Similarity reduction, consistent Riccati expansion solvability and novel solutions for a generalized variable-coefficient modified KdV equation with external-force term

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Abstract

In this study, the generalized modified variable-coefficient KdV equation with external-force term (gvcmKdV) arising in fluid mechanics, plasma physics and ocean dynamics is studied for integrability by using consistent Riccati expansion (CRE) solvability and reduced to nonlinear integrable fourth order ordinary differential equation by Clarkson and Kruskal (CK) similarity reduction method. By using the solutions of Riccati equations given before in literature many novel solitary and periodic wave solutions obtained for the gvcmKdV.

1 Introduction

The KdV-type equation is a key for many investigated methods in nonlinear partial differential equations and it was a challenge for modifying many methods like symmetry groups, Bäcklund transformation, Painléve analysis, trail equation method,...etc. [1–17]. One of those challenge KdV type equations is the generalized variable-coefficients modified KdV equation (gvcmKdV) with external-force term given by [18]

$$v_t + f_1 v_{xxx} + \left(f_2 v^2 + f_3 v + f_4\right) v_x + f_5 v + f_6 = 0, \tag{1}$$

where $f_i, i = 1, ..., 6$, are arbitrary functions of t. The gvcmKdV describe the blocking events in atmospheric and oceanic dynamic systems [19,20] moreover, equation (1) contains many types of KdV equation as an instant internal solitary waves (ISWs) in ocean, pressure pulses in fluid-filled tubes of special value in arterial dynamics, trapped quasi-one-dimensional Bose-Einstein condensates, ion-acoustic solitary waves in plasmas and the effect of a bump on wave propagation in a fluid-filled elastic tube [21–25].

From the previous importance for the gvcmKdV we are going to study the integrability of this equation using the consistent Riccati expansion (CRE) solvability and reducing it to nonlinear ordinary differential equation of fourth order using classical Clarkson and Kruskal (CK) direct similarity reduction method, then in both integrability and reduction many novel solitary and periodic wave solutions will obtained.

2 Integrability of gvcmKdV equation

In this section we are going to discuss the integrability property of the gvcmKdV equation by using the consistent Riccati expansion solvability [25-26] as follows:

Assume that a nonlinear partial differential equation is given by

$$Q(x,t,v)=0,\ x=\{x_1,x_2,x_3,...\} \tag{2}$$

we assume that Q has a solution in the form

$$v\left(x,t\right)=\sum_{k=0}^{n} v_k\left(x,t\right) R^k\left(w\left(x,t\right)\right), \tag{3}$$

where $v_k\left(x,t\right)$ are arbitrary functions in the independent variables x,t and $R(w)$ is a solution of the Riccati equation

$$R_w\left(w\right)=a+b\ R\left(w\right)+c\ R^2\left(w\right), \tag{4}$$

where a,b and c are arbitrary constants and $w=w(x,t)$ is an unknown function to be determined. The integer n can be obtained by balance method then by substitution from (3) in (2) using (4) a polynomial in $R(w)$ is obtained, then by finishing all coefficients of $R(w)$ a system of partial differential equations is obtained.

Definition 1 *If the partial differential system obtained by finishing all powers of the polynomial $R\left(w\right)$ is consistent, or, not over-determined, we call that the expansion (3) is a CRE and equation (2) is integrable according to CRE solvable.*

Now, By applying the previous steps on the gvcmKdV equation and from the balance method we get $n=1$, therefore we can assume that the solution of equation (1) takes the form

$$v(x,t)=v_0(x,t)+v_1(x,t)R(w(x,t)), \tag{5}$$

where $v_0(x,t)$, $v_1(x,t)$ are arbitrary functions. By back substitution from (5) into (1) using (4) a polynomial of fourth order in R is constructed, by equating the coefficients of R with zero then, a partial differential system of only five equations is obtained. From the previous definition, we have found that the obtained system is consistent, or, not over-determined with the following solutions for it

$$
\begin{aligned}
v_0(x,t) &= \frac{1}{2f_2(t)}\left(k_1 b f_2(t)\sqrt{\frac{-6f_1(t)}{f_2(t)}} - f_3(t)\right), v_1(x,t) = k_1 c\sqrt{\frac{-6f_1(t)}{f_2(t)}} \quad (6)\\
w(x,t) &= k_1\left(x - \frac{1}{4}\int\left(8ack_1^2 f_1(t)f_2(t) - 2b^2k_1^2 f_1(t)f_2(t) - f_3^2(t) + 4f_2(t)f_4(t)\right)dt\right), \quad (7)
\end{aligned}
$$

with integrability conditions

$$
\begin{aligned}
f_5(t) &= \frac{f_1(t)f_2'(t) - f_2(t)f_1'(t)}{2f_1(t)f_2(t)},\\
f_6(t) &= \frac{2f_1(t)f_2(t)f_3'(t) - f_1(t)f_3(t)f_2'(t) - f_2(t)f_3(t)f_1'(t)}{4f_1(t)f_2^2(t)}, \quad (8)
\end{aligned}
$$

where $'$ means differentiation with respect to time t. Moreover, by using the known solutions for the Riccati equation (4) see [17], the following new solutions for the gvcmKdV equation are obtained:

Type I: If $b^2 - 4ac > 0$ and $bc \neq 0$ (or $ac \neq 0$), the solutions of Eq. (1) are

$$
\begin{aligned}
v_1(x,t) &= v_0(x,t) - \frac{k_1}{2}\sqrt{\frac{-6f_1(t)}{f_2(t)}}\left(b + \sqrt{b^2-4ac}\tanh\left(\frac{\sqrt{b^2-4ac}}{2}w\right)\right), \quad (9)\\
v_2(x,t) &= v_0(x,t) - \frac{k_1}{2}\sqrt{\frac{-6f_1(t)}{f_2(t)}}\left(b + \sqrt{b^2-4ac}\coth\left(\frac{\sqrt{b^2-4ac}}{2}w\right)\right), \quad (10)\\
v_3(x,t) &= v_0(x,t) + \frac{k_1}{2}\sqrt{\frac{-6f_1(t)}{f_2(t)}}\left(-b \pm \frac{\sqrt{b^2-4ac}\left(\sqrt{A^2+B^2} \mp A\cosh(\sqrt{b^2-4ac}\,w)\right)}{A\sinh(\sqrt{b^2-4ac}\,w) + B}\right) \quad (11)\\
v_4(x,t) &= v_0(x,t) + \frac{2ak_1 c\sqrt{\frac{-6f_1(t)}{f_2(t)}}\cosh\left(\frac{\sqrt{b^2-4ac}}{2}w\right)}{\sqrt{b^2-4ac}\sinh\left(\frac{\sqrt{b^2-4ac}}{2}w\right) - b\cosh\left(\frac{\sqrt{b^2-4ac}}{2}w\right)}. \quad (12)\\
v_5(x,t) &= v_0(x,t) + \frac{\pm 2ak_1 c\sqrt{\frac{-6f_1(t)}{f_2(t)}}\sinh\left(\frac{\sqrt{b^2-4ac}}{2}w\right)}{\pm\sqrt{b^2-4ac}\left(\cosh(\sqrt{b^2-4ac}\,w) \pm 1\right) \mp b\sinh(\sqrt{b^2-4ac}\,w)}, \quad (13)
\end{aligned}
$$

where A, B are two non-zero real constants.

Type II: If $a = 0$ and $bc \neq 0$, the solutions of Eq. (1) are

$$
\begin{aligned}
v_6(x,t) &= v_0(x,t) - \frac{k_1 bd\sqrt{\frac{-6f_1(t)}{f_2(t)}}}{d + \cosh(bw) - \sinh(bw)}, \quad (14)\\
v_7(x,t) &= v_0(x,t) - \frac{k_1 b\sqrt{\frac{-6f_1(t)}{f_2(t)}}\left(\cosh(bw) + \sinh(bw)\right)}{d + \cosh(bw) + \sinh(bw)}, \quad (15)
\end{aligned}
$$

where d is an arbitrary constant.

Type III: If $b^2-4ac<0$ and $bc\neq 0$ (or $ac\neq 0$), the solutions of Eq. (1) are

$$v_8(x,t) = v_0(x,t)+\frac{k_1}{2}\sqrt{\frac{-6f_1(t)}{f_2(t)}}(-b+\sqrt{4ac-b^2}\tan(\frac{\sqrt{4ac-b^2}}{2}w)), \tag{16}$$

$$v_9(x,t) = v_0(x,t)-\frac{k_1}{2}\sqrt{\frac{-6f_1(t)}{f_2(t)}}(b+\sqrt{4ac-b^2}\cot(\frac{\sqrt{4ac-b^2}}{2}w)), \tag{17}$$

$$v_{10}(x,t) = v_0(x,t)+\frac{k_1}{2}\sqrt{\frac{-6f_1(t)}{f_2(t)}}(-b\pm\frac{\sqrt{4ac-b^2}(\sqrt{A^2-B^2}\mp A\cos(\sqrt{4ac-b^2}w))}{A\sin(\sqrt{4ac-b^2}w)+B}) \tag{18}$$

$$v_{11}(x,t) = v_0(x,t)-\frac{2ak_1c\sqrt{\frac{-6f_1(t)}{f_2(t)}}\cos(\frac{\sqrt{4ac-b^2}}{2}w)}{\sqrt{4ac-b^2}\sin(\frac{\sqrt{b^2-4ac}}{2}w)+b\cos(\frac{\sqrt{4ac-b^2}}{2}w)}. \tag{19}$$

$$v_{12}(x,t) = v_0(x,t)+\frac{2akc\sqrt{\frac{-6f_1(t)}{f_2(t)}}\sin(\frac{\sqrt{4ac-b^2}}{2}w)}{\sqrt{4ac-b^2}\cos(\frac{\sqrt{4ac-b^2}}{2}w)-b\sin(\frac{\sqrt{b^2-4ac}}{2}w)}. \tag{20}$$

$$v_{13}(x,t) = v_0(x,t)+\frac{2ak_1c\sqrt{\frac{-6f_1(t)}{f_2(t)}}\sin(\frac{\sqrt{b^2-4ac}}{2}w)}{\sqrt{b^2-4ac}(\cos(\sqrt{4ac-b^2}w)\pm 1)-b\sin(\sqrt{4ac-b^2}w)}. \tag{21}$$

$$v_{14}(x,t) = v_0(x,t)+\frac{2ak_1c\sqrt{\frac{-6f_1(t)}{f_2(t)}}\cos(\frac{\sqrt{b^2-4ac}}{2}w)}{\sqrt{b^2-4ac}(\sin(\sqrt{4ac-b^2}w)\pm 1)+b\cos(\sqrt{4ac-b^2}w)}. \tag{22}$$

with $v_0(x,t)$ given by (6).

3 CK similarity reduction method

The classical Clarkson and Kruskal (CK) direct similarity reduction method was first introduced in 1989 [28] and then enlarged and modified by many authors in [7,9,21,22] in this paper we have used the classic CK method because the other modified CK methods specially the connected CK with homogeneous balance method could not use to reduce equation (1).

In the following the main steps of the classic CK method:

1) If ϑ is a partial differential equation given by

$$\vartheta(t,x,v_x,v_t,v_{xx},v_{xxx},f_1(t),f_2(t),...,f_6\,(t))=0, \tag{23}$$

where $v=v(x,t)$, $f_i(t), i=1,...,6$ are arbitrary functions in t.

2) Assume that

$$v(x,t)=\sigma(x,t)+\gamma(x,t)V(\varsigma), \tag{24}$$

where $\varsigma=\varsigma(x,t)$ is a similarity variable and $\sigma(x,t)$ and $\gamma(x,t)$ are arbitrary functions in x,t to be determined later.Collect all coefficients of $V(\varsigma)$ and equate it with the coefficient of the most linear term multiply with arbitrary function $\Theta_j(\varsigma), j=1,2,...k$ a partial differential system in σ,γ and ς is given. To solve the determined system we can use the following rules.

a) If $\sigma(x,t)=\gamma(x,t)\ \Theta(\varsigma)+\sigma_0(x,t)$, then we can assume that $\Theta(\varsigma)=0$,

b) If $\gamma(x,t)=\gamma_0(x,t)\Theta(\varsigma)$, then $\Theta(\varsigma)$ can be assumed as constant.

c) If $\Theta(\varsigma)=\varsigma_0(x,t)$, then we can take $\Theta(\varsigma)=\varsigma$.

Finally, equation (23) is reduced to a nonlinear ordinary differential equation with constant coefficients.

4 Reduction and solutions for gvcmKdV

In this section we have substitute from (23) into (1) and the following partial differential equation is given

$$\begin{aligned} &f_1\gamma\varsigma_x^3V''' + 3f_1\left(\gamma_x\varsigma_x^2+\gamma\varsigma_x\varsigma_{xx}\right)V'' + [f_4\gamma\varsigma_x + f_1 3\left(\gamma_{xx}\varsigma_x+\gamma_x\varsigma_{xx}\right)+\gamma\varsigma_{xxx}) + f_2\gamma\varsigma_x\sigma^2 + f_3\sigma\gamma\varsigma_x]V' \\ &+\left(2f_2\gamma\sigma\varsigma_x + f_3\gamma^2\varsigma_x\right)VV' + f_2\gamma^3\varsigma_xV^2V' + \left(f_2\left(2\gamma\gamma_x\sigma+\gamma^2\sigma_x\right)+f_3\gamma\gamma_x\right)V^2 \\ &+\left(\gamma_t + f_4\gamma_x + f_1\gamma_{xxx} + f_2\gamma_x\sigma^2 + f_5\gamma + f_3\gamma\sigma_x + f_3\gamma\sigma_x + 2f_2\gamma\sigma\sigma_x\right)V \\ &+f_2\gamma^2\gamma_xV^3 + \sigma_t + f_1\sigma_{xxx} + \left(f_2\sigma^2+f_3\sigma+f_4\right)\sigma_x + f_5\sigma + f_6 = 0 \qquad (25)\end{aligned}$$

To make the above equation ordinary differential equation in V we need to make the coefficients constants or functions on ς therefore, we take the coefficient of V''' as a normalized coefficients and equate other coefficients in (4) with $f_1\gamma\varsigma_x^3\Theta_i\left(\varsigma\right), i=1,...,8$ as follows:

$$\begin{aligned} &f_2\gamma^3\varsigma_x = f_1\gamma\varsigma_x^3\Theta_1\left(\varsigma\right), \qquad f_2\gamma^2\gamma_x = f_1\gamma\varsigma_x^3\Theta_2\left(\varsigma\right), \qquad 3f_1\left(\gamma_x\varsigma_x^2+\gamma\varsigma_x\varsigma_{xx}\right) = f_1\gamma\varsigma_x^3\Theta_3\left(\varsigma\right), \\ &2f_2\gamma\sigma\varsigma_x + f_3\gamma^2\varsigma_x = f_1\gamma\varsigma_x^3\Theta_4\left(\varsigma\right), \qquad f_2\left(2\gamma\gamma_x\sigma+\gamma^2\sigma_x\right)+f_3\gamma\gamma_x = f_1\gamma\varsigma_x^3\Theta_5\left(\varsigma\right), \\ &f_4\gamma\varsigma_x + f_1 3\left(\gamma_{xx}\varsigma_x+\gamma_x\varsigma_{xx}\right)+\gamma\varsigma_{xxx} + f_2\gamma\varsigma_x\sigma^2 + f_3\sigma\gamma\varsigma_x = f_1\gamma\varsigma_x^3\Theta_6\left(\varsigma\right), \\ &\gamma_t + f_4\gamma_x + f_1\gamma_{xxx} + f_2\gamma_x\sigma^2 + f_5\gamma + f_3\gamma\sigma_x + f_3\gamma\sigma_x + 2f_2\gamma\sigma\sigma_x = f_1\gamma\varsigma_x^3\Theta_7\left(\varsigma\right), \\ &\sigma_t + f_1\sigma_{xxx} + \left(f_2\sigma^2+f_3\sigma+f_4\right)\sigma_x + f_5\sigma + f_6 = f_1\gamma\varsigma_x^3\Theta_8\left(\varsigma\right). \qquad (26)\end{aligned}$$

By using the assumptions $(a-c)$ in the previous section we obtain

$$\Theta_1\left(\varsigma\right)=b_2,\ \Theta_4\left(\varsigma\right)=b_3,\ \Theta_6\left(\varsigma\right)=1,\ \ \Theta_2\left(\varsigma\right)=\Theta_3\left(\varsigma\right)=\Theta_5\left(\varsigma\right)=\Theta_7\left(\varsigma\right)=\Theta_8\left(\varsigma\right)=0, \qquad (27)$$

with σ,γ and ς in the form

$$\sigma(x,t)=c_1e^{-\int f_5(t)dt}, \qquad \gamma(x,t)=\left(c_2-\int f_6(t)e^{\int f_5(t)dt}dt\right)e^{-\int f_5(t)dt}. \quad (28)$$

with integrability conditions

$$f_2(t)=b_2k^2f_1(t)e^{2\int f_5(t)dt}, \quad f_3=\frac{k^2f_1}{c_1^2}e^{\int f_5(t)dt}\left(c_1b_3+2b_2\left(\int f_6e^{\int f_5(t)dt}dt-c_2\right)\right) \qquad (29)$$

By back substitution from (28) into (24) the similarity solution of gvcmKdV takes the form

$$v(x,t) = e^{-\int f_5(t)dt}\left(c_1 V(\varsigma) - \int f_6(t) e^{\int f_5(t)dt} dt + c_2\right) \tag{30}$$

with the following similarity variable

$$\varsigma = k\left(x + \frac{1}{c_1^2}\int\left(f_1 k^2\left(c_1^2 + b_2 c_2^2 - c_1 c_2 b_3 + [b_2(1-2c_2) + b_3 c_1]\int f_6 e^{\int f_5(t)dt} dt\right) - c_1^2 f_4\right) dt + C\right), \tag{31}$$

where C is an integration constant. Then equation (25) becomes

$$V''' + b_2 V^2 V' + b_3 V V' + V' = 0, \tag{32}$$

Integrate equation (32) twice with respect to ς, we obtain

$$V'^2 + \frac{b_2}{6}V^4 + \frac{b_3}{2}V^3 + V^2 = A_1 V + A_2, \tag{33}$$

where A_1 and A_2 are integration constants. Equation (33) is a Riccati equation [29] has many solutions we will contrast our attention on solutions contains all variable coefficients it means that $b_2 \neq 0$ and $b_3 \neq 0$, so we have the following solution for equation (33)

$$V_1 = \frac{6\sec(\varsigma)}{\sqrt{b_3^2 - 6b_2} - b_3\sec(\varsigma)}, \tag{34}$$

$$V_2 = \frac{6\csc(\varsigma)}{\sqrt{b_3^2 - 6b_2} - b_3\csc(\varsigma)}, \tag{35}$$

with $A_1 = 0$ and $A_2 = 0$ then back substitution from (34-35) into (30) the following new solutions for the gvcmKdV are obtained

$$v_1(x,t) = e^{-\int f_5(t)dt}\left(\frac{6c_1\sec(\varsigma)}{\sqrt{b_3^2 - 6b_2} - b_3\sec(\varsigma)} - \int f_6(t) e^{\int f_5(t)dt} dt + c_2\right), \tag{36}$$

$$v_2(x,t) = e^{-\int f_5(t)dt}\left(\frac{6c_1\csc(\varsigma)}{\sqrt{b_3^2 - 6b_2} - b_3\csc(\varsigma)} - \int f_6(t) e^{\int f_5(t)dt} dt + c_2\right), \tag{37}$$

with ς given by equation (31) under integrability conditions (29).

5 Conclusion

In this paper the gvcmKdV is studied by two different techniques the CRE solvability and the CK direct reduction method after this study we have the following concluding remarks:

1) The nonlinear partial differential system (PDEs) obtained from The CRE solvability is consistent with solutions given by equations (5-6) under integrability conditions (8) therefore the gvcmKdV is integrable in this case.

2) The PDEs system obtained from the CRE is solvable if we take $v_0(x,t) = \gamma(x,t)$, $v_1(x,t) = \sigma(x,t)$ and $w = \varsigma$ with integrability conditions (29) and in this case Riccati equation (4) will be satisfy with $a = -\frac{b_3^2-6b_2}{2b_2^2}\sqrt{-\frac{b_2}{6}}, b = \frac{1}{b_2}\sqrt{-\frac{b_1 b_3}{6}}$ and $c = \sqrt{\frac{-b_2}{6}}$ therefore solutions (9-23) are obtained for the CK direct similarity method also.

3) In this study we have used the classic CK because the external force term $f_6(t)$ make other modifications in literature [7,9,21,22] could not be applied.

4) If we put $b_3 = 0$ in Riccati equation (33) many single and combined non-degenerate Jacobi elliptic solutions [29] could be obtained for the gvcmKdV in this case the obtained solutions recover solutions obtained for constants coefficients modified KdV equation in [20].

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