Well-posedness and attractors of the multi-dimensional hyperviscous magnetohydrodynamics equations*

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Abstract The multi-dimensional hyperviscous magnetohydrodynamics equation is considered in this paper. The well-posedness of the multi-dimensional hyperviscous magnetohydrodynamics equation is proved. Global attractor of the multi-dimensional hyperviscous magnetohydrodynamics equations is proved in $H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}}$.

Key words Magnetohydrodynamics equations; Strong solution; Global attractor

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1 Introduction

In this paper, we consider the following multi-dimensional hyperviscous magnetohydrodynamic(MHD) equations:

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla(p+\frac{|b|^2}{2}) = f(x), \quad (x,t)\in D\times(0,T), \\ \partial_t b + (-\Delta)^\alpha b + (u\cdot\nabla)b - (b\cdot\nabla)u = g(x), \quad (x,t)\in D\times(0,T), \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \quad (x,t)\in D\times(0,T), \\ u|_{\partial D} = b|_{\partial D} = 0, \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0. \end{cases} \tag{1.1}$$

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Where $D \subseteq \mathbb{R}^n$ is a bounded domain with the boundary ∂D and $n \geq 3$. u, b are the fluid velocity and magnetic fields, respectively. $f(x), g(x)$ are the external body force. p is the pressure. $\alpha > 0$ and the fractional Laplacian operator $(-\Delta)^\alpha$ is given by via the Fourier transform

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi),$$

here, $\hat{f}$ represents the Fourier transform of f. For simplicity, we set $\alpha = \frac{1}{2} + \frac{n}{4}$ and $n \geq 3$.

The magnetohydrodynamics system govern the dynamics of the velocity field and magnetic field in electrically conducting fluids. In recent years, generalized magnetohydrodynamics equations were investigated by many authors in [14, 18, 19]. In [18], global smooth solution for the system (1.1) was proved for the initial conditions in $H^{\frac{5}{2}}$, $n = 3$ and $\alpha \geq \frac{5}{4}$. Regularity criteria for the three dimensional magnetohydrodynamics equations were established (see [3, 7, 8]).

Attractors of the dissipative partial differential equations were proved in [2, 12, 13, 17, 20]. In [15, 16], Song etc. have proved the global attractor and uniform attractor for the three-dimensional Navier-Stokes equations with damping. In [10,11], Liu etc. have proved global attractor and uniform attractor for the three-dimensional magnetohydrodynamics equations with damping. Based on the existence of global attractor, the existence of inertial manifolds for the hyperviscous Navier-Stokes equations was proved by using the the spatial averaging method for $\alpha \geq \frac{3}{2}$ in [6]. Meanwhile, Li and Sun have proved the the existence of inertial manifolds for the hyperviscous Navier-Stokes equations by using the extend slightly spatial averaging method for $\alpha \geq \frac{5}{4}$ in [9].

The long time behaviours of the multi-dimensional hyperviscous magnetohydrodynamics equation can be described by the called global attractors. Sobolev regularity and Gevrey regularity of the global attractor for the three dimensional magnetohydrodynamic-α model were proved in [1]. Global existence and finite dimensional global attractor for the three dimensional viscous magnetohydrodynamic-α model were proved in [4, 5].

To obtain the existence of global attractors for the multi-dimensional hyperviscous magnetohydrodynamics equations, we overcome the main difficulty lies in dealing with the nonlinear term $(u\cdot\nabla)u$, $(u\cdot\nabla)b$, $(b\cdot\nabla)u$, $(b\cdot\nabla)b$. In order to get the global attractor in $H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}}$, we overcome the main difficulty lies in the estimation of $||A^{1+\frac{n}{2}}u(t)||$, $||A^{1+\frac{n}{2}}b(t)||$,$||u_t||_{\frac{1}{2}+\frac{n}{4}}$,$||b_t||_{\frac{1}{2}+\frac{n}{4}}$, $||\bar{u}||^2_{\frac{1}{2}+\frac{n}{4}}$ and $||\bar{b}||^2_{\frac{1}{2}+\frac{n}{4}}$.

This paper is organized as follows. In section 2, we give some preliminaries and Theorem 2.1-Theorem 2.3, and get the well-posedness for the system (2.3). In section 3, the existence of uniform estimate is proved. In section 4, existence of global attractor for the system (2.3) is proved in $H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}}$.

2 Preliminaries

In this paper, the inner products and norms are given by

$$(u,v)=\int_D u\cdot vdx,\quad \forall u,v\in L^2,\quad ((u,v))=\int_D \nabla u\cdot\nabla vdx,\quad \forall u,v\in H_0^1,$$

and $||\cdot||^2=(\cdot,\cdot)$, $||\nabla\cdot||^2=((\cdot,\cdot))$. Let $P:L^2(D)\to H$ be the Helmhdtz-Leray orthogonal projection operator. Let $A=-P\Delta$ denote the stokes operators such that $Au=-P\Delta u=-\Delta u$, $Ab=-P\Delta b=-\Delta b$, for all $u,b\in D(A)$. Let $D(A')=H^{1+\frac{n}{2}}\cap H^{\frac{1}{2}+\frac{n}{4}}$ and $D(A)=H^2\cap H_0^1$. Let $H^s=D(A^{\frac{s}{2}})$, $s>0$ and the norm is defined by $||\cdot||_s=||A^{\frac{s}{2}}\cdot||$, and the Sobolev space H^s is given by

$$H^s=\{u\in H:||u||_{H^s}^2=\sum_{j\in\mathbb{Z}^3\setminus\{0\}}|j|^{2s}|\hat{u}_j|^2<\infty\}. \tag{2.1}$$

The space H is defined by

$$\begin{aligned}\mathcal{V}&=\{u\in C_0^\infty(D,\mathbb{R}^n):\mathrm{div}u=0\},\\ H&=\text{ the closure of }\mathcal{V}\text{ in }L^2(D).\end{aligned}$$

Let the bilinear form is defined by

$$B(w_1,w_2)=P((w_1\cdot\nabla)w_2),\quad\text{for}\quad w_1,w_2\in H^1. \tag{2.2}$$

Then the system (1.1) is rewritten as the following forms

$$\begin{cases}u_t+A^{\frac{1}{2}+\frac{n}{4}}u+B(u,u)-B(b,b)=f,\quad u|_{t=0}=u_0,\\ b_t+A^{\frac{1}{2}+\frac{n}{4}}b+B(u,b)-B(b,u)=g,\quad b|_{t=0}=b_0.\end{cases} \tag{2.3}$$

Theorem 2.1. Assume that $f,g\in H^{-\frac{1}{2}-\frac{n}{4}}$. For any $u_0\in H$ and $b_0\in H$, the system (2.3) has a unique weak solutions such that $u\in L^\infty(0,T;H)\cap L^2(0,T;H^{\frac{1}{2}+\frac{n}{4}})$ and $b\in L^\infty(0,T;H)\cap L^2(0,T;H^{\frac{1}{2}+\frac{n}{4}})$ with $u_t\in L^2(0,T;H^{-\frac{1}{2}-\frac{n}{4}})$ and $b_t\in L^2(0,T;H^{-\frac{1}{2}-\frac{n}{4}})$, for any $T>0$.
Proof. Multiplying the first equation of system (2.3) with u and the second equation of system (2.3) with b, respectively. Integrating their results on D and adding up their results, then we have

$$\frac{d}{dt}(||u||^2+||b||^2)+||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}}\le||f||^2_{-\frac{1}{2}-\frac{n}{4}}+||g||^2_{-\frac{1}{2}-\frac{n}{4}}. \tag{2.4}$$

Then we deduce

$$||u||^2+||b||^2\le e^{-t}(||u_0||^2+||b_0||^2)+(||f||^2_{-\frac{1}{2}-\frac{n}{4}}+||g||^2_{-\frac{1}{2}-\frac{n}{4}})(1-e^{-t}). \tag{2.5}$$

Integrating (2.4) on $[0,t]$, then we get

$$\int_0^t (||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}})ds \leq ||u_0||^2 + ||b_0||^2 + t(||f||^2_{-\frac{1}{2}-\frac{n}{4}} + ||g||^2_{-\frac{1}{2}-\frac{n}{4}}). \tag{2.6}$$

Applying the Sobolev imbedding, it yields

$$\begin{aligned}(B(u,u),\phi) &\leq C||\phi||_{L^{\frac{4n}{n-2}}}||\nabla u||_{L^{\frac{4n}{n+2}}}||u|| \\ &\leq C||\phi||_{\frac{1}{2}+\frac{n}{4}}||u||_{\frac{1}{2}+\frac{n}{4}}||u||. \end{aligned} \tag{2.7}$$

It follows that $||B(u,u)||_{-\frac{1}{2}-\frac{n}{4}} \leq C||u||_{\frac{1}{2}+\frac{n}{4}}||u||$. By inequalities (2.5) and (2.6), we get $B(u,u)$ is bounded in $L^2(0,T;H^{-\frac{1}{2}-\frac{n}{4}})$. Similarly,

$$(B(b,b),\phi) \leq C||\phi||_{\frac{1}{2}+\frac{n}{4}}||b||_{\frac{1}{2}+\frac{n}{4}}||b||. \tag{2.8}$$

It follows that $||B(b,b)||_{-\frac{1}{2}-\frac{n}{4}} \leq C||b||_{\frac{1}{2}+\frac{n}{4}}||b||$. By inequalities (2.5) and (2.6), we get $B(b,b)$ is bounded in $L^2(0,T;H^{-\frac{1}{2}-\frac{n}{4}})$. Similarly,

$$\begin{aligned}(B(u,b),\phi)-(B(b,u),\phi) &\leq C||\phi||_{L^{\frac{4n}{n-2}}}||\nabla b||_{L^{\frac{4n}{n+2}}}||u|| + C||\phi||_{L^{\frac{4n}{n-2}}}||\nabla u||_{L^{\frac{4n}{n+2}}}||b|| \\ &\leq C||\phi||_{\frac{1}{2}+\frac{n}{4}}||b||_{\frac{1}{2}+\frac{n}{4}}||u|| + C||\phi||_{\frac{1}{2}+\frac{n}{4}}||u||_{\frac{1}{2}+\frac{n}{4}}||b||. \end{aligned} \tag{2.9}$$

It follows that $||B(u,b)||_{-\frac{1}{2}-\frac{n}{4}} + ||B(b,u)||_{-\frac{1}{2}-\frac{n}{4}} \leq C||b||_{\frac{1}{2}+\frac{n}{4}}||u|| + C||u||_{\frac{1}{2}+\frac{n}{4}}||b||$. Finally, we deduce u_t, b_t are bounded in $L^2(0,T;H^{-\frac{1}{2}-\frac{n}{4}})$. By using the standard Galerkin method, priori estimates and compactness argument, we prove the existence of global weak solutions for system (2.3). Let (u_1,b_1) and (u_2,b_2) be two solutions for system (2.3). Let $\bar{u}=u_1-u_2$ and $\bar{b}=b_1-b_2$. We get the following form.

$$\begin{cases} \bar{u}_t + A^{\frac{1}{2}+\frac{n}{4}}\bar{u} + B(u_1,\bar{u}) + B(\bar{u},u_2) - B(b_1,\bar{b}) - B(\bar{b},b_2) = 0, \\ \bar{b}_t + A^{\frac{1}{2}+\frac{n}{4}}\bar{b} + B(u_1,\bar{b}) + B(\bar{u},b_2) - B(b_1,\bar{u}) - B(\bar{b},u_2) = 0. \end{cases} \tag{2.10}$$

Multiplying the first equation of system (2.10) with $\bar{u}$ and the second equation of system (2.10) with $\bar{b}$, respectively. Integrating their results on D and summing up their results, then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||\bar{u}||^2 + ||\bar{b}||^2) + ||\bar{u}||^2_{\frac{1}{2}+\frac{n}{4}} + ||\bar{b}||^2_{\frac{1}{2}+\frac{n}{4}} \\ &= \int_D (-\bar{u}\nabla u_2\bar{u} + b_1\nabla\bar{b}\bar{u} + \bar{b}\nabla b_2\bar{u} - \bar{u}\nabla b_2\bar{b} + b_1\nabla\bar{u}\bar{b} + \bar{b}\nabla u_2\bar{b})dx \\ &\leq C||\bar{u}||_{L^{\frac{4n}{n-2}}}||\nabla u_2||_{L^{\frac{4n}{n+2}}}||\bar{u}|| + C||b_1||_{L^{\frac{4n}{n-2}}}||\nabla\bar{b}||_{L^{\frac{4n}{n+2}}}||\bar{u}|| + C||\bar{b}||_{L^{\frac{4n}{n-2}}}||\nabla b_2||_{L^{\frac{4n}{n+2}}}||\bar{u}|| \\ &+ C||b_1||_{L^{\frac{4n}{n-2}}}||\nabla\bar{u}||_{L^{\frac{4n}{n+2}}}||\bar{b}|| + C||\bar{b}||_{L^{\frac{4n}{n-2}}}||\nabla u_2||_{L^{\frac{4n}{n+2}}}||\bar{b}|| \\ &\leq C||\bar{u}||_{\frac{1}{2}+\frac{n}{4}}||u_2||_{\frac{1}{2}+\frac{n}{4}}||\bar{u}|| + C||b_1||_{\frac{1}{2}+\frac{n}{4}}||\bar{b}||_{\frac{1}{2}+\frac{n}{4}}||\bar{u}|| + C||\bar{b}||_{\frac{1}{2}+\frac{n}{4}}||b_2||_{\frac{1}{2}+\frac{n}{4}}||\bar{u}||\end{aligned}$$

$$
\begin{aligned}
&+C||b_1||_{\frac{1}{2}+\frac{n}{4}}||\bar{u}||_{\frac{1}{2}+\frac{n}{4}}||\bar{b}||+C||\bar{b}||_{\frac{1}{2}+\frac{n}{4}}||u_2||_{\frac{1}{2}+\frac{n}{4}}||\bar{b}|| \\
&\le \frac{1}{2}(||\bar{u}||^2_{\frac{1}{2}+\frac{n}{4}}+||\bar{b}||^2_{\frac{1}{2}+\frac{n}{4}}) \\
&+C(||u_2||^2_{\frac{1}{2}+\frac{n}{4}}+||b_1||^2_{\frac{1}{2}+\frac{n}{4}}+||b_2||^2_{\frac{1}{2}+\frac{n}{4}})(||\bar{u}||^2+||\bar{b}||^2). \qquad (2.11)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\frac{d}{dt}(||\bar{u}||^2+||\bar{b}||^2)+||\bar{u}||^2_{\frac{1}{2}+\frac{n}{4}}+||\bar{b}||^2_{\frac{1}{2}+\frac{n}{4}} \\
&\le C(||u_2||^2_{\frac{1}{2}+\frac{n}{4}}+||b_1||^2_{\frac{1}{2}+\frac{n}{4}}+||b_2||^2_{\frac{1}{2}+\frac{n}{4}})(||\bar{u}||^2+||\bar{b}||^2). \qquad (2.12)
\end{aligned}
$$

Applying the Gronwall inequality, it yields

$$
\begin{aligned}
&||\bar{u}||^2+||\bar{b}||^2 \le e^{(C\int_0^t(||u_2||^2_{\frac{1}{2}+\frac{n}{4}}+||b_1||^2_{\frac{1}{2}+\frac{n}{4}}+||b_2||^2_{\frac{1}{2}+\frac{n}{4}})d\tau)}(||\bar{u}(0)||^2+||\bar{b}(0)||^2) \\
&\le Ce^{C(||u_1(0)||^2+||u_2(0)||^2+||b_1(0)||^2+||b_2(0)||^2)+Ct(||f||^2_{-\frac{1}{2}-\frac{n}{4}}+||g||^2_{-\frac{1}{2}-\frac{n}{4}})}(||\bar{u}(0)||^2+||\bar{b}(0)||^2). \qquad (2.13)
\end{aligned}
$$

If $\bar{u}(0)=u_1(0)-u_2(0)=0$ and $\bar{b}(0)=b_1(0)-b_2(0)=0$, we deduce $\bar{u}(t)=\bar{b}(t)=0$. This finishes the proof of the Theorem 2.1.

Theorem 2.2. Assume that $f,g\in L^2$. For any $u_0\in H^{\frac{1}{2}+\frac{n}{4}}$ and $b_0\in H^{\frac{1}{2}+\frac{n}{4}}$, the system (2.3) has a strong solutions such that $u\in L^\infty(0,T;H^{\frac{1}{2}+\frac{n}{4}})\cap L^2(0,T;H^{1+\frac{n}{2}})$ and $b\in L^\infty(0,T;H^{\frac{1}{2}+\frac{n}{4}})\cap L^2(0,T;H^{1+\frac{n}{2}})$ for any $T>0$.

Proof. Multiplying the first equation of system (2.3) by $-\Delta u$ and the second equation of system (2.3) by $-\Delta b$, integrating their results on D, respectively. Adding up their results, then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\nabla u||^2+||\nabla b||^2)+||u||^2_{\frac{3}{2}+\frac{n}{4}}+||b||^2_{\frac{3}{2}+\frac{n}{4}} \\
&\le |\int_D(u\cdot\nabla)u\Delta udx|+|\int_D(b\cdot\nabla)b\Delta udx|+|\int_D(u\cdot\nabla)b\Delta bdx| \\
&+|\int_D(b\cdot\nabla)u\Delta bdx|+|(f,\Delta u)|+|(g,\Delta b)| \\
&\le C||u||_{L^{\frac{4n}{n-2}}}||\nabla u||||\Delta u||_{L^{\frac{4n}{n+2}}}+C||b||_{L^{\frac{4n}{n-2}}}||\nabla b||||\Delta u||_{L^{\frac{4n}{n+2}}}+C||u||_{L^{\frac{4n}{n-2}}}||\nabla b||||\Delta b||_{L^{\frac{4n}{n+2}}} \\
&+C||b||_{L^{\frac{4n}{n-2}}}||\nabla u||||\Delta b||_{L^{\frac{4n}{n+2}}}+C||f||_{L^{\frac{4n}{3n-2}}}||\Delta u||_{L^{\frac{4n}{n+2}}}+C||g||_{L^{\frac{4n}{3n-2}}}||\Delta b||_{L^{\frac{4n}{n+2}}} \\
&\le C||u||_{\frac{1}{2}+\frac{n}{4}}||\nabla u||||u||_{\frac{3}{2}+\frac{n}{4}}+C||b||_{\frac{1}{2}+\frac{n}{4}}||\nabla b||||u||_{\frac{3}{2}+\frac{n}{4}}+C||u||_{\frac{1}{2}+\frac{n}{4}}||\nabla b||||b||_{\frac{3}{2}+\frac{n}{4}} \\
&+C||b||_{\frac{1}{2}+\frac{n}{4}}||\nabla u||||b||_{\frac{3}{2}+\frac{n}{4}}+C||f||_{L^{\frac{4n}{3n-2}}}||u||_{\frac{3}{2}+\frac{n}{4}}+C||g||_{L^{\frac{4n}{3n-2}}}||b||_{\frac{3}{2}+\frac{n}{4}} \\
&\le \frac{1}{2}(||u||^2_{\frac{3}{2}+\frac{n}{4}}+||b||^2_{\frac{3}{2}+\frac{n}{4}})+C(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}})(||\nabla u||^2+||\nabla b||^2) \\
&+C||f||^2_{L^{\frac{4n}{3n-2}}}+C||g||^2_{L^{\frac{4n}{3n-2}}}. \qquad (2.14)
\end{aligned}
$$

Then it yields

$$
\begin{aligned}
&\frac{d}{dt}(||\nabla u||^2+||\nabla b||^2)+||u||^2_{\frac{3}{2}+\frac{n}{4}}+||b||^2_{\frac{3}{2}+\frac{n}{4}}\\
&\le C(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}})(||\nabla u||^2+||\nabla b||^2)+C||f||^2_{L^{\frac{4n}{3n-2}}}+C||g||^2_{L^{\frac{4n}{3n-2}}}. \qquad (2.15)
\end{aligned}
$$

By the Gronwall inequality, we have

$$
||\nabla u||^2+||\nabla b||^2+\int_0^t(||u||^2_{\frac{3}{2}+\frac{n}{4}}+||b||^2_{\frac{3}{2}+\frac{n}{4}})ds\le C. \qquad (2.16)
$$

Multiplying the first equation of system (2.3) by $A^{\frac{1}{2}+\frac{n}{4}}u$ and the second equation of system (2.3) by $A^{\frac{1}{2}+\frac{n}{4}}b$, integrating their results on D, respectively. Adding up their results and (2.16), then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}})+||u||^2_{1+\frac{n}{2}}+||b||^2_{1+\frac{n}{2}}\\
&\le|\int_D(u\cdot\nabla)uA^{\frac{1}{2}+\frac{n}{4}}udx|+|\int_D(b\cdot\nabla)bA^{\frac{1}{2}+\frac{n}{4}}udx|+|\int_D(u\cdot\nabla)bA^{\frac{1}{2}+\frac{n}{4}}bdx|\\
&+|\int_D(b\cdot\nabla)uA^{\frac{1}{2}+\frac{n}{4}}bdx|+|(f,A^{\frac{1}{2}+\frac{n}{4}}u)|+|(g,A^{\frac{1}{2}+\frac{n}{4}}b)|\\
&\le C||u||_{L^\infty}||\nabla u||||A^{\frac{1}{2}+\frac{n}{4}}u||+C||b||_{L^\infty}||\nabla b||||A^{\frac{1}{2}+\frac{n}{4}}u||+C||u||_{L^\infty}||\nabla b||||A^{\frac{1}{2}+\frac{n}{4}}b||\\
&+C||b||_{L^\infty}||\nabla u||||A^{\frac{1}{2}+\frac{n}{4}}b||+C||f||||A^{\frac{1}{2}+\frac{n}{4}}u||+C||g||||A^{\frac{1}{2}+\frac{n}{4}}b||\\
&\le\frac{1}{4}(||u||^2_{1+\frac{n}{2}}+||b||^2_{1+\frac{n}{2}})+C||u||^2_{L^\infty}||\nabla u||^2+C||b||^2_{L^\infty}||\nabla b||^2+C||u||^2_{L^\infty}||\nabla b||^2\\
&+C||b||^2_{L^\infty}||\nabla u||^2+C||f||^2+C||g||^2\\
&\le\frac{1}{4}(||u||^2_{1+\frac{n}{2}}+||b||^2_{1+\frac{n}{2}})+C||u||^{\frac{8}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||u||^{\frac{2n-4}{n+2}}_{1+\frac{n}{2}}+C||b||^{\frac{8}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||b||^{\frac{2n-4}{n+2}}_{1+\frac{n}{2}}+C||f||^2+C||g||^2\\
&\le\frac{1}{2}(||u||^2_{1+\frac{n}{2}}+||b||^2_{1+\frac{n}{2}})+C||u||^2_{\frac{1}{2}+\frac{n}{4}}+C||b||^2_{\frac{1}{2}+\frac{n}{4}}+C||f||^2+C||g||^2. \qquad (2.17)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\frac{d}{dt}(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}})+||u||^2_{1+\frac{n}{2}}+||b||^2_{1+\frac{n}{2}}\\
&\le C||u||^2_{\frac{1}{2}+\frac{n}{4}}+C||b||^2_{\frac{1}{2}+\frac{n}{4}}+C||f||^2+C||g||^2. \qquad (2.18)
\end{aligned}
$$

Integrating (2.18) on $[0,t]$, it yields

$$
||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}}+\int_0^t(||u||^2_{1+\frac{n}{2}}+||b||^2_{1+\frac{n}{2}})ds\le C. \qquad (2.19)
$$

Now, we introduce the main result as follows.

Theorem 2.3. Assume that $f,g\in L^2$ and $(u_0,b_0)\in H^{\frac{1}{2}+\frac{n}{4}}\times H^{\frac{1}{2}+\frac{n}{4}}$. The operator $\{S(t)\}_{t\ge0}$ of system (2.3) satisfies

$$
S(t)(u_0,b_0)=(u(t),b(t)).
$$

$\{S(t)\}_{t\geq 0}$ is defined in the space $H^{\frac{1}{2}+\frac{n}{4}}\times H^{\frac{1}{2}+\frac{n}{4}}$. The system (2.3) has a $(H^{\frac{1}{2}+\frac{n}{4}}\times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}}\times H^{1+\frac{n}{2}})-$global attractor satisfies the following

(i) The global attractor $\mathcal{A}$ is invariant and compact in $H^{1+\frac{n}{2}}\times H^{1+\frac{n}{2}}$.

(ii) The global attractor $\mathcal{A}$ attracts bounded subset of $H^{\frac{1}{2}+\frac{n}{4}}\times H^{\frac{1}{2}+\frac{n}{4}}$ in relation to the norm topology of $H^{1+\frac{n}{2}}\times H^{1+\frac{n}{2}}$.

3 Uniform estimate

Firstly, we will show the uniform estimates of strong solutions for the system (2.3) as $t\to\infty$. In order to get the existence of attractors, we will prove the following estimates.

Lemma 3.1. Let $(u_0,b_0)\in H^{\frac{1}{2}+\frac{n}{4}}\times H^{\frac{1}{2}+\frac{n}{4}}$ and $f,g\in L^2$. There exists a constant t_0 such that

$$||u(t)||^2+||b(t)||^2\leq C, \tag{3.1}$$

$$\int_t^{t+1}(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}})ds\leq C. \tag{3.2}$$

Proof. Multiplying the first equation of system (2.3) by u and the second equation of system (2.3) by b, integrating their results on D, respectively. Adding up their results, then we have

$$\frac{d}{dt}(||u||^2+||b||^2)+||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}}\leq ||f||^2_{-\frac{1}{2}-\frac{n}{4}}+||g||^2_{-\frac{1}{2}-\frac{n}{4}}. \tag{3.3}$$

Applying the Poincar$\acute{e}$ and Gronwall's inequalities, then there exists a positive constant γ such that

$$||u(t)||^2+||b(t)||^2\leq e^{-\gamma t}(||u(0)||^2+||b(0)||^2)+\frac{1}{\gamma}(||f||^2_{-\frac{1}{2}-\frac{n}{4}}+||g||^2_{-\frac{1}{2}-\frac{n}{4}}). \tag{3.4}$$

For (3.4), it is easy to get

$$\limsup_{t\to+\infty}(||u(t)||^2+||b(t)||^2)\leq\frac{1}{\gamma}(||f||^2_{-\frac{1}{2}-\frac{n}{4}}+||g||^2_{-\frac{1}{2}-\frac{n}{4}}). \tag{3.5}$$

Then there exists a $t_0=t_0(||u(0)||,||b(0)||)$ such that

$$||u(t)||^2+||b(t)||^2\leq\frac{2}{\gamma}(||f||^2_{-\frac{1}{2}-\frac{n}{4}}+||g||^2_{-\frac{1}{2}-\frac{n}{4}})\leq C. \tag{3.6}$$

Integrating (3.3) on $[t,t+1]$ and (3.5), it yields

$$\int_t^{t+1}(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}})ds\leq C.$$

Lemma 3.2. Let $(u_0, b_0) \in H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $f, g \in L^2$. There exists a constant t_1 such that

$$||\nabla u(t)||^2 + ||\nabla b(t)||^2 \le C. \tag{3.7}$$

Proof. By the above Theorem 2.2 and (3.2), then there exists a positive constant $t_1 = t_0+1$ such that

$$||\nabla u(t)||^2 + ||\nabla b(t)||^2 \le C.$$

Lemma 3.3. Let $(u_0, b_0) \in H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $f, g \in L^2$. There exists a constant t_1 such that

$$\int_t^{t+1} (||u(s)||^2_{\frac{3}{2}+\frac{n}{4}} + ||b(s)||^2_{\frac{3}{2}+\frac{n}{4}}) ds \le C. \tag{3.8}$$

Proof. By the inequality (2.15), then we deduce

$$\begin{aligned} &\frac{d}{dt}(||\nabla u||^2 + ||\nabla b||^2) + ||u||^2_{\frac{3}{2}+\frac{n}{4}} + ||b||^2_{\frac{3}{2}+\frac{n}{4}} \\ &\le C(||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}})(||\nabla u||^2 + ||\nabla b||^2) + ||f||^2_{L^{\frac{4n}{3n-2}}} + ||g||^2_{L^{\frac{4n}{3n-2}}}. \end{aligned} \tag{3.9}$$

Applying the Gronwall inequality and the Theorem 2.2, we get

$$||\nabla u||^2 + ||\nabla b||^2 + \int_0^t (||u||^2_{\frac{3}{2}+\frac{n}{4}} + ||b||^2_{\frac{3}{2}+\frac{n}{4}}) ds \le C. \tag{3.10}$$

Finally, then there exists a positive constant t_1 such that

$$\int_t^{t+1} (||u(s)||^2_{\frac{3}{2}+\frac{n}{4}} + ||b(s)||^2_{\frac{3}{2}+\frac{n}{4}}) ds \le C.$$

Lemma 3.4. Let $(u_0, b_0) \in H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $f, g \in L^2$. There exists a constant t_2 such that

$$||u(t)||^2_{\frac{1}{2}+\frac{n}{4}} + ||b(t)||^2_{\frac{1}{2}+\frac{n}{4}} \le C. \tag{3.11}$$

Proof. By using the above Theorem 2.2, we get (3.11).

Lemma 3.5. Let $(u_0, b_0) \in H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $f, g \in L^2$. There exists a constant t_3 such that

$$||u_t(s)||^2 + ||b_t(s)||^2 \le C, \quad \forall s \ge t_3. \tag{3.12}$$

Proof. We multiply the first equation of system (2.3) by u_t and the second equation of system (2.3) by b_t, integrate their results on D, respectively. Adding up their results, it yields

$$||u_t||^2 + ||b_t||^2 + \frac{1}{2}\frac{d}{dt}(||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}})$$

$$= -\int_D (u\nabla u)u_t dx + \int_D (b\nabla b)u_t dx + \int_D f u_t dx - \int_D (u\nabla b)b_t dx + \int_D (b\nabla u)b_t dx + \int_D g b_t dx$$
$$\le \frac{1}{2}(||u_t||^2 + ||b_t||^2) + C(||f||^2 + ||g||^2) + C(||u\nabla u||^2 + ||b\nabla b||^2 + ||u\nabla b||^2 + ||b\nabla u||^2)$$
$$\le \frac{1}{2}(||u_t||^2 + ||b_t||^2) + C(||f||^2 + ||g||^2) + C||u||^2_{L^{\frac{4n}{n-2}}}||\nabla u||^2_{L^{\frac{4n}{n+2}}} + C||b||^2_{L^{\frac{4n}{n-2}}}||\nabla b||^2_{L^{\frac{4n}{n+2}}}$$
$$+ C||u||^2_{L^{\frac{4n}{n-2}}}||\nabla b||^2_{L^{\frac{4n}{n+2}}} + C||b||^2_{L^{\frac{4n}{n-2}}}||\nabla u||^2_{L^{\frac{4n}{n+2}}}$$
$$\le \frac{1}{2}(||u_t||^2 + ||b_t||^2) + C(||f||^2 + ||g||^2) + C||u||^2_{\frac{1}{2}+\frac{n}{4}}||u||^2_{\frac{1}{2}+\frac{n}{4}}$$
$$+ C||b||^2_{\frac{1}{2}+\frac{n}{4}}||b||^2_{\frac{1}{2}+\frac{n}{4}} + C||u||^2_{\frac{1}{2}+\frac{n}{4}}||b||^2_{\frac{1}{2}+\frac{n}{4}}$$
$$\le \frac{1}{2}(||u_t||^2 + ||b_t||^2) + C(||f||^2 + ||g||^2)$$
$$+ C(||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}})(||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}}). \tag{3.13}$$

Then we get

$$||u_t||^2 + ||b_t||^2 + \frac{d}{dt}(||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}})$$
$$\le C(||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}})(||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}}) + C(||f||^2 + ||g||^2). \tag{3.14}$$

Integrating (3.14) on $[t, t+1]$, it yields

$$\int_t^{t+1} (||u_t(s)||^2 + ||b_t(s)||^2) ds \le C. \tag{3.15}$$

Applying ∂_t to the first equation of system (2.3) and multiplying the L_2-inner product by u_t. Similarly, we apply ∂_t to the second equation of system (2.3) and multiply the L_2-inner product by b_t. Then we get

$$\frac{1}{2}\frac{d}{dt}(||u_t||^2 + ||b_t||^2) + ||u_t||^2_{\frac{1}{2}+\frac{n}{4}} + ||b_t||^2_{\frac{1}{2}+\frac{n}{4}}$$
$$= -\int_D (u_t\nabla u)u_t dx + \int_D (b_t\nabla b)u_t dx - \int_D (u_t\nabla b)b_t dx + \int_D (b_t\nabla u)b_t dx$$
$$\le C||u_t||_{L^{\frac{4n}{n-2}}}||\nabla u||_{L^{\frac{4n}{n+2}}}||u_t|| + C||b_t||_{L^{\frac{4n}{n-2}}}||\nabla b||_{L^{\frac{4n}{n+2}}}||u_t||$$
$$+ C||u_t||_{L^{\frac{4n}{n-2}}}||\nabla b||_{L^{\frac{4n}{n+2}}}||b_t|| + C||b_t||_{L^{\frac{4n}{n-2}}}||\nabla u||_{L^{\frac{4n}{n+2}}}||b_t||$$
$$\le C||u_t||_{\frac{1}{2}+\frac{n}{4}}||u||_{\frac{1}{2}+\frac{n}{4}}||u_t|| + C||b_t||_{\frac{1}{2}+\frac{n}{4}}||b||_{\frac{1}{2}+\frac{n}{4}}||u_t||$$
$$+ C||u_t||_{\frac{1}{2}+\frac{n}{4}}||b||_{\frac{1}{2}+\frac{n}{4}}||b_t|| + C||b_t||_{\frac{1}{2}+\frac{n}{4}}||u||_{\frac{1}{2}+\frac{n}{4}}||b_t||$$
$$\le \frac{1}{2}(||u_t||^2_{\frac{1}{2}+\frac{n}{4}} + ||b_t||^2_{\frac{1}{2}+\frac{n}{4}}) + C(||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}})(||u_t||^2 + ||b_t||^2). \tag{3.16}$$

Then we have

$$\frac{d}{dt}(||u_t||^2 + ||b_t||^2) + ||u_t||^2_{\frac{1}{2}+\frac{n}{4}} + ||b_t||^2_{\frac{1}{2}+\frac{n}{4}}$$

$$\leq C(||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}})(||u_t||^2 + ||b_t||^2). \tag{3.17}$$

Integrating (3.17) on $[s, t+1]$ for $t < s < t+1$, we get

$$\begin{aligned} &||u_t(t+1)||^2 + ||b_t(t+1)||^2 \\ &\leq ||u_t(s)||^2 + ||b_t(s)||^2 + C\int_s^{t+1}(||u_t(\tau)||^2 + ||b_t(\tau)||^2)d\tau. \end{aligned} \tag{3.18}$$

Integrating (3.18) on $[t, t+1]$ with respect to s and (3.15), it yields

$$||u_t(t+1)||^2 + ||b_t(t+1)||^2 \leq C\int_t^{t+1}(||u_t(s)||^2 + ||b_t(s)||^2)ds \leq C, \quad for\ t \geq t_2. \tag{3.19}$$

Let $t_3 = t_2 + 1$, (3.12) is proved.

Lemma 3.6. Let $(u_0, b_0) \in H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $f, g \in L^2$. There exists a constant t_3 such that

$$||A^{\frac{1}{2}+\frac{n}{4}}u(t)|| + ||A^{\frac{1}{2}+\frac{n}{4}}b(t)|| \leq C. \tag{3.20}$$

Proof. For system (2.3), then we get

$$\begin{aligned} &||A^{\frac{1}{2}+\frac{n}{4}}u(t)|| + ||A^{\frac{1}{2}+\frac{n}{4}}b(t)|| \\ &\leq ||u_t|| + ||b_t|| + ||B(u,u)|| + ||B(b,b)|| + ||B(u,b)|| + ||B(b,u)|| + ||f|| + ||g|| \\ &\leq ||u_t|| + ||b_t|| + ||f|| + ||g|| + C||u||_{L^{\frac{4n}{n-2}}}||\nabla u||_{L^{\frac{4n}{n+2}}} + C||b||_{L^{\frac{4n}{n-2}}}||\nabla b||_{L^{\frac{4n}{n+2}}} \\ &+ C||u||_{L^{\frac{4n}{n-2}}}||\nabla b||_{L^{\frac{4n}{n+2}}} + C||b||_{L^{\frac{4n}{n-2}}}||\nabla u||_{L^{\frac{4n}{n+2}}} \\ &\leq ||u_t|| + ||b_t|| + ||f|| + ||g|| + C||u||^2_{\frac{1}{2}+\frac{n}{4}} + C||b||^2_{\frac{1}{2}+\frac{n}{4}} + C||u||_{\frac{1}{2}+\frac{n}{4}}||b||_{\frac{1}{2}+\frac{n}{4}} \\ &\leq ||u_t|| + ||b_t|| + ||f|| + ||g|| + C||u||^2_{\frac{1}{2}+\frac{n}{4}} + C||b||^2_{\frac{1}{2}+\frac{n}{4}}. \end{aligned} \tag{3.21}$$

By (3.11) and (3.12), we get

$$||A^{\frac{1}{2}+\frac{n}{4}}u(t)|| + ||A^{\frac{1}{2}+\frac{n}{4}}b(t)|| \leq C.$$

Lemma 3.7. Let $(u_0, b_0) \in H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $f, g \in L^2$. There exists a constant t_4 such that

$$\int_t^{t+1}(||u_t(s)||^2_{\frac{1}{2}+\frac{n}{4}} + ||b_t(s)||^2_{\frac{1}{2}+\frac{n}{4}})ds \leq C, for\ \ \forall t \geq t_3, \tag{3.22}$$

$$||\nabla u_t||^2 + ||\nabla b_t||^2 \leq C, for\ \ t \geq t_4. \tag{3.23}$$

Proof. Integrating (3.17) on $[t, t+1]$, using the Lemma 3.4 and Lemma 3.5, then there exists a constant t_3

$$\int_t^{t+1}(||u_t(s)||^2_{\frac{1}{2}+\frac{n}{4}} + ||b_t(s)||^2_{\frac{1}{2}+\frac{n}{4}})ds$$

$$\leq ||u_t(t)||^2+||b_t(t)||^2+C\int_t^{t+1}(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}})(||u_t||^2+||b_t||^2)ds$$
$$\leq C,\quad for\ \ \forall t\geq t_3. \tag{3.24}$$

Applying ∂_t to the first equation of system (2.3) and multiplying the L_2-inner product by $-\Delta u_t$. Similarly, we apply ∂_t to the second equation of system (2.3) and multiply the L_2-inner product by $-\Delta b_t$. Then we have

$$\frac{1}{2}\frac{d}{dt}(||\nabla u_t||^2+||\nabla b_t||^2)+||u_t||^2_{\frac{3}{2}+\frac{n}{4}}+||b_t||^2_{\frac{3}{2}+\frac{n}{4}}$$
$$\leq |\int_D(u_t\nabla u)\Delta u_tdx|+|\int_D(u\nabla u_t)\Delta u_tdx|+|\int_D(b_t\nabla b)\Delta u_tdx|+|\int_D(b\nabla b_t)\Delta u_tdx|$$
$$+|\int_D(u_t\nabla b)\Delta b_tdx|+|\int_D(u\nabla b_t)\Delta b_tdx|+|\int_D(b_t\nabla u)\Delta b_tdx|+|\int_D(b\nabla u_t)\Delta b_tdx|$$
$$\leq C||u_t||||\nabla u||_{L^{\frac{4n}{n-2}}}||\Delta u_t||_{L^{\frac{4n}{n+2}}}+C||u||_{L^{\frac{4n}{n-2}}}||\nabla u_t||||\Delta u_t||_{L^{\frac{4n}{n+2}}}$$
$$+C||b_t||||\nabla b||_{L^{\frac{4n}{n-2}}}||\Delta u_t||_{L^{\frac{4n}{n+2}}}+C||b||_{L^{\frac{4n}{n-2}}}||\nabla b_t||||\Delta u_t||_{L^{\frac{4n}{n+2}}}$$
$$+C||u_t||||\nabla b||_{L^{\frac{4n}{n-2}}}||\Delta b_t||_{L^{\frac{4n}{n+2}}}+C||u||_{L^{\frac{4n}{n-2}}}||\nabla b_t||||\Delta b_t||_{L^{\frac{4n}{n+2}}}$$
$$+C||b_t||||\nabla u||_{L^{\frac{4n}{n-2}}}||\Delta b_t||_{L^{\frac{4n}{n+2}}}+C||b||_{L^{\frac{4n}{n-2}}}||\nabla u_t||||\Delta b_t||_{L^{\frac{4n}{n+2}}}$$
$$\leq C||\nabla u_t||||u||_{\frac{3}{2}+\frac{n}{4}}||u_t||_{\frac{3}{2}+\frac{n}{4}}+C||u||_{\frac{1}{2}+\frac{n}{4}}||\nabla u_t||||u_t||_{\frac{3}{2}+\frac{n}{4}}$$
$$+C||\nabla b_t||||b||_{\frac{3}{2}+\frac{n}{4}}||u_t||_{\frac{3}{2}+\frac{n}{4}}+C||b||_{\frac{1}{2}+\frac{n}{4}}||\nabla b_t||||u_t||_{\frac{3}{2}+\frac{n}{4}}$$
$$+C||\nabla u_t||||b||_{\frac{3}{2}+\frac{n}{4}}||b_t||_{\frac{3}{2}+\frac{n}{4}}+C||u||_{\frac{1}{2}+\frac{n}{4}}||\nabla b_t||||b_t||_{\frac{3}{2}+\frac{n}{4}}$$
$$+C||\nabla b_t||||u||_{\frac{3}{2}+\frac{n}{4}}||b_t||_{\frac{3}{2}+\frac{n}{4}}+C||b||_{\frac{1}{2}+\frac{n}{4}}||\nabla u_t||||b_t||_{\frac{3}{2}+\frac{n}{4}}$$
$$\leq\frac{1}{2}(||u_t||^2_{\frac{3}{2}+\frac{n}{4}}+||b_t||^2_{\frac{3}{2}+\frac{n}{4}})$$
$$+C(||\nabla u_t||^2+||\nabla b_t||^2)(||u||^2_{\frac{3}{2}+\frac{n}{4}}+||b||^2_{\frac{3}{2}+\frac{n}{4}}+||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}}). \tag{3.25}$$

Then it is easy to get

$$\frac{d}{dt}(||\nabla u_t||^2+||\nabla b_t||^2)$$
$$\leq C(||\nabla u_t||^2+||\nabla b_t||^2)(||u||^2_{\frac{3}{2}+\frac{n}{4}}+||b||^2_{\frac{3}{2}+\frac{n}{4}}+||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}}). \tag{3.26}$$

Using the uniform Gronwall lemma, we have

$$||\nabla u_t(t+1)||^2+||\nabla b_t(t+1)||^2$$
$$\leq C\int_t^{t+1}(||\nabla u_t||^2+||\nabla b_t||^2)d\tau exp^{\int_t^{t+1}(||u||^2_{\frac{3}{2}+\frac{n}{4}}+||b||^2_{\frac{3}{2}+\frac{n}{4}}+||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}})d\tau}. \tag{3.27}$$

Using the Gagliardo-Nirenberg inequality and Young inequality, we have

$$\int_t^{t+1}(||\nabla u_t||^2+||\nabla b_t||^2)d\tau\leq C\int_t^{t+1}(||u_t||^{\frac{8}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||u_t||^{\frac{2n-4}{n+2}}+||b_t||^{\frac{8}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||b_t||^{\frac{2n-4}{n+2}})d\tau$$

$$\leq C\int_t^{t+1}(||u_t||^2_{\frac{1}{2}+\frac{n}{4}}+||u_t||^2+||b_t||^2_{\frac{1}{2}+\frac{n}{4}}+||b_t||^2)d\tau. \quad (3.28)$$

By the (3.22), (3.28) and Lemma 3.4-Lemma 3.5, we have

$$||\nabla u_t||^2+||\nabla b_t||^2\leq C, \quad for \quad t\geq t_4=t_3+1. \quad (3.29)$$

Lemma 3.8. Let $(u_0,b_0)\in H^{\frac{1}{2}+\frac{n}{4}}\times H^{\frac{1}{2}+\frac{n}{4}}$ and $f,g\in L^2$. There exists a constant t_5 such that

$$||u_t||^2_{\frac{1}{2}+\frac{n}{4}}+||b_t||^2_{\frac{1}{2}+\frac{n}{4}}\leq C, \quad for \quad t\geq t_5. \quad (3.30)$$

Proof. Applying ∂_t to the first equation of system (2.3) and multiplying the L_2-inner product by $A^{\frac{1}{2}+\frac{n}{4}}u_t$. Similarly, we apply ∂_t to the second equation of system (2.3) and multiply the L_2-inner product by $A^{\frac{1}{2}+\frac{n}{4}}b_t$. Then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||u_t||^2_{\frac{1}{2}+\frac{n}{4}}+||b_t||^2_{\frac{1}{2}+\frac{n}{4}})+||u_t||^2_{1+\frac{n}{2}}+||b_t||^2_{1+\frac{n}{2}}\\
&\leq|\int_D(u_t\nabla u)A^{\frac{1}{2}+\frac{n}{4}}u_tdx|+|\int_D(u\nabla u_t)A^{\frac{1}{2}+\frac{n}{4}}u_tdx|+|\int_D(b_t\nabla b)A^{\frac{1}{2}+\frac{n}{4}}u_tdx|\\
&+|\int_D(b\nabla b_t)A^{\frac{1}{2}+\frac{n}{4}}u_tdx|+|\int_D(u_t\nabla b)A^{\frac{1}{2}+\frac{n}{4}}b_tdx|+|\int_D(u\nabla b_t)A^{\frac{1}{2}+\frac{n}{4}}b_tdx|\\
&+|\int_D(b_t\nabla u)A^{\frac{1}{2}+\frac{n}{4}}b_tdx|+|\int_D(b\nabla u_t)A^{\frac{1}{2}+\frac{n}{4}}b_tdx|\\
&\leq C||u_t||_{L^\infty}||\nabla u||||A^{\frac{1}{2}+\frac{n}{4}}u_t||+C||u||_{L^{\frac{4n}{n-2}}}||\nabla u_t||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}u_t||\\
&+C||b_t||_{L^\infty}||\nabla b||||A^{\frac{1}{2}+\frac{n}{4}}u_t||+C||b||_{L^{\frac{4n}{n-2}}}||\nabla b_t||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}u_t||\\
&+C||u_t||_{L^\infty}||\nabla b||||A^{\frac{1}{2}+\frac{n}{4}}b_t||+C||u||_{L^{\frac{4n}{n-2}}}||\nabla b_t||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}b_t||\\
&+C||b_t||_{L^\infty}||\nabla u||||A^{\frac{1}{2}+\frac{n}{4}}b_t||+C||b||_{L^{\frac{4n}{n-2}}}||\nabla u_t||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}b_t||\\
&\leq C||u_t||^{\frac{4}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||u_t||^{\frac{n-2}{n+2}}_{1+\frac{n}{2}}||\nabla u||||u_t||_{1+\frac{n}{2}}+C||u||_{\frac{1}{2}+\frac{n}{4}}||u_t||_{\frac{1}{2}+\frac{n}{4}}||u_t||_{1+\frac{n}{2}}\\
&+C||b_t||^{\frac{4}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||b_t||^{\frac{n-2}{n+2}}_{1+\frac{n}{2}}||\nabla b||||u_t||_{1+\frac{n}{2}}+C||b||_{\frac{1}{2}+\frac{n}{4}}||b_t||_{\frac{1}{2}+\frac{n}{4}}||u_t||_{1+\frac{n}{2}}\\
&+C||u_t||^{\frac{4}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||u_t||^{\frac{n-2}{n+2}}_{1+\frac{n}{2}}||\nabla b||||b_t||_{1+\frac{n}{2}}+C||u||_{\frac{1}{2}+\frac{n}{4}}||b_t||_{\frac{1}{2}+\frac{n}{4}}||b_t||_{1+\frac{n}{2}}\\
&+C||b_t||^{\frac{4}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||b_t||^{\frac{n-2}{n+2}}_{1+\frac{n}{2}}||\nabla u||||b_t||_{1+\frac{n}{2}}+C||b||_{\frac{1}{2}+\frac{n}{4}}||u_t||_{\frac{1}{2}+\frac{n}{4}}||b_t||_{1+\frac{n}{2}}\\
&\leq\frac{1}{4}(||u_t||^2_{1+\frac{n}{2}}+||b_t||^2_{1+\frac{n}{2}})+C||u_t||^{\frac{8}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||u_t||^{\frac{2n-4}{n+2}}_{1+\frac{n}{2}}+C||b_t||^{\frac{8}{n+2}}_{\frac{1}{2}+\frac{n}{4}}||b_t||^{\frac{2n-4}{n+2}}_{1+\frac{n}{2}}\\
&+C(||u_t||^2_{\frac{1}{2}+\frac{n}{4}}+||b_t||^2_{\frac{1}{2}+\frac{n}{4}})(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}})\\
&\leq\frac{1}{2}(||u_t||^2_{1+\frac{n}{2}}+||b_t||^2_{1+\frac{n}{2}})+C(||u_t||^2_{\frac{1}{2}+\frac{n}{4}}+||b_t||^2_{\frac{1}{2}+\frac{n}{4}})(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}}+1). \quad (3.31)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{d}{dt}(||u_t||^2_{\frac{1}{2}+\frac{n}{4}}+||b_t||^2_{\frac{1}{2}+\frac{n}{4}})+||u_t||^2_{1+\frac{n}{2}}+||b_t||^2_{1+\frac{n}{2}}\\
&\le C(||u_t||^2_{\frac{1}{2}+\frac{n}{4}}+||b_t||^2_{\frac{1}{2}+\frac{n}{4}})(||u||^2_{\frac{1}{2}+\frac{n}{4}}+||b||^2_{\frac{1}{2}+\frac{n}{4}}+1)\\
&\le C(||u_t||^2_{\frac{1}{2}+\frac{n}{4}}+||b_t||^2_{\frac{1}{2}+\frac{n}{4}}).
\end{aligned}\tag{3.32}$$

By using the Gronwall lemma, then we get

$$||u_t||^2_{\frac{1}{2}+\frac{n}{4}}+||b_t||^2_{\frac{1}{2}+\frac{n}{4}}\le C,\quad for\quad t\ge t_5=t_4+1.$$

Lemma 3.9. $\{S(t)\}_{t\ge 0}$ is Lipschitz continuous in $H^{\frac{1}{2}+\frac{n}{4}}\times H^{\frac{1}{2}+\frac{n}{4}}$.
Proof. Let (u_1,b_1) and (u_2,b_2) be two solutions of the system (2.3) with initial values (u_{01},b_{01}) and (u_{02},b_{02}). We set $\bar u=u_1-u_2$ and $\bar b=b_1-b_2$. We multiply the inner product with $A^{\frac{1}{2}+\frac{n}{4}}\bar u$ and $A^{\frac{1}{2}+\frac{n}{4}}\bar b$, respectively. Adding up their results, then we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\bar u||^2_{\frac{1}{2}+\frac{n}{4}}+||\bar b||^2_{\frac{1}{2}+\frac{n}{4}})+||\bar u||^2_{1+\frac{n}{2}}+||\bar b||^2_{1+\frac{n}{2}}\\
&\le\int_D|\bar u\nabla u_1A^{\frac{1}{2}+\frac{n}{4}}\bar u|dx+\int_D|u_2\nabla\bar uA^{\frac{1}{2}+\frac{n}{4}}\bar u|dx+\int_D|\bar b\nabla b_1A^{\frac{1}{2}+\frac{n}{4}}\bar u|dx\\
&+\int_D|b_2\nabla\bar bA^{\frac{1}{2}+\frac{n}{4}}\bar u|dx+\int_D|u_1\nabla\bar bA^{\frac{1}{2}+\frac{n}{4}}\bar b|dx+\int_D|\bar u\nabla b_2A^{\frac{1}{2}+\frac{n}{4}}\bar b|dx\\
&+\int_D|\bar b\nabla u_1A^{\frac{1}{2}+\frac{n}{4}}\bar b|dx+\int_D|b_2\nabla\bar uA^{\frac{1}{2}+\frac{n}{4}}\bar b|dx=\sum_{i=1}^{8}L_i.
\end{aligned}\tag{3.33}$$

For L_1, using the Gagliardo-Nirenberg inequality, it yields

$$\begin{aligned}
L_1&\le C||\bar u||_{L^{\frac{4n}{n-2}}}||A^{\frac{1}{2}+\frac{n}{4}}\bar u||||\nabla u_1||_{L^{\frac{4n}{n+2}}}\\
&\le C||\bar u||_{\frac{1}{2}+\frac{n}{4}}||u_1||_{\frac{1}{2}+\frac{n}{4}}||\bar u||_{1+\frac{n}{2}}\\
&\le\frac{1}{8}||\bar u||^2_{1+\frac{n}{2}}+C||u_1||^2_{\frac{1}{2}+\frac{n}{4}}||\bar u||^2_{\frac{1}{2}+\frac{n}{4}}.
\end{aligned}\tag{3.34}$$

For L_2, using the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
L_2&\le C||\nabla\bar u||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}\bar u||||u_2||_{L^{\frac{4n}{n-2}}}\\
&\le C||\bar u||_{\frac{1}{2}+\frac{n}{4}}||u_2||_{\frac{1}{2}+\frac{n}{4}}||\bar u||_{1+\frac{n}{2}}\\
&\le\frac{1}{8}||\bar u||^2_{1+\frac{n}{2}}+C||u_2||^2_{\frac{1}{2}+\frac{n}{4}}||\bar u||^2_{\frac{1}{2}+\frac{n}{4}}.
\end{aligned}\tag{3.35}$$

For L_3-L_4, similarly, it is easy to get

$$L_3+L_4\le C||\bar b||_{L^{\frac{4n}{n-2}}}||\nabla b_1||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}\bar u||+C||b_2||_{L^{\frac{4n}{n-2}}}||\nabla\bar b||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}\bar u||$$

$$\leq C||\bar{b}||_{\frac{1}{2}+\frac{n}{4}}||b_1||_{\frac{1}{2}+\frac{n}{4}}||\bar{u}||_{1+\frac{n}{2}} + C||\bar{b}||_{\frac{1}{2}+\frac{n}{4}}||b_2||_{\frac{1}{2}+\frac{n}{4}}||\bar{u}||_{1+\frac{n}{2}}$$
$$\leq \frac{1}{4}||\bar{u}||^2_{1+\frac{n}{2}} + C||b_1||^2_{\frac{1}{2}+\frac{n}{4}}||\bar{b}||^2_{\frac{1}{2}+\frac{n}{4}} + C||b_2||^2_{\frac{1}{2}+\frac{n}{4}}||\bar{b}||^2_{\frac{1}{2}+\frac{n}{4}}. \tag{3.36}$$

For $L_5 - L_8$, we also have

$$\begin{aligned}
\sum_{i=5}^{8} L_i &\leq C||u_1||_{L^{\frac{4n}{n-2}}}||\nabla\bar{b}||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}\bar{b}|| + ||\bar{u}||_{L^{\frac{4n}{n-2}}}||\nabla b_2||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}\bar{b}|| \\
&+ ||\bar{b}||_{L^{\frac{4n}{n-2}}}||\nabla u_1||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}\bar{b}|| + ||b_2||_{L^{\frac{4n}{n-2}}}||\nabla\bar{u}||_{L^{\frac{4n}{n+2}}}||A^{\frac{1}{2}+\frac{n}{4}}\bar{b}|| \\
&\leq C||\bar{b}||_{\frac{1}{2}+\frac{n}{4}}||u_1||_{\frac{1}{2}+\frac{n}{4}}||\bar{b}||_{1+\frac{n}{2}} + C||\bar{u}||_{\frac{1}{2}+\frac{n}{4}}||b_2||_{\frac{1}{2}+\frac{n}{4}}||\bar{b}||_{1+\frac{n}{2}} \\
&+ C||\bar{b}||_{\frac{1}{2}+\frac{n}{4}}||u_1||_{\frac{1}{2}+\frac{n}{4}}||\bar{b}||_{1+\frac{n}{2}} + C||\bar{u}||_{\frac{1}{2}+\frac{n}{4}}||b_2||_{\frac{1}{2}+\frac{n}{4}}||\bar{b}||_{1+\frac{n}{2}} \\
&\leq \frac{1}{2}||\bar{b}||^2_{1+\frac{n}{2}} + C||u_1||^2_{\frac{1}{2}+\frac{n}{4}}||\bar{b}||^2_{\frac{1}{2}+\frac{n}{4}} + C||b_2||^2_{\frac{1}{2}+\frac{n}{4}}||\bar{u}||^2_{\frac{1}{2}+\frac{n}{4}}.
\end{aligned} \tag{3.37}$$

Putting (3.34)-(3.37) into (3.33), it yields

$$\begin{aligned}
&\frac{d}{dt}(||\bar{u}||^2_{\frac{1}{2}+\frac{n}{4}} + ||\bar{b}||^2_{\frac{1}{2}+\frac{n}{4}}) + ||\bar{u}||^2_{1+\frac{n}{2}} + ||\bar{b}||^2_{1+\frac{n}{2}} \\
&\quad \leq C(||u_1||^2_{\frac{1}{2}+\frac{n}{4}} + ||u_2||^2_{\frac{1}{2}+\frac{n}{4}} + ||b_1||^2_{\frac{1}{2}+\frac{n}{4}} + ||b_2||^2_{\frac{1}{2}+\frac{n}{4}})(||\bar{u}||^2_{\frac{1}{2}+\frac{n}{4}} + ||\bar{b}||^2_{\frac{1}{2}+\frac{n}{4}}).
\end{aligned} \tag{3.38}$$

Applying the Gronwall's inequality, we have

$$\begin{aligned}
&||\bar{u}(t)||^2_{\frac{1}{2}+\frac{n}{4}} + ||\bar{b}(t)||^2_{\frac{1}{2}+\frac{n}{4}} \\
&\leq (||\bar{u}(0)||^2_{\frac{1}{2}+\frac{n}{4}} + ||\bar{b}(0)||^2_{\frac{1}{2}+\frac{n}{4}})exp\{C\int_0^t(||u_1||^2_{\frac{1}{2}+\frac{n}{4}} + ||u_2||^2_{\frac{1}{2}+\frac{n}{4}} + ||b_1||^2_{\frac{1}{2}+\frac{n}{4}} + ||b_2||^2_{\frac{1}{2}+\frac{n}{4}})ds\}.
\end{aligned} \tag{3.39}$$

By Lemma 3.1 and Lemma 3.4, this completes the proof of the Lemma 3.9.

4 Global attractors

In order to get the existence of global attractor, we need to prove the following lemma.
Lemma 4.1. Assume that $\mathcal{A}$ is a $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}})-$global attractor for $\{S(t)\}_{t\geq 0}$. $\mathcal{A}$ is a $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})-$global attractor if and only if

(i) $\{S(t)\}_{t\geq 0}$ is a bounded $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})-$absorbing set.

(ii) $\{S(t)\}_{t\geq 0}$ is $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})-$asymptotically compact.

Firstly, we will prove the operator $\{S(t)\}_{t\geq 0}$ has a $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}})-$global attractor, then by using the above Lemma 4.1, we get the attractor is a $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})-$global attractor. Let

$$B_1 = \{u, b \in H^{\frac{1}{2}+\frac{n}{4}} : ||u||^2_{\frac{1}{2}+\frac{n}{4}} + ||b||^2_{\frac{1}{2}+\frac{n}{4}} \leq C\}$$

and

$$B_2 = \{u, b \in D(A') : ||A^{\frac{1}{2}+\frac{n}{4}}u(t)||^2 + ||A^{\frac{1}{2}+\frac{n}{4}}b(t)||^2 \le C\}.$$

By the above Lemma 3.4, we deduce that B_1 is bounded absorbing set of $\{S(t)\}_{t\ge 0}$ in the space $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}})$. By the above Lemma 3.6, we get that B_2 is bounded absorbing set of $\{S(t)\}_{t\ge 0}$ in the space $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})$. By Lemma 3.6, the $\{S(t)\}_{t\ge 0}$ is $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}})$-asymptotically compact. Inspired by [15–17], we get a $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}})$-global attractor $\mathcal{A}$. Finally, we will show $\{S(t)\}_{t\ge 0}$ is $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})$-asymptotically compact. We need the following Lemma.

Lemma 4.2. Let $(u_0, b_0) \in H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $f, g \in L^2$, the dynamical system $\{S(t)\}_{t\ge 0}$ is $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})$-asymptotically compact.

Proof. Let (u_{0n}, b_{0n}) denote bounded in $H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $t_n \to \infty$. We will show $\{S(t_n)(u_{0n}, b_{0n})\}$ has a convergent subsequence in $H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}}$. Let

$$(u_n(t), b_n(t)) = S(t)(u_{0n}, b_{0n}), \quad (\bar{u}_n(t_n), \bar{b}_n(t_n)) = (\frac{\partial u_n}{\partial t}|_{t=t_n}, \frac{\partial b_n}{\partial t}|_{t=t_n}).$$

For the first equation and the second equation of system (2.3), we get

$$\begin{aligned} A^{\frac{1}{2}+\frac{n}{4}}u_n(t_n) &= f - \bar{u}_n(t_n) - B(u_n(t_n), u_n(t_n)) + B(b_n(t_n), b_n(t_n)), \\ A^{\frac{1}{2}+\frac{n}{4}}b_n(t_n) &= g - \bar{b}_n(t_n) - B(u_n(t_n), b_n(t_n)) + B(b_n(t_n), u_n(t_n)). \end{aligned}$$

By Lemma 3.6 and Lemma 3.8, then there exists a positive constant $T > 0$ such that for any $t \ge T$,

$$||\frac{\partial u_n}{\partial t}(t)||^2_{\frac{1}{2}+\frac{n}{4}} + ||\frac{\partial b_n}{\partial t}(t)||^2_{\frac{1}{2}+\frac{n}{4}} \le C, \quad ||A^{\frac{1}{2}+\frac{n}{4}}u_n(t)|| + ||A^{\frac{1}{2}+\frac{n}{4}}b_n(t)|| \le C. \tag{4.1}$$

When $t_n \to \infty$, there exists a $N > 0$ such that $t_n \ge T$ for every $n \ge N$. Applying (4.1), we deduce for $n \ge N$,

$$||\bar{u}_n(t_n)||_{\frac{1}{2}+\frac{n}{4}} + ||\bar{b}_n(t_n)||_{\frac{1}{2}+\frac{n}{4}} \le C, \quad ||A^{\frac{1}{2}+\frac{n}{4}}u_n(t_n)|| + ||A^{\frac{1}{2}+\frac{n}{4}}b_n(t_n)|| \le C. \tag{4.2}$$

Applying the compactness of embedding $H^{\frac{1}{2}+\frac{n}{4}} \hookrightarrow L^2$ and $D(A') \hookrightarrow H^{\frac{1}{2}+\frac{n}{4}}$ and (4.2), then there exist $(\bar{u}, \bar{b}) \in H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}$ and $(\hat{u}, \hat{b}) \in D(A') \times D(A')$ such that

$$u_n(t_n) \to \hat{u} \ \text{ strongly in } \ H^{\frac{1}{2}+\frac{n}{4}}, \tag{4.3}$$

$$b_n(t_n) \to \hat{b} \ \text{ strongly in } \ H^{\frac{1}{2}+\frac{n}{4}}, \tag{4.4}$$

$$\bar{u}_n(t_n) \to \bar{u} \ \text{ strongly in } \ L_2, \tag{4.5}$$

$$\bar{b}_n(t_n) \to \bar{b} \ \text{ strongly in } \ L_2. \tag{4.6}$$

Then, by Sobolev inequality, we have

$$\begin{aligned}
&||(u_n(t_n)\cdot\nabla)u_n(t_n)-(\hat{u}\cdot\nabla)\hat{u}||^2\\
&\leq C(||(u_n(t_n)\cdot\nabla)(u_n(t_n)-\hat{u})||^2+||(u_n(t_n)-\hat{u})\cdot\nabla\hat{u}||^2)\\
&\leq C(||u_n(t_n)||^2_{L^{\frac{4n}{n-2}}}||\nabla(u_n(t_n)-\hat{u})||^2_{L^{\frac{4n}{n+2}}}+||u_n(t_n)-\hat{u}||^2_{L^{\frac{4n}{n-2}}}||\nabla\hat{u}||^2_{L^{\frac{4n}{n+2}}})\\
&\leq C(||u_n(t_n)||^2_{\frac{1}{2}+\frac{n}{4}}||u_n(t_n)-\hat{u}||^2_{\frac{1}{2}+\frac{n}{4}}+||u_n(t_n)-\hat{u}||^2_{\frac{1}{2}+\frac{n}{4}}||\hat{u}||_{\frac{1}{2}+\frac{n}{4}})\to 0,\\
&\text{as } n\to\infty. \qquad (4.7)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&||(b_n(t_n)\cdot\nabla)b_n(t_n)-(\hat{b}\cdot\nabla)\hat{b}||^2\\
&\leq C(||(b_n(t_n)\cdot\nabla)(b_n(t_n)-\hat{b})||^2+||(b_n(t_n)-\hat{b})\cdot\nabla\hat{b}||^2)\\
&\leq C(||b_n(t_n)||^2_{L^{\frac{4n}{n-2}}}||\nabla(b_n(t_n)-\hat{b})||^2_{L^{\frac{4n}{n+2}}}+||b_n(t_n)-\hat{b}||^2_{L^{\frac{4n}{n-2}}}||\nabla\hat{b}||^2_{L^{\frac{4n}{n+2}}})\\
&\leq C(||b_n(t_n)||^2_{\frac{1}{2}+\frac{n}{4}}||b_n(t_n)-\hat{b}||^2_{\frac{1}{2}+\frac{n}{4}}+||b_n(t_n)-\hat{b}||^2_{\frac{1}{2}+\frac{n}{4}}||\hat{b}||^2_{\frac{1}{2}+\frac{n}{4}})\\
&\to 0, \quad \text{as } \ n\to\infty, \qquad (4.8)\\
&||(u_n(t_n)\cdot\nabla)b_n(t_n)-(\hat{u}\cdot\nabla)\hat{b}||^2\\
&\leq C(||(u_n(t_n)\cdot\nabla)(b_n(t_n)-\hat{b})||^2+||(u_n(t_n)-\hat{u})\cdot\nabla\hat{b}||^2)\\
&\leq C(||u_n(t_n)||^2_{L^{\frac{4n}{n-2}}}||\nabla(b_n(t_n)-\hat{b})||^2_{L^{\frac{4n}{n+2}}}+||u_n(t_n)-\hat{u}||^2_{L^{\frac{4n}{n+2}}}||\nabla\hat{b}||^2_{L^{\frac{4n}{n-2}}})\\
&\leq C(||u_n(t_n)||^2_{\frac{1}{2}+\frac{n}{4}}||b_n(t_n)-\hat{b}||^2_{\frac{1}{2}+\frac{n}{4}}+||u_n(t_n)-\hat{u}||^2_{\frac{1}{2}+\frac{n}{4}}||\hat{b}||^2_{\frac{1}{2}+\frac{n}{4}})\\
&\to 0, \quad \text{as } \ n\to\infty, \qquad (4.9)
\end{aligned}$$

and

$$\begin{aligned}
&||(b_n(t_n)\cdot\nabla)u_n(t_n)-(\hat{b}\cdot\nabla)\hat{u}||^2\\
&\leq C(||(b_n(t_n)\cdot\nabla)(u_n(t_n)-\hat{u})||^2+||(b_n(t_n)-\hat{b})\cdot\nabla\hat{u}||^2)\\
&\leq C(||b_n(t_n)||^2_{L^{\frac{4n}{n-2}}}||\nabla(u_n(t_n)-\hat{u})||^2_{L^{\frac{4n}{n+2}}}+||b_n(t_n)-\hat{b}||^2_{L^{\frac{4n}{n-2}}}||\nabla\hat{u}||^2_{L^{\frac{4n}{n+2}}})\\
&\leq C(||b_n(t_n)||^2_{\frac{1}{2}+\frac{n}{4}}||u_n(t_n)-\hat{u}||^2_{\frac{1}{2}+\frac{n}{4}}+||b_n(t_n)-\hat{b}||^2_{\frac{1}{2}+\frac{n}{4}}||\hat{u}||^2_{\frac{1}{2}+\frac{n}{4}})\\
&\to 0, \quad \text{as } \ n\to\infty. \qquad (4.10)
\end{aligned}$$

The (4.7)-(4.10) imply that

$$-(u_n(t_n)\cdot\nabla)u_n(t_n)+(b_n(t_n)\cdot\nabla)b_n(t_n)\to-(\hat{u}\cdot\nabla)\hat{u}+\hat{b}\cdot\nabla)\hat{b} \ \text{ strongly in } \ L_2, \quad (4.11)$$

$$-(u_n(t_n)\cdot\nabla)b_n(t_n)+(b_n(t_n)\cdot\nabla)u_n(t_n)\to-(\hat{u}\cdot\nabla)\hat{b}+\hat{b}\cdot\nabla)\hat{u} \ \text{ strongly in } \ L_2. \quad (4.12)$$

Applying (4.5), (4.6), (4.11) and (4.12), then we get

$$A^{\frac{1}{2}+\frac{n}{4}}u_n(t_n)\to f-\bar{u}-B(\hat{u},\hat{u})+B(\hat{b},\hat{b}), \ \text{ strongly in } \ L_2, \qquad (4.13)$$

$$A^{\frac{1}{2}+\frac{n}{4}}b_n(t_n) \to g - \bar{b} - B(\hat{u},\hat{b}) + B(\hat{b},\hat{u}) \text{ strongly in } L_2, \tag{4.14}$$

as $n \to \infty$. We get $\{S(t)\}_{t\geq 0}$ is $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})$-asymptotically compact.
Proof of Theorem 2.3 Applying Lemma 3.6, we get $B_2 = \{u, b \in D(A') : ||A^{\frac{1}{2}+\frac{n}{4}}u||^2 + ||A^{\frac{1}{2}+\frac{n}{4}}b||^2 \leq C\}$ denotes a bounded $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})-$absorbing set. Nextly, applying the Lemma 4.2, we obtain the $\{S(t)\}_{t\geq 0}$ is $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})$-asymptotically compact. Finally, by Lemma 4.1, $\mathcal{A}$ is a $(H^{\frac{1}{2}+\frac{n}{4}} \times H^{\frac{1}{2}+\frac{n}{4}}, H^{1+\frac{n}{2}} \times H^{1+\frac{n}{2}})-$global attractor.

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