(\varepsilon)\big)+\frac{Ml(\varepsilon)}{a_1(\varepsilon)}. \tag{15}$$

Using Lemma 3, we get

$$\alpha_1\Big(\|y_\varepsilon(t;t_0,x_0)\|\Big) \le \frac{a_2(\varepsilon)}{a_1(\varepsilon)}E_\alpha\big(-c(t-t_0)^\alpha\big)\alpha_2(\|x_0\|)+c(\varepsilon). \tag{16}$$

It follows from Lemma 11 that

$$\|y_\varepsilon(t;t_0,x_0)\| \le \alpha_1^{-1}\Big(2KE_\alpha\big(-c(t-t_0)^\alpha\big)\alpha_2(\|x_0\|)\Big)+\alpha_1^{-1}\Big(2c(\varepsilon)\Big). \tag{17}$$

Now, we will prove that (i), (ii) and (iii) are satisfied.
Let $c_2>0$.
From (17), we get

$$\|y_\varepsilon(t;t_0,x_0)\| \le \alpha_1^{-1}\Big(2K\alpha_2(\|x_0\|)\Big)+\alpha_1^{-1}\Big(2c(\varepsilon)\Big). \tag{18}$$

We have

$$\lim_{r\to 0}\alpha_1^{-1}\big(2K\alpha_2(r)\big)=0,$$

and

$$\lim_{\varepsilon\to 0}\alpha_1^{-1}\big(2c(\varepsilon)=0.$$

then, there exist $c_1>0$ and $\hat{\varepsilon}\in(0,\varepsilon^*]$ such that

$$\alpha_1^{-1}\big(2K\alpha_2(\|x_0\|)\big)<\frac{c_2}{2},\ \forall\ \|x_0\|<c_1$$

and

$$\alpha_1^{-1}\big(2c(\varepsilon)<\frac{c_2}{2},\ \forall\ \varepsilon\in(0,\hat{\varepsilon}).$$

Then,

$$\|y_\varepsilon(t;t_0,x_0)\|<c_2,\ \forall\ t\ge t_0,\ \forall\ \|x_0\|<c_1,\ \forall\ \varepsilon\in(0,\hat{\varepsilon}).$$

Hence, (i) is satisfied.
Let $d_1>0$,
We have

$$\lim_{\varepsilon\to 0}\alpha_1^{-1}\big(2c(\varepsilon)\big)=0,$$

then, there exists $\hat{\varepsilon}_1>0$ such that

$$\alpha_1^{-1}\big(2c(\varepsilon)\big)<1,\ \forall\ \varepsilon\in(0,\hat{\varepsilon}_1).$$

We have

$$\alpha_1^{-1}\big(2K\alpha_2(\|x_0\|)\big)<\alpha_1^{-1}\big(2K\alpha_2(d_1)\big),\quad\forall\ \|x_0\|<d_1,$$

then for $d_2=\alpha_1^{-1}\big(2K\alpha_2(c_1)\big)+1$, we get

$$\|y_\varepsilon(t;t_0,x_0)\|<d_2,\ \forall\ \ \|x_0\|<d_1,\ \forall\ \varepsilon\in(0,\hat{\varepsilon}_1).$$

Hence, (ii) is satisfied.
Let $\delta_1>0$, $\delta_2>0$.
Let $x_0\in R^n$ such that $\|x_0\|<\delta_1$. It follows from (17) that

$$\|y_\varepsilon(t;t_0,x_0)\| \le \alpha_1^{-1}\Big(2KE_\alpha\big(-c(t-t_0)^\alpha\big)\alpha_2(\delta_1)\Big)+\alpha_1^{-1}\Big(2c(\varepsilon)\Big).$$

We have

$$\lim_{s\to+\infty}E_\alpha\big(-cs^\alpha\big)=0,$$

then there exist $T>0$ and $\hat{\varepsilon}_2>0$ such that

$$\alpha_1^{-1}\Big(2KE_\alpha\big(-c(t-t_0)^\alpha\big)\alpha_2(\delta_1)\Big)<\frac{\delta_2}{2},\quad\forall\ t-t_0\ge T,$$

and

$$\alpha_1^{-1}\big(2c(\varepsilon)\big)<\frac{\delta_2}{2},\quad\forall\ \varepsilon\in(0,\hat{\varepsilon}_2).$$

Thus,

$$\|y_\varepsilon(t;t_0,x_0)\| < \delta_2, \quad \forall\, t-t_0 \geq T,\ \forall \varepsilon \in (0,\hat{\varepsilon}_2).$$

Hence, (iii) is satisfied.
Therefore, the system (5) is ε^*-practically globally uniformly asymptotically stable with respect to y. $\square$

Remark 2. In the case when $\alpha_1(s) = c_1 s^p$ and $\alpha_1(s) = c_2 s^p$ with $c_1, c_2 > 0$ and $p \geq 1$, we get the practical Mittag Leffler stability with respect to y for the system (5).

4 | STABILITY ANALYSIS FOR A CLASS OF TRIANGULAR SYSTEMS

In this section, we consider the following triangular system depending on a parameter $\varepsilon > 0$:

$$\begin{aligned} {}^C D^\alpha_{t_0,t} y(t) &= Ay + g_1(t,y,z,\varepsilon), && (19)\\ {}^C D^\alpha_{t_0,t} z(t) &= g_2(t,z,\varepsilon),\ t \geq t_0, && (20)\end{aligned}$$

where $y \in R^m$, $z \in R^p$. For the study of stability of system (19)-(20), we make the following Hypothesis:
(H_1) : (20) is ε_0-practically uniformly bounded.
(H_2) : The function g_1 satisfies for all $t \geq 0$, $\varepsilon > 0$ and $y \in R^m$, $z \in R^p$

$$\|g_1(t,y,z,\varepsilon)\| \leq \delta_1(\varepsilon)\nu_1(t) + \delta_2(\varepsilon)\|y\|, \tag{21}$$

such that $\delta_1(\varepsilon)$, $\delta_2(\varepsilon) > 0$ and $\delta_1(\varepsilon)$, $\delta_2(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$ and ν_1 is a nonnegative continuous function.
(H_1') : (20) is ε_h-practically globally uniformly asymptotically stable.
(H_2') : The function g_1 satisfies for all $t \geq 0$, $\varepsilon > 0$ and $y \in R^m$, $z \in R^p$

$$\|g_1(t,y,z,\varepsilon)\| \leq \delta_3(\varepsilon)\nu_2(t) + \delta_4(\varepsilon)\|y\| + \delta_5(\varepsilon)\|z\|, \tag{22}$$

such that $\delta_3(\varepsilon)$, $\delta_4(\varepsilon) > 0$, $\delta_5(\varepsilon) > 0$ and $\delta_3(\varepsilon)$, $\delta_4(\varepsilon)$, $\delta_5(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$ and ν_2 is a nonnegative continuous function.
For the partial stability of system (19)-(20), we have the following results.

Theorem 3. Suppose that (H_2) holds, the system (19)-(20) is ε_1-practically globally uniformly asymptotically stable with respect to y for some $\varepsilon_1 > 0$ if there exists a symmetric positive definite matrix P, $\eta > 0$ and $\lambda \in]0,\eta[$ such that

$$A^T P + PA + \eta I < 0, \tag{23}$$

and

$$t \mapsto \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\Big(-\frac{(\eta-\lambda)}{\lambda_{max}(P)}(t-s)^\alpha\Big)\nu_1^2(s)ds \textit{ is a bounded function.}$$

Proof. We have from (32)

$$\begin{aligned} 2y^T Pg(t,y,z,\varepsilon) &\leq 2\|y\|\|P\|\|g(t,y,z,\varepsilon)\|\\ &\leq 2\|y\|\|P\|\big(\delta_1(\varepsilon)\nu_1(t) + \delta_2(\varepsilon)\|y\|\big). \end{aligned} \tag{24}$$

Let $0 < \lambda_1 < \lambda$, we have

$$2\|y\|\|P\|\delta_1(\varepsilon)\nu_1(t) \leq \lambda_1\|y\|^2 + \frac{\|P\|^2\delta_1(\varepsilon)^2\nu_1^2(t)}{\lambda_1}. \tag{25}$$

Then,

$$2y^T Pg(t,y,z,\varepsilon) \leq \big(\lambda_1 + 2\delta_2(\varepsilon)\|P\|\big)\|y\|^2 + \frac{\|P\|^2\delta_1(\varepsilon)^2\nu_1^2(t)}{\lambda_1}.$$

It follows from $\lim\limits_{\varepsilon\to 0}\delta_2(\varepsilon) = 0$ that there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in]0,\varepsilon_1]$, we have

$$2y^T Pg(t,y,z,\varepsilon) \leq \lambda\|y\|^2 + \frac{\|P\|^2\delta_1(\varepsilon)^2\nu_1^2(t)}{\lambda_1}. \tag{26}$$

Let $\varepsilon \in]0, \varepsilon_1]$. We consider the Lyapunov-like function $V(t, y, z) = y^T P y$.
Using Lemma 1, we get

$$\begin{aligned} {}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t; t_0, x_0)) &\leq 2 y_\varepsilon(t; t_0, x_0)^T P\, {}^C D^{\alpha}_{t_0,t} y_\varepsilon(t; t_0, x_0) \\ &\leq y_\varepsilon(t; t_0, x_0)^T \big(A^T P + PA\big) y_\varepsilon(t; t_0, x_0) \\ &\quad + 2 y_\varepsilon(t; t_0, x_0)^T P g(t, y_\varepsilon(t; t_0, x_0), z_\varepsilon(t; t_0, x_0), \varepsilon). \end{aligned} \tag{27}$$

Using (32), (26) and (27), we get

$${}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t; t_0, x_0)) \leq -\sigma \|y_\varepsilon(t; t_0, x_0)\|^2 + \frac{\|P\|^2 \delta_1(\varepsilon)^2 \nu_1^2(t)}{\lambda_1},$$

with $\sigma = \eta - \lambda$.
Then, the assumptions of Theorem 1 are satisfied.
Consequently, the system (19)-(20) is ε_1-practically globally uniformly asymptotically stable with respect to y. □

Theorem 4. Suppose that (H_1), (H_2') hold, the system (19)-(20) is ε_1-practically globally uniformly asymptotically stable with respect to y for some $\varepsilon_1 > 0$ if there exists a symmetric positive definite matrix P, $\eta > 0$ and $\lambda \in]0, \eta[$ such that

$$A^T P + PA + \eta I < 0, \tag{28}$$

and

$$t \mapsto \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\Big(-\frac{(\eta-\lambda)}{\lambda_{max}(P)}(t-s)^{\alpha}\Big)\nu_2^2(s) ds \ is\ a\ bounded\ function.$$

Proof. Let $0 < \lambda_1 < \lambda$, as the same in Theorem 3, there exists ε_1 such that we have the following estimation:

$$2y^T P g(t, y, z, \varepsilon) \leq \lambda \|y\|^2 + \frac{\|P\|^2 \delta_1(\varepsilon)^2 \nu_2^2(t) + \|P\|^2 \delta_3(\varepsilon)^2 \|z\|^2}{\lambda_1}, \ \forall \varepsilon \in (0, \varepsilon_1]. \tag{29}$$

Now, we start by proving the points (i), (ii), and (iii) for the practical uniform stability.
First step: We will prove the practical uniform stability of system (19)-(20).
Let $c_2 > 0$.
It follows from (H_1) that there exist $d > 0$, $\varepsilon_0' > 0$ such that

$$z_\varepsilon(t; t_0, x_0) < d, \ \forall t \geq t_0, \ \forall \|z_0\| < 1, \ \forall \varepsilon \in (0, \varepsilon_0').$$

Let $\varepsilon_1 = min(\varepsilon_0, \varepsilon_0')$.
We consider the Lyapunov-like function $V(t, x) = y^T P y$.
We get for $\|x_0\| < 1$,

$${}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t; t_0, x_0)) \leq -\sigma \|y_\varepsilon(t; t_0, x_0)\|^2 + \frac{\|P\|^2 \delta_1(\varepsilon)^2 \nu_2^2(t) + \|P\|^2 d^2 \delta_3(\varepsilon)^2}{\lambda_1},$$

with $\sigma = \eta - \lambda$.
As the same in Theorem 1, we obtain the practical uniform stability of the system (19)-(20).
Second step: We will prove the practical uniform boundedness and the practical uniform attractivity of the system (19)-(20).
Let $c_1 > 0$.
It follows from (H_1) that there exist $d_2 > 0$, $\varepsilon_1' > 0$ such that

$$z_\varepsilon(t; t_0, x_0) < d_2, \ \forall t \geq t_0, \ \forall \|z_0\| < c_1, \ \forall \varepsilon \in (0, \varepsilon_1').$$

In this case, we obtain for $\varepsilon < min(\varepsilon_0, \varepsilon_1')$, and $\|x_0\| < c_1$

$${}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t; t_0, x_0)) \leq -\sigma \|y_\varepsilon(t; t_0, x_0)\|^2 + \frac{\|P\|^2 \delta_1(\varepsilon)^2 \nu_2^2(t) + \|P\|^2 d_2^2 \delta_3(\varepsilon)^2}{\lambda_1}, \tag{30}$$

with $\sigma = \eta - \lambda$.
As the same in Theorem 1, we obtain from (30) the practical uniform boundedness and the practical uniform attractivity of the system (19)-(20).

□

In the case when the system (20) is ε^*-practically globally uniformly asymptotically stable for some $\varepsilon^* > 0$, we deduce from Theorem 3 and Theorem 4 the following results about the global practical stability of the system (19)-(20).

Theorem 5. Suppose that (H_2), (H_1') hold, the system (19)-(20) is ε_1-practically globally uniformly asymptotically stable for some $\varepsilon_1 > 0$ if there exists a symmetric positive definite matrix P, $\eta > 0$ and $\lambda \in]0, \eta[$ such that

$$A^T P + PA + \eta I < 0, \tag{31}$$

and

$$t \mapsto \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\Big(-\frac{(\eta-\lambda)}{\lambda_{max}(P)}(t-s)^{\alpha}\Big)\nu_1^2(s)ds \textit{ is a bounded function.}$$

Theorem 6. Suppose that (H_1'), (H_2') hold, the system (19)-(20) is ε_1-practically globally uniformly asymptotically stable for some $\varepsilon_1 > 0$ if there exists a symmetric positive definite matrix P, $\eta > 0$ and $\lambda \in]0, \eta[$ such that

$$A^T P + PA + \eta I < 0, \tag{32}$$

and

$$t \mapsto \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\Big(-\frac{(\eta-\lambda)}{\lambda_{max}(P)}(t-s)^{\alpha}\Big)\nu_2^2(s)ds \textit{ is a bounded function.}$$

5 | EXAMPLES

In this section, three examples are given to illustrate the effectiveness of the proposed theoretical results.

Example 1. Consider the following fractional-order system

$$\begin{aligned} {}^C D^{\alpha}_{t_0,t} x_1 &= -x_1 + e^{-t^2}\cos(x_2)x_1 + \tfrac{\varepsilon e^{-t}}{1+x_1^2+x_2^2+x_3^2}, \\ {}^C D^{\alpha}_{t_0,t} x_2 &= -x_2 + \sin(x_3)x_2 + \tfrac{\varepsilon e^{-t}}{1+x_1^2+x_2^2+x_3^2}, \\ {}^C D^{\alpha}_{t_0,t} x_3 &= 2x_3, \end{aligned} \tag{33}$$

where, $0 < \alpha < 1$ and $x(t) = \big(x_1(t), x_2(t), x_3(t)\big) \in R^3$.

Consider the Lyapunov-like function: $V(t,x) = \dfrac{x_1^2 + x_2^2}{2}$.

By Lemma 1, we get

$$\begin{aligned} {}^C D^{\alpha}_{t_0,t} V(t, x(t;t_0,x_0)) &\le x_1(t;t_0,x_0)\,{}^C D^{\alpha}_{t_0,t} x_1(t;t_0,x_0) + x_2(t;t_0,x_0)\,{}^C D^{\alpha}_{t_0,t} x_2(t;t_0,x_0) \\ &\le \varepsilon e^{-t} \end{aligned} \tag{34}$$

Then, we get from Theorem 1 the practical uniform stability and the practical uniform boundedness with respect to (x_1, x_2) of the system (33).

Example 2. Consider the following fractional-order system

$$\begin{aligned} {}^C D^{\alpha}_{t_0,t} x_1 &= -2x_1 + x_2 + \varepsilon x_1 + \tfrac{\varepsilon^2\cos(t)}{1+x_3^2}, \\ {}^C D^{\alpha}_{t_0,t} x_2 &= x_1 - x_2 + \varepsilon x_2 + \tfrac{\varepsilon^2 e^{-t}}{1+x_1^2+x_3^2}, \\ {}^C D^{\alpha}_{t_0,t} x_3 &= a x_3 + \tfrac{\varepsilon}{1+t^2}, \end{aligned} \tag{35}$$

where, $a \in R$, $0 < \alpha < 1$ and $x(t) = \big(x_1(t), x_2(t), x_3(t)\big) \in R^3$.

This system has the form of (19)-(20) with $y = (x_1, x_2)$, $z = x_3$,

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix},$$

$g_1(t, x_1, x_2, x_3, \varepsilon) = \varepsilon(x_1, x_2) + \varepsilon^2(\frac{\cos(t)}{1+x_3^2}, \frac{e^{-t}}{1+x_1^2+x_3^2}$ and $g_2(t, x_1, x_2, x_3, \varepsilon) = a x_3 + \frac{\varepsilon}{1+t^2}$.

The function g_1 satisfies (H_2) with $\delta_1(\varepsilon) = \varepsilon^2$, $\delta_2(\varepsilon) = \varepsilon$ and $\nu_1(t) = \sqrt{2}$.

Select $P = 2I$, we get $A^T P + PA + I < 0$ then, the assumptions of Theorem 3 are satisfied. Hence (35) is ε_1-"1-practically globally uniformly asymptotically stable with respect to (x_1, x_2) for some $\varepsilon_1 > 0$.

In addition, if $a < 0$, then (H_1') hold. In this case we have the system (35) is ε_1-"1-practically globally uniformly asymptotically stable for some $\varepsilon_1 > 0$.

Example 3. Consider the following fractional-order system

$$\begin{array}{ll} {}^C D^{\alpha}_{t_0,t} x_1 & = -2x_1 + x_2 + \varepsilon^2 x_1 + \varepsilon x_3 e^{-t} + \frac{\varepsilon \sin(t)}{1+t^2}, \\ {}^C D^{\alpha}_{t_0,t} x_2 & = x_1 - x_2 + \varepsilon^2 x_2 +_3 +\varepsilon \cos(t), \\ {}^C D^{\alpha}_{t_0,t} x_3 & = a x_3 + \varepsilon^2 e^{-t}, \end{array} \tag{36}$$

where, $a \in R$, $0 < \alpha < 1$ and $x(t) = \big(x_1(t), x_2(t), x_3(t)\big) \in R^3$.
This system has the form of (19)-(20) with $y = (x_1, x_2)$, $z = x_3$,

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix},$$

$g_1(t, x_1, x_2, x_3, \varepsilon) = \varepsilon^2(x_1, x_2) + \varepsilon^2 x_3(e^{-t}, 1) + \varepsilon(\frac{\sin(t)}{1+t^2}, \cos(t))$ and $g_2(t, x_1, x_2, x_3, \varepsilon) = a x_3 + \varepsilon^2 e^{-t}$.
The function g_1 satisfies (H_2') with $\delta_3(\varepsilon) = \varepsilon$, $\delta_4(\varepsilon) = \varepsilon^2$, $\delta_5(\varepsilon) = \varepsilon$ and $\nu_1(t) = \sqrt{2}$.
If $a \le 0$, (H_1) holds, so we get from Theorem 4 the practical global uniform stability with respect to (x_1, x_2) for the system (36).
Furthermore, if $a < 0$, (H_1') holds and we get from Theorem 6 the practical global uniform stability for the system (36).

6 | CONCLUSION

In this paper, by using Lyapunov-like function, sufficient conditions are derived to ensure the partial practical stability of fractional-order nonlinear systems depending on a parameter. Some examples are given to show the effectiveness of the contributed results.

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