New exact solution and simulation for the coupled nonlinear Schrödinger equations with variable coefficients

Yuting Qiu[1]*, Ping Gao[1]

[1]School of Mathematics and Information science, Guangzhou University, Guangzhou, 510006

Email: victoriayutingqiu@163.com, 1554744857@qq.com

Abstract

In this paper, we study the coupled nonlinear Schrödinger equation with variable coefficients (VCNLS) by means of modified Sine-Gordon equation method, the subsistence of some novel bright-dark solitons and dark-dark solitons are obtained. Moreover, some figures are simulated by computer to show the solutions are soliton solutions and how the evolution of soliton is determined by different values of variable group velocity dispersion terms, which can be used to simulate various phenomena.

Keywords: Modified Sine-Gordon equation method; coupled nonlinear Schrödinger equation; exact solutions; simulation.

1. Introduction

It is well familiar that various types of physical phenomena in nature can be expressed in terms of nonlinear partial differential equations (NPDEs). The solutions of these equations have crucial impact in physical, mathematics and engineering fields. In particular, the analysis of solitary wave solutions participates as very significant role in the study of some physical models. One of the most significant NPDEs is the nonlinear Schrödinger (NLS) equation, it can describe the nonlinear dynamics of surface gravity waves in oceans and is an approximation of the fully nonlinear equations, it contains the two basic ingredients of surface wave dynamics: nonlinearity and dispersion (in an oceanographic context). Besides, the equation can be used to describe many different physical systems, such as nonlinear optics, Bose-Einstein condensates(BECs), plasma waves and so on.

In this paper, we will consider the following coupled nonlinear Schrödinger equation with variable coefficients (VCNLS):

$$\begin{aligned} &\mathrm{i}\psi_{1t} + a(t)\psi_{1xx} + \left(b_2(t)\left|\psi_1\right|^2 + b_2(t)\left|\psi_2\right|^2\right)\psi_1 + v(t)\psi_1 = 0,\\ &\mathrm{i}\psi_{2t} + a(t)\psi_{2xx} + \left(b_1(t)\left|\psi_1\right|^2 + b_2(t)\left|\psi_2\right|^2\right)\psi_2 + v(t)\psi_2 = 0. \end{aligned} \tag{1}$$

which can be applied to describe the interaction among the modes in nonlinear optics and some other branches of nonlinear science such as BECs and so on [1, 2, 3, 4, 5, 6]. Where ψ_1 and ψ_2 are complex envelopes of the propagating beam of the two modes, and x, t are the spatial coordinate and retarded time respectively. The coefficients $a(t)$ represent the group velocity dispersion, $b1(t)$, $b2(t)$ mean nonlinear interactions and $v(t)$ means external potential [7].

When $a(t) = 1/2$, $b_1(t) = b_2(t) = e$, $v(t) = 0$, then System VCNLS reduces to the classical coupled Schrödinger Systems (CNLS):

$$\begin{aligned} &\mathrm{i}q_{1T} + \tfrac{1}{2}q_{1XX} + e\left(|q_1|^2 + |q_2|^2\right)q_1 = 0,\\ &\mathrm{i}q_{2T} + \tfrac{1}{2}q_{2XX} + e\left(|q_1|^2 + |q_2|^2\right)q_2 = 0. \end{aligned} \tag{2}$$

*Corresponding author. Email: victoriayutingqiu@163.com

which has been solved to get soliton solutions on trivial background through Hirota bilinear method in [8, 9], Darboux transformation in [10, 11]. The CNLS with variable coefficients attracts much attention for the past few decades. Biswas has done lots of work on solitons in the birefringent fibers or with Hamiltonian perturbations [12, 13, 14]. The variation of the coefficients would make the solitons travel in different external potentials. For the scalar Schrödinger system, there are plenty of results on solitons and well-posedness results, which can be referred to [2, 4, 15, 16].

In what follows, based on the modified Sine-Gordon equation method [17], we derive some bright-dark solitons and dark-dark solitons for VCNLS and obtain the following results: (i) Choosing different values for variable group velocity dispersion term, we obtained the bell-shaped, parabolic, cubic and periodic solitons, respectively. (ii) Interaction between the two solutions is investigated and obtained each soliton shape keeps invariant after interaction. Remarkably, nonlinear wave equations with variable coefficients are viewed as generalizations of evolution equations with constant coefficients. Therefore, considering the system (1) will give richer knowledge on dynamics in nonlinear media described by the system (2).

2. The method

Let us consider a form of a nonlinear partial differential equation

$$H_k\left(x, t, \frac{\partial\psi_k}{\partial x}, \frac{\partial\psi_k}{\partial t}, \frac{\partial^2\psi_k}{\partial x^2}, \ldots, a(t), b_1(t), b_2(t), v(t)\right) = 0 \tag{3}$$

where $a(t)$, $b_k(t)$ and $v(t)$ are arbitrary functions in t and $k = 1, 2$. In the following, we offer the main steps of this method:

Step 1: Use the following assumptions:

$$\begin{aligned} \psi_1(x,t) &= U_1(\zeta)exp(i(\alpha x - \theta(t))), \\ \psi_2(x,t) &= U_2(\zeta)exp(i(\alpha x - \theta(t))). \end{aligned} \tag{4}$$

where $U_1(\zeta)$ and $U_1(\zeta)$ are the new dependant functions, $\zeta = \mu(x - \lambda(t))$ is the new independent variable, $\lambda(t)$ is an arbitrary function of t and μ and α are the frequency and the width of the soliton respectively.

Step 2: Collect the coeffients of $U_1(\zeta)$and $U_1(\zeta)$ and their derivatives,and then assume the imaginary part is equal to zero.

Step 3: Take the coefficient of the largest linear term as the normalization coefficient.

Step 4: The derivatives and powers of $U_1(\zeta)$ and $U_1(\zeta)$ are equal to the term multiplied by a constant, so the arbitrary functions will be determined, and the Eq.(3) is transformed into the following nonlinear ordinary differential system.

$$Q_i\left(U_i(\zeta), U_i'(\zeta), U_i''(\zeta), \ldots\right) = 0 \tag{5}$$

Step 5: Use the solutions of the Sine-Gordon equation [18, 19]by assuming that

$$\begin{aligned} U_1(\xi) &= \sum_{i=1}^{n} \cos^{i-1}(w(\xi)) \times \left[B_i \sin(w(\xi)) + A_i \cos(w(\xi))\right] + A_0, \\ U_2(\xi) &= \sum_{j=1}^{m} \cos^{j-1}(w(\xi)) \times \left[E_j \sin(w(\xi)) + D_j \cos(w(\xi))\right] + D_0. \end{aligned} \tag{6}$$

where A_i, B_i, D_i, E_i, A_0 and D_0 are arbitrary constants and n and m are determined by balancing the most dispersive term and the greatest nonlinear term in Eq.(5), and

$$w'(\xi) = \sin(w(\xi)) \tag{7}$$

and

$$\sin(w(\xi)) = \mathrm{sech}(\xi) \quad \text{or} \quad \cos(w(\xi)) = \tanh(\xi) \tag{8}$$

Step 6: Equating the coefficients of $\sin^i(w(\xi))$ and $\cos^i(w(\xi))$ to zero and an algebraic system for the constant A_i, B_i, D_i, E_i, A_0 and D_0 are obtained, by solving them with a Maple program and back-substituting into Eqs.(6) and Eqs.(4) via Eqs.(8), novel soliton solutions are obtained for the system of Eq.(3).

Advantages of the method: The Sine-Gordon equation method has limitations and is suitable for some constant coefficient systems, but modified Sine-Gordon equation method is applicable to systems with variable coefficients containing imaginary parts. As a result, some spanking new solutions might be originated via this method and this method can use computational software like Maple or Mathematica to reduce the amount of computation.

3. Exact solutions for VCNLS

By substituting the assumptions in Eqs.(4) into Eqs.(1), we obtain

$$\begin{aligned} &a(t)\mu^2 U_1''(\xi) + i\mu\left(2\alpha a(t) - \lambda'\right) U_1'(\xi) + \left(\theta' - \alpha^2 a(t)\right) U_1(\xi) + \left(b_1(t)U_1^2(\xi) + b_2(t)U_2^2(\xi)\right) U_1(\xi) + v(t)U_1(\xi) = 0,\\ &a(t)\mu^2 U_2''(\xi) + i\mu\left(2\alpha a(t) - \lambda'\right) U_2'(\xi) + \left(\theta' - \alpha^2 a(t)\right) U_2(\xi) + \left(b_1(t)U_1^2(\xi) + b_2(t)U_2^2(\xi)\right) U_2(\xi) + v(t)U_2(\xi) = 0. \end{aligned} \tag{9}$$

to make Eqs.(9) real, the terms $U_1'(\xi)$ and $U_2'(\xi)$ must be eliminated, so according to Step 2, we get $\lambda(t) = 2\alpha\int a(t)dt + \lambda_0$, and then follow Step 3, that is, take the coefficients of $U_1''(\xi)$ and $U_2''(\xi)$ as the normalized coefficients, and we get

$$\theta(t) = \int\left(\mu^2 c_1 + \alpha^2\right)a(t)dt + \theta_0 \tag{10}$$

$$\lambda(t) = 2\alpha\int a(t)dt + \lambda_0 \tag{11}$$

$$b_1(t) = c_2\mu^2 a(t) \tag{12}$$

$$b_2(t) = c_3\mu^2 a(t) \tag{13}$$

$$v(t) = c_4\mu^2 a(t) \tag{14}$$

where c_1, c_2, c_3 and c_4 are constants and λ_0 and θ_0 is an integration constant. Therefore,

$$\xi = \mu\left(x - 2\alpha\int a(t)dt\right) \tag{15}$$

And, Eqs.(9) can be simplified as follows

$$\begin{aligned} &U_1''(\xi) + c_1U_1(\xi) + \left(c_2U_1^2(\xi) + c_3U_2^2(\xi)\right)U_1(\xi) + c_4U_1(\xi) = 0,\\ &U_2''(\xi) + c_1U_2(\xi) + \left(c_2U_1^2(\xi) + c_3U_2^2(\xi)\right)U_2(\xi) + c_4U_2(\xi) = 0. \end{aligned} \tag{16}$$

By balancing the dispersive and nonlinear terms in $Eqs.$(16) we get $m + 2 = 2m + m$ and $n + 2 = 2n + n$, i.e. $m = n = 1$, so according to Step 5, we assume that

$$\begin{aligned} U_1(\xi) &= B_1\sin(w(\xi)) + A_1\cos(w(\xi)) + A_0,\\ U_2(\xi) &= E_1\sin(w(\xi)) + D_1\cos(w(\xi)) + D_0. \end{aligned} \tag{17}$$

Substituting Eqs.(17) and the necessary derivatives into Eq.(16) using Eqs.(7), applying trigonometric identities and collecting the coefficients of $\sin(w(\xi))$ and $\cos(w(\xi))$ that are containing independent combinations to zero, and we obtain the following independent parametric equations:

$\sin(w(\xi))$:

$$c_1B_1 + c_4B_1 + 3c_2B_1A_0^2 + c_3B_1D_0^2 + 2c_3E_1D_0A_0 = 0 \tag{18}$$

$\cos(w(\xi))$:

$$c_1A_1 + c_4A_1 + 3c_2A_1A_0^2 + c_3A_1D_0^2 + 2c_3D_1D_0A_0 + c_2A_1^3 + c_3A_1D_1^2 = 0 \tag{19}$$

$\sin^3(w(\xi))$:

$$-B_1 + c_2B_1^3 + c_3B_1E_1^2 = 0 \tag{20}$$

$\sin(w(\xi))\cos^2(w(\xi))$:

$$B_1 + 3c_2B_1A_1^2 + c_3B_1D_1^2 + 2c_3A_1E_1D_1 = 0 \tag{21}$$

$\sin^2(w(\xi))\cos(w(\xi))$:

$$-2A_1 + 3c_2A_1B_1^2 + c_3A_1E_1^2 + 2c_3E_1D_1B_1 - c_2A_1^3 - c_3A_1D_1^2 = 0 \tag{22}$$

$\sin^2(w(\xi))$:

$$3c_2A_0B_1^2 + c_3A_0E_1^2 + 2c_3E_1D_0B_1 = 0 \tag{23}$$

$\sin(w(\xi))\cos(w(\xi))$:

$$6c_2B_1A_1A_0 + 2c_3D_1D_0B_1 + 2c_3A_1E_1D_0 + 2c_3A_0E_1D_1 = 0 \tag{24}$$

$\cos^2(w(\xi))$:

$$3c_2A_0A_1^2 + 2c_3A_1D_1D_0 + c_3A_0D_1^2 = 0 \tag{25}$$

constants:

$$c_1A_0 + c_4A_0 + c_2A_0^3 + c_3A_0D_0^2 = 0 \tag{26}$$

Solving Eqs.(18)-(26), we obtain the following cases and solutions using Eqs.(8).

Case 1: When $A_0 = B_1 = 0,\ c_2 = -\frac{2}{A_1^2}, c_3 = 0, c_4 = -c_1 + 2$, we get the following bright-dark solitons:

$$\begin{aligned}
\psi_1(x,t) &= A_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)dt\right)\right) exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)dt + \theta_0\right)\right),\\
\psi_2(x,t) &= \left[E_1 \operatorname{sech}\left(\mu\left(x - 2\alpha\int a(t)dt\right)\right) + D_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)dt\right)\right) + D_0\right] exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)dt + \theta_0\right)\right).
\end{aligned} \tag{27}$$

where A_1, D_1, E_1 and c_1 are arbitrary constants.

Case 2: When $A_0 = B_1 = D_0 = E_1 = 0, c_3 = -\frac{2}{D_1^2}, c_4 = -c_1$, we get the following dark-dark solitons:

$$\begin{aligned}
\psi_1(x,t) &= A_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)dt\right)\right) exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)dt + \theta_0\right)\right),\\
\psi_2(x,t) &= D_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)dt\right)\right) exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)dt + \theta_0\right)\right).
\end{aligned} \tag{28}$$

where A_1, D_1, c_1 and c_2 are arbitrary constants.

Case 3: When $A_0 = A_1 = D_0 = E_1 = 0, c_2 = -\frac{2}{B_1^2}, c_3 = -\frac{2}{D_1^2}, c_4 = -c_1$, we get the following bright-dark solitons:

$$\begin{aligned}
\psi_1(x,t) &= B_1 \operatorname{sech}\left(\mu\left(x - 2\alpha\int a(t)dt\right)\right) exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)dt + \theta_0\right)\right),\\
\psi_2(x,t) &= D_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)dt\right)\right) exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)dt + \theta_0\right)\right).
\end{aligned} \tag{29}$$

where B_1, D_1, and c_1 are arbitrary constants.

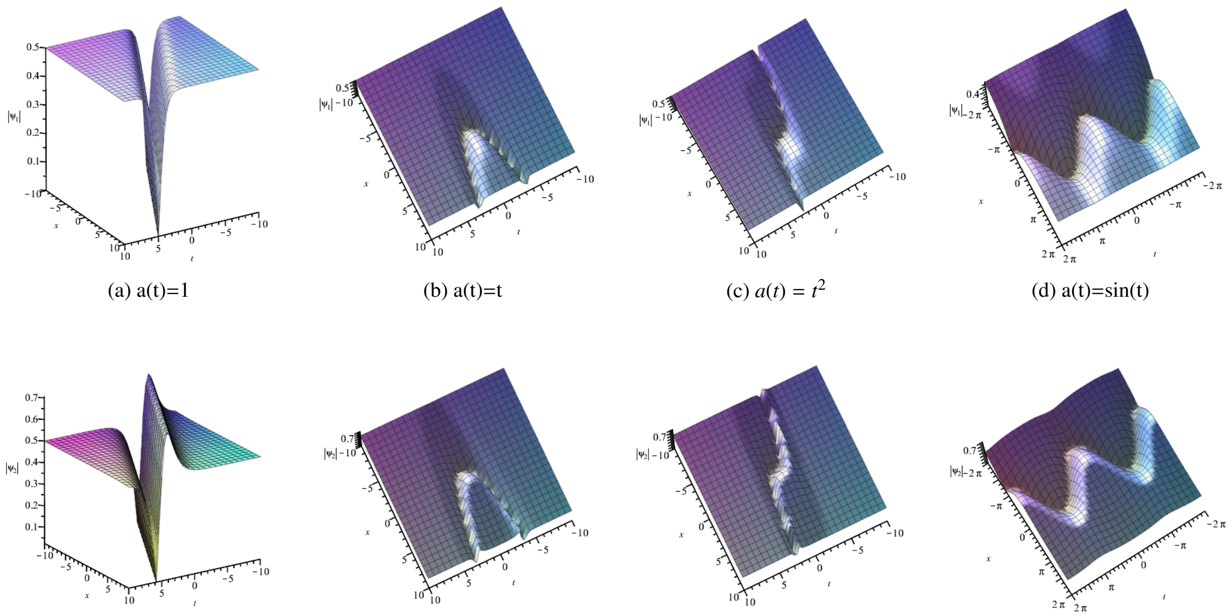
Case 4: When $A_0 = B_1 = D_0 = D_1 = 0, c_2 = -\frac{c_4+c_1}{A_1^2}, c_3 = -\frac{c_4+c_1-2}{E_1^2}$, we get the following dark-bright solitons:

$$\begin{aligned}\psi_1(x,t) &= A_1 \tanh\left(\mu\left(x - 2\alpha\int a(t)dt\right)\right) exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)dt + \theta_0\right)\right),\\ \psi_2(x,t) &= \left[E_1 \operatorname{sech}\left(\mu\left(x - 2\alpha\int a(t)dt\right)\right) + D_0\right] exp\left(i\left(\alpha x - \int\left(\mu^2 c_1 + \alpha^2\right)a(t)dt + \theta_0\right)\right).\end{aligned} \tag{30}$$

where A_1, E_1, c_1 and c_4 are arbitrary constants.

4. Simulation and physical explanation

In this segment, we will illustrate the figure and designate the acquired solutions to the VCNLS equations. The solutions (27)-(30) come in terms of hyperbolic function. Next, we study the evolution behavior of the dark-bright soliton solutions given by Eqs.(27), the bright-dark soliton solutions given by Eqs.(29), and interaction of the two solutions given by Eqs.(29), illustrated in the following figures.



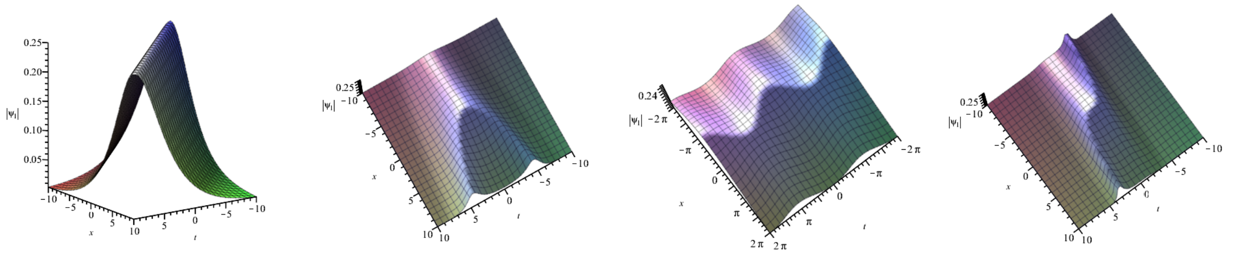
(a) a(t)=1 (b) a(t)=t (c) $a(t) = t^2$ (d) a(t)=sin(t)

(e) a(t)=1 (f) a(t)=t (g) $a(t) = t^2$ (h) a(t)=sin(t)

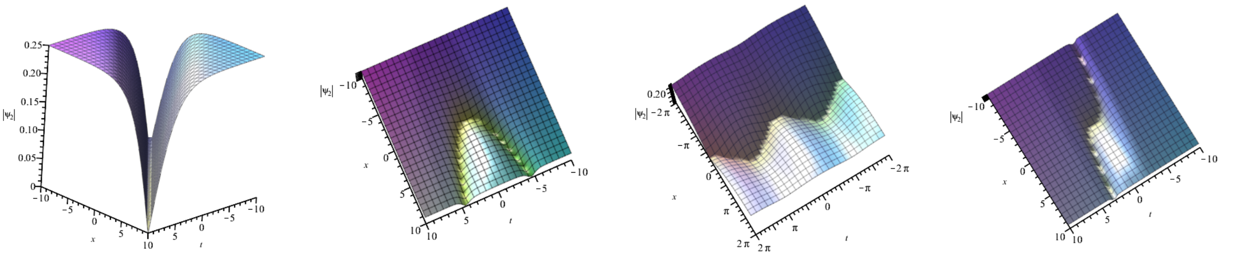
Figure 1: Evolution of dark-bright soliton solutions of Eqs.(27), plotted for different values of $a(t)$.

In fig.1, we shows the soliton solutions evolution of Eqs.(27) with different variable coefficients 1, t, t^2, sin(t). Fig.1 (a) and (e) depict the result with $a(t) = 1$, which shows a dark bell-shaped soliton and a singular soliton. When $a(t) = t$, we obtain a parabolic cubic soliton, as shown in (b) and (f). When $a(t) = t^2$, we obtain a cubic soliton,as shown in (c) and (g). Periodical-oscillating soliton is obtained when we choose $a(t) = sin(t)$ as depicted in (d) and (h).

In fig.2, we shows the soliton solution evolutionis of Eqs.(29) with different variable coefficients 1, t, t^2, sin(t). Fig.2 (a) and (e) depict the result with $a(t) = 1$, which shows a bright bell-shaped solition and a dark bell-shaped soliton. When $a(t) = t$, we obtain a parabolic cubic soliton, as shown in (b) and (f). When $a(t) = t^2$, we obtain a cubic soliton, as shown in (c) and (g). Periodical-oscillating soliton is obtained when we choose $a(t) = sin(t)$ as depicted in (d) and (h).

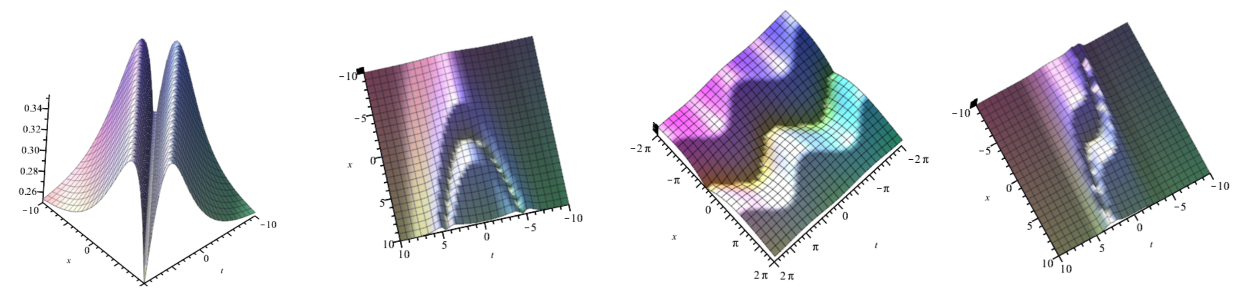


(a) a(t)=1 (b) a(t)=t (c) $a(t) = t^2$ (d) a(t)=sin(t)



(e) a(t)=1 (f) a(t)=t (g) $a(t) = t^2$ (h) a(t)=sin(t)

Figure 2: Evolution of bright-dark soliton solution of Eqs.(29), plotted for different values of $a(t)$.



(a) a(t)=1 (b) a(t)=t (c) $a(t) = t^2$ (d) a(t)=sin(t)

Figure 3: Evolution and interaction of Eqs.(29)with the same parameters as Fig.2

In fig.3, we can obtain the similar results. Fig.3 demonstrates that each soliton shape keeps invariant after interaction, which denotes that the interaction is elasticwe. We can see that the solitons show a periodic property but the solitons are not symmetrical in the t direction, and have the bell-shaped, parabolic, cubic or periodical-oscillating shapes.

5. Conclusion

In this paper, $Eqs.(1)$, the coupled nonlinear Schrödinger equation with variable coefficients, has been investigated. Via modified Sine-Gordon equation method,some dark-bright soliton solutions and dark-dark soliton soiutions have been obtained. Then, we have discussed the effects of $a(t)$, which is the group velocity dispersion. For bright–dark Soliton Solutions (27) and (29), we have chosen $a(t)$ as the constant, linear, quadratic and trigonometric functions, respectively, and the bell-shaped, parabolic, cubic and quasi-parabolic solitons have been obtained correspondingly, as shown in figs.1 and 2. We also investigated the evolution and interaction between the two solutions, and obtained that each solution shape keeps invariant after interaction and a periodic property in the t direction, as presented in fig.3. It is revealed that the method provides an authoritative mathematical instrument for solving nonlinear wave equations in mathematical physics and engineering problems.

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