HyersUlamRassias stability of some non-linear fractional integral equations using Bielecki metric

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Abstract: We apply the Bielecki metric on the space $\mathscr{C}([a,b])$, to analyze the different types of stabilities of non-linear fractional integral equation corresponding to fractional boundary value problems. Sufficient conditions are obtained to prove stability results for fractional non-linear Volterra and Fredholm integral equations, given by Ulam, Hyer and Rassias. We extend the respective stability results to the fractional integral equations where the domain of integration is an unbounded interval. We provide numerical examples which asserts our stability results.

Keywords: Bilecki metric, fractional integral equation, HyerUlamRassias stability

1 Introduction

The applications of the integral equations of fractional order is found in many discipline including biological science, physical and chemical sciences, aerodynamics, control theory and signal processing etc. The integral equations in the mathematical modeling of systems or process have been stimulated for extensive research in their respective disciplines. Authors Konjik et al. (2011); Valerjo et al. (2014) have studied some recent and specific pioneer applications of fractional calculus. Comparatively, the analysis of these new stability results are very less. To mention few, one can refer to Haihuva Wang (2017); Joan Hoffacker (2011).

The question arised by Ulam (1940) has become a genesis for stability problems of functional equations. Ulam's question can be phrased as "Under what conditions, the set of all additive mapping between metric groups is dense in the set of all approximate additive mappings between metric groups$''$. Hyers (1941) settled this question when the metric group is Banach space.

Let B_1, B_2 be two Banach spaces and $\epsilon > 0$. If $g : B_1 \to B_2$ satisfies

$$\|g(b+w) - g(b) - g(w)\| \leq \epsilon, \text{ for every } b, w \in B_1,$$

then we can find unique additive mapping $h : B_1 \to B_2$ such that

$$\|g(b) - h(b)\| \leq \epsilon, \text{ for all } b \in B_1,$$

where $h(b+w) = h(b) + h(w)$ for all $b, w \in B_1$.

This condition is called as Hyers-Ulam stability for functional equations. Rassias (1978) generalized it for linear mappings. These latest stability results for several functional equations are discussed by Hyers et al. (1998) and Jung (2001, 2004). Among the class of integral equations, Wei et al. (2012) used standard norm to provide sufficient conditions for these new stability results and Castro et al. (2018) used Bielcki metric to obtain such stability results.

In this paper, we give attention to the non-linear fractional Volterra integral equation with delay α, which are equivalent to the fractional boundary problem:

$$g(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t, s, g(s), g(\alpha(s)))ds - \frac{b}{a+b} \left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g(s), g(\alpha(s)))ds - c \right]. \tag{1}$$

Here $t \in \mathbb{J} := [0, T]$ for some $T > 0$, f is continuous on $\mathbb{J}^2 \times \mathbb{C}^2$, α is continuous delay function and a, b are the constants involved in the fractional boundary conditions. The solutions of above integral equation are solutions of fractional boundary value problems.

Lemma 1: *[Benchohra et al. (2008)] Let $0 < p < 1$, and f be continuous $\mathbb{J}^2 \times \mathbb{C}^2$. A function $g(t) \in \mathscr{C}(\mathbb{J}, \mathbb{C})$ is a solution of the fractional integral equation*

$$g(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t, s, g(s), g(\alpha(s)))ds - \frac{b}{a+b}\left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g(s), g(\alpha(s)))ds - c\right]$$

if and only if $g(t)$ is a solution the fractional boundary value problem

$$\begin{aligned} {}^{c}\mathbf{D}^{\alpha} g(t) &= f(t, g(t)), t \in \mathbb{J} \\ ag(0) + bg(T) &= c. \end{aligned}$$

2 preliminaries

The different types of stabilities given by Ulam, Hyer and Rassias can be defined to integral equations as follows.

Definition 2.1: Let $\omega(t)$ be an increasing function on $\mathbb{J}$. If an arbitrary function g satisfies

$$\left| g(t) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t, s, g(s), g(\alpha(s)))ds - \frac{b}{a+b}\left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g(s), g(\alpha(s)))ds - c\right]\right| \leq \omega(t)$$

for all $t \in \mathbb{J}$, there is a solution g_0 of (1) with

$$|g(t) - g_0(t)| \leq C\omega(t),$$

for all $t \in \mathbb{J}$, then we say that (1) possess HyersUlamRassias stability. Where C is positive constant and independent of g and g_0.

Definition 2.2: Let ϵ be a non negative number. Then (1) possess HyersUlam stability if an arbitrary function g satisfies

$$\left| g(t) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t, s, g(s), g(\alpha(s)))ds - \frac{b}{a+b}\left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g(s), g(\alpha(s)))ds - c\right]\right| \leq \epsilon$$

for all $t \in \mathbb{J}$, there is a solution g_0 of (1) and a positive constant C such that

$$|g(t) - g_0(t)| \leq C\epsilon,$$

for all $t \in \mathbb{J}$ (where C is independent of g and g_0).

Definition 2.3: Suppose $\omega(t)$ be a non decreasing function on $\mathbb{J}$ and $\epsilon \geq 0$. Then (1) is having semi-Hyers-Ulam stability if an arbitrary function g satisfies

$$\left| g(t) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t, s, g(s), g(\alpha(s)))ds - \frac{b}{a+b}\left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g(s), g(\alpha(s)))ds - c\right]\right| \leq \epsilon$$

for all $t \in \mathbb{J}$, there is a solution g_0 of (1) and a positive constant C independent of g and g_0 such that

$$|g(t) - g_0(t)| \leq C\omega(t),$$

for all $t \in \mathbb{J}$.

Let us recall the following well known concepts.

Definition 2.4: A generalized metric δ on a set A is defined as a function from $A \times A$ to $[0, \infty)$ satisfying following conditions.

1. $\delta(a, a') = 0$ if and only if $a = a'$;
2. $\delta(a, a') = \delta(a', a)$ for all $a, a' \in A$;
3. $\delta(a, a'') \leq \delta(a, a') + \delta(a', a'')$ for all $a, a', a'' \in A$.

Theorem 2: *Let (A, δ) be complete metric space and the operator $\mathscr{T}$ be contraction on A with a Lipschitz constant $L < 1$. If there is any non-negative integer n_0 such that $\delta(\mathscr{T}^{n_0+1}a, \mathscr{T}^{n_0}a) < \infty$ for some $a \in A$, then the following properties hold good:*

1. *the sequence of elements $(\mathscr{T}^n a), n = 1, 2, 3, \cdots$ in A converges to a unique fixed point a^* in $A^* = \{y \in A : \delta(\mathscr{T}^{n_0}a, y) < \infty\}$;*
2. *If $y \in A^*$, then*

$$\delta(y, a^*) \leq \frac{1}{1-L}\delta(\mathscr{T}y, y). \tag{2}$$

3 HyersUlamRassias Stability for fractional Volterra integral equations in the finite interval case using Bielecki metric

We are interested in the continuous functions space $\mathscr{C}(\mathbb{J}, \mathbb{C})$, equipped with a bielecki metric given by

$$\delta_p(g, \phi) = \sup_{x \in \mathbb{J}} \frac{|g(x) - \phi(x)|}{e^{px}}, \forall g, \phi \in \mathscr{C}(\mathbb{J}, \mathbb{C}). \tag{3}$$

In general,

$$\delta(g,\phi)=\sup_{x\in\mathbb{J}}\frac{|g(x)-\phi(x)|}{\omega(x)},\forall g,\phi\in\mathscr{C}(\mathbb{J},\mathbb{C}). \tag{4}$$

Where $\omega:[a,b]\to(0,\infty)$ is a nondecreasing and continuous function. The completeness of the space $\mathscr{C}(\mathbb{J},\mathbb{C})$ with respect to the Bielecki metric is given by Rolewicz (1987). We denote $\mathscr{C}_p(\mathbb{J},\mathbb{C})$ for the set of all continuous functions with respect to the metric (3).

Theorem 3: *Let $0<q<p<1$ and $p+q=1$. Let $K=\frac{1}{\Gamma(p)}\frac{1-q}{p-q}T^{p-q}$ for some $T>0$. Let α be a continuous delay functions on $\mathbb{J}$ satisfies $\alpha(x)\le x$. Let L and η be any constants with $KL\eta\left(1+\frac{b}{a+b}\right)<1$ and $\omega:\mathbb{J}\to(0,\infty)$ be any increaing function with the property that*

$$\left(\int_a^x(\omega(t))^{\frac{1}{p}}\,dt\right)^p\le\eta\omega(x),\ \text{for all } x\in\mathbb{J}. \tag{5}$$

If $f:\mathbb{J}^2\times\mathbb{C}^2\to\mathbb{C}$ is continuous and satisfies the Lipschitz condition

$$\begin{aligned}&|f(x,t,g(t),g(\alpha(t))-f(x,t,h(t),h(\alpha(t))|\\&\le L|g(t)-h(t)|,\forall x,t\in\mathbb{J},\end{aligned} \tag{6}$$

then the following fractional integral equation

$$g(t)=\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds-\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int_0^T(T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds-c\right]$$

has the HyersUlamRassias stability. That is, if $g\in\mathscr{C}(\mathbb{J},\mathbb{C})$ is such that

$$\left|g(t)-\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds+\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int_0^T(T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds-c\right]\right|\le\omega(t), \tag{7}$$

for every $t\in\mathbb{J}$ and $KL\eta\left(1+\frac{b\omega(T)/\omega(0)}{a+b}\right)<1$, then there is unique solution $g_0\in\mathscr{C}(\mathbb{J},\mathbb{C})$ such that

$$g_0(t)=\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}f(t,s,g_0(s),g_0(\alpha(s)))ds-\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int_0^T(T-s)^{p-1}f(T,s,g_0(s),g_0(\alpha(s)))ds-c\right]$$

and

$$|g(x)-g_0(x)|\le\frac{1}{1-KL\eta\left(1+\frac{b\omega(T)/\omega(0)}{a+b}\right)}\omega(x), \tag{8}$$

for all $x\in\mathbb{J}$.

Proof: We define the map $\mathscr{T}:\mathscr{C}(\mathbb{J},\mathbb{C})\to\mathscr{C}(\mathbb{J},\mathbb{C})$ by

$$(\mathscr{T}g)(t)=\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds-\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int_0^T(T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds-c\right],$$

for all $g\in\mathscr{C}(\mathbb{J},\mathbb{C})$. Our claim is to show that $\mathscr{T}$ is contraction with respect to the Bielecki metric. We take $g,h\in\mathscr{C}(\mathbb{J},\mathbb{C})$. Using Holder's inequality, we have

$$\begin{aligned}
&\delta(\mathscr{T}g,\mathscr{T}h)\\
&=\sup_{t\in\mathbb{J}}\frac{|(\mathscr{T}g)(t)-(\mathscr{T}h)(t)|}{\omega(t)}\\
&=\sup_{t\in\mathbb{J}}\frac{1}{\omega(t)}\left|\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds\right.\\
&\quad-\frac{b}{a+b}\frac{1}{\Gamma(p)}\int_0^T(T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds\\
&\quad+\frac{bc}{a+b}-\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}f(t,s,h(s),h(\alpha(s)))ds\\
&\quad+\frac{b}{a+b}\frac{1}{\Gamma(p)}\int_0^T(T-s)^{p-1}f(T,s,h(s),h(\alpha(s)))ds\\
&\quad\left.-\frac{bc}{a+b}\right|\\
&\le\sup_{t\in\mathbb{J}}\frac{1}{\omega(t)}\frac{1}{\Gamma(p)}\left\{\int_0^t(t-s)^{p-1}\right.\\
&\quad|f(t,s,g(s),g(\alpha(s)))-f(t,s,h(s),h(\alpha(s)))|\,ds\\
&\quad+\frac{b}{a+b}\int_0^T(T-s)^{p-1}\\
&\quad\left.|f(T,s,g(s),g(\alpha(s)))-f(T,s,h(s),h(\alpha(s)))|\,ds\right\}\\
&\le\sup_{t\in\mathbb{J}}\frac{1}{\omega(t)}\frac{L}{\Gamma(p)}\left\{\int_0^t(t-s)^{p-1}\,|g(s)-h(s)|\,ds\right.\\
&\quad\left.+\frac{b}{a+b}\int_0^T(T-s)^{p-1}\,|g(s)-h(s)|\,ds\right\}\\
&=\frac{L}{\Gamma(p)}\sup_{t\in\mathbb{J}}\frac{1}{\omega(t)}\left\{\int_0^t(t-s)^{p-1}w(s)\frac{|g(s)-h(s)|}{w(s)}ds\right.
\end{aligned}$$

$$+\frac{b}{a+b}\int_0^T (T-s)^{p-1}w(s)\frac{|g(s)-h(s)|}{w(s)}ds\Bigg\}$$
$$\le \frac{L\delta(g,h)}{\Gamma(p)}\sup_{t\in\mathbb{J}}\frac{1}{\omega(t)}\Bigg\{\int_0^t (t-s)^{p-1}w(s)ds$$
$$+\frac{b}{a+b}\int_0^T (T-s)^{p-1}w(s)ds\Bigg\}$$
$$\le \frac{L\delta(g,h)}{\Gamma(p)}\sup_{t\in\mathbb{J}}\frac{1}{\omega(t)}\Bigg\{\left(\int_0^t (t-s)^{\frac{p-1}{q}}\right)^q$$
$$\left(\int_0^t |\omega(s)|^{\frac{1}{p}}ds\right)^p+\frac{b}{a+b}\left(\int_0^T (T-s)^{\frac{p-1}{q}}\right)^q$$
$$\left(\int_0^T |\omega(s)|^{\frac{1}{p}}ds\right)^p\Bigg\}$$
$$\le L\delta(g,h)K\sup_{t\in\mathbb{J}}\frac{1}{\omega(t)}\Bigg\{\left(\int_0^t |\omega(s)|^{\frac{1}{p}}ds\right)^p+\frac{b}{a+b}$$
$$\left(\int_0^T |\omega(s)|^{\frac{1}{p}}ds\right)^p\Bigg\}$$
$$\le L\delta(g,h)K\left\{\sup_{t\in\mathbb{J}}\eta\frac{\omega(t)}{\omega(t)}+\frac{b}{a+b}\sup_{t\in\mathbb{J}}\eta\frac{\omega(T)}{\omega(0)}\right\}$$
$$\le LK\eta\left(1+\frac{b\omega(T)/\omega(0)}{a+b}\right)\delta(g,h).$$

Thus, by hypothesis and Theorem 2, we have the HyerUlam stability for (1). If we again apply the Theorem 2, we get that

$$\delta(g,g_0)\le\frac{1}{1-KL\eta\left(1+\frac{b\omega(T)/\omega(0)}{a+b}\right)}\delta(\mathscr{T}g,g).$$

Using the metric δ and (7),

$$\sup_{x\in\mathbb{J}}\frac{|g(x)-g_0(x)|}{\omega(x)}\le\frac{1}{1-KL\eta\left(1+\frac{b\omega(T)/\omega(0)}{a+b}\right)}.$$

By supremum property (8) follows.

Corrolary 4: *Let $0<q<p<1$ and $p+q=1$. Let $T>0$ and $K=\frac{1}{\Gamma(p)}\frac{1-q}{p-q}T^{p-q}$. Let α be a continuous delay functions satisfies $\alpha(x)\le x$ for all $x\in\mathbb{J}$. If $f:\mathbb{J}^2\times\mathbb{C}^2\to\mathbb{C}$ is a continuous Lipschitz function with Lipschitz constant $L>0$ satisfying (6) and $g\in\mathscr{C}(\mathbb{J},\mathbb{C})$ is such that*

$$\Bigg|g(t)-\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds-$$
$$\frac{b}{a+b}\Bigg[\frac{1}{\Gamma(p)}\int_0^T (T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds$$
$$-c\Bigg]\Bigg|\le e^{pt},$$

for all $t\in\mathbb{J}$ and $KL\left(e^T-1\right)^p\left(1+\frac{be^T}{a+b}\right)<1$, then there is unique solution g_0 of (1) such that

$$g_0(t)=\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(t,s,g_0(s),g_0(\alpha(s)))ds-$$
$$\frac{b}{a+b}\Bigg[\frac{1}{\Gamma(p)}\int_0^T (T-s)^{p-1}f(T,s,g_0(s),g_0(\alpha(s)))ds$$
$$-c\Bigg]$$

and

$$|g(t)-g_0(t)|\le\frac{e^{pt}}{1-KL\left(e^T-1\right)^p\left(1+\frac{be^T}{a+b}\right)},$$

for all $t\in\mathbb{J}$.

Proof: If $\omega(t)=e^{pt}$ and $\eta\ge\left(e^T-1\right)^p$, then above result follows from the inequality

$$\left(\int_0^t \left(e^{p\tau}\right)^{\frac{1}{p}}d\tau\right)^p\le\eta e^t,\forall t\in\mathbb{J}.$$

In Theorem 3, the sufficient condition (5) can be replaced as follows.

Theorem 5: *Let $0<p<1$ Let α be a continuous delay functions on $\mathbb{J}$ satisfies $\alpha(x)\le x$. Let L and η be any constants with $L\eta\left(1+\frac{b}{a+b}\right)<1$ and $\omega:\mathbb{J}\to(0,\infty)$ be a non-decreasing function with the property that*

$$\left|\frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1}\omega(t)dt\right|\le\eta\omega(x), \tag{9}$$

for all $x\in\mathbb{J}$. If $f:\mathbb{J}^2\times\mathbb{C}^2\to\mathbb{C}$ continuous with the Lipschitz condition

$$|f(x,t,g(t),g(\alpha(t))-f(x,t,h(t),h(\alpha(t))|$$
$$\le L|g(t)-h(t)|, \tag{10}$$

for all $x,t\in\mathbb{J}$, then the HyersUlamRassias stability for the integral equation (1) is obtained. That is, if $g\in\mathscr{C}(\mathbb{J},\mathbb{C})$ is such that

$$\Bigg|g(t)-\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds+$$
$$\frac{b}{a+b}\Bigg[\frac{1}{\Gamma(p)}\int_0^T (T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds$$
$$-c\Bigg]\Bigg|\le\omega(t), \tag{11}$$

for every $t \in \mathbb{J}$ *and* $L\eta\left(1+\frac{b\omega(T)/\omega(0)}{a+b}\right)<1$, *then there is unique solution* $g_0 \in \mathscr{C}(\mathbb{J},\mathbb{C})$ *such that*

$$g_0(t)=\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(t,s,g_0(s),g_0(\alpha(s)))ds$$
$$-\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int_0^T (T-s)^{p-1}f(T,s,g_0(s),g_0(\alpha(s)))ds\right.$$
$$\left.-c\right]$$

and

$$|g(x)-g_0(x)|\leq \frac{1}{1-L\eta\left(1+\frac{b\omega(T)/\omega(0)}{a+b}\right)}\omega(x), \tag{12}$$

for all $x\in\mathbb{J}$.

4 HyersUlam and Semi HyersUlam stabilities for fractional integral equations with delay in the finite interval case using Bielecki metric

We provide sufficient conditions to obtain Semi HyersUlam and HyersUlam stabilities for (1). We proceed with metrics defined in (3) and (4).

Theorem 6: *Let* $0<q<p<1$ *and* $p+q=1$. *Let* $K=\frac{1}{\Gamma(p)}\frac{1-q}{p-q}T^{p-q}$ *for* $T>0$. *Let* $\omega(t)=e^{pt}$ *and* α *be a continuous delay function with* $\alpha(t)\leq t$ *for all* $t\in\mathbb{J}$. *Suppose* $f:\mathbb{J}^2\times\mathbb{C}^2\to\mathbb{C}$ *is a continuous Lipschitz function with the positive Lipschitz constant L and*

$$|f(t,s,g(s),g(\alpha(s))-f(t,s,h(s),h(\alpha(s))|$$
$$\leq L|g(t)-h(t)| \tag{13}$$

If $g\in\mathscr{C}_p(\mathbb{J})$ *is such that*

$$\left|g(t)-\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds\right.$$
$$+\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int_0^T (T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds\right.$$
$$\left.\left.-c\right]\right|\leq\epsilon, \forall t\in\mathbb{J}, \tag{14}$$

where $\epsilon>0$ *and* $KL(e^T-1)^p\left(1+\frac{be^T}{a+b}\right)<1$, *then there is unique solution* $g_0\in\mathscr{C}(\mathbb{J},\mathbb{C})$ *such that*

$$g_0(t)=\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(t,s,g_0(s),g_0(\alpha(s)))ds-$$
$$\frac{b}{a+b}\left(\frac{1}{\Gamma(p)}\int_0^T (T-s)^{p-1}f(T,s,g_0(s),g_0(\alpha(s)))ds\right.$$
$$\left.-c\right).$$

Moreover

$$|g(t)-g_0(t)|\leq\frac{(a+b)\epsilon}{a+b-KL(e^T-1)^p(a+b+be^T)}e^{pt}$$

for all $t\in\mathbb{J}$. *That is above sufficient conditions give the semi HyersUlam stability for (1).*

Proof: We define the map $\mathscr{T}:\mathscr{C}_p(\mathbb{J})\to\mathscr{C}_p(\mathbb{J})$ by

$$(\mathscr{T}g)(t)=\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds$$
$$-\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int_0^T (T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds\right.$$
$$\left.-c\right],$$

for all $g\in\mathscr{C}(\mathbb{J},\mathbb{C})$. By applying the same process as in the above theorem, $\mathscr{T}$ becomes contractive with respect to the extended Bielecki metric (3) provided $KL\left(e^T-1\right)^p\left(1+\frac{be^T}{a+b}\right)<1$. Hence, the semi-Hyers-Ulam stability for the integral equation (1) is ensured by the Banach fixed point theorem.

Corrolary 7: *Let* $0<q<p<1$ *and* $p+q=1$. *Let* $K=\frac{1}{\Gamma(p)}\frac{1-q}{p-q}T^{p-q}$ *for some* $T>0$. *Let* α *be a continuous delay functions satisfies* $\alpha(t)\leq t$ *for all* $t\in\mathbb{J}$. *Suppose* $f:\mathbb{J}^2\times\mathbb{C}^2\to\mathbb{C}$ *is a continuous function with the Lipschitz constant* $L>0$ *satisfy (13). If* $y\in\mathscr{C}_p(\mathbb{J})$ *satisfies (14), with* $\epsilon>0$ *and* $KL\left(e^T-1\right)^p\left(1+\frac{be^T}{a+b}\right)<1$, *then there is unique solution* $g_0\in\mathscr{C}_p(\mathbb{J})$ *inwhich*

$$g_0(t)=\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(t,s,g_0(s),g_0(\alpha(s)))ds$$
$$-\frac{b}{a+b}\left(\frac{1}{\Gamma(p)}\int_0^T (T-s)^{p-1}f(T,s,g_0(s),g_0(\alpha(s)))ds\right.$$
$$\left.-c\right),$$

and

$$|g(t)-g_0(t)|\leq\frac{(a+b)e^T}{a+b-KL\left(e^T-1\right)^p\left(a+b+be^T\right)}\epsilon$$

for all $t\in\mathbb{J}$.

Thus the HyersUlam stability under above conditions for (1) is achieved. Next, we shall consider the generalized metric (4).

Theorem 8: *Let* $0<q<p<1$ *and* $p+q=1$. *Let* $K=\frac{1}{\Gamma(p)}\frac{1-q}{p-q}T^{p-q}$ *for some* $T>0$. *Assume the delay function* α *be continuous with* $\alpha(t)\leq t$ *for all* $t\in\mathbb{J}$ *and* $\omega:\mathbb{J}\to(0,\infty)$ *be an increasing function. Suppose that there is* $\eta\in\mathbb{R}$ *such that*

$$\left(\int_a^t(\omega(s))^{\frac{1}{p}}ds\right)^p\leq\eta\omega(t),\ \textit{for all } t\in\mathbb{J}. \tag{15}$$

If $f : \mathbb{J}^2 \times \mathbb{C}^2 \to \mathbb{C}$ is a continuous Lipschitz function such that

$$|f(t, s, g(s), g(\alpha(s)) - f(t, s, h(s), h(\alpha(s))| \\ \leq L|g(t) - h(t)|, \text{ for all } t \in \mathbb{J} \tag{16}$$

with $L > 0$. If $g \in \mathscr{C}(\mathbb{J}, \mathbb{C})$ is such that

$$\left| g(t) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t, s, g(s), g(\alpha(s)))ds + \frac{b}{a+b} \left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g(s), g(\alpha(s)))ds - c \right] \right| \leq \epsilon, \tag{17}$$

for all $t \in \mathbb{J}$, where $\epsilon > 0$ and $KL\eta \left(1 + \frac{b}{a+b}\right) < 1$, then there is unique solution $g_0 \in \mathscr{C}_p(\mathbb{J})$ such that

$$g_0(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t, s, g_0(s), g_0(\alpha(s)))ds - \frac{b}{a+b} \left(\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g_0(s), g_0(\alpha(s)))ds - c \right).$$

Moreover,

$$|g(x) - g_0(x)| \leq \frac{(a+b)\epsilon}{a + b - KL\eta(a + b + b\omega(T)/\omega(0))} \omega(x)$$

for all $x \in \mathbb{J}$. That is, semi-Hyers-Ulam stability for (1) is obtained.

Proof: Define $\mathscr{T} : \mathscr{C}(\mathbb{J}, \mathbb{C}) \to \mathscr{C}(\mathbb{J}, \mathbb{C})$ by

$$(\mathscr{T}u)(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t, s, g(s), g(\alpha(s)))ds - \frac{b}{a+b} \left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g(s), g(\alpha(s)))ds - c \right],$$

for all $g \in \mathscr{C}(\mathbb{J}, \mathbb{C})$, for all $t \in \mathbb{J}$.

We can prove the contractive property of $\mathscr{T}$ with respect to the generalized Bielecki mentric by applying the same process used in the above theorem and the fact that $KL\eta \left(1 + \frac{b\omega(T)/\omega(0)}{a+b}\right) < 1$. As $\mathscr{T}$ becomes contraction, the Banach fixed point theorem gives the desired result.

Corrolary 9: *Let $0 < q < p < 1$ and $p + q = 1$. Let $K = \frac{1}{\Gamma(p)} \frac{1-q}{p-q} T^{p-q}$ with $T > 0$. Let ω be non decreasing function $\mathbb{J}$ with the condition (15) and assume the delay function α be continuous with $\alpha(t) \leq t$ for all $t \in \mathbb{J}$. Suppose $f : \mathbb{J}^2 \times \mathbb{C}^2 \to \mathbb{C}$ (16) is a Lipschitz function with the Lipschitz constant with $L > 0$. If $g \in \mathscr{C}(\mathbb{J}, \mathbb{C})$ satisfies (17), with $\epsilon > 0$ and $KL\eta \left(1 + \frac{b\omega(T)/\omega(0)}{a+b}\right) < 1$, then there is unique solution g_0 such that*

$$g_0(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t, s, g_0(s), g_0(\alpha(s)))ds - \frac{b}{a+b} \left(\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g_0(s), g_0(\alpha(s)))ds - c \right),$$

and

$$|g(x) - g_0(x)| \leq \frac{(a+b)\omega(T)/\omega(0)}{a + b - KL\eta(a + b + b\omega(T)/\omega(0))} \epsilon$$

for all $x \in \mathbb{J}$. This implies that (1) has Hyers-Ulam stability.

Remark 1: In a similar way, we can discuss above stability results to obtain sufficient conditions for the fractional integral equation of the form,

$$g(t) = \frac{1}{\Gamma(p)} \int_0^T (t-s)^{p-1} f(t, s, g(s), g(\alpha(s)))ds - \frac{b}{a+b} \left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T, s, g(s), g(\alpha(s)))ds - c \right], \forall t \in \mathbb{J}. \tag{18}$$

Here both terms are considered to be Fredholm, f is continuous on $\mathbb{J}^2 \times \mathbb{C}^2$ and α is a delay function. Since $t \in \mathbb{J}$, then

$$\left(\int_0^t (t-s)^{\frac{p-1}{q}} \right)^q \leq \left(\int_0^T (t-s)^{\frac{p-1}{q}} \right)^q.$$

Hence, all the above stability results will be true for the Fredholm integral equations of fractional order(18) if the sufficient conditions are unaltered.

5 HyersUlamRassias stability for the integral equation of fractional order in the infinite interval case using Bielecki metric

In this section, we shall analyze the HyersUlamRassias stability for the given fractional integral equation (1), when the interval is unbounded. Here, we allow the parameter t from 0 to ∞. In this regard, we consider the integral equations of the form,

$$g(t)=\frac{1}{\Gamma(p)}\int\limits_0^t (t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds$$
$$-\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int\limits_0^T (T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds\right.$$
$$\left.-c\right],$$

for all $t\in[0,\infty)$. We consider the Bielecki metric over unbounded interal defined by

$$\delta_B(g,\phi)=\sup_{t\in[0,\infty)}\frac{|g(t)-\phi(t)|}{\omega(t)},$$

where $\omega:[0,\infty)\to(\mu,\nu)$ (where $\mu,\nu>0$) is an increasing bounded continuous function. We have $\mathscr{C}(\mathbb{J},\mathbb{C})$ is complete with respect to δ_B. One can see Cadariu et al. (2012); Rolewicz (1987).

Theorem 10: *Let $0<p<1$. Let $\alpha:[0,\infty)\to[0,\infty)$ be a continuous delay functions with $\alpha(x)\le x,\forall x\in[0,\infty)$ and $\omega:[0,\infty)\to(\mu,\nu),\mu,\nu>0$ be a non-decreasing bounded continuous function. Choose $\eta\in\mathbb{R}$ with*

$$\left|\frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1}\omega(t)dt\right|\le\eta\omega(x),\tag{19}$$

for all $x\in[0,\infty)$. If $f:[0,\infty)^2\times\mathbb{C}^2\to\mathbb{C}$ is a continuous Lipschitz function with Lipschitz constant $L>0$ or

$$|f(x,t,g(t),g(\beta(t))-f(x,t,h(t),h(\beta(t))|$$
$$\le L|g(t)-h(t)|\tag{20}$$

then the integral equation

$$g(t)=\frac{1}{\Gamma(p)}\int\limits_0^t (t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds$$
$$-\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int\limits_0^T (T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds\right.$$
$$\left.-c\right]\tag{21}$$

for all $t\in[0,\infty)$, has the HyersUlamRassias stability. That is, if $y\in\mathscr{C}_B([0,\infty))$ is such that

$$\left|g(t)-\frac{1}{\Gamma(p)}\int\limits_0^t (t-s)^{p-1}f(t,s,g(s),g(\alpha(s)))ds\right.$$
$$+\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int\limits_0^T (T-s)^{p-1}f(T,s,g(s),g(\alpha(s)))ds\right.$$
$$\left.\left.-c\right]\right|\le\omega(t),$$

for every $t\in[0,\infty)$ and $L\eta\left(1+\frac{b\nu/\omega(0)}{a+b}\right)<1$, then there is unique solution g_0 in $\mathscr{C}_B(\mathbb{R})$ such that

$$g_0(t)=\frac{1}{\Gamma(p)}\int\limits_0^t (t-s)^{p-1}f(t,s,g_0(s),g_0(\alpha(s)))ds$$
$$-\frac{b}{a+b}\left(\frac{1}{\Gamma(p)}\int\limits_0^T (T-s)^{p-1}f(T,s,g_0(s),g_0(\alpha(s)))ds\right.$$
$$\left.-c\right)\tag{22}$$

and

$$|g(x)-g_0(x)|\le\frac{1}{1-L\eta\left(1+\frac{b\nu/\omega(0)}{a+b}\right)}\omega(x),\tag{23}$$

for all $x\in[0,\infty)$. That is the Volterra integral equation of fractional order(21) has the HyersUlamRassias stability.

Proof: Define $I_n=[0,0+n]$ for each $n\in\mathbb{N}$. By the Theorem 5, we can find unique bounded continuous function $g_{0,n}:I_n\to\mathbb{C}$ such that

$$g_{0,n}(t)=\frac{1}{\Gamma(p)}\int\limits_0^t (t-s)^{p-1}f(t,s,g_{0,n}(s),g_{0,n}(\alpha(s)))ds$$
$$-\frac{b}{a+b}\left[\frac{1}{\Gamma(p)}\int\limits_0^T (T-s)^{p-1}f(T,s,g_{0,n}(s),g_{0,n}(\alpha(s)))ds\right.$$
$$\left.-c\right],\tag{24}$$

and

$$|g(t)-g_{0,n}(t)|\le\frac{1}{1-L\eta\left(1+\frac{b\omega(n)/\omega(0)}{a+b}\right)}\omega(t),\tag{25}$$

for each $t\in I_n$. Since $I_n\subset I_{n+1}$ and the uniqueness of $g_{0,n}$ implies that

$$g_{0,n}(t)=g_{0,n+j}(t)\text{ for each }j=1,2,3,\cdots.\tag{26}$$

For each $t\in[0,\infty)$, let $n_t=\min\{n\in\mathbb{N}/t\in I_n\}$ and define

$$g_0(t)=g_{0,n_t}(t).\tag{27}$$

Let us prove that, this g_0 is continuous on $[0,\infty)$. For any $t_\alpha\in[0,\infty)$, Choose an integer $n_\alpha=n_{t_\alpha}$. Then t_α lies in the interior of $I_{n_\alpha+1}$. We can find $\epsilon>0$ such that $g_0(t)=g_{0,n_\alpha+1}(t)$ for all $t\in(t_\alpha-\epsilon,t_\alpha+\epsilon)$. Thus, continuity of g_0 follows from continuity of $g_{n_\alpha+1}$.

Now, we prove that g_0 satisfies the Volterra fractional integral equation (21) and the inequality (25). For, Choose an arbitrary $t\in[0,\infty)$ and n_t so that $t\in I_{n_t}$. By (24) and (27),we get

$$g_0(t)$$
$$=g_{0,n_t}(t)$$

$$= \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t,s,g_{0,n_t}(s), g_{0,n_t}(\alpha(s)))ds - \frac{b}{a+b}$$
$$\left(\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T,s,g_{0,n_t}(s), g_{0,n_t}(\alpha(s)))ds \right.$$
$$\left. -c \right)$$
$$= \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(t,s,g_0(s), g_0(\alpha(s)))ds$$
$$- \frac{b}{a+b} \left(\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} f(T,s,g_0(s), g_0(\alpha(s)))ds \right.$$
$$\left. -c \right).$$

Thus we obtain the last equality in above equation by $n_x \leq n_t$ and $n_{\alpha(x)} \leq n_t$, for any $x \in I_{n_t}$. Using (26) and (27), we have

$$g_{0,n_t}(x) = g_{0,n_x}(x) = g_0(x)$$

and

$$g_{0,n_t}(\alpha(x)) = g_{0,n_x}(\alpha(x)) = g_0(\alpha(x)).$$

Also, $t \in I_{n_t}$ for every $t \in [0,\infty)$. Using (27) and boundedness of ω, we have

$$|g(t) - g_0(t)| = |g(t) - g_{0,n_t}(t)| \leq \frac{1}{1 - L\eta \left(1 + \frac{b\nu/\omega(0)}{a+b}\right)} \omega(t),$$

for each $t \in [0,\infty)$. To prove the uniqueness, let h_0 be any other function which satisfies (22) and (23) for all $t \in [0,\infty)$. For any arbitrary $t \in [0,\infty)$, the restriction of g_0 and h_0 on I_{n_t} satisfy (22) and (23) for all $t \in [0,\infty)$. By the uniqueness of the solution of $g_{n_t} = g_0|_{I_{n_t}}$ gives that

$$\begin{aligned} g_0(t) &= g_0|_{I_{n_t}}(t) \\ &= h_0|_{I_{n_t}}(t) \\ &= h_0(t). \end{aligned}$$

Thus the proof is completed.

6 Example

Here, we show few examples where above stability results are possible.

Example 11: *Set* $T = -\frac{p}{q} ln\left(\frac{p}{p+q}\right)$, $0 < q < p < 1$, $K = \frac{1}{\Gamma(p)} \frac{1-q}{p-q} T^{p-q}$. *Let* $Z(s,t)$ *be a polynomial in* s *and* t, $\alpha(x) = sinx, \omega(t) = e^{-pt}$, *Let* $L = \min\{\frac{T}{T+1}, \frac{1}{K}\}$. *Assume the function* f *with the Lipschitz constant* L *as follows:*

$$f(t,s,g(s),g(\alpha(s)) = Z(s,t) + L(g(s) + g(sins)).$$

If $g(t)$ *is a continuous function on* $\mathbb{J}$ *with*

$$\left| g(t) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \right.$$
$$(Z(s,t) + L(g(s) + g(sins)))\, ds$$
$$+ \frac{b}{a+b} \left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} \right.$$
$$(P(s,T) + L(g(s) + g(sins)))\, ds - c]| \leq e^{-pt},$$

for all $t \in \mathbb{J}$. *Also, we have,*

$$\left(\int_0^x (\omega(t))^{\frac{1}{p}}\, dt \right)^p = \left(\int_0^x e^{\frac{-pt}{q}}\, dt \right)^q \leq e^{-px},$$

for all $x \in \mathbb{J}$. *Here* $\eta = 1$. *Therefore, by applying Theorem 3, there is unique solution* g_0 *of (1) satisfying*

$$|g(t) - g_0(t)| \leq \frac{1}{1 - L\left(1 + \frac{be^{-pT}}{a+b}\right)} e^{-pt},$$

for all $t \in \mathbb{J}$.

Example 12: *Let* $I = [0,\infty)$ *and* $T \in I$. *Assume that* $Q(s,t)$ *is any polynomial* s *and* t, *and* y *is a continuous functions on* I, *with*

$$\left| g(t) - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left(Q(s,t) + L(g(s) + g(3^s))\right) ds \right.$$
$$+ \frac{b}{a+b} \left[\frac{1}{\Gamma(p)} \int_0^T (T-s)^{p-1} \right.$$
$$(Q(s,T) + L(g(s) + g(3^s)))\, ds - c]| \leq e^{-pt},$$

for all $t \in I$. *We take* $f = (Q(s,t) + L(g(s) + g(3^s)))$ *with Lipschitz constant* $L > 0$, $\omega(t) = e^{-pt}$. *Then*

$$\left| \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} \omega(t) dt \right| \leq \eta(p) e^{-px}, \text{ for all } x \in [0,\infty).$$

provided $\eta(p) = \left(\int_0^\infty a^{p-1} e^{pa} da\right)$. *Therefore, by applying Theorem 10, there is unique solution* g_0 *of (21) satisfying*

$$|g(x) - g_0(x)| \leq \frac{1}{1 - L\eta(p)\left(1 + \frac{b}{a+b}\right)} \omega(x),$$

for all $x \in [0,\infty)$.

7 Conclusion

We obtained some new stability results for the class of fractional integral equations having the boundary condition in its corresponding differential equation. Novel stability ideas are used to obtain sufficient conditions. Using Banach fixed point theorem, we have proved the HyersUlamRassias, Hyers-Ulam and semi Hyers Ulam stabilites for system of fractional non-linear integral equations satisfying the given boundary condition with delay on a closed and bounded interval. Further, we extended same results to unbounded intervals. Our results show that, there is a close analytic solution to the system which are stable in the sense of HyersUlam and HyersUlamRassias. In future work, these new concepts of stability results and Bielecki metric could be used to stabilize the system of impulsive fractional integral equations.

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