Large-time behavior of solutions to the inflow problem of the non-isentropic micropolar fluid model

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Abstract

We investigate the asymptotic behavior of solutions to the initial boundary value problem for the micropolar fluid model in a half line $\mathbb{R}_+ := (0, \infty)$. Inspired by the relationship between micropolar fluid and Navier-Stokes, we prove that the composite wave consisting of the transonic boundary layer solution, the 1-rarefaction wave, the viscous 2-contact wave and the 3-rarefaction wave for the inflow problem on the micropolar fluid model is time-asymptotically stable under some smallness conditions. Meanwhile, we obtain the global existence of solutions based on the basic energy method.

Key words. micropolar fluid model, composite wave, inflow problem, stability.

AMS subject classifications. 35Q35, 76D33, 35M33, 35B35.

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1 Introduction

The 1-D compressible viscous micropolar fluid model in the half line $\mathbb{R}_+ =: (0,+\infty)$ reads in Eulerian coordinates:

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \quad x>0,\ t>0, \\ \partial_t(\rho u) + \partial_x(\rho u^2+p) = \mu\partial_x^2 u, \quad x>0,\ t>0, \\ \partial_t(\rho\omega) + \partial_x(\rho u\omega) + A\omega = A\partial_x^2\omega, \quad x>0,\ t>0, \\ \partial_t[\rho(e+\frac{u^2}{2})] + \partial_x[\rho u(e+\frac{u^2}{2})+pu] = \mu\partial_x(u\partial_x u) + \kappa\partial_x^2\theta + (\partial_x\omega)^2 + \omega^2, \quad x>0,\ t>0, \end{cases} \tag{1.1}$$

where ρ, u, ω and θ represent the mass density, velocity, microrotation velocity and temperature of the fluid respectively. Here we assume A, μ, κ are positive constants. Assuming that the fluid is perfect and polytropic, for pressure p and internal energy e we have the state equations:

$$p = R\rho\theta, \qquad e = \frac{R}{\gamma-1}\theta, \tag{1.2}$$

where R and $\gamma > 1$ are positive constants.

We consider the system (1.1) with the initial values

$$(\rho,u,\omega,\theta)(x,0) = (\rho_0,u_0,\omega_0,\theta_0)(x), \quad \inf_{x\in\mathbb{R}_+}\rho_0(x) > 0, \quad \inf_{x\in\mathbb{R}_+}\theta_0(x) > 0. \tag{1.3}$$

Assume that initial data at the far field $x=+\infty$ is constant, namely

$$\lim_{x\to+\infty}(\rho_0,u_0,\omega_0,\theta_0)(x) = (\rho_+,u_+,\omega_+,\theta_+) \tag{1.4}$$

and the boundary values for ρ, u, ω and θ at $x=0$ are given by

$$(\rho,u,\omega,\theta)(0,t) = (\rho_-,u_-,\omega_-,\theta_-), \quad \forall\, t\ge 0, \tag{1.5}$$

where $\rho_- > 0$, $u_- > 0$, $\theta_- > 0$, ω_- are constants and the following compatibility conditions hold

$$\rho_0(0) = \rho_-, \quad u_0(0) = u_-, \quad \omega_0(0) = \omega_-, \quad \theta_0(0) = \theta_-. \tag{1.6}$$

The boundary conditions to the half-place problem (1.1) can be proposed as one of the following three cases.

Case 1. Outflow problem(negative velocity on the boundary):

$$u(0,t) = u_- < 0, \quad \theta(0,t) = \theta_-. \tag{1.7}$$

Case 2. Impermeable wall problem(zero velocity on the boundary):

$$u(0,t) = 0, \quad \theta(0,t) = \theta_-. \tag{1.8}$$

Case 3. Inflow problem(positive velocity on the boundary):

$$\rho(0,t) = \rho_-, \quad u(0,t) = u_- > 0, \quad \theta(0,t) = \theta_-. \tag{1.9}$$

Notice that in case 1 and case 2 the density ρ_- could not be given, but in case 3, ρ_- must be imposed due to the well-posedness theory of the hyperbolic equation $(1.1)_1$.

The micropolar fluid model was firstly introduced by Eringen in 1966 ([7]). The micropolar fluid model enables us to consider some complex fluids such as suspensions, animal blood, liquid crystals which cannot be described properly by classical Navier-Stokes equation. For more background, please refer to [19] and references therein. Much attention has been paid to this model by mathematicians.

First, some of them made a series of efforts in [4, 19, 1, 30] studying the existence of strong solutions and weak solutions for the micropolar fluid model. Second, Mujaković proved the regularity to an initial boundary value problem for the micropolar fluid model in [23] and [26]. For the large time behavior and stability of solutions, Mujaković and other authors made a series work in [24, 25, 27, 33]. In addition, there are other authors who contribute to the studying of this model, such as for one dimensional compressible micropolar fluid model, Liu and Yin [15] have obtained the stability of solution of contact discontinuity for the Cauchy problem. For three-dimensional model, Chen and Huang [2, 3] obtained the blowup criterion of solutions for this model. Liu and Zhang [16] proved the optimal time decay of the three dimensional compressible flow. In Liu and Zhang's recent work [17], they also obtained the large time behavior of solutions for the compressible micropolar fluid model with a potential external force in $\mathbb{R}^3$. Recently, Duan [6] proved existence and uniqueness of global strong solution of compressible micropolar fluid model in one dimensional space with density dependent viscosity and temperature dependent heat conductivity. The stability of rarefaction waves for one dimensional compressible viscous micropolar fluid model was obtained by Jin and Duan in [13].

We assume microrotation velocity $\omega = 0$ for the large time behavior of solutions to the initial boundary value problem (1.1), (1.3), (1.4), (1.5), (1.6), then the micropolar fluid model (1.1) can be reduced to the following single Navier-Stokes system in the form

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \mu \partial_x^2 u, \\ \partial_t[\rho(e + \frac{u^2}{2})] + \partial_x[\rho u(e + \frac{u^2}{2}) + pu] = \mu \partial_x(u \partial_x u) + \kappa \partial_x^2 \theta. \end{cases} \tag{1.10}$$

Moreover, when the dissipation effects are neglected for the large time behavior, Navier-Stokes system (1.10) can be reduced to the following Euler system in the form of

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t[\rho(e + \frac{u^2}{2})] + \partial_x[\rho u(e + \frac{u^2}{2}) + pu] = 0. \end{cases} \tag{1.11}$$

It is well known that the Euler system (1.11) is a typical example of the hyperbolic conservation laws. The Riemann solutions for Euler system (1.11) contains three basic wave patterns, that is, two nonlinear waves, called shock wave and rarefaction wave and one linear wave, called contact discontinuity and their linear combinations. Later, not only basic wave patterns but also a new wave, which is called a boundary layer solution(BL-solution for brevity) [20] may appear in the initial boundary value problem. Because the large time behavior of solutions to the Cauchy problem (on the isentropic or nonisentropic Navier-Stokes system) are basically described by the viscous versions of three basic wave patterns. There have been a lot of mathematical studies about basic wave patterns and BL-solution to the isentropic or nonisentropic Navier-Stokes system, please rereferring to [8, 9, 10, 11, 12, 14, 18, 20, 21, 22, 28, 29, 31]

Now we review some recent work on the inflow and outflow problems of the micropolar fluid model. Yin [5, 35] respectively obtained the stability of BL-solution to the inflow and outflow problems. In the previous paper [35], Yin proved the stability of the composite wave consisting of the subsonic BL-solution, the viscous 2-contact wave and the 3-rarefaction wave under the condition that the amplitude of the contact wave and the BL-solution is small enough but the 3-rarefaction wave is not necessarily small for the one dimensional compressible micropolar fluid model.

Therefore,there is a natural question: How is the asymptotic stability of the composite wave consisting of the transonic BL-solution, the 1-rarefaction wave, the viscous 2-contact wave and the

3-rarefaction wave for the inflow problem on micropolar fluid model (1.1) to the Riemann problem on Euler system (1.11) in the setting of $\omega(x,t) = 0$ under the condition that $\omega_\pm = 0$? We will give the positive answer on this problem in this paper. As far as we know, this is the first work on the stability of composite wave of the transonic BL-solution, the 1-rarefaction wave, the viscous 2-contact wave and 3-rarefaction wave for the compressible micropolar fluid model. It is worthwhile to point out that the four wave patterns are different from the Cauchy problem due to the boundary effect. Correspondingly, some new mathematical difficulties occur due to the degeneracy of the transonic BL-solution and its interactions with other wave patterns in the composite wave.

In order to study the large time behavior of solutions to (1.1), (1.3), (1.4), (1.5) and (1.6), it is more convenient to use the following Lagrangian coordinate transformation:

$$x \Rightarrow \int_{(0,0)}^{(x,t)} \rho(y,\tau)dy - \rho u(y,\tau)d\tau, \qquad t \Rightarrow t.$$

Thus the system (1.1) can be transformed into the following moving boundary problem of micropolar fluid model in the Lagrangian coordinates:

$$\begin{cases} \partial_t v - \partial_x u = 0, \quad x > \sigma_- t, \ t > 0, \\ \partial_t u + \partial_x p = \mu \partial_x \left(\dfrac{\partial_x u}{v} \right), \quad x > \sigma_- t, \ t > 0, \\ \partial_t \omega + Av\omega = A\partial_x \left(\dfrac{\partial_x \omega}{v} \right), \quad x > \sigma_- t, \ t > 0, \\ \partial_t (e + \dfrac{u^2}{2}) + \partial_x (pu) = \mu \partial_x (\dfrac{u\partial_x u}{v}) + \kappa \partial_x (\dfrac{\partial_x \theta}{v}) + \dfrac{(\partial_x \omega)^2}{v} + v\omega^2, \quad x > \sigma_- t, \ t > 0, \\ (v,u,\omega,\theta)(x = \sigma_- t, t) = (v_-, u_-, 0, \theta_-), \quad u_- > 0, \\ (v,u,\omega,\theta)(x,0) = (v_0,u_0,\omega_0,\theta_0)(x) \to (v_+,u_+,0,\theta_+), \quad as \quad x \to +\infty, \end{cases} \tag{1.12}$$

where $v(x,t) = \frac{1}{\rho(x,t)}$ represents the specific volume of the fluid, and the boundary moves with the constant speed $\sigma_- = -\frac{u_-}{v_-} < 0$. Now we have that for the perfect gas,

$$p = \frac{R\theta}{v}. \tag{1.13}$$

In order to fix the moving boundary $x = \sigma_- t$, we introduce a new variable $\xi = x - \sigma_- t$. Then we have the half-space problem

$$\begin{cases} \partial_t v - \sigma_- \partial_\xi v - \partial_\xi u = 0, \quad \xi > 0, \ t > 0, \\ \partial_t u - \sigma_- \partial_\xi u + \partial_\xi p = \mu \partial_\xi \left(\dfrac{\partial_\xi u}{v} \right), \quad \xi > 0, \ t > 0, \\ \partial_t \omega - \sigma_- \partial_\xi \omega + Av\omega = A\partial_\xi \left(\dfrac{\partial_\xi \omega}{v} \right), \quad \xi > 0, \ t > 0, \\ \partial_t (e + \dfrac{u^2}{2}) - \sigma_- \partial_\xi (e + \dfrac{u^2}{2}) + \partial_\xi (pu) = \mu \partial_\xi (\dfrac{u\partial_\xi u}{v}) + \kappa \partial_\xi (\dfrac{\partial_\xi \theta}{v}) + \dfrac{(\partial_\xi \omega)^2}{v} + v\omega^2, \quad \xi > 0, \ t > 0, \\ (v,u,\omega,\theta)(\xi = 0, t) = (v_-, u_-, 0, \theta_-), \quad u_- > 0, \\ (v,u,\omega,\theta)(\xi,0) = (v_0,u_0,\omega_0,\theta_0)(\xi) \to (v_+,u_+,0,\theta_+), \quad as \quad \xi \to +\infty. \end{cases} \tag{1.14}$$

We next assume, as usual in thermodynamics, that by any given two of the five thermodynamical variables, v, p, e, the temperature θ (> 0) and entropy s, the remaining three variables can be expressed. Without lose of generality, we define the entropy s as follows

$$s = R \ln v + \frac{R}{\gamma - 1} \ln \theta + 1, \tag{1.15}$$

which obeys the second law of thermodynamics

$$\theta ds = de + pdv.$$

Then due to (1.15), the initial data $s(v_0(x), \theta_0(x))$ is expressed by $(v_0(x), \theta_0(x))$ as follows

$$s(v_0(x), \theta_0(x)) = R \ln v_0(x) + \frac{R}{\gamma - 1} \ln \theta_0(x) + 1. \tag{1.16}$$

Thus $s_+ = \lim\limits_{x \to +\infty} s(v_0(x), \theta_0(x))$ satisfying

$$s_+ = s(v_+, \theta_+) = R \ln v_+ + \frac{R}{\gamma - 1} \ln \theta_+ + 1. \tag{1.17}$$

The rest of the paper is arranged as follows. In the Section 2, we give some preliminaries of the Navier-Stokes system, then we reformulate the original system (1.1) and introduce our main theorem concerning the global existence and asymptotic stability of solutions. The proof of Theorem 2.1 is concluded in Section 3. In the Appendix, we present the details which are left in the proofs of the previous sections for completeness of the paper.

Notation: Throughout the paper, we denote positive constants (generally large) and (generally small) independent of t by C and c, respectively. And the character "C" and "c" may take different values in different places. $L^p = L^p(\mathbb{R}_+)$ $(1 \leq p \leq \infty)$ denotes the usual Lebesgue space on $[0, \infty)$ with its norm $\|\cdot\|_{L^p}$, and when $p = 2$, we write $\|\cdot\|_{L^2(\mathbb{R}_+)} = \|\cdot\|$. $H^s = H^s(\mathbb{R}_+)$ denotes the usual s-th order Sobolev space with its norm $\|f\|_{H^s(\mathbb{R}_+)} = (\sum\limits_{i=0}^{s} \|\partial^i f\|^2)^{\frac{1}{2}}$.

2 Some preliminaries of the Navier-Stokes system

Since we expect the large time behavior of micropolar fluid model (1.14) behaves as the same as that of Navier-Stokes system, we assume $\omega(x, t) = 0$ for the large time behavior. Therefore, when time $t \to +\infty$, the micropolar fluid model (1.12) and (1.14) respectively become the following Navier-Stokes system

$$\begin{cases} \partial_t v - \partial_x u = 0, \quad x > \sigma_- t, \ t > 0, \\ \partial_t u + \partial_x p = \mu \partial_x \left(\dfrac{\partial_x u}{v} \right), \quad x > \sigma_- t, \ t > 0, \\ \partial_t (e + \dfrac{u^2}{2}) + \partial_x (pu) = \mu \partial_x (\dfrac{u \partial_x u}{v}) + \kappa \partial_x (\dfrac{\partial_x \theta}{v}), \quad x > \sigma_- t, \ t > 0, \\ (v, u, \theta)(x = \sigma_- t, t) = (v_-, u_-, \theta_-), \quad u_- > 0, \\ (v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x) \to (v_+, u_+, \theta_+), \quad as \quad x \to +\infty, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \partial_t v - \sigma_- \partial_\xi v - \partial_\xi u = 0, \quad \xi > 0, \ t > 0, \\ \partial_t u - \sigma_- \partial_\xi u + \partial_\xi p = \mu \partial_\xi \left(\dfrac{\partial_\xi u}{v} \right), \quad \xi > 0, \ t > 0, \\ \partial_t (e + \dfrac{u^2}{2}) - \sigma_- \partial_\xi (e + \dfrac{u^2}{2}) + \partial_\xi (pu) = \mu \partial_\xi (\dfrac{u \partial_\xi u}{v}) + \kappa \partial_\xi (\dfrac{\partial_\xi \theta}{v}), \quad \xi > 0, \ t > 0, \\ (v, u, \theta)(\xi = 0, t) = (v_-, u_-, \theta_-), \quad u_- > 0, \\ (v, u, \theta)(\xi, 0) = (v_0, u_0, \theta_0)(\xi) \to (v_+, u_+, \theta_+), \quad as \quad \xi \to +\infty. \end{cases} \tag{2.2}$$

Since Navier-Stokes system (2.1) and (2.2) have been studied by Qin and Wang in [31] which obtained the existence (or nonexistence) of the boundary layer solution (BL-solution) for the inflow problem when the right end state (v_+, u_+, θ_+) belonged to the subsonic, transonic, and supersonic regions, respectively, and proved the asymptotic stability of not only the single contact wave but also the composite wave consisting of the subsonic BL-solution, the contact wave, and the rarefaction wave. From now on, in order to prove the composite wave consisting of the transonic BL-solution, the 1-rarefaction wave, the viscous 2-contact wave, and the 3-rarefaction wave for the inflow problem on the micropolar fluid model (1.14) is time-asymptotically stable, we firstly review some known results about Navier-Stokes system in [32] which will be used repeatedly in this paper.

For any given right state (v_+, u_+, θ_+), we can define wave curves (BL-solution curve, 1-rarefaction wave curve, viscous 2-contact wave curve and 3-rarefaction wave curve) in terms of (v, u, θ) with $v > 0$ and $\theta > 0$ in the phase space as follows:

* Transonic boundary layer curve:

$$BL(v_+, u_+, \theta_+) \equiv \left\{ (v, u, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \;\middle|\; \frac{u}{v} = -\sigma_- = \frac{u_+}{v_+}, \quad (u, \theta) \in \Sigma(u_+, \theta_+) \right\},$$

where $(v_+, u_+, \theta_+) \in \Gamma^+_{trans} := \{(u, \theta) | u = \sqrt{R\gamma\theta} > 0\}$ is the transonic region defined in Section 2.1 with positive gas velocity and $\Sigma(u_+, \theta_+)$ is the trajectory at the point (u_+, θ_+) defined in Case II of Proposition 2.1 below.

* Contact wave curve:

$$CD(v_+, u_+, \theta_+) \equiv \left\{ (v, u, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \;\middle|\; p = p_+, \ u = u_+, \ v \neq v_+ \right\},$$

where $p_+ = \frac{R\theta_+}{v_+}$.

* i-rarefaction wave curve (i=1,3):

$$R_i(v_+, u_+, \theta_+) \equiv \left\{ (v, u, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \;\middle|\; s(v, \theta) = s_+, \quad u = u_+ - \int_{v_+}^{v} \lambda_i(z, s_+) dz \right\},$$

and $\lambda_i = \lambda_i(v, s)$ (i=1, 3) is respectively the first and third characteristic speed given in (2.3).

In this paper, we expect to prove that if the left state $(v_-, u_-, \theta_-) \in BL{-}R_1{-}CD{-}R_3(v_+, u_+, \theta_+)$, then there exist a unique state $(v_*, u_*, \theta_*) \in \Gamma^+_{trans}$ and a unique state (v_m, u_m, θ_m) and (v^*, u^*, θ^*) such that $(v_-, u_-, \theta_-) \in BL(v_*, u_*, \theta_*)$, $(v_*, u_*, \theta_*) \in R_1(v_m, u_m, \theta_m)$, $(v_m, u_m, \theta_m) \in CD(v^*, u^*, \theta^*)$, and $(v^*, u^*, \theta^*) \in R_3(v_+, u_+, \theta_+)$ and the superposition of the BL-solution, 1-rarefaction wave, viscous 2-contact wave and 3-rarefaction wave for the inflow problem on the micropolar fluid model (1.14) is asymptotically stable provided that the wave strength $\delta = \mid (v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-) \mid$ is suitably small and the conditions in Theorem 2.1 hold.

2.1 BL-solutions

The characteristic speeds of the hyperbolic part of (2.1) are

$$\lambda_1 = -\sqrt{\frac{\gamma p}{v}}, \qquad \lambda_2 = 0, \qquad \lambda_3 = \sqrt{\frac{\gamma p}{v}}. \tag{2.3}$$

The first and the third characteristic field is genuinely nonlinear, which may have nonlinear waves, shock wave and rarefaction wave, while the second characteristic field is linearly degenerate, where contact discontinuity may occur. See [34].

The sound speed $C(v, \theta)$ and the Mach number $M(v, u, \theta)$ are defined by

$$C(v, \theta) = v\sqrt{\frac{\gamma p}{v}} = \sqrt{R\gamma\theta}$$

and

$$M(v,u,\theta)=\frac{|u|}{\sqrt{R\gamma\theta}}.$$

Let $C_+=C(v_+,\theta_+)=\sqrt{R\gamma\theta_+}$ and $M_+=\frac{|u_+|}{C_+}$ be the sound speed and the Mach number at the far field $x=+\infty$, respectively. The phase plane $\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+$ of (v,u,θ) can be divided into three subsets:

$$\begin{aligned}
\Omega_{sub}&:=\{(v,u,\theta)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+;\quad M(v,u,\theta)<1\},\\
\Gamma_{trans}&:=\{(v,u,\theta)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+;\quad M(v,u,\theta)=1\},\\
\Omega_{super}&:=\{(v,u,\theta)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+;\quad M(v,u,\theta)>1\},
\end{aligned}$$

where Ω_{sub}, Γ_{trans} and Ω_{super} are called the subsonic, transonic and supersonic regions, respectively. If we add the alternative condition $u>0$ or $u<0$, then we have six connected subsets $\Omega^{\pm}_{sub}$, $\Gamma^{\pm}_{trans}$ and $\Omega^{\pm}_{super}$.

When $(v_-,u_-,\theta_-)\in\Gamma^+_{trans}\bigcup\Omega^+_{sub}$, we have $\lambda_1(v_-,\theta_-)\le\sigma_-<0$, hence the existence of the traveling wave solution

$$\begin{cases}(V^B,U^B,\Theta^B)(\xi),\qquad \xi=x-\sigma_-t,\\ (V^B,U^B,\Theta^B)(0)=(v_-,u_-,\theta_-),\quad (V^B,U^B,\Theta^B)(+\infty)=(v_+,u_+,\theta_+).\end{cases}\tag{2.4}$$

to (2.1) or the stationary solution (BL-solution) to (2.2) is expected. From (2.4), BL-solution $(V^B,U^B,\Theta^B)(\xi)$ satisfies the following ODE system:

$$\begin{cases}-\sigma_-\partial_\xi V^B-\partial_\xi U^B=0,\quad \xi>0,\\
-\sigma_-\partial_\xi U^B+\partial_\xi P^B=\mu\partial_\xi\left(\dfrac{\partial_\xi U^B}{V^B}\right),\quad \xi>0,\\
-\sigma_-\partial_\xi\left(\dfrac{R}{\gamma-1}\Theta^B+\dfrac{(U^B)^2}{2}\right)+\partial_\xi(P^BU^B)=\mu\partial_\xi\left(\dfrac{U^B\partial_\xi U^B}{V^B}\right)+\kappa\partial_\xi\left(\dfrac{\partial_\xi\Theta^B}{V^B}\right),\quad \xi>0,\\
(V^B,U^B,\Theta^B)(0)=(v_-,u_-,\theta_-),\quad (V^B,U^B,\Theta^B)(+\infty)=(v_+,u_+,\theta_+),\end{cases}\tag{2.5}$$

where $P^B=p(V^B,\Theta^B)=\frac{R\Theta^B}{V^B}$. Integrating the system $(2.5)_1$ over $(\xi,+\infty)$, and then taking $\xi=0$ in the resulting equality, it is easy to get

$$\sigma_-=-\frac{u_-}{v_-}=-\frac{U^B}{V^B}=-\frac{u_+}{v_+}.\tag{2.6}$$

Then the existence and uniqueness for the ODE system (2.5) are given as follows. For later use, we only list some useful properties of solutions for (2.5).

Proposition 2.1. *(See [31].) Assume that $v_\pm>0$, $u_->0$, $\theta_\pm>0$ and define $\delta^B=|(u_+-u_-,\theta_+-\theta_-)|$. If $u_+\le0$, then there is no solution to (2.5). If $u_+>0$, then there exists a suitable small constant $\delta_0>0$ such that if $0<\delta^B\le\delta_0$, then note the following cases.*

Case I. Supersonic case: $M_+>1$. Then there is no solution to (2.5).

Case II. Transonic case: $M_+=1$. Then there exists a unique trajectory Σ tangent to the line

$$\mu u_+(U^B-u_+)-\kappa(\gamma-1)(\Theta^B-\theta_+)=0$$

at the point (u_+,θ_+). For each $(u_-,\theta_-)\in\Sigma(u_+,\theta_+)$, there exists a unique solution (U^B,Θ^B) satisfying

$$U^B_\xi>0,\qquad \Theta^B_\xi>0,$$

and

$$\left|\frac{d^n}{d\xi^n}(U^B-u_+,\Theta^B-\theta_+)\right|\le C\frac{(\delta^B)^{n+1}}{(1+\delta^B\xi)^{n+1}},\qquad n=0,1,2,\ldots.\tag{2.7}$$

Case III. Subsonic case: $M_+ < 1$. Then there exists a center-stable manifold $\mathcal{M}$ tangent to the line

$$(1 + a_2c_2u_+)(U^B - u_+) - a_2(\Theta^B - \theta_+) = 0$$

on the opposite directions at the point (u_+, θ_+), where a_2 and c_2 are some positive constants, see [31] for their definitions. Only when $(u_-, \theta_-) \in \mathcal{M}(u_+, \theta_+)$, does there exist a unique solution $(U^B, \Theta^B) \subset \mathcal{M}(u_+, \theta_+)$ satisfying

$$\left| \frac{d^n}{d\xi^n}(U^B - u_+, \Theta^B - \theta_+) \right| \leq C\delta^B e^{-c\xi}, \qquad n = 0, 1, 2, \ldots. \tag{2.8}$$

2.2 Viscous contact wave

If $(v_-, u_-, \theta_-) \in CD(v_+, u_+, \theta_+)$, i.e.,

$$u_- = u_+, \quad p_- = p_+, \tag{2.9}$$

then the following Riemann problem of the Euler system

$$\begin{cases} \partial_t v - \partial_x u = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ \partial_t u + \partial_x p = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ \partial_t (e + \dfrac{u^2}{2}) + \partial_x (pu) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ (v, u, \theta)(x, 0) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases} \end{cases} \tag{2.10}$$

admits a single contact discontinuity solution

$$(v, u, \theta)(x, t) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \quad t > 0, \\ (v_+, u_+, \theta_+), & x > 0, \quad t > 0. \end{cases} \tag{2.11}$$

From [11], we know that the viscous version of the above contact discontinuity, called viscous contact wave $\left(V^{CD}, U^{CD}, \Theta^{CD}\right)(x, t)$, could be defined by

$$\begin{cases} \Theta^{CD}(x, t) = \Theta^{Sim}(\dfrac{x}{\sqrt{1+t}}), \\ V^{CD}(x, t) = \dfrac{R\Theta^{CD}(x, t)}{p_+}, \\ U^{CD}(x, t) = u_+ + \dfrac{\kappa(\gamma - 1)}{R\gamma} \dfrac{\partial_x \Theta^{CD}(x, t)}{\Theta^{CD}(x, t)}, \end{cases} \tag{2.12}$$

where $\Theta^{Sim}(\eta)$ $(\eta = \frac{x}{\sqrt{1+t}})$ is the unique self-similar solution of the nonlinear diffusion equation

$$\partial_t \Theta = \frac{\kappa(\gamma - 1)p_+}{R^2\gamma} \partial_x \left(\frac{\partial_x \Theta}{\Theta} \right), \quad \Theta(\pm\infty, t) = \theta_\pm. \tag{2.13}$$

Thus the viscous contact wave defined in (2.13) satisfies the following property:

$$(1+t)^{\frac{3}{2}} \left|\partial_x^3 \Theta^{CD}\right| + (1+t) \left|\partial_x^2 \Theta^{CD}\right| + (1+t)^{\frac{1}{2}} \left|\partial_x \Theta^{CD}\right| + \left|\Theta^{CD} - \theta_\pm\right| = O(1)\delta^{CD} e^{-\frac{c_0 x^2}{1+t}}, \text{as } x \to \pm\infty, \tag{2.14}$$

where $\delta^{CD}=|\theta_+-\theta_-|$ is the amplitude of the viscous contact wave and c_0 is some positive constant. Note that $\xi=x-\sigma_-t$, then the viscous contact wave $\left(V^{CD},U^{CD},\Theta^{CD}\right)(\xi,t)$ satisfies

$$\begin{cases}\partial_t V^{CD}-\sigma_-\partial_\xi V^{CD}-\partial_\xi U^{CD}=0,\\ \partial_t U^{CD}-\sigma_-\partial_\xi U^{CD}+\partial_\xi P^{CD}=\mu\partial_\xi\left(\dfrac{\partial_\xi U^{CD}}{V^{CD}}\right)+\bar{Q}_1,\\ \dfrac{R}{\gamma-1}(\partial_t\Theta^{CD}-\sigma_-\partial_\xi\Theta^{CD})+P^{CD}\partial_\xi U^{CD}=\mu\dfrac{(\partial_\xi U^{CD})^2}{V^{CD}}+\kappa\partial_\xi\left(\dfrac{\partial_\xi\Theta^{CD}}{V^{CD}}\right)+\bar{Q}_2,\end{cases}\tag{2.15}$$

where $P^{CD}:=p(V^{CD},\Theta^{CD})=\frac{R\Theta^{CD}}{V^{CD}}$ and the error terms $\bar{Q}_1,\bar{Q}_2$ are given by

$$\begin{aligned}\bar{Q}_1=&\partial_t U^{CD}-\sigma_-\partial_\xi U^{CD}-\mu\partial_\xi\left(\frac{\partial_\xi U^{CD}}{V^{CD}}\right)=O(1)\left(|\partial_\xi\Theta^{CD}|^3+|\partial_\xi^3\Theta^{CD}|+|\partial_\xi^2\Theta^{CD}||\partial_\xi\Theta^{CD}|\right)\\ =&O(1)\delta^{CD}(1+t)^{-\frac{3}{2}}e^{-\frac{c_0(\xi+\sigma_-t)^2}{1+t}},\ \text{ as }\ |\xi+\sigma_-t|\to+\infty,\end{aligned}\tag{2.16}$$

$$\begin{aligned}\bar{Q}_2=&-\mu\frac{(\partial_\xi U^{CD})^2}{V^{CD}}=O(1)\left(|\partial_\xi\Theta^{CD}|^4+|\partial_\xi^2\Theta^{CD}|^2\right)\\ =&O(1)\delta^{CD}(1+t)^{-2}e^{-\frac{c_0(\xi+\sigma_-t)^2}{1+t}},\ \text{ as }\ |\xi+\sigma_-t|\to+\infty.\end{aligned}\tag{2.17}$$

2.3 Rarefaction wave

If $(v_-,u_-,\theta_-)\in R_i(v_+,u_+,\theta_+)(i=1,3)$, then there exists a i-rarefaction wave $\left(v^{r_i},u^{r_i},\theta^{r_i}\right)\left(\frac{x}{t}\right)$ which is the global (in time) weak solution of the following Riemann problem

$$\begin{cases}\partial_t v-\partial_x u=0,\quad t>0,\quad x\in\mathbb{R},\\ \partial_t u+\partial_x p(v,\theta)=0,\quad t>0,\quad x\in\mathbb{R},\\ \dfrac{R}{\gamma-1}\partial_t\theta+p(v,\theta)\partial_x u=0,\quad t>0,\quad x\in\mathbb{R},\\ (v,u,\theta)(x,0)=\begin{cases}(v_-,u_-,\theta_-), & x<0,\\ (v_+,u_+,\theta_+), & x>0.\end{cases}\end{cases}\tag{2.18}$$

In order to construct the smooth approximated rarefaction wave, we consider the Riemann problem on the Burgers equation

$$\begin{cases}\partial_t\bar{w}+\bar{w}\partial_x\bar{w}=0,\\ \bar{w}(x,0)=\bar{w}_0(x)=\begin{cases}w_-, & x<0,\\ w_+, & x>0\end{cases}\end{cases}\tag{2.19}$$

for $w_-<w_+$. It is well-known that the Riemann problem (2.19) admits a continuous weak solution $\bar{w}(\frac{x}{t})$ connecting w_- and w_+, taking the form of

$$\bar{w}\left(\frac{x}{t}\right)=\begin{cases}w_-,\ x\le w_-t,\\ \dfrac{x}{t},\ w_-t<x<w_+t,\\ w_+,\ w_+t\le x.\end{cases}\tag{2.20}$$

Moreover, $\bar{w}(\frac{x}{t})$ is approximated by a smooth function $w(x,t)$ satisfying

$$\begin{cases}\partial_t w+w\partial_x w=0,\\ w(x,0)=w_0(x)=\begin{cases}w_-, & x<0,\\ w_-+C_q(w_+-w_-)\displaystyle\int_0^x y^q e^{-y}dy, & x>0,\end{cases}\end{cases}\tag{2.21}$$

where $q \geq 14$ is a constant, C_q is a constant such that $C_q \displaystyle\int_0^\infty y^q e^{-y} dy = 1$. The solution to the Burgers equation can be expressed by

$$w(x,t) = w_0(x_0(x,t)), x = x_0(x,t) + w_0(x_0(x,t))t.$$

And from [32] we know for any positive constant $\sigma_0 > 0$ and for $x \geq 0$

$$\begin{aligned}
| w(x,t) - w_+ | =& | w_0(x_0(x,t)) - w_+ | \\
&= C_q(w_+ - w_-) \int_{x_0(x,t)}^\infty y^q e^{-y} dy \\
&= C_q(w_+ - w_-) \int_{x - w_0(x_0(x,t))t}^\infty y^q e^{-y} dy \\
&\leq C_q(w_+ - w_-) \int_{x - w_+ t}^\infty y^q e^{-y} dy \\
&\leq C_q(w_+ - w_-) e^{-\sigma_0 t}, x \geq (2\sigma_0 + w_+)t.
\end{aligned} \tag{2.22}$$

Then the solution $w(x,t)$ of the Burgers equation (2.21) have the following properties:

Lemma 2.1. *Let $0 < w_- < w_+$, $\delta^r := w_+ - w_-$, then Burgers equation (2.21) has a unique smooth solution $w(x,t)$ which satisfies the following properties:*

(i) $w_- \leq w(x,t) < w_+$, $\partial_x w \geq 0$ for $x \in \mathbb{R}$ and $t \geq 0$.

(ii) For any p $(1 \leq p \leq \infty)$, there exists a positive constant $C_{p,q}$ such that for $t \geq 0$

$$\|\partial_x w(t)\|_{L^p} \leq C_{p,q} \min\{\delta^r, (\delta^r)^{\frac{1}{p}} t^{-1+\frac{1}{p}}\},$$

$$\left\|\partial_x^2 w(t)\right\|_{L^p} \leq C_{p,q} \min\{\delta^r, (\delta^r)^{\frac{1}{p}+\frac{1}{q}} (1+t)^{-1+\frac{1}{q}}\}.$$

(iii) When $x \leq w_- t$, $w(x,t) \equiv w_-$.

(iv) $\lim\limits_{t\to\infty} \sup\limits_{x\in\mathbb{R}} \left|w(x,t) - \bar{w}(\frac{x}{t})\right| = 0$.

Thus we construct the smooth approximated rarefaction wave $\left(V^{R_i}, U^{R_i}, \Theta^{R_i}\right)(x,t)(i = 1,3)$ by

$$\begin{cases}
S^{R_i}(x,t) = s(V^{R_i}(x,t), \Theta^{R_i}(x,t)) = s_+, \\
\lambda_i(V^{R_i}(x,t), s_+) = w(x, 1+t), \\
U^{R_i}(x,t) = u_+ - \displaystyle\int_{v_+}^{V^{R_i}(x,t)} \lambda_i(z, s_+) dz.
\end{cases} \tag{2.23}$$

Note that $\xi = x - \sigma_- t$, then the smoothed i-rarefaction wave $\left(V^{R_i}, U^{R_i}, \Theta^{R_i}\right)(\xi,t)(i = 1,3)$ defined above satisfies

$$\begin{cases}
\partial_t V^{R_i} - \sigma_- \partial_\xi V^{R_i} - \partial_\xi U^{R_i} = 0, \quad \xi > 0, \ t > 0, \\
\partial_t U^{R_i} - \sigma_- \partial_\xi U^{R_i} + \partial_\xi P^{R_i} = 0, \quad \xi > 0, \ t > 0, \\
\dfrac{R}{\gamma - 1}(\partial_t \Theta^{R_i} - \sigma_- \partial_\xi \Theta^{R_i}) + P^{R_i} \partial_\xi U^{R_i} = 0, \quad \xi > 0, \ t > 0, \\
(V^{R_i}, U^{R_i}, \Theta^{R_i})(\xi = 0, t) = (v_-, u_-, \theta_-), \quad (V^{R_i}, U^{R_i}, \Theta^{R_i})(\xi = +\infty, t) = (v_+, u_+, \theta_+),
\end{cases} \tag{2.24}$$

where $P^{R_i} := p(V^{R_i}, \Theta^{R_i}) = \frac{R\Theta^{R_i}}{V^{R_i}}$.

Lemma 2.2. *Let $\delta^{R_i} = |(v_+, u_+, \theta_+) - (v_-, u_-, \theta_-)|$. Then the smooth approximate rarefaction wave $\left(V^{R_i}, U^{R_i}, \Theta^{R_i}\right)(\xi,t)(i = 1,3)$ satisfies the following properties [32]:*

(i) $\partial_\xi U^{R_i} \geq 0$ for $\xi \in \mathbb{R}_+$ and $t \geq 0$.

(ii) *For any* $1 \le p \le +\infty$, *there exists a constant* $C_{p,q}$ *such that for* $t \ge 0$,

$$\left\|\partial_\xi \left(V^{R_i}, U^{R_i}, \Theta^{R_i}\right)\right\|_{L^p(\mathbb{R}_+)} \le C_{p,q} \min\{\delta^{R_i}, (\delta^{R_i})^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\},$$

$$\left\|\partial_\xi^2 \left(V^{R_i}, U^{R_i}, \Theta^{R_i}\right)\right\|_{L^p(\mathbb{R}_+)} \le C_{p,q} \min\{\delta^{R_i}, (\delta^{R_i})^{\frac{1}{p}+\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}\}.$$

(iii) *For* $\forall \sigma_0 > 0$, *if* $\xi \ge [-\sigma_- + \lambda_1(v_+, \theta_+) + 2\sigma_0](1+t)$, *then*

$$\mid \partial_\xi^n \{(V^{R_1}, U^{R_1}, \Theta^{R_1}) - (v_+, u_+, \theta_+,)\} \mid \le C\delta^{R_1} e^{-\sigma_0 t}, n = 0, 1, 2 \cdots.$$

(iv) *If* $\xi + \sigma_- t \le \lambda_3(v_-, \theta_-)(1+t)$, *then* $\left(V^{R_3}, U^{R_3}, \Theta^{R_3}\right)(\xi, t) \equiv (v_-, u_-, \theta_-)$.

(v) $\lim\limits_{t\to\infty} \sup\limits_{\xi\in\mathbb{R}_+} \left|\left(V^{R_i}, U^{R_i}, \Theta^{R_i}\right)(\xi, t) - (v, u, \theta)\left(\frac{\xi}{1+t}\right)\right| = 0.$

2.4 Composite waves and main results

Define the composite wave $(V, U, \Theta)(\xi, t)$ by

$$\begin{pmatrix} V \\ U \\ \Theta \end{pmatrix}(\xi, t) = \begin{pmatrix} V^B + V^{R_1} + V^{CD} + V^{R_3} \\ U^B + U^{R_1} + U^{CD} + U^{R_3} \\ \Theta^B + \Theta^{R_1} + \Theta^{CD} + \Theta^{R_3} \end{pmatrix}(\xi, t) - \begin{pmatrix} v_* + v_m + v^* \\ u_* + u_m + u^* \\ \theta_* + \theta_m + \theta^* \end{pmatrix}, \tag{2.25}$$

where $(V^B, U^B, \Theta^B)(\xi, t)$ is the transonic BL-solution (Case II) defined in Proposition 2.1 with the right state (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*), $(V^{R_1}, U^{R_1}, \Theta^{R_1})(\xi, t)$ is the 1-rarefaction wave defined in (2.12) with the end states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v_*, u_*, θ_*) and (v_m, u_m, θ_m), respectively, $(V^{CD}, U^{CD}, \Theta^{CD})(\xi, t)$ is the smoothed viscous 2-contact wave defined in (2.23) with the states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v_m, u_m, θ_m) and (v^*, u^*, θ^*), and $(V^{R_3}, U^{R_3}, \Theta^{R_3})(\xi, t)$ is the 3-rarefaction wave defined in (2.12) with the end states (v_-, u_-, θ_-) and (v_+, u_+, θ_+) replaced by (v^*, u^*, θ^*) and (v_+, u_+, θ_+).

Now we state the main result of our paper as follows.

Theorem 2.1. *For any given* $[v_\pm, u_\pm, \omega_\pm, \theta_\pm]$ *with* $v_\pm > 0$, $u_- > 0$ *and* $\theta_\pm > 0$, *we suppose that* $u_+ > 0$, $\omega_\pm = 0$ *and* $(v_-, u_-, \theta_-) \in BL{-}R_1{-}CD{-}R_3(v_+, u_+, \theta_+)$. *Let* $[V, U, \Theta](\xi, t)$ *be the composite wave consisting of the transonic BL-solution, 1-rarefaction wave, the viscous 2-contact wave, and the 3-rarefaction wave defined in* (2.25). *There exist positive constants* $\delta_0 > 0$ *and* $C_0 > 0$, *such that if*

$$[v_0(\xi) - V(\xi, 0), u_0(\xi) - U(\xi, 0), w_0(\xi) - 0, \theta_0(\xi) - \Theta(\xi, 0)](\xi) \in H^1(\mathbb{R}_+)$$

and the wave strength $\delta =\mid (v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-) \mid$ *satisfy*

$$\|[v_0(\xi) - V(\xi, 0), u_0(\xi) - U(\xi, 0), w_0(\xi) - 0, \theta_0(\xi) - \Theta(\xi, 0)](\xi)\|^2_{H^1(\mathbb{R}_+)} + \delta \le \delta_0, \tag{2.26}$$

then the micropolar fluid model to the inflow problem (1.12) *or to the half-space problem* (1.14) *admits a unique global solution* $[v, u, \omega, \theta](\xi, t)$ *satisfying*

$$[v - V, u - U, \omega, \theta - \Theta] \in C(0, +\infty; H^1(\mathbb{R}_+))$$

and

$$\sup_{t\ge0} \|[v - V, u - U, \omega, \theta - \Theta]\|_{H^1(0,+\infty)} \le C_0 \delta_0^{\frac{1}{2}}. \tag{2.27}$$

Moreover, it holds that

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}_+} |[v - V, u - U, \omega, \theta - \Theta]| = 0. \tag{2.28}$$

Remark 2.1. *In Theorem 2.1, we assume that* $\delta =\mid (v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-) \mid$ *is suitably small. This assumption is equivalent to one that the amplitudes of the four waves are all suitably small.*

Remark 2.2. *This model can also be generalized to general gases.*

3 Global existence and large time behavior

3.1 Wave interaction estimates

From (2.5), (2.15), (2.24) and (2.25), by a careful calculation, we have

$$\begin{cases} \partial_t V - \sigma_- \partial_\xi V - \partial_\xi U = 0, \quad \xi > 0, \ t > 0, \\ \partial_t U - \sigma_- \partial_\xi U + \partial_\xi P = \mu \partial_\xi \left(\dfrac{\partial_\xi U}{V} \right) + Q_1, \quad \xi > 0, \ t > 0, \\ \dfrac{R}{\gamma - 1} (\partial_t \Theta - \sigma_- \partial_\xi \Theta) + P \partial_\xi U = \mu \dfrac{(\partial_\xi U)^2}{V} + \kappa \partial_\xi \left(\dfrac{\partial_\xi \Theta}{V} \right) + Q_2, \quad \xi > 0, \ t > 0, \\ (V, U, \Theta)(\xi = 0, t) = (v_- + V^{CD} - v_m, u_- + U^{CD} - u_m, \theta_- + \Theta^{CD} - \theta_m)(\xi = 0, t), \\ (V, U, \Theta)(\xi = +\infty, t) = (v_+, u_+, \theta_+), \end{cases} \tag{3.1}$$

where $P := p(V, \Theta) = \frac{R\Theta}{V}$ and the error terms Q_1, Q_2 are given by

$$Q_1 = \partial_\xi \left(P - P^B - P^{R_1} - P^{CD} - P^{R_3} \right) - \mu \left[\partial_\xi \left(\frac{\partial_\xi U}{V} \right) - \partial_\xi \left(\frac{\partial_\xi U^B}{V^B} \right) - \partial_\xi \left(\frac{\partial_\xi U^{CD}}{V^{CD}} \right) \right] + \bar{Q}_1,$$

$$\begin{aligned} Q_2 = & \left(P \partial_\xi U - P^B \partial_\xi U^B - P^{R_1} \partial_\xi U^{R_1} - P^{CD} \partial_\xi U^{CD} - P^{R_3} \partial_\xi U^{R_3} \right) \\ & - \mu \left[\frac{(\partial_\xi U)^2}{V} - \frac{(\partial_\xi U^B)^2}{V^B} - \frac{(\partial_\xi U^{CD})^2}{V^{CD}} \right] - \kappa \left[\partial_\xi \left(\frac{\partial_\xi \Theta}{V} \right) - \partial_\xi \left(\frac{\partial_\xi \Theta^B}{V^B} \right) - \partial_\xi \left(\frac{\partial_\xi \Theta^{CD}}{V^{CD}} \right) \right] + \bar{Q}_2, \end{aligned}$$

where $\bar{Q}_1$ and $\bar{Q}_2$ are the error terms defined in (2.16) and (2.17) to the viscous contact wave.

In order to control the interaction terms coming from different wave patterns, we give the following lemma which will be important in the energy estimate.

Lemma 3.1. *(wave interaction estimates)[32].*

$$\int_{\mathbb{R}_+} | V_\xi^B (V^{R_1} - v_*) | + | V_\xi^{R_1} (V^B - v_*) | \, d\xi = O(1) \delta^{\frac{1}{8}} (1 + t)^{-\frac{13}{16}},$$

$$\int_{\mathbb{R}_+} | V_\xi^B (V^{CD} - v_m) | + | V_\xi^{CD} (V^B - v_*) | \, d\xi = O(1) \delta (1 + t)^{-1},$$

$$\int_{\mathbb{R}_+} | V_\xi^B (V^{R_3} - v^*) | + | V_\xi^{R_3} (V^B - v_*) | \, d\xi = O(1) \delta^{\frac{1}{8}} (1 + t)^{-\frac{7}{8}},$$

$$\int_{\mathbb{R}_+} | V_\xi^{CD} (V^{R_1} - v_m) | + | V_\xi^{R_1} (V^{CD} - v_m) | \, d\xi = O(1) \delta e^{-ct},$$

$$\int_{\mathbb{R}_+} | V_\xi^{CD} (V^{R_3} - v^*) | + | V_\xi^{R_3} (V^{CD} - v^*) | \, d\xi = O(1) \delta e^{-ct},$$

$$\int_{\mathbb{R}_+} | V_\xi^{R_1} (V^{R_3} - v^*) | + | V_\xi^{R_3} (V^{R_1} - v_m) | \, d\xi = O(1) \delta e^{-ct},$$

$$\int_{\mathbb{R}_+} | V_\xi^B V_\xi^{CD} | \, d\xi = O(1) \delta (1 + t)^{-2}, \quad \int_{\mathbb{R}_+} | V_\xi^B V_\xi^{R_1} | \, d\xi = O(1) \delta (1 + t)^{-1},$$

$$\int_{\mathbb{R}_+} | V_\xi^B V_\xi^{R_3} | \, d\xi = O(1) \delta (1 + t)^{-1}, \quad \int_{\mathbb{R}_+} | V_\xi^{CD} V_\xi^{R_1} | \, d\xi = O(1) \delta e^{-ct},$$

$$\int_{\mathbb{R}_+} | V_\xi^{CD} V_\xi^{R_3} | \, d\xi = O(1) \delta e^{-ct}, \quad \int_{\mathbb{R}_+} | V_\xi^{R_1} V_\xi^{R_3} | \, d\xi = O(1) \delta e^{-ct}.$$

3.2 Reformulation of the problem

We first define the perturbation as

$$[\varphi, \psi, \omega, \zeta](\xi, t) = [v - V, u - U, \omega - 0, \theta - \Theta] (\xi, t).$$

Then from (2.2) and (3.1), it is easy to obtain that $[\varphi, \psi, \omega, \zeta](\xi, t)$ satisfies

$$
\begin{cases}
\partial_t \varphi - \sigma_- \partial_\xi \varphi - \partial_\xi \psi = 0, \qquad \xi > 0,\ t > 0, \\
\partial_t \psi - \sigma_- \partial_\xi \psi + \partial_\xi (p - P) = \mu \partial_\xi \left(\dfrac{\partial_\xi u}{v} - \dfrac{\partial_\xi U}{V} \right) - Q_1, \quad \xi > 0,\ t > 0, \\
\partial_t \omega - \sigma_- \partial_\xi \omega + A v \omega = A \partial_\xi \left(\dfrac{\partial_\xi \omega}{v} \right), \qquad \xi > 0,\ t > 0, \\
\dfrac{R}{\gamma - 1} (\partial_t \zeta - \sigma_- \partial_\xi \zeta) + (p \partial_\xi u - P \partial_\xi U) = \kappa \partial_\xi \left(\dfrac{\partial_\xi \theta}{v} - \dfrac{\partial_\xi \Theta}{V} \right) \\
\qquad\qquad + \mu \left(\dfrac{(\partial_\xi u)^2}{v} - \dfrac{(\partial_\xi U)^2}{V} \right) + \dfrac{(\partial_\xi \omega)^2}{v} + v \omega^2 - Q_2, \quad \xi > 0,\ t > 0, \\
[\varphi, \psi, \omega, \zeta](\xi, 0) = [\varphi_0, \psi_0, \omega_0, \zeta_0](\xi) \\
\qquad = [v_0(\xi) - V(\xi, 0), u_0(\xi) - U(\xi, 0), \omega_0(\xi) - 0, \theta_0(\xi) - \Theta(\xi, 0)] \to (0, 0, 0, 0), \quad as \ \ \xi \to +\infty, \\
[\varphi, \psi, \omega, \zeta](0, t) = (v_- - V, u_- - U, 0, \theta_- - \Theta)(0, t).
\end{cases}
\tag{3.2}
$$

The key to the proof of the global existence part of Theorem 2.1 is to derive the uniform *a priori* estimates of solutions to the half-space problem (3.2). Our *a priori* assumption is defined as follows:

$$
\sup_{0 \le \tau \le t} \| [\varphi, \psi, \omega, \zeta](\tau) \|_{H^1(\mathbb{R}_+)} \le \varepsilon_1, \tag{3.3}
$$

where ε_1 is a small positive constant.

Proposition 3.1. *(A priori estimates). Assume all the conditions listed in Theorem 2.1 hold. Let $[\varphi, \psi, \omega, \zeta](\xi, t)$ be a solution to the half-space problem (3.2) on $0 \le t \le T$ for some positive constant T. There are constants $\delta_0 > 0$ and $C > 0$, such that if $[\varphi, \psi, \omega, \zeta] \in C(0, T; H^1(\mathbb{R}_+))$ and*

$$
\| [\varphi_0, \psi_0, \omega_0, \zeta_0](\xi) \|^2_{H^1(\mathbb{R}_+)} + \delta \le \delta_0, \tag{3.4}
$$

then for all $t \in [0, T]$, the solution $[\varphi, \psi, \omega, \zeta](\xi, t)$ satisfies

$$
\begin{aligned}
&\sup_{0 \le \tau \le t} \| [\varphi, \psi, \omega, \zeta](\tau) \|^2_{H^1(\mathbb{R}_+)} + \int_0^t \left(\| \partial_\xi \varphi \|^2 + \| \partial_\xi [\psi, \zeta] \|^2_{H^1(\mathbb{R}_+)} + \| \omega \|^2_{H^2(\mathbb{R}_+)} \right) d\tau \\
\le & C \| [\varphi_0, \psi_0, \omega_0, \zeta_0] \|^2_{H^1(\mathbb{R}_+)} + C \delta^{\frac{1}{8}}.
\end{aligned}
\tag{3.5}
$$

From *a priori* assumption (3.3), it is easy to get

$$
\| [\varphi, \psi, \omega, \zeta] \|_{L^\infty} \le \sqrt{2} \varepsilon_1, \tag{3.6}
$$

where the following Sobolev inequality

$$
\| h(\xi) \|_{L^\infty} \le \sqrt{2} \| h \|^{1/2} \| h_\xi \|^{1/2} \text{ for } h(\xi) \in H^1(\mathbb{R}_+) \tag{3.7}
$$

is used.

3.3 Energy estimates

Lemma 3.2. *(boundary estimates)[31]. There exists a positive constant C such that for any $t > 0$,*

$$
\int_0^t |(\varphi, \psi, \zeta)(0, \tau)|^2 d\tau \le C \delta, \tag{3.8}
$$

$$
\int_0^t (| \psi \partial_\xi \psi | + | \zeta \partial_\xi \zeta |)(0, \tau) d\tau \le C \delta \int_0^t (\| \partial_\xi [\psi, \zeta] \|^2 + \| \partial_\xi^2 [\psi, \zeta] \|^2) d\tau + C \delta, \tag{3.9}
$$

$$\int_0^t (| \partial_\tau\varphi\psi | + (\partial_\xi\varphi)^2)(0,\tau)d\tau \le C\delta + \nu\int_0^t \| \partial_\xi^2\psi \|^2 \, d\tau + C_\nu \int_0^t \| \partial_\xi\psi \|^2 \, d\tau, \tag{3.10}$$

$$\int_0^t (| \partial_\tau\psi\partial_\xi\psi | + (\partial_\xi\psi)^2)(0,\tau)d\tau \le C\delta + \nu\int_0^t \| \partial_\xi^2\psi \|^2 \, d\tau + C_\nu \int_0^t \| \partial_\xi\psi \|^2 \, d\tau, \tag{3.11}$$

$$\int_0^t (| \partial_\tau\zeta\partial_\xi\zeta | + (\partial_\xi\psi)^2)(0,\tau)d\tau \le C\delta + \nu\int_0^t \| \partial_\xi^2\zeta \|^2 \, d\tau + C_\nu \int_0^t \| \partial_\xi\zeta \|^2 \, d\tau, \tag{3.12}$$

where ν is a positive small constant to be determined later, and C_ν is a positive constant depending on ν.

Lemma 3.3. *Assume the conditions in Proposition* 3.1 *hold, then we have the following energy estimate for $t \in [0,T]$,*

$$\begin{aligned}
&\|[\psi,\varphi,\zeta,\omega]\|^2 + \int_0^t \left(\|\partial_\xi [\psi,\zeta,\omega]\|^2 + \|\omega\|^2\right) d\tau + \int_0^t \|\sqrt{[\partial_\xi U^{R_1}, \partial_\xi U^{R_3}, \partial_\xi U^B]}[\varphi,\zeta]\|^2 d\tau \\
\le& C\,\|[\psi_0,\zeta_0,\varphi_0,\omega_0]\|^2 + C\delta^{\frac{1}{8}} + C\delta^{\frac{1}{8}}\int_0^t \|\partial_\xi\varphi\|^2 d\tau \\
&+ C\delta\int_0^t\int_{\mathbb{R}_+} (1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}} (\varphi^2+\zeta^2) d\xi d\tau.
\end{aligned} \tag{3.13}$$

Proof. Multiplying $(3.2)_1$, $(3.2)_2$, $(3.2)_3$ and $(3.2)_4$ by $-R\Theta\left(\frac{1}{v}-\frac{1}{V}\right)$, ψ, ω and $\frac{\zeta}{\theta}$, respectively, then taking the summation of the resulting equations, we obtain

$$\begin{aligned}
&\partial_t\left(\frac{1}{2}\psi^2 + R\Theta\Phi\left(\frac{v}{V}\right) + \frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right) + \frac{\omega^2}{2}\right) + \partial_\xi H_1 + \mu\frac{\Theta(\partial_\xi\psi)^2}{v\theta} + \kappa\frac{\Theta}{v\theta^2}(\partial_\xi\zeta)^2 + Av\omega^2 + \frac{A}{v}(\partial_\xi\omega)^2 \\
&\quad + P\left(\partial_\xi U^{R_1} + \partial_\xi U^{R_3} + \partial_\xi U^B\right)\left[\Phi\left(\frac{\theta V}{v\Theta}\right) + \gamma\Phi\left(\frac{v}{V}\right)\right] = Q_3 - Q_1\psi - \frac{\zeta}{\theta}Q_2 + \frac{\zeta}{\theta}\left[\frac{(\partial_\xi\omega)^2}{v} + v\omega^2\right],
\end{aligned} \tag{3.14}$$

where

$$\Phi(s) = s - 1 - \ln s,$$

$$\begin{aligned}
H_1 =& -\sigma_-\left(\frac{1}{2}\psi^2 + R\Theta\Phi\left(\frac{v}{V}\right) + \frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right) + \frac{\omega^2}{2}\right) + (p-P)\,\psi \\
& - \mu\left(\frac{\partial_\xi u}{v} - \frac{\partial_\xi U}{V}\right)\psi - \kappa\frac{\zeta}{\theta}\left(\frac{\partial_\xi\theta}{v} - \frac{\partial_\xi\Theta}{V}\right) - A\frac{\omega\partial_\xi\omega}{v},
\end{aligned}$$

$$\begin{aligned}
Q_3 =& -P\partial_\xi U^{CD}\left[\Phi\left(\frac{\theta V}{v\Theta}\right) + \gamma\Phi\left(\frac{v}{V}\right)\right] \\
&+ \left[\mu\frac{(\partial_\xi U)^2}{V} + \kappa\partial_\xi\left(\frac{\partial_\xi\Theta}{V}\right) + Q_2\right]\left[(\gamma-1)\Phi\left(\frac{v}{V}\right) - \Phi\left(\frac{\Theta}{\theta}\right)\right] \\
&+ \kappa\frac{\partial_\xi\Theta}{\theta^2 v}\zeta\partial_\xi\zeta + \kappa\frac{\Theta\varphi\partial_\xi\zeta}{\theta^2 vV}\partial_\xi\Theta - \kappa\frac{\zeta\varphi}{\theta^2 vV}(\partial_\xi\Theta)^2 + \mu\frac{\partial_\xi U}{vV}\varphi\partial_\xi\psi - \mu\frac{(\partial_\xi U)^2}{v\theta V}\varphi\zeta + 2\mu\frac{\partial_\xi U}{v\theta}\zeta\partial_\xi\psi.
\end{aligned}$$

Then integrating the resulting identity (3.14) over $\mathbb{R}_+ \times [0,t]$, we thus arrive at

$$\begin{aligned}
&\int_{\mathbb{R}_+}\left(\frac{\psi^2}{2}+R\Theta\Phi\left(\frac{v}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{\omega^2}{2}\right)d\xi+\mu\int_0^t\int_{\mathbb{R}_+}\frac{\Theta(\partial_\xi\psi)^2}{v\theta}d\xi d\tau+\kappa\int_0^t\int_{\mathbb{R}_+}\frac{\Theta(\partial_\xi\zeta)^2}{v\theta^2}d\xi d\tau\\
&+\int_0^t\int_{\mathbb{R}_+}\left[Av\omega^2+\frac{A}{v}(\partial_\xi\omega)^2\right]d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}P\left(\partial_\xi U^{R_1}+\partial_\xi U^{R_3}+\partial_\xi U^B\right)\left[\Phi\left(\frac{\theta V}{v\Theta}\right)+\gamma\Phi\left(\frac{v}{V}\right)\right]d\xi d\tau\\
=&\int_{\mathbb{R}_+}\left(\frac{\psi^2}{2}+R\Theta\Phi\left(\frac{v}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{\omega^2}{2}\right)(\xi,0)d\xi+\int_0^t H_1(0,\tau)d\tau+\int_0^t\int_{\mathbb{R}_+}Q_3 d\xi d\tau\\
&-\int_0^t\int_{\mathbb{R}_+}Q_1\psi d\xi d\tau-\int_0^t\int_{\mathbb{R}_+}\frac{\zeta}{\theta}Q_2 d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}\frac{\zeta}{\theta}\left[\frac{(\partial_\xi\omega)^2}{v}+v\omega^2\right]d\xi d\tau.
\end{aligned}\tag{3.15}$$

From the definition of $\Phi(\cdot)$ and the smallness of perturbation solutions $[\varphi,\psi,\omega,\zeta]$, we have

$$\frac{\psi^2}{2}+R\Theta\Phi\left(\frac{v}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{\omega^2}{2}=O(1)(\varphi^2+\psi^2+\omega^2+\zeta^2),\tag{3.16}$$

$$\Phi\left(\frac{\theta V}{v\Theta}\right)+\gamma\Phi\left(\frac{v}{V}\right)=O(1)(\varphi^2+\zeta^2).\tag{3.17}$$

Since $\partial_\xi U^{R_1}\geq 0$, $\partial_\xi U^{R_3}\geq 0$, $\partial_\xi U^B\geq 0$, we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}P\left(\partial_\xi U^{R_1}+\partial_\xi U^{R_3}+\partial_\xi U^B\right)\left[\Phi\left(\frac{\theta V}{v\Theta}\right)+\gamma\Phi\left(\frac{v}{V}\right)\right]d\xi d\tau\\
\geq& c\int_0^t\int_{\mathbb{R}_+}\left(\partial_\xi U^{R_1}+\partial_\xi U^{R_3}+\partial_\xi U^B\right)(\varphi^2+\zeta^2)d\xi d\tau,
\end{aligned}\tag{3.18}$$

where we have used (3.17).

By applying the *a priori* assumption (3.3), $(3.2)_6$, (2.16), (2.17), Cauchy-Schwarz's inequality with $0<\nu<1$, (3.16), (3.17), (3.6), Sobolev's inequality (3.7) and Lemma 3.2, we obtain the estimates for the right hand side of (3.15) as follows:

$$\int_{\mathbb{R}_+}\left(\frac{\psi^2}{2}+R\Theta\Phi\left(\frac{v}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{\omega^2}{2}\right)(\xi,0)d\xi\leq c\|[\varphi_0,\psi_0,\omega_0,\zeta_0]\|^2,\tag{3.19}$$

$$\int_0^t H_1(0,\tau)d\tau\leq\nu\int_0^t(\|\,\partial_\xi[\psi,\zeta]\,\|^2+\|\,\partial_\xi^2[\psi,\zeta]\,\|^2)d\tau+C_\nu\delta,\tag{3.20}$$

$$\int_0^t\int_{\mathbb{R}_+}\frac{\zeta}{\theta}\left[\frac{(\partial_\xi\omega)^2}{v}+v\omega^2\right]d\xi d\tau\leq C\int_0^t\|\zeta\|_\infty(\|\partial_\xi\omega\|^2+\|\omega\|^2)d\tau\leq C\varepsilon_1\int_0^t(\|\partial_\xi\omega\|^2+\|\omega\|^2)d\tau,\tag{3.21}$$

and

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+} Q_3 d\xi d\tau\\
\le&\nu\int_0^t\|\partial_\xi[\psi,\zeta]\|^2d\tau+(C_\nu+C)\underbrace{\int_0^t\int_{\mathbb{R}_+}\left|\left(\partial_\xi^2\Theta^B,(\partial_\xi\Theta^B)^2,(\partial_\xi V^B)^2,(\partial_\xi U^B)^2\right)\right|(\varphi^2+\zeta^2)d\xi d\tau}_{I_1}\\
&+(C_\nu+C)\underbrace{\int_0^t\int_{\mathbb{R}_+}\sum_{i=1,3}\left|\left(\partial_\xi^2\Theta^{R_i},(\partial_\xi\Theta^{R_i})^2,(\partial_\xi V^{R_i})^2,(\partial_\xi U^{R_i})^2\right)\right|(\varphi^2+\zeta^2)d\xi d\tau}_{I_2}\\
&+(C_\nu+C)\underbrace{\int_0^t\int_{\mathbb{R}_+}\left|\left(\partial_\xi^2\Theta^{CD},(\partial_\xi\Theta^{CD})^2\right)\right|(\varphi^2+\zeta^2)d\xi d\tau}_{I_3}+C\underbrace{\int_0^t\int_{\mathbb{R}_+}|Q_2|(\varphi^2+\zeta^2)d\xi d\tau}_{I_4}.
\end{aligned}
\tag{3.22}
$$

For I_1, we have

$$
\begin{aligned}
I_1\le&C\int_0^t\int_{\mathbb{R}_+}\left(|(\varphi,\zeta)|^2(0,\tau)+\xi\|\partial_\xi[\varphi,\zeta]\|^2\mid\frac{\delta^4}{(1+\delta\xi)^4}\right)d\xi d\tau\\
\le&C\delta\int_0^t|(\varphi,\zeta)|^2(0,\tau)d\tau+C\delta\int_0^t\int_{\mathbb{R}_+}\|\partial_\xi[\varphi,\zeta]\|^2d\xi d\tau\\
\le&C\delta+C\delta\int_0^t\|\partial_\xi[\varphi,\zeta]\|^2d\tau,
\end{aligned}
\tag{3.23}
$$

where we have used (2.7), (3.8) and the fact that

$$
|f(\xi)|=|f(0)+\int_0^\xi\partial_\xi f dy|\le|f(0)|+\sqrt{\xi}\|\partial_\xi f\|. \tag{3.24}
$$

From Lemma 2.2, we have

$$
\begin{aligned}
I_2\le&\int_0^t\sum_{i=1,3}\left(\|\partial_\xi[V^{R_i},U^{R_i},\Theta^{R_i}]\|^2+\|\partial_\xi^2\Theta^{R_i}\|_{L^1}\right)\|[\varphi,\zeta]\|_{L^\infty}^2d\tau\\
\le&C\delta^{\frac18}\int_0^t(1+\tau)^{-\frac{13}{16}}\|[\varphi,\zeta]\|\|\partial_\xi[\varphi,\zeta]\|d\tau\le C\delta^{\frac18}+C\delta^{\frac18}\int_0^t\|\partial_\xi[\varphi,\zeta]\|^2d\tau.
\end{aligned}
\tag{3.25}
$$

From the properties of the viscous 2-contact wave, we can get

$$
I_3\le C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\zeta^2)d\xi d\tau. \tag{3.26}
$$

Similar to the estimates of I_2 , we have

$$
I_4\le C\delta^{\frac18}+C\delta^{\frac18}\int_0^t\|\partial_\xi[\varphi,\zeta]\|^2d\tau, \tag{3.27}
$$

Thus substituting (3.23)-(3.27) into (3.22), we have

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}_+}Q_3d\xi d\tau\le&[\nu+C_\nu+C\delta^{\frac18}]\int_0^t\|\partial_\xi[\varphi,\psi,\zeta]\|^2d\tau+C\delta^{\frac18}\\
&+(C_\nu+C)\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\zeta^2)d\xi d\tau.
\end{aligned}
\tag{3.28}
$$

Now we estimate the last two terms as follows:

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}_+} Q_1 \psi d\xi d\tau \leq & C \int_0^t \|\psi\|_{L^\infty} \|Q_1\|_{L^1} d\tau \\
\leq & C \int_0^t \|\psi\|^{\frac{1}{2}} \|\partial_\xi \psi\|^{\frac{1}{2}} \delta^{\frac{1}{8}} (1+t)^{-\frac{13}{16}} d\tau \\
\leq & C\delta^{\frac{1}{8}} + C\delta^{\frac{1}{8}} \int_0^t \|\partial_\xi \psi\|^2 d\tau,
\end{aligned} \tag{3.29}
$$

and

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}_+} Q_2 \frac{\zeta}{\theta} d\xi d\tau \leq & C \int_0^t \|\zeta\|_{L^\infty} \|Q_2\|_{L^1} d\tau \\
\leq & C \int_0^t \|\zeta\|^{\frac{1}{2}} \|\partial_\xi \zeta\|^{\frac{1}{2}} \delta^{\frac{1}{8}} (1+t)^{-\frac{13}{16}} d\tau \\
\leq & C\delta^{\frac{1}{8}} + C\delta^{\frac{1}{8}} \int_0^t \|\partial_\xi \zeta\|^2 d\tau.
\end{aligned} \tag{3.30}
$$

Substituting the above estimates into (3.15), letting ν, and δ be suitably small, we obtain (3.13) and thus complete the proof of Lemma 3.3. $\square$

Lemma 3.4. *Assume the conditions in Proposition* 3.1 *hold, then we have the following energy estimate for* $t \in [0, T]$,

$$
\begin{aligned}
\|\partial_\xi \varphi\|^2 + \int_0^t \|\partial_\xi \varphi\|^2 d\tau \leq C \|[\psi_0, \zeta_0, \omega_0]\|^2 + C\|\varphi_0\|_{H^1}^2 + C\delta^{\frac{1}{8}} \\
+ \nu \int_0^t \|\partial_\xi^2 \psi\|^2 d\tau + C\delta \int_0^t \int_{\mathbb{R}_+} (1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_- \tau)^2}{1+\tau}} (\varphi^2 + \psi^2 + \zeta^2) d\xi d\tau.
\end{aligned} \tag{3.31}
$$

Proof. We first differentiate $(3.2)_1$ with respect to ξ and then obtain

$$
\partial_t \partial_\xi \varphi - \sigma_- \partial_\xi^2 \varphi - \partial_\xi^2 \psi = 0. \tag{3.32}
$$

Then multiplying $(3.2)_2$ and (3.32) by $-v\partial_\xi \varphi$ and $\mu \partial_\xi \varphi$, respectively, and integrating the resulting equalities over $\mathbb{R}_+ \times [0, t]$, one has

$$
\begin{aligned}
& -\int_0^t \int_{\mathbb{R}_+} \partial_t \psi v \partial_\xi \varphi d\xi d\tau + \sigma_- \int_0^t \int_{\mathbb{R}_+} \partial_\xi \psi v \partial_\xi \varphi d\xi d\tau \\
& - \int_0^t \int_{\mathbb{R}_+} \partial_\xi (p - P) v \partial_\xi \varphi d\xi d\tau + \mu \int_0^t \int_{\mathbb{R}_+} \partial_\xi^2 \psi \partial_\xi \varphi d\xi d\tau \\
= & -\mu \int_0^t \int_{\mathbb{R}_+} \partial_\xi (v^{-1}) \partial_\xi u v \partial_\xi \varphi d\xi d\tau + \int_0^t \int_{\mathbb{R}_+} \mu \partial_\xi^2 U (\frac{1}{V} - \frac{1}{v}) v \partial_\xi \varphi d\xi d\tau \\
& + \int_0^t \int_{\mathbb{R}_+} \mu \partial_\xi U \partial_\xi (V^{-1}) v \partial_\xi \varphi d\xi d\tau + \int_0^t \int_{\mathbb{R}_+} Q_1 v \partial_\xi \varphi d\xi d\tau,
\end{aligned} \tag{3.33}
$$

and

$$
\mu \int_0^t \int_{\mathbb{R}_+} \left(\partial_t \partial_\xi \varphi - \sigma_- \partial_\xi^2 \varphi - \partial_\xi^2 \psi \right) \partial_\xi \varphi d\xi d\tau = 0. \tag{3.34}
$$

The summation of (3.33) and (3.34) further implies

$$
\begin{aligned}
&-\int_{\mathbb{R}_+}\psi v\partial_\xi\varphi d\xi+\frac{\mu}{2}\int_{\mathbb{R}_+}(\partial_\xi\varphi)^2d\xi+\int_0^t\int_{\mathbb{R}_+}P(\partial_\xi\varphi)^2d\xi d\tau\\
=&-\int_{\mathbb{R}_+}\psi_0(\xi)v_0(\xi)\partial_\xi\varphi_0(\xi)d\xi+\frac{\mu}{2}\int_{\mathbb{R}_+}(\partial_\xi\varphi_0(\xi))^2d\xi\underbrace{+\frac{\mu|\sigma_-|}{2}\int_0^t(\partial_\xi\varphi)^2(0,\tau)d\tau}_{I_5}\underbrace{-\int_0^t\int_{\mathbb{R}_+}\psi\partial_t v\partial_\xi\varphi d\xi d\tau}_{I_6}\\
&\underbrace{-\int_0^t\int_{\mathbb{R}_+}\psi v\partial_t\partial_\xi\varphi d\xi d\tau}_{I_7}\underbrace{+\int_0^t\int_{\mathbb{R}_+}R\partial_\xi\left[\frac{\zeta}{v}\right]v\partial_\xi\varphi d\xi d\tau}_{I_8}\underbrace{-\int_0^t\int_{\mathbb{R}_+}R\varphi\partial_\xi\left[\frac{\Theta}{vV}\right]v\partial_\xi\varphi d\xi d\tau}_{I_9}\\
&\underbrace{-\sigma_-\int_0^t\int_{\mathbb{R}_+}\partial_\xi\psi v\partial_\xi\varphi d\xi d\tau}_{I_{10}}\underbrace{-\mu\int_0^t\int_{\mathbb{R}_+}\partial_\xi(v^{-1})\partial_\xi uv\partial_\xi\varphi d\xi d\tau}_{I_{11}}\underbrace{+\int_0^t\int_{\mathbb{R}_+}\mu\partial_\xi U\partial_\xi(V^{-1})v\partial_\xi\varphi d\xi d\tau}_{I_{12}}\\
&\underbrace{+\int_0^t\int_{\mathbb{R}_+}\mu\partial_\xi^2U(\frac{1}{V}-\frac{1}{v})v\partial_\xi\varphi d\xi d\tau}_{I_{13}}\underbrace{+\int_0^t\int_{\mathbb{R}_+}Q_1v\partial_\xi\varphi d\xi d\tau}_{I_{14}}.
\end{aligned}
\tag{3.35}
$$

By applying the *a priori* assumption (3.3), Cauchy-Schwarz's inequality with $0<\nu<1$, Sobolev's inequality (3.7) and Lemma 3.2, we turn to estimate $I_i(5\le i\le 14)$ as follows

$$
|I_5|\le\nu\int_0^t\|\partial_\xi^2\psi\|^2d\tau+C_\nu\int_0^t\|\partial_\xi\psi\|^2d\tau+C\delta,
$$

$$
\begin{aligned}
|I_6|\le&C\int_0^t\int_{\mathbb{R}_+}|\psi\partial_\xi\psi\partial_\xi\varphi|\,d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}|\psi\partial_\xi U\partial_\xi\varphi|\,d\xi d\tau\\
&+C\int_0^t\int_{\mathbb{R}_+}\left|\psi(\partial_\xi\varphi)^2\right|d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}|\psi\partial_\xi V\partial_\xi\varphi|\,d\xi d\tau\\
\le&C_\nu\delta^{\frac18}+C(\varepsilon_1+\nu+\delta^{\frac18})\int_0^t\|\partial_\xi[\psi,\varphi]\|^2d\tau\\
&+C_\nu\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}\psi^2d\xi d\tau,
\end{aligned}
$$

$$
\begin{aligned}
|I_7|\le&\int_0^t|(\psi v\partial_\xi\psi)(0,\tau)+\sigma_-(\psi v\partial_\xi\varphi)(0,\tau)|d\tau+C\int_0^t\int_{\mathbb{R}_+}(\partial_\xi\psi)^2d\xi d\tau\\
&+C\int_0^t\int_{\mathbb{R}_+}|\psi\partial_\xi v\partial_\xi[\varphi,\psi]|\,d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}|\partial_\xi\varphi\partial_\xi\psi|\,d\xi d\tau\\
\le&C_\nu\delta^{\frac18}+\nu\int_0^t\|\partial_\xi[\psi,\varphi,\partial_\xi\psi]\|^2d\tau+(C_\nu+C)\int_0^t\|\partial_\xi\psi\|^2d\tau\\
&+C_\nu\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}\psi^2d\xi d\tau,
\end{aligned}
$$

$$
\begin{aligned}
|I_8|+|I_9|+|I_{10}|\le&(\nu+C\varepsilon_1)\int_0^t\|\partial_\xi\varphi\|^2d\tau+C_\nu\int_0^t\|\partial_\xi[\zeta,\psi]\|^2d\tau+C_\nu\int_0^t\int_{\mathbb{R}_+}(\varphi^2+\zeta^2)(\partial_\xi\Theta)^2d\xi d\tau\\
\le&C\delta^{\frac18}+(\nu+C\varepsilon_1+C\delta^{\frac18})\int_0^t\|\partial_\xi[\varphi,\zeta]\|^2d\tau+C_\nu\int_0^t\|\partial_\xi[\zeta,\psi]\|^2d\tau\\
&+C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\zeta^2)d\xi d\tau,
\end{aligned}
$$

$$
\begin{aligned}
|I_{11}+I_{12}+I_{13}| \le & C\int_0^t\int_{\mathbb{R}_+}(\partial_\xi\varphi)^2|\partial_\xi\psi|d\xi d\tau + C\int_0^t\int_{\mathbb{R}_+}(\partial_\xi\varphi)^2|\partial_\xi U|d\xi d\tau \\
& + C\int_0^t\int_{\mathbb{R}_+}(|\partial_\xi V\partial_\xi U| + |\partial_\xi^2 U|)|\varphi\partial_\xi\varphi|d\xi d\tau + \int_0^t\int_{\mathbb{R}_+}|\partial_\xi V\partial_\xi\psi\partial_\xi\varphi|d\xi d\tau \\
\le & C(\varepsilon_1+\delta^{\frac{1}{8}})\int_0^t\|\partial_\xi[\psi,\partial_\xi\psi,\varphi]\|^2 d\tau + C\delta,
\end{aligned}
$$

and

$$
|I_{14}| \le \nu\int_0^t\|\partial_\xi\varphi\|^2 d\tau + C_\nu\int_0^t\|Q_1\|^2 d\tau \le \nu\int_0^t\|\partial_\xi\varphi\|^2 d\tau + C_\nu\delta.
$$

Substituting the above estimates for I_i $(5\le i\le 14)$ and (3.13) into (3.35), letting ν,δ and ε_1 be suitably small, and using Cauchy-Schwarz's inequality, we obtain (3.31). Thus we complete the proof of Lemma 3.4. □

Lemma 3.5. *Assume the conditions in Proposition* 3.1 *hold, then we have the following energy estimate for $t\in[0,T]$,*

$$
\begin{aligned}
\|\partial_\xi[\psi,\omega,\zeta]\|^2 + \int_0^t\|\partial_\xi^2[\psi,\omega,\zeta]\|^2 d\tau \le & C\|[\varphi_0,\psi_0,\zeta_0,\omega_0]\|^2_{H^1(\mathbb{R}_+)} + C\delta^{\frac{1}{8}} \\
& + C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau. \quad (3.36)
\end{aligned}
$$

Proof. Multiplying $(3.2)_2$ by $-\partial_\xi^2\psi$, and integrating the resulting equality over $\mathbb{R}_+\times[0,t]$, one has

$$
\begin{aligned}
& \frac{1}{2}\int_{\mathbb{R}_+}(\partial_\xi\psi)^2 d\xi + \mu\int_0^t\int_{\mathbb{R}_+}\frac{(\partial_\xi^2\psi)^2}{v}d\xi d\tau \\
= & \frac{1}{2}\int_{\mathbb{R}_+}(\partial_\xi\psi_0)^2 d\xi \underbrace{-\int_0^t(\partial_\xi\psi\partial_\tau\psi)(0,\tau)d\tau}_{I_{15}} \underbrace{+\frac{\sigma_-}{2}\int_0^t(\partial_\xi\psi)^2(0,\tau)d\tau}_{I_{16}} \\
& \underbrace{+\int_0^t\int_{\mathbb{R}_+}\partial_\xi\left(p-P\right)\partial_\xi^2\psi d\xi d\tau}_{I_{17}} \underbrace{+\mu\int_0^t\int_{\mathbb{R}_+}\frac{\partial_\xi\psi\partial_\xi\varphi}{v^2}\partial_\xi^2\psi d\xi d\tau}_{I_{18}} \\
& \underbrace{+\mu\int_0^t\int_{\mathbb{R}_+}\frac{\partial_\xi\psi\partial_\xi V}{v^2}\partial_\xi^2\psi d\xi d\tau}_{I_{19}} \underbrace{-\mu\int_0^t\int_{\mathbb{R}_+}\partial_\xi\left(\frac{\partial_\xi U}{v}-\frac{\partial_\xi U}{V}\right)\partial_\xi^2\psi d\xi d\tau}_{I_{20}} \underbrace{+\int_0^t\int_{\mathbb{R}_+}Q_1\partial_\xi^2\psi d\xi d\tau}_{I_{21}}. \quad (3.37)
\end{aligned}
$$

We now turn to compute I_i $(15\le i\le 21)$ term by term. For brevity, we directly give the following computations:

$$
|I_{15}|+|I_{16}| \le \nu\int_0^t\|\partial_\xi^2\psi\|^2 d\tau + C_\nu\int_0^t\|\partial_\xi\psi\|^2 d\tau + C\delta,
$$

$$
\begin{aligned}
|I_{17}| \le & C\int_0^t\int_{\mathbb{R}_+}\left|\partial_\xi[\zeta,\varphi]\partial_\xi^2\psi\right|d\xi d\tau + C\int_0^t\int_{\mathbb{R}_+}\left|[\zeta,\varphi]\partial_\xi[\varphi,V]\partial_\xi^2\psi\right|d\xi d\tau \\
\le & C\delta^{\frac{1}{8}} + (C\varepsilon_1+\nu)\int_0^t\|\partial_\xi[\varphi,\partial_\xi\psi]\|^2 d\tau + (C_\nu + C\delta^{\frac{1}{8}})\int_0^t\|\partial_\xi[\zeta,\varphi]\|^2 d\tau \\
& + C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\zeta^2)d\xi d\tau,
\end{aligned}
$$

$$
|I_{18}|+|I_{19}| \le C(\delta^{\frac{1}{8}}+\varepsilon_1)\int_0^t\|\partial_\xi[\psi,\partial_\xi\psi]\|^2 d\tau,
$$

$$
\begin{aligned}
|I_{20}| \leq & C\int_0^t\int_{\mathbb{R}_+}\left|\partial_\xi^2 U\varphi\partial_\xi^2\psi\right|d\xi d\tau + C\int_0^t\int_{\mathbb{R}_+}\left|\partial_\xi U\partial_\xi\varphi\partial_\xi^2\psi\right|d\xi d\tau\\
\leq & C\delta^{\frac{1}{8}} + C\delta^{\frac{1}{8}}\int_0^t \|\partial_\xi[\varphi,\partial_\xi\psi]\|^2 d\tau,
\end{aligned}
$$

and

$$
|I_{21}| \leq \nu\int_0^t \|\partial_\xi^2\psi\|^2 d\tau + C_\nu\int_0^t \|Q_1\|^2 d\tau \leq \nu\int_0^t \|\partial_\xi^2\psi\|^2 d\tau + C_\nu\delta.
$$

Plug the above estimations for I_i ($15 \leq i \leq 21$) into (3.37), and recall (3.31) and (3.13), then choose $\delta > 0$ and $\nu > 0$ suitably small, to derive

$$
\begin{aligned}
\|\partial_\xi\psi\|^2 + \int_0^t \|\partial_\xi^2\psi\|^2 d\tau \leq & C\,\|[\zeta_0,\omega_0]\|^2 + C\|[\varphi_0,\psi_0]\|^2_{H^1(\mathbb{R}_+)} + C\delta^{\frac{1}{8}}\\
& + C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau. \qquad (3.38)
\end{aligned}
$$

Multiplying $(3.2)_3$ by $-\partial_\xi^2\omega$, and integrating the resulting equality over $\mathbb{R}_+\times[0,t]$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}_+}(\partial_\xi\omega)^2 d\xi + A\int_0^t\int_{\mathbb{R}_+}\frac{(\partial_\xi^2\omega)^2}{v}d\xi d\tau + \frac{|\sigma_-|}{2}\int_0^t(\partial_\xi\omega)^2(0,\tau)d\tau\\
=&\frac{1}{2}\int_{\mathbb{R}_+}(\partial_\xi\omega_0)^2 d\xi \underbrace{+A\int_0^t\int_{\mathbb{R}_+}\frac{\partial_\xi\omega\partial_\xi\varphi}{v^2}\partial_\xi^2\omega d\xi d\tau}_{I_{22}}\underbrace{+A\int_0^t\int_{\mathbb{R}_+}\frac{\partial_\xi\omega\partial_\xi V}{v^2}\partial_\xi^2\omega d\xi d\tau}_{I_{23}}\underbrace{+A\int_0^t\int_{\mathbb{R}_+}v\omega\partial_\xi^2\omega d\xi d\tau}_{I_{24}},
\end{aligned}
\qquad (3.39)
$$

where we have used $\sigma_- < 0$ to deal with the boundary term.

To obtain the estimates for I_i ($22 \leq i \leq 24$), we use Cauchy-Schwarz's inequality with $0 < \nu < 1$, Sobolev's inequlity (3.7) and the *a priori* assumption (3.3) to obtain

$$
\begin{aligned}
&|I_{22}| + |I_{23}| + |I_{24}|\\
\leq & C\int_0^t \|\partial_\xi\omega\|_\infty\|\partial_\xi\varphi\|\|\partial_\xi^2\omega\|d\tau + C(\delta^{\frac{1}{8}}+\nu)\int_0^t \|\partial_\xi[\omega,\partial_\xi\omega]\|^2 d\tau + C_\nu\int_0^t \|\omega\|^2 d\tau\\
\leq & C\int_0^t (\|\partial_\xi\omega\| + \|\partial_\xi^2\omega\|)\|\partial_\xi\varphi\|\|\partial_\xi^2\omega\|d\tau + C(\delta^{\frac{1}{8}}+\nu)\int_0^t \|\partial_\xi[\omega,\partial_\xi\omega]\|^2 d\tau + C_\nu\int_0^t \|\omega\|^2 d\tau\\
\leq & C(\varepsilon_1+\nu+\delta^{\frac{1}{8}})\int_0^t \|\partial_\xi[\omega,\partial_\xi\omega]\|^2 d\tau + C_\nu\int_0^t \|\omega\|^2 d\tau.
\end{aligned}
$$

Plug the above estimations into (3.39), and recall (3.13), (3.31), (3.38), then choose $\varepsilon_1 > 0$, $\delta > 0$ and $\nu > 0$ suitably small, to derive

$$
\begin{aligned}
\|\partial_\xi\omega\|^2 + \int_0^t \|\partial_\xi^2\omega\|^2 d\tau \leq & C\,\|\zeta_0\|^2 + C\|[\varphi_0,\psi_0,\omega_0]\|^2_{H^1(\mathbb{R}_+)} + C\delta^{\frac{1}{8}}\\
& + C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau. \qquad (3.40)
\end{aligned}
$$

Multiplying $(3.2)_4$ by $-\partial_x^2\zeta$, almost similar to the estimate of $\|\partial_\xi\psi\|^2(t)$, we can obtain

$$
\begin{aligned}
\|\partial_\xi\zeta\|^2 + \int_0^t \|\partial_\xi^2\zeta\|^2 d\tau \leq & C\|[\varphi_0,\psi_0,\omega_0,\zeta_0]\|^2_{H^1(\mathbb{R}_+)} + C\delta^{\frac{1}{8}}\\
& + C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau. \qquad (3.41)
\end{aligned}
$$

Summing up (3.41), (3.40) and (3.38), we get the desired estimate (3.36). Thus we complete the proof of Lemma 3.5. □

Proof of Proposition 3.1. Now, we are ready to prove Proposition 3.1. Combining Lemma 3.3-3.5 with Lemma 4.2 in the appendix, and if the wave strength δ and the constants ε_1 are small enough, then for all $t \in [0,T]$, we have

$$\begin{aligned}
&\|[\varphi,\psi,\omega,\zeta](t)\|^2_{H^1(\mathbb{R}_+)} + \int_0^t \left(\|\partial_\xi\varphi\|^2 + \|\partial_\xi\left[\psi,\zeta\right]\|^2_{H^1(\mathbb{R}_+)} + \|\omega\|^2_{H^2(\mathbb{R}_+)}\right) d\tau \\
\leq & C\|[\varphi_0,\psi_0,\omega_0,\zeta_0]\|^2_{H^1(\mathbb{R}_+)} + C\delta^{\frac{1}{8}},
\end{aligned} \tag{3.42}$$

which gives desired estimate (3.5). □

Proof of Theorem 2.1. We are now in a position to complete the proof of Theorem 2.1. In view of the energy estimates obtained in Proposition 3.1, one sees that

$$\sup_{0\leq\tau\leq t}\|[\varphi,\psi,\omega,\zeta](\tau)\|^2_{H^1(\mathbb{R}_+)} \leq C\|[\varphi_0,\psi_0,\omega_0,\zeta_0]\|^2_{H^1(\mathbb{R}_+)} + C\delta^{\frac{1}{8}}. \tag{3.43}$$

Notice that δ are parameters independent of ε_1. By letting δ be small enough, the global existence of solution of the half-space problem (3.2) then follows from the standard continuation argument based on the local existence and the *a priori* estimate (3.5). Moreover, (3.43) and (2.26) imply (2.27). Our intention next is to prove the large time behavior as (2.28). For this, we first justify the following limits:

$$\lim_{t\to+\infty}\|\partial_\xi[\varphi,\psi,\omega,\zeta](t)\|^2_{L^2} = 0. \tag{3.44}$$

To prove (3.44), we get from (3.2), (3.5), (2.14), Lemma 2.2 and (2.8) that

$$\begin{aligned}
\int_0^{+\infty}\left|\frac{d}{dt}\|\partial_\xi[\varphi,\psi,\omega,\zeta]\|^2\right| dt =& 2\int_0^{+\infty}|(\partial_t\partial_\xi[\varphi,\psi,\omega,\zeta],\partial_\xi[\varphi,\psi,\omega,\zeta])|\, dt \\
\leq & C + C\int_0^{+\infty}\|\partial_\xi\left[\varphi,\psi,\omega,\zeta,\partial_\xi\left[\psi,\zeta,\omega\right]\right]\|^2 dt < +\infty.
\end{aligned} \tag{3.45}$$

Consequently, (3.45) together with (3.5) gives (3.44). Then (2.28) follows from (3.44) and Sobolev's inequality (3.7). This ends the proof of Theorem 2.1.

□

4 Appendix

In this appendix, we will give some basic results used in the paper. Lemma 4.1 and Lemma 4.2 are borrowed from [10] and [31], and we omit some details here.

Lemma 4.1. *Suppose that $h(\xi,t)$ satisfies*

$$h \in L^\infty(0,T;L^2(\mathbb{R}_+)), \quad \partial_\xi h \in L^2(0,T;L^2(\mathbb{R}_+)), \quad \partial_t h - \sigma_-\partial_\xi h \in L^2(0,T;H^{-1}(\mathbb{R}_+)),$$

then the following estimate holds:

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{\alpha(\xi+\sigma_-\tau)^2}{1+\tau}}h^2 d\xi d\tau \leq & C_\alpha\Big[\|h(\xi,0)\|^2 + \int_0^t h^2(0,\tau)d\tau + \int_0^t\|\partial_\xi h\|^2 d\tau \\
&+ \int_0^t\langle\partial_t h - \sigma_-\partial_\xi h, hg^2\rangle_{H^{-1}\times H^1}d\tau\Big],
\end{aligned} \tag{4.1}$$

where

$$g(\xi,t) = -(1+t)^{-\frac{1}{2}}\int_{\xi+\sigma_- t}^{+\infty}e^{-\frac{\alpha x^2}{1+t}}dx,$$

and $\alpha > 0$ is the constant to be determined later.

We now give the following estimates concerning the delicate term $\int_0^t \int_{\mathbb{R}_+} (1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_- \tau)^2}{1+\tau}} (\varphi^2 + \psi^2 + \zeta^2) d\xi d\tau$ by using Lemma 4.1.

Lemma 4.2. *Under the conditions of Proposition 3.1, then there exists a constant $C > 0$ such that*

$$\begin{aligned}\int_0^t \int_{\mathbb{R}_+} (1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_- \tau)^2}{1+\tau}} (\varphi^2 + \zeta^2 + \psi^2) d\xi d\tau \leq & C + C \int_0^t \left(\|\partial_\xi[\varphi, \psi, \zeta, \omega]\|^2 + \|\omega\|^2 \right) d\tau \\ & + C \int_0^t \|\partial_\xi^2[\psi, \zeta]\|^2 d\tau \end{aligned} \tag{4.2}$$

provided that the wave strength δ is small enough.

Proof. For any $\nu > 0$, the proof of inequality (4.2) consists of the following two parts:

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+} [(R\zeta - P\varphi)^2 + \psi^2](1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_- \tau)^2}{1+\tau}} d\xi d\tau \\ \leq & C + C\delta^{\frac{1}{8}} \int_0^t \int_{\mathbb{R}_+} (\varphi^2 + \zeta^2)(1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_- \tau)^2}{1+\tau}} d\xi d\tau \\ & + C \int_0^t \left(\|\partial_\xi[\varphi, \psi, \zeta, \omega]\|^2 + \|\omega\|^2 \right) d\tau + \nu \int_0^t \|\partial_\xi^2[\psi, \zeta]\|^2 d\tau, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+} (R\zeta + (\gamma-1)P\varphi)^2 (1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_- \tau)^2}{1+\tau}} d\xi d\tau \\ \leq & C + C\delta^{\frac{1}{8}} \int_0^t \int_{\mathbb{R}_+} (\varphi^2 + \zeta^2)(1+\tau)^{-1} e^{-\frac{c_0(\xi+\sigma_- \tau)^2}{1+\tau}} d\xi d\tau \\ & + C_\nu \int_0^t \left(\|\partial_\xi[\varphi, \psi, \zeta, \omega]\|^2 + \|\omega\|^2 \right) d\tau + \nu \int_0^t \|\partial_\xi^2 \zeta\|^2 d\tau. \end{aligned} \tag{4.4}$$

In fact, multiplying inequality (4.3) by $\gamma - 1$ and adding the resulting inequality to (4.4) and taking δ suitably small thus implies (4.2) easily.

We firstly prove (4.3). Define

$$\eta(\xi, t) = -(1+t)^{-1} \int_{\xi+\sigma_- t}^{+\infty} e^{-\frac{c_0 x^2}{1+t}} dx.$$

We rewrite $(3.2)_2$ as follows:

$$(\partial_t \psi - \sigma_- \partial_\xi \psi) + \partial_\xi \left(\frac{R\zeta - P\varphi}{v} \right) = \mu \partial_\xi \left(\frac{\partial_\xi u}{v} - \frac{\partial_\xi U}{V} \right) - Q_1. \tag{4.5}$$

Multiplying (4.5) by $(R\zeta - P\varphi)\,v\eta$, integrating the resulting equation over $\mathbb{R}_+\times[0,t]$ leads to

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_{\mathbb{R}_+}(R\zeta-P\varphi)^2\,\partial_\xi\eta d\xi d\tau\\
=&-\frac{1}{2}\int_0^t[(R\zeta-P\varphi)^2\,\eta](0,\tau)d\tau+\mu\int_0^t\left[(\frac{\partial_\xi u}{v}-\frac{\partial_\xi U}{V})\,(R\zeta-P\varphi)\,v\eta\right](0,\tau)d\tau\\
&+\sigma_-\int_0^t[\psi\,(R\zeta-P\varphi)\,v\eta](0,\tau)d\tau+\int_{\mathbb{R}_+}\psi\,(R\zeta-P\varphi)\,v\eta d\xi-\int_{\mathbb{R}_+}\psi_0\,(R\zeta_0-P(\xi,0)\varphi_0)\,v_0\eta(\xi,0)d\xi\\
&\underbrace{-\int_0^t\int_{\mathbb{R}_+}[\partial_t\,(R\zeta-P\varphi)-\sigma_-\partial_\xi\,(R\zeta-P\varphi)]\,\psi v\eta d\xi d\tau}_{\mathcal{K}_1}-\int_0^t\int_{\mathbb{R}_+}\psi\,(R\zeta-P\varphi)\,(\partial_t v-\sigma_-\partial_\xi v)\eta d\xi d\tau\\
&-\int_0^t\int_{\mathbb{R}_+}\psi\,(R\zeta-P\varphi)\,v(\partial_t\eta-\sigma_-\partial_\xi\eta)d\xi d\tau-\int_0^t\int_{\mathbb{R}_+}\frac{\partial_\xi(V+\varphi)}{v}\,(R\zeta-P\varphi)^2\,\eta d\xi d\tau\\
&+\mu\int_0^t\int_{\mathbb{R}_+}\left(\frac{\partial_\xi u}{v}-\frac{\partial_\xi U}{V}\right)\partial_\xi\,[(R\zeta-P\varphi)\,v\eta]\,d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}Q_1\,(R\zeta-P\varphi)\,v\eta d\xi d\tau.
\end{aligned}\tag{4.6}$$

For the delicate term $\mathcal{K}_1$, it can be rewritten as

$$\begin{aligned}
\mathcal{K}_1=&-\int_0^t\int_{\mathbb{R}_+}[(R\partial_t\zeta-R\sigma_-\partial_\xi\zeta)-(\partial_t P-\sigma_-\partial_\xi P)\varphi-P(\partial_t\varphi-\sigma_-\partial_\xi\varphi)]\,\psi v\eta d\xi d\tau\\
=&-\frac{1}{2}\int_0^t\left(\gamma Pv\eta\psi^2\right)(0,\tau)d\tau-\frac{1}{2}\int_0^t\int_{\mathbb{R}_+}\gamma Pv\partial_\xi\eta\psi^2 d\xi d\tau-\frac{1}{2}\int_0^t\int_{\mathbb{R}_+}\gamma\partial_\xi(Pv)\eta\psi^2 d\xi d\tau\\
&+\int_0^t\int_{\mathbb{R}_+}(\partial_t P-\sigma_-\partial_\xi P)\varphi\psi v\eta d\xi d\tau-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\Bigg[-(p-P)\partial_\xi u+\kappa\partial_\xi\left(\frac{\partial_\xi\theta}{v}-\frac{\partial_\xi\Theta}{V}\right)\\
&+\mu\left(\frac{(\partial_\xi u)^2}{v}-\frac{(\partial_\xi U)^2}{V}\right)+\frac{(\partial_\xi\omega)^2}{v}+v\omega^2-Q_2\Bigg]\psi v\eta d\xi d\tau\\
=&-\frac{1}{2}\int_0^t\left(\gamma Pv\eta\psi^2\right)(0,\tau)d\tau+\kappa(\gamma-1)\int_0^t\left[\left(\frac{\partial_\xi\theta}{v}-\frac{\partial_\xi\Theta}{V}\right)\psi v\eta\right](0,\tau)d\tau\\
&-\frac{1}{2}\int_0^t\int_{\mathbb{R}_+}\gamma Pv\partial_\xi\eta\psi^2 d\xi d\tau-\frac{1}{2}\int_0^t\int_{\mathbb{R}_+}\gamma\partial_\xi(Pv)\eta\psi^2 d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}(\partial_t P-\sigma_-\partial_\xi P)\varphi\psi v\eta d\xi d\tau\\
&-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\left[-(p-P)\partial_\xi u+\mu\left(\frac{(\partial_\xi u)^2}{v}-\frac{(\partial_\xi U)^2}{V}\right)+\frac{(\partial_\xi\omega)^2}{v}+v\omega^2-Q_2\right]\psi v\eta d\xi d\tau\\
&+\kappa(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\left(\frac{\partial_\xi\theta}{v}-\frac{\partial_\xi\Theta}{V}\right)\partial_\xi(\psi v\eta)d\xi d\tau,
\end{aligned}\tag{4.7}$$

where in the second identity we have used $(3.2)_1$ and $(3.2)_4$.

Since

$$\partial_\xi\eta(\xi,t)=(1+t)^{-1}e^{-\frac{c_0(\xi+\sigma_- t)^2}{1+t}},$$

combining (4.6) and (4.7), we have

$$\frac{1}{2}\int_0^t\int_{\mathbb{R}_+}[(R\zeta-P\varphi)^2+\gamma Pv\psi^2](1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}d\xi d\tau=\int_0^t H_2(0,\tau)d\tau+Q_3,\tag{4.8}$$

where

$$H_2(0,\tau) = -\frac{1}{2}[(R\zeta - P\varphi)^2\,\eta](0,\tau) + \mu\left[(\frac{\partial_\xi u}{v} - \frac{\partial_\xi U}{V})\,(R\zeta - P\varphi)\,v\eta\right](0,\tau)$$
$$+\,\sigma_-[\psi\,(R\zeta - P\varphi)\,v\eta](0,\tau) - \frac{1}{2}\left(\gamma P v\eta\psi^2\right)(0,\tau) + \kappa(\gamma-1)\left[\left(\frac{\partial_\xi\theta}{v} - \frac{\partial_\xi\Theta}{V}\right)\psi v\eta\right](0,\tau), \tag{4.9}$$

and

$$\begin{aligned}
Q_3 = &\int_{\mathbb{R}_+}\psi\,(R\zeta - P\varphi)\,v\eta d\xi - \int_{\mathbb{R}_+}\psi_0\,(R\zeta_0 - P(\xi,0)\varphi_0)\,v_0\eta(\xi,0)d\xi - \int_0^t\int_{\mathbb{R}_+}\psi\,(R\zeta - P\varphi)\,\partial_\xi u\eta d\xi d\tau\\
&- \int_0^t\int_{\mathbb{R}_+}\psi\,(R\zeta - P\varphi)\,v(\partial_t\eta - \sigma_-\partial_\xi\eta)d\xi d\tau - \int_0^t\int_{\mathbb{R}_+}\frac{\partial_\xi(V+\varphi)}{v}\,(R\zeta - P\varphi)^2\,\eta d\xi d\tau\\
&+ \mu\int_0^t\int_{\mathbb{R}_+}\left(\frac{\partial_\xi u}{v} - \frac{\partial_\xi U}{V}\right)\partial_\xi\left[(R\zeta - P\varphi)\,v\eta\right]d\xi d\tau + \int_0^t\int_{\mathbb{R}_+}Q_1\,(R\zeta - P\varphi)\,v\eta d\xi d\tau\\
&- \frac{1}{2}\int_0^t\int_{\mathbb{R}_+}\gamma\partial_\xi(Pv)\eta\psi^2 d\xi d\tau + \int_0^t\int_{\mathbb{R}_+}(\partial_t P - \sigma_-\partial_\xi P)\varphi\psi v\eta d\xi d\tau\\
&- (\gamma-1)\int_0^t\int_{\mathbb{R}_+}\left[-(p-P)\partial_\xi u + \mu\left(\frac{(\partial_\xi u)^2}{v} - \frac{(\partial_\xi U)^2}{V}\right) + \frac{(\partial_\xi\omega)^2}{v} + v\omega^2 - Q_2\right]\psi v\eta d\xi d\tau\\
&+ \kappa(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\left(\frac{\partial_\xi\theta}{v} - \frac{\partial_\xi\Theta}{V}\right)\partial_\xi(\psi v\eta)d\xi d\tau.
\end{aligned} \tag{4.10}$$

From Lemma 3.2(boundary estimates), we have

$$\int_0^t H_2(0,\tau)d\tau \le \nu\int_0^t(\|\partial_\xi[\psi,\zeta]\|^2 + \|\partial_\xi^2[\psi,\zeta]\|^2)d\tau + C_\nu\delta. \tag{4.11}$$

Note that $\|\eta(\cdot,t)\|_\infty \le C(1+t)^{-\frac{1}{2}}$ and by applying (2.8), (2.14), Cauchy-Schwarz's inequality, Sobolev's inequality (3.7) and Young's inequality, we can successfully estimate Q_3. Then combining this with (4.11) and (4.8), we obtain (4.3).

Next we prove inequality (4.4) by using Lemma 4.1. Let $h = R\zeta + (\gamma-1)P\varphi$, then from $(3.2)_1$ and $(3.2)_4$, we have

$$\begin{aligned}
&\int_0^t\langle\partial_t h - \sigma_-\partial_\xi h, hg^2\rangle_{H^{-1}\times H^1}d\tau\\
=&\underbrace{-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}(p-P)\,\partial_\xi\psi hg^2 d\xi d\tau}_{\mathcal{K}_2}\underbrace{+(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\left[(\partial_t P - \sigma_-\partial_\xi P)\,\varphi - (p-P)\,\partial_\xi U\right]hg^2 d\xi d\tau}_{\mathcal{K}_3}\\
&\underbrace{-(\gamma-1)\int_0^t\left[\kappa\left(\frac{\partial_\xi\theta}{v} - \frac{\partial_\xi\Theta}{V}\right)hg^2\right](0,\tau)d\tau}_{\mathcal{K}_4}\underbrace{-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\kappa\left(\frac{\partial_\xi\theta}{v} - \frac{\partial_\xi\Theta}{V}\right)\partial_\xi(hg^2)d\xi d\tau}_{\mathcal{K}_5}\\
&\underbrace{+(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\mu\left(\frac{(\partial_\xi u)^2}{v} - \frac{(\partial_\xi U)^2}{V}\right)hg^2 d\xi d\tau}_{\mathcal{K}_6}\underbrace{+(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\left(\frac{(\partial_\xi\omega)^2}{v} + v\omega^2\right)hg^2 d\xi d\tau}_{\mathcal{K}_7}\\
&\underbrace{-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}Q_2 hg^2 d\xi d\tau}_{\mathcal{K}_8}.
\end{aligned}$$

Noticing that $\|g(\cdot,t)\|_{L^\infty}\le C_\alpha$, we can directly estimate $\mathcal{K}_i(3\le i\le 8)$. In order to estimate $\mathcal{K}_2$, by the mass equation $(3.2)_1$ and $p-P=\frac{R\zeta-P\varphi}{v}$, we have

$$
\begin{aligned}
\mathcal{K}_2 =&-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\frac{h-\gamma P\varphi}{v}hg^2(\partial_t\varphi-\sigma_-\partial_\xi\varphi)d\xi d\tau\\
=&-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\left\{\frac{h^2g^2}{v}(\partial_t\varphi-\sigma_-\partial_\xi\varphi)-\frac{\gamma Phg^2}{2v}[\partial_t(\varphi^2)-\sigma_-\partial_\xi(\varphi^2)]\right\}d\xi d\tau\\
=&-(\gamma-1)\int_{\mathbb{R}_+}\frac{2h^2g^2\varphi-\gamma Phg^2\varphi^2}{2v}d\xi+(\gamma-1)\int_{\mathbb{R}_+}\left[\frac{2h^2g^2\varphi-\gamma Phg^2\varphi^2}{2v}\right](\xi,0)d\xi\\
&-(\gamma-1)\sigma_-\int_0^t\left[\frac{2h^2g^2\varphi-\gamma Phg^2\varphi^2}{2v}\right](0,\tau)d\tau-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\frac{\gamma g^2\varphi^2h}{2v}(\partial_tP-\sigma_-\partial_\xi P)d\xi d\tau\\
&-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\frac{\gamma Ph\varphi^2-2h^2\varphi}{v}g(\partial_tg-\sigma_-\partial_\xi g)d\xi d\tau\\
&+(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\frac{\gamma Ph\varphi^2-2h^2\varphi}{2v^2}g^2(\partial_tv-\sigma_-\partial_\xi v)d\xi d\tau\\
&-(\gamma-1)\int_0^t\int_{\mathbb{R}_+}\frac{\gamma Pg^2\varphi^2-4g^2h\varphi}{2v}(\partial_th-\sigma_-\partial_\xi h)d\xi d\tau.
\end{aligned}
\tag{4.12}
$$

Now each term in (4.12) can be estimated directly, and the detailed proof can be seen in [10]. Note that here we need to compute the boundary terms. Hence after taking $\alpha=\frac{c_0}{2}$, estimate (4.4) thus easily follows from Lemma 4.1 and Lemma 3.2(boundary estimates).

□

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