Received 6 February 2020; | Revised XX XX 2020; | Accepted XX XX 2020

DOI: xxx/xxxx

ARTICLE TYPE

Asymptotic Behavior of Discrete Kuramoto Model on Graphs

Chun-Bo Lian | Bin Ge* | Kai-Xin Xing

School of Mathematical Sciences, Harbin Engineering University, Harbin, P.R. China

Correspondence
*Ge Bin, School of Mathematical Sciences, Harbin Engineering University, Harbin, 150001, P.R. China. Email: gebin04523080261@163.com

In this paper, we study the asymptotic behavior of the discrete Kuramoto model on graphs. The main research method is: by using the theory of graph limits, we rigorously justify that the solutions of the initial value problems (IVPs) for the discrete Kuramoto model with external drive convergence to the solution of the initial value problem for its continuum limit on deterministic graphs, W-random graphs and SW graphs.

KEYWORDS:
Kuramoto model, Asymptotic behavior, Graph limit, Deterministic graphs, Random graphs, Continuum limit

1 | INTRODUCTION

1.1 | General background

Recently, the complex networks had been widely considered which could be applied in diverse disciplines such as molecular biology [1], ecology, neuroscience [2,3], nonlinear dynamics, physics [4,5,6], and sociology [7] etc.. Network dynamic behavior research is a hot field of complex network research. Each node in a complex network has its own dynamic behavior, if each node in the network represents a dynamic system and there is a connection between nodes, it means that the dynamic system described by two nodes has mutual coupling effect, and such a network is called a dynamic network. Network structure has an important influence on network dynamic behavior, at present, the most studied network structures include ER random network model, NW small world network model and scale-free network model.

The node oscillator used in this paper is Kuramoto model. In the model, each oscillator represents a phase oscillator making sinusoidal motion, there is a weak coupling between the oscillators, and the phase of each oscillator is affected by the phase of adjacent oscillators. The Kuramoto model [8] was first proposed by Yoshiki Kuramoto in 1975. Since its introduction in 1975, the Kuramoto model has imposed itself as a standard mathematical model to describe the large variety of synchronization phenomena encountered in natural and human-made systems. For the discrete Kuramoto model, many scholars have studied the synchronization of its, but there is little research on the asymptotic behavior of the discrete Kuramoto model. Discrete behavior can be seen almost everywhere in real life, and the research on discrete model has more important practical significance. Therefore, it is a challenge to study the asymptotic behavior of the discrete Kuramoto model in the discrete case under several different network structures.

1.2 | The discrete Kuremoto model with external drive

In this part, we introduce the discrete Kuremoto model with external drive in this paper.

Let $G_n = \langle V(G_n), E(G_n)\rangle$ be an undirected graph on n nodes, $V(G_n) = [n]$ and $|E(G_n)| = \mathrm{O}(|V(G_n)|^2)$ stand for the sets of nodes and edges of G_n respectively, where $|\cdot|$ denotes the cardinality of a set.

The discrete Kuramoto model with external drive on G_n is given by the following equations:

$$\begin{cases} \dfrac{d}{dt}u_{ni}(t) = \omega + \dfrac{1}{n} \displaystyle\sum_{j:(i,j)\in E(G_n)} a_{nij} \sin(u_{nj} - u_{ni}) + h\sin(u_{ni}), \\ u_{ni}(0) = g(x_i), i \in [n], \end{cases} \tag{1}$$

where u_{ni} is the phase of ith oscillator, $u_n(t) = (u_{n1}(t), u_{n2}(t), \cdots u_{nn}(t))$ is a step function on I. Here and below, I denotes $[0,1]$, ω is the intrinsic frequency, n is the number of oscillators, a_{nij} stands for the $n \times n$ adjacency matrix of the graph G_n. If G_n is a weighted graphs,

$$a_{nij} = \begin{cases} \varpi_{nij}, & (i,j) \in E(G_n), \\ 0, & \text{otherwise.} \end{cases}$$

On the contrary,

$$a_{nij} = \begin{cases} 1, & (i,j) \in E(G_n), \\ 0, & \text{otherwise,} \end{cases}$$

$h\sin(u_{ni})$ are external driving function, h is the strength of the external force driving $h\sin(u_{ni})$, $h > 0$, g is a bounded measurable function on I. The sum on the right-hand side of (1) models the nonlinear diffusion across edges of G_n.

1.3 | The continuum limit of the discrete Kuromoto model

We are interested in the dynamical behavior of the discrete Kuramoto model oscillators with a large of oscillators. In order to study the solution of the discrete Kuramoto model as $n \to \infty$, we will use its continuous limit.

Let $W \in \mathcal{W}_0$, a class of symmetric measurable functions, W represents the continuous counterparts of the adjacency matrix a_{nij} of graphs $\{G_n\}$ in the large n limit (cf.[9]). This function is called graph. After interpreting the right-hand side of (1) as a Riemann sum and sending $n \to \infty$, the continuum limit of the discrete Kuromoto model is given by the following equations:

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = \omega + \displaystyle\int_I W(x,y)\sin(u(y,t) - u(x,t))\,dy + h\sin(u(x,t)), \\ u(x,0) = g(x), x \in I, \end{cases} \tag{2}$$

where $u(x,t)$ describes the evolution of the continuum of oscillators distributed over I, $g \in L^\infty(I)$, $g(x)$ is a step function. Throughout this paper, we use bold font to denote vector-valued functions, for example, $\mathbf{u}(t) = u(\cdot,t) \in L^\infty(I)$.

1.4 | Motivations and problems

The expression of the classical Kuramoto model proposed by Yoshiki Kuramoto in 1975 is shown below

$$\begin{cases} \dot{\theta}_i = \omega_i + \dfrac{K}{N} \displaystyle\sum_{i=1}^{N} \sin(\theta_j - \theta_i), i = 1, \cdots, N, \\ \theta_i(0) = \theta_{i0}, \end{cases}$$

where $\theta_i = \theta_i(t) \in R$ is the phase of the ith oscillator, ω_i is the natural frequency of oscillator i, K stands for coupling strength, N is the number of oscillators. Because the Kuramoto model can properly analyze the behavior of oscillators in complex systems, and at the same time it is convenient for researchers to carry out numerical calculation and theoretical analysis, the Kuramoto model is widely used by researchers.

In 1985, Ermentrout first proposed the Kuramoto model with random frequencies that had a continuous limit as the oscillator went to infinity[10]. In these literature[11,12], the authors pointed out that the continuous limit is a very useful tool for analyzing non-locally coupled dynamical systems.

In recent years, with the rise of complex network research, the Kuramoto model has been studied on complex networks. It has been found that the network topology results have a significant impact on the dynamic behavior of the Kuramoto model, see[13,14]. Medvedev derived the continuum limit in the form of the nonlinear heat equation and rigorous justified that the solutions of the initial value problems for the nonlinear heat equation on discrete domains converge to the solution of the IVP for its continuum limit on deterministic and W-random graphs respectively. In [15], the author showed that the model has a family of q-twisted state solutions when the number of oscillators in the network goes to infinity the Kuramoto model on SW graphs. In [16], Medvedev showed that the solution of the IVP of the continuous model is the limit of solutions of the IVPs for the discrete Kuramoto model on sparse random graphs.

Based on the above research background, we find that the research on the asymptotic behavior of the generalized Kuramoto model on the complex networks is very rare. Considering that the real system in reality contains some other influencing factors, in order to be closer to reality, we will consider the Kuramoto model with external drive and study the asymptotic behavior of the discrete Kuramoto model with external drive on deterministic graphs, W-random graphs and SW graphs.

1.5 | Organization of paper

The organization of the paper is as follows. In the next section, we introduce some mathematical concepts, some important theorems and conclusions involved in this paper. In Section 3, we study the asymptotic behavior of the discrete Kuramoto model on deterministic graphs. The deterministic graphs are divided into two classes of convergent graph sequences: simple graphs and weighted graphs. We prove the solutions of the initial value problems for the discrete Kuramoto model converge to the solution of the IVP for its continuum limit on simple graphs and weighted graphs respectively. For sequences of simple graphs converging to {0,1}-valued graphs, we find the rate of convergence depends on the regularity of the boundary of support of the graph limits. In Section 4, we study the asymptotic behavior of the discrete Kuramoto model on W-random graphs by random sequences. We prove convergence of solutions of the discrete Kuramoto model on W-random graphs by random sequences and find the rate of convergence depends on the the Central Limit Theorem (CLT) and holds for all graphs W. In Section 5, we study the asymptotic behavior of the discrete Kuramoto model on SW graphs. We prove prove convergence of solutions of the discrete Kuramoto model on SW graphs and find the rate of convergence depends on the the Central Limit Theorem (CLT) and the auxiliary IVPs. In Section 6, we conclude with brief discussion.

2 | PRELIMINARIES

In this section, we introduce some mathematical concepts, some important theorems and conclusions involved in this paper.

Definition 2.1. Let $u_{ni}(t)$ be the solutions of the IVP (1), $u(x,t)$ be the solution of the IVP (2). If

$$u_{ni}(t) \to u(x,t), \text{ as } n \to +\infty,$$

uniformly for $t \in [0,T]$, $T > 0$. $u(x,t)$ is called the asymptotic solution.

2.1 | Graph limits

In this paper, the limit theory of graphs is used to provide strict mathematical proof for the continuous limit of the discrete Kuramoto model. So next we review several notions and results concerning the theory of graph limits that we will need in the remainder paper, we mainly follow [17,9,18].

Let $G_n = (V(G_n), E(G_n))$, $n \in \mathbb{N}$ be a sequence of dense graphs, that is, $|E(G_n)| = \mathrm{O}\left(|V(G_n)|^2\right)$. The convergence of the graph sequence $\{G_n\}$ is defined according to the homomorphism densities

$$t(F, G_n) = \frac{\hom(F, G_n)}{|V(G_n)|^{|V(F)|}}, \tag{3}$$

where, $F = (V(F), E(F))$ is a simple graph (without loops and multiple edges is called simple) and $\hom(F, G_n)$ denotes the number of homomorphisms (i.e., adjacency preserving maps $V(F) \to V(G_n)$). In other words, (3) is the probability that a random map h: $V(F) \to V(G_n)$ to be a homomorphism.

Definition 2.2. The sequence of graphs $\{G_n\}$ is called convergent, if $t(F, G_n)$ has a limit for every simple graph F.

The fact proved that convergent graph sequences have a limit object, which can be expressed as measurable functions $W : I^2 \to I$. Such functions are called graphs. The set of all graphs is denoted by $\mathcal{W}_0$.

Theorem 2.3. Let $W \in \mathcal{W}_0$, for every simple graph F, there is a sequence of graphs G_n satisfying

$$t(F, G_n) \to t(F, W) = \int\limits_{I^{|V(F)|}} \prod_{(i,j)\in E(F)} W(x_i, x_j)\, dx.$$

The cut-norm is the key to describe the metric properties of graphs. For any graph $W \in W_0$

$$\|W\|_1 = \sup_{S,T\in[0,1]} \left| \int\limits_{S\times T} W(x,y)\, dxdy \right|$$

is called the cut-norm of W. We define the cut-distance between two graphs W and U by

$$\delta_1(U, W) = \inf_{\phi} \left\|U - W^{\phi}\right\|_1,$$

where $W^{\phi}(x,y) = W(\phi(x), \phi(y))$ and all invertible maps ϕ: $[0,1] \to [0,1]$, such that ϕ and its inverse are measurable-preserving. The infinum on the whole ϕ is used to keep the cut-distance between graphs unchanging in regard to isomorphisms of the graph, at the same time, some other transformations that do not change the asymptotic nature of the graph sequences. A sequence of graphs $\{G_n\}$ is convergent if and only if it is Cauchy in the δ_1 distance.

2.2 | Knowledge and conclusions about random variables

Let $X_1, X_2, \cdots X_n, \cdots$ be a set of independent random variables, $E(X)$ is the expectation of X, $Var[X]$ is the variance of X.

Definition 2.4. (Convergence in distribution[19]) A sequence $X_1, X_2, \cdots X_n, \cdots$ of real-valued random variables is said to converge in distribution, or converge weakly, or converge in law to a random variable X if

$$\lim_{n\to\infty} F_n(x) = F(x),$$

for every number $x \in \mathbb{R}$ at which F is continuous. Here F_n and F are the cumulative distribution functions of random variables X_n and X respectively. Convergence in distribution may be denoted as $X_n \Rightarrow X$ or $X_n \to_d X$. If X is standard normal we can write $X_n \to_d N(0,1)$.

Definition 2.5. (Convergence in probability[19]) A sequence $\{X_n\}$ of random variables convergence in probability towards the random variable X if for all $\varepsilon > 0$

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$$

Convergence in probability is denoted as $X_n \to_P X$. If random elements $\{X_n\}$ on a separable metric space (S, d), convergence in probability is defined similarly by

$$\forall \varepsilon > 0,\ P(d(X_n, X) \geq \varepsilon) \to 0.$$

Theorem 2.6. (Lyapunov Central Limit Theorem[19]) Suppose $\{X_1, X_2, \cdots\}$ is a sequence of independent random variables, each with finite expected value u_i and variance σ_i^2. Define $S_n^2 = \sum\limits_{i=1}^{n} \sigma_i^2$. If for some $\delta > 0$, Lapunov's condition

$$\lim_{n\to\infty} \frac{1}{S_n^{2+\delta}} \sum_{i=1}^{n} E\left[|X_i - u_i|^{2+\delta}\right] = 0,$$

is satisfied, then a sum of $\frac{X_i - u_i}{S_n}$ converges in distribution to a standard normal random variable, as $n \to \infty$:

$$\frac{1}{S_n} \sum_{i=1}^{n} (X_i - u_i) \to_d N(0,1).$$

In practice it is usually easiest to check Lyapunov's condition for $\delta = 1$.

2.3 | The well-posedness of the IVP

To facilitate the study the relation between solutions of the discrete Kuramoto model and its continuous limit, we need to show the well-posedness of the IVP for (2).

Theorem 2.7. Assume that $W \in L^{\infty}\left(I^2\right)$ and $g \in L^{\infty}(I)$. For any $T > 0$, then there has a unique solution of the IVP for (2) $\mathbf{u} \in C^1\left(\mathbb{R}; L^{\infty}(I)\right)$ satisfies the initial condition $\mathbf{u}(0) = g(x)$.

Proof. The contraction mapping principle (see [20]) is used to prove Theorem 2.7. We notice that the IVP for (2) can be rewritten as the following integral equation:

$$\mathbf{u} = K\mathbf{u}, \tag{4}$$

where

$$[K\mathbf{u}](x,t) = \mathbf{g} + \int_0^t \left(\int_I W(x,y) \sin\left(u(y,s) - u(x,s)\right) dy + h \sin\left(u(x,s)\right) \right) ds.$$

Denote

$$\tau = \frac{1}{2\left(2\|W\| + h\right)} > 0. \tag{5}$$

Let M_g be a metric subspace of $C\left(0,\tau; L^{\infty}(I)\right)$ formed by the functions $\mathbf{u}$. Then (4) is the fixed point equation for the mapping $K : M_g \to M_g$.

For any $\mathbf{u}, \mathbf{v} \in M_g$, we have

$$\begin{aligned}
&\|K\mathbf{u} - K\mathbf{v}\|_{M_g} = \max_{t\in[0,\tau]} \|K\mathbf{u} - K\mathbf{v}\|_{L^{\infty}(I)} \\
\le& \max_{t\in[0,\tau]} \left\| \int_0^t \left(\int_I |W(x,y)| |\sin(u(y,s) - u(x,s)) - \sin(v(y,s) - v(x,s))| dy \right.\right. \\
&\left.\left. + h|u(x,s) - v(x,s)| dy \right) ds \right\|_{L^{\infty}(I)} \\
\le& \max_{t\in[0,\tau]} \left\| \int_0^t \left(\int_I |W(x,y)| |u(y,s) - u(x,s) - v(y,s) + v(x,s)| dy \right.\right. \\
&\left.\left. + h|u(x,s) - v(x,s)| dy \right) ds \right\|_{L^{\infty}(I)} \\
\le& \tau \max_{t\in[0,\tau]} \left\{ \left\| \int_I |W(x,y)| |u(y,t) - v(y,t)| dy \right\|_{L^{\infty}(I)} \right. \\
&\left. + \left\| \int_I |W(x,y)| |u(x,t) - v(x,t)| dy \right\|_{L^{\infty}(I)} + \left\| h|u(x,t) - v(x,t)| \right\|_{L^{\infty}(I)} \right\} \\
=& \tau \max_{t\in[0,\tau]} \left\{ \left\| \int_I |W(x,y)| |u(y,t) - v(y,t)| dy \right\|_{L^{\infty}(I)} \right. \\
&\left. + (\|W\|_{L^{\infty}(I^2)} + h) \|\mathbf{u}(t) - \mathbf{v}(t)\|_{L^{\infty}(I)} \right\} \\
=& \tau \max_{t\in[0,\tau]} \{ \|W\|_{L^{\infty}(I^2)} \|\mathbf{u}(t) - \mathbf{v}(t)\|_{L^{\infty}(I)} \\
&+ (\|W\|_{L^{\infty}(I^2)} + h) \|\mathbf{u}(t) - \mathbf{v}(t)\|_{L^{\infty}(I)} \} \\
=& (2\|W\|_{L^{\infty}(I^2)} + h) \tau \max_{t\in[0,\tau]} \|\mathbf{u}(t) - \mathbf{v}(t)\|_{L^{\infty}(I)}.
\end{aligned}$$

Thus, using (5), we have

$$\|K\mathbf{u} - K\mathbf{v}\|_{Mg} \le \frac{1}{2} \|\mathbf{u} - \mathbf{v}\|_{Mg}.$$

From the Banach contraction mapping principle, it is straightforward to show that the IVP for (2) has a unique solution $\overline{\mathbf{u}} \in M_g \subset C(0,\tau; L^\infty(I))$. With $\overline{\mathbf{u}}(\tau)$ as the initial condition, the local solution is generalized to $[0, 2\tau]$, and by repeating this argument to $[0,T]$ for any $T>0$, we can prove the existence and uniqueness of the solution of the IVP for (2). □

3 | ASYMPTOTIC BEHAVIOR OF THE DISCRETE KURAMOTO MODEL ON DETERMINISTIC GRAPHS

3.1 | Simple graphs

Definition 3.1. $G_n = \langle V(G_n), E(G_n)\rangle$ is called a simple graph with $V(G_n) = [n]$ and

$$E(G_n) = \left\{(i,j) \in [n]^2 : (I_{ni} \times I_{nj}) \cap W^+ \neq \emptyset\right\}.$$

Assume that $W\colon I^2 \to \{0,1\}$ is a symmetric measurable function. We define the support of W as

$$W^+ = \left\{(x,y) \in I^2 : W(x,y) \neq 0\right\},$$

and its boundary is given by ∂W^+. Fix $n \in \mathbb{N}$, divide I into n subintervals

$$I_{n1} = \left[0, \frac{1}{n}\right), I_{n2} = \left[\frac{1}{n}, \frac{2}{n}\right), \cdots, I_{nn} = \left[\frac{n-1}{n}, 1\right), \tag{6}$$

For convenience, (2) can be rewritten as

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \omega + \displaystyle\int_I W(x,y)\sin(u(y,t) - u(x,t))\,dy + h\sin(u(x,t)), \\ u(x,0) = g(x), x \in I. \end{cases} \tag{7}$$

On $\{G_n\}$, the corresponding discrete Kuramoto model of (7) is shown below:

$$\begin{cases} \dfrac{d}{dt} u_{ni}(t) = \omega + \dfrac{1}{n} \displaystyle\sum_{j:(i,j)\in E(G_n)} \sin(u_{nj} - u_{ni}) + h\sin(u_{ni}), \\ u_{ni}(0) = g(x_i), i \in [n]. \end{cases} \tag{8}$$

In order to study the relationship between the solutions of discrete models (8) and the solution of the IVP for continuous models (7). In this subsection, we assign $g(x_i)$ the average value of $g(x)$ on I_{ni}:

$$g(x_i) = n \int_{I_{ni}} g(x)dx. \tag{9}$$

And we define a step-function $\mathbf{u}_n$ such that

$$u_n(x,t) = u_{ni}(t),\ x \in I_{ni}.$$

Suppose that $u_n(x,t)$ satisfies the following system of differential equations:

$$\begin{cases} \dfrac{\partial}{\partial t} u_n(x,t) = \omega + \displaystyle\int_I \hat{W}_n(x,y)\sin(u_n(y,t) - u_n(x,t))\,dy + h\sin(u_n(x,t)), \\ u_n(x,0) = g_n(x), \end{cases} \tag{10}$$

where

$$g_n(x) = g(x_i) \text{ if } x \in I_{ni}, i \in [n],$$

and $\widehat{W}_n(x,y)$ is the step function such that for $(x,y) \in I_{ni} \times I_{nj}, (i,j) \in [n]^2$,

$$\hat{W}_n(x,y) = \begin{cases} 1, & (I_{ni} \times I_{nj}) \cap W^+ \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.2. Let $\mathbf{u}$ and $\mathbf{u_n}$ stand for the solutions of IVP (7) and (10) respectively. Use $2b = \overline{\dim}_B \partial W^+$ (cf. [21]) to represent the upper box-counting dimension of ∂W^+ and assume that $b \in [0.5, 1)$. For any $\varepsilon > 0$ and all sufficiently large n, then we have the following relation:

$$\|\mathbf{u} - \mathbf{u_n}\|_{C(0,T;L^2(I))} \leq C_1 n^{-(1-b-\varepsilon)}, \tag{11}$$

where constant C_1 is independent of n.

Proof. Denote $\xi_n(x,t) = u_n(x,t) - u(x,t)$. By subtracting (7) from (10), we have

$$\begin{aligned}\frac{\partial \xi_n}{\partial t} &= \int_I \hat{W}_n(x,y)\sin(u_n(y,t) - u_n(x,t))dy + h\sin(u_n(x,t)) \\ &\quad - \int_I W(x,y)\sin(u(y,t) - u(x,t))dy - h\sin(u(x,t)) \\ &= \int_I \hat{W}_n(x,y)[\sin(u_n(y,t) - u_n(x,t)) - \sin(u(y,t) - u(x,t))]dy \\ &\quad + \int_I \Big(\hat{W}_n(x,y) - W(x,y)\Big)\sin(u(y,t) - u(x,t))dy \\ &\quad + h[\sin(u_n(x,t)) - \sin(u(x,t))].\end{aligned} \tag{12}$$

Next, we multiply $\xi_n(x,t)$ on both sides of (12) and integrate over I to obtain

$$\begin{aligned}&\frac{1}{2}\int_I \frac{\partial}{\partial t}\xi_n(x,t)^2 dx \\ =&\int_{I^2} \hat{W}_n(x,y)[\sin(u_n(y,t) - u_n(x,t)) - \sin(u(y,t) - u(x,t))]\xi_n(x,t)dxdy \\ &+ \int_{I^2} (\hat{W}_n(x,y) - W(x,y))\sin(u(y,t) - u(x,t))\xi_n(x,t)dxdy \\ &+ \int_I h[\sin(u_n(x,t)) - \sin(u(x,t))]\xi_n(x,t)dx.\end{aligned} \tag{13}$$

For the first term on the right-hand side of (13), we use $\|\hat{W}\|_{L^\infty(I^2)} = 1$, the triangle inequality, the Cauchy-Schwarz inequality and continuity of $sin(\cdot)$ to obtain

$$\begin{aligned}&|\int_{I^2} \hat{W}_n(x,y)\{\sin(u_n(y,t) - u_n(x,t)) - \sin(u(y,t) - u(x,t))\}\xi_n(x,t)dxdy| \\ \leq&\int_{I^2} |(\xi_n(y,t) - \xi_n(x,t))\xi_n(x,t)|dxdy \\ =&\int_{I^2} |\xi_n(y,t)\xi_n(x,t) - [\xi_n(x,t)]^2|dxdy \\ \leq&\int_{I^2} |\xi_n(y,t)\xi_n(x,t)| + |\xi_n(x,t)|^2 dxdy \\ \leq&[\int_I |\xi_n(y,t)|^2 dy]^{\frac{1}{2}}[\int_I |\xi_n(x,t)|^2 dx]^{\frac{1}{2}} + \int_I |\xi_n(x,t)|^2 dx \\ =&\int_I |\xi(\cdot,t)|^2 dx + \int_I |\xi(\cdot,t)|^2 dx \\ =&2\int_I |\xi(\cdot,t)|^2 dx = 2[(\int_I |\xi(\cdot,t)|^2 dx)^{\frac{1}{2}}]^2 = 2\|\xi_n\|^2_{L^2[I]}.\end{aligned} \tag{14}$$

Using the Cauchy-Schwarz inequality and the bound on $sin(\cdot)$, $|sin(\cdot)| \leq 1$, we estimate the second term on the right-hand side of (13)

$$\begin{aligned}
&\left|\int\limits_{I^2}\left(\hat{W}_n(x,y)-W(x,y)\right)\sin(u(y,t)-u(x,t))\xi_n(x,t)dxdy\right| \\
\leq &\operatorname*{\mathit{ess}\,sup}_{(x,y,t)\in I^2\times[0,T]}|\sin(u(y,t)-u(x,t))|\times\left|\int\limits_{I^2}(\hat{W}_n(x,y)-W(x,y))\xi_n(x,t)dxdy\right| \\
\leq &\left[\int\limits_{I^2}(\hat{W}_n(x,y)-W(x,y))^2dxdy\right]^{\frac{1}{2}}\times\left[\int\limits_{I}(\xi_n(x,t))^2dx\right]^{\frac{1}{2}} \\
\leq &\|W-\hat{W}_n\|_{L^2(I^2)}\|\xi_{\mathbf{n}}\|_{L^2(I)}.
\end{aligned} \tag{15}$$

Using continuity of $\sin(\cdot)$, we estimate the third term on the right-hand side of (13)

$$\begin{aligned}
&\left|\int\limits_{I}h[\sin(u_n(x,t))-\sin(u(x,t))]\xi_n(x,t)dx\right| \\
\leq &h\left|\int\limits_{I}[u_n(x,t)-u(x,t)]\xi_n(x,t)dx\right| \\
= &h\left|\int\limits_{I}[\xi_n(x,t)]^2dx\right| = h\|\xi_{\mathbf{n}}\|^2_{L^2(I)}.
\end{aligned} \tag{16}$$

Using (14), (15) and (16), from (13) we have

$$\begin{aligned}
\frac{d}{dt}\|\xi_{\mathbf{n}}\|^2_{L^2(I)} \leq &4\|\xi_{\mathbf{n}}\|^2_{L^2(I)}+2\|W-\hat{W}_n\|_{L^2(I^2)}\|\xi_{\mathbf{n}}\|_{L^2(I)}+h\|\xi_{\mathbf{n}}\|^2_{L^2(I)} \\
= &(4+h)\|\xi_{\mathbf{n}}\|^2_{L^2(I)}+2\|W-\hat{W}_n\|_{L^2(I^2)}\|\xi_{\mathbf{n}}\|_{L^2(I)}.
\end{aligned} \tag{17}$$

For notational simplicity, let $\varepsilon > 0$ be arbitrary but fixed, denote

$$\phi_\varepsilon(t) = \sqrt{\|\xi_{\mathbf{n}}\|^2_{L^2(I)}+\varepsilon},$$

(17) can be rewritten as

$$\frac{d}{dt}\phi_\varepsilon(t)^2 \leq (4+h)\phi_\varepsilon(t)^2+2\|W-\hat{W}_n\|_{L^2(I^2)}\phi_\varepsilon(t). \tag{18}$$

Because $\phi_\varepsilon(t) > 0$ on $[0,T]$, from (18), we obtain

$$\frac{d}{dt}\phi_\varepsilon(t) \leq \frac{(4+h)}{2}\phi_\varepsilon(t)+\left\|W-\hat{W}_n\right\|_{L^2(I^2)},\ t\in[0,T].$$

By Gronwall's inequality, we have

$$\sup_{t\in[0,T]}\phi_\varepsilon(t) \leq \left(\phi_\varepsilon(0)+\frac{\|W-\hat{W}_n\|_{L^2(I^2)}}{2+\frac{h}{2}}\right)\exp\{(2+\frac{h}{2})T\}. \tag{19}$$

Due to $\varepsilon > 0$ is arbitrary, from (19), we have

$$\sup_{t\in[0,T]}\|\xi_{\mathbf{n}}(t)\|_{L^2(I)} \leq \left(\|\mathbf{g}-\mathbf{g_n}\|_{L^2(I)}+\frac{\|W-\hat{W}_n\|_{L^2(I^2)}}{2+\frac{h}{2}}\right)\exp\{(2+\frac{h}{2})T\}. \tag{20}$$

It remains to estimate $\|W-\hat{W}_n\|_{L^2(I^2)}$. We need to the following definitions: the set of discrete cells $I_{ni}\times I_{nj}$ which covers the boundary of the support of W

$$J(n) = \left\{(i,j)\in[n]^2 : (I_{ni}\times I_{nj})\cap\partial W^+\neq\emptyset\right\} \text{ and } C(n) = |J(n)|.$$

According to one of a few isovalent definitions of the upper box-counting dimension of a subset of R^n, we have

$$2b = \overline{\dim}_B\partial W^+ = \overline{\lim_{\delta\to 0}}\frac{\log N_\delta(\partial W^+)}{-\log\delta},$$

where $N_\delta\left(\partial W^+\right)$ is the number of cells of a $(\delta\times\delta)$-mesh that intersect ∂W^+ (see[21]). Hence, for any $\varepsilon>0$ and all sufficiently large n, we obtain

$$C\left(n\right)\le n^{2(b+\varepsilon)}.$$

Because W and $\hat{W}_n$ coincide on all cells $I_{ni}\times I_{nj}$ for which $(i,j)\notin J\left(n\right)$, for any $\varepsilon>0$ and all sufficiently large n, we have

$$\|W-\hat{W}_n\|^2_{L^2(I^2)}=\int\limits_{I^2}\left(W-\hat{W}_n\right)^2dxdy\le C(n)n^{-2}\le n^{-2(1-b-\varepsilon)}. \tag{21}$$

Finally, we use the relation (9) to obtain

$$\|\mathbf{g}-\mathbf{g_n}\|^2_{L^2(I)}=O(n^{-1}). \tag{22}$$

Combing (20), (21) and (22), we obtain (11). □

3.2 | Weighted graphs

In this section, we study the Kuramoto model on convergent sequences of weighted graphs.

Assume that $W\colon I^2\to[-1,1]$ is a symmetric measurable function. Let $P_n=\{I_{ni},i\in[n]\}$ (see (6) and

$$X_n=\left\{\frac{1}{n},\frac{2}{n},\cdots\frac{n}{n}\right\}.$$

Let's start with the description of the weighted graph sequences generated by a given graph on W that we will be used in this subsection.

Define W/P_n is the complete graph on n nodes

$$W/P_n=\left\langle[n]\,,[n]\times[n]\,,\bar{W}_n\right\rangle,$$

such that each edge of W/P_n is supplied with the weight

$$\left(\bar{W}_n\right)_{ij}=n^2\int\limits_{I_i\times I_j}W\left(x,y\right)dxdy.$$

In the rest of this subsection, we prove convergence of the discrete Kuramoto model on W/P_n to the continuum Kuramoto model on the graph on W (cf. 7). In addition, we prove that the above problems correspond to the discretizations of (7) using Galerkin method.

First of all, let

$$H_n=span\left\{\phi_1,\phi_2,\cdots,\phi_n\right\},$$

be a finite-dimensional subspace of $L^2\left(I\right)$, here ϕ_n=$\chi_{I_{ni}}$ is the characteristic function of $I_{ni}=\left[(i-1)\,n^{-1},in^{-1}\right)$.

Next, We construct the Galerkin approximate of the solution for (7) as shown below:

$$u_n\left(x,t\right)=\sum_{k=1}^{n}u_{nk}\left(t\right)\phi_k\left(x\right)\in H_n. \tag{23}$$

We replace $u(x,t)$ in (7) with (23) and project the resultant equation on H_n to obtain the following IVP for the unknown coefficients $u_{nk}\left(t\right),k\in[n]$ on W/P_n:

$$\begin{cases}\dfrac{d}{dt}u_{ni}\left(t\right)=\omega+\dfrac{1}{n}\displaystyle\sum_{j=1}^{n}\left(\bar{W}_n\right)_{ij}\sin\left(u_{nj}\left(t\right)-u_{ni}\left(t\right)\right)+h\sin\left(u_{ni}\right),\\ u_{ni}\left(0\right)=g_{ni},i\in[n]\,.\end{cases} \tag{24}$$

We notice that the Galerkin equation (24) can be rewritten as the following integral equation:

$$\begin{cases}\dfrac{\partial}{\partial t}u_n\left(x,t\right)=\omega+\displaystyle\int\limits_I W_n\left(x,y\right)\sin\left(u_n\left(y,t\right)-u_n\left(x,t\right)\right)dy+h\sin\left(u_n\left(x,t\right)\right),\\ u_n\left(x,0\right)=g_n\left(x\right),x\in I,\end{cases} \tag{25}$$

where W_n and g_n are the step functions

$$\begin{aligned} W_n\left(x,y\right) &= \left(\bar{W}_n\right)_{ij} \quad \text{for } (x,y) \in I_{ni} \times I_{nj}. \\ g_n\left(x\right) &= g_{ni} \qquad\quad \text{for } x \in I_{ni}. \end{aligned}$$

Theorem 3.3. Let $\mathbf{u}$ and $\mathbf{u}_n$ be the solutions of (7) and (25) respectively. Assume $W \in L^\infty\left(I^2\right)$ and $g \in L^\infty\left(I\right)$. Then we have the following relation:

$$\|\mathbf{u}-\mathbf{u_n}\|_{C(0,T;L^2(I))} \to 0, \text{ as } n \to +\infty. \tag{26}$$

Proof. Denote $\xi_n\left(x,t\right) = u_n\left(x,t\right) - u\left(x,t\right)$. By subtracting (7) from (25), we have

$$\begin{aligned} \frac{\partial \xi_n}{\partial t} &= \int_I W_n(x,y)\sin\left(u_n(y,t)-u_n(x,t)\right)dy + h\sin(u_n(x,t)) \\ &\quad - \int_I W(x,y)\sin(u(y,t)-u(x,t))dy - h\sin(u(x,t)) \\ &= \int_I W_n(x,y)\left[\sin(u_n(y,t)-u_n(x,t)) - \sin(u(y,t)-u(x,t))\right]dy \\ &\quad + \int_I \Big(W_n(x,y)-W(x,y)\Big)\sin(u(y,t)-u(x,t))dy \\ &\quad + h[\sin(u_n(x,t)) - \sin(u(x,t))]. \end{aligned} \tag{27}$$

Next, we multiply $\xi_n(x,t)$ on both sides of (27) and integrate over I

$$\begin{aligned} &\frac{1}{2}\int_I \frac{\partial}{\partial t}\xi_n(x,t)^2 dx \\ &= \int_{I^2} W_n(x,y)[\sin(u_n(y,t)-u_n(x,t)) - \sin(u(y,t)-u(x,t))] \\ &\quad \times \xi_n(x,t)dxdy \\ &\quad + \int_{I^2} \Big(W_n(x,y)-W(x,y)\Big)\sin(u(y,t)-u(x,t))\xi_n(x,t)dxdy \\ &\quad + \int_I h[\sin(u_n(x,t)) - \sin(u(x,t))]\xi_n(x,t)dx. \end{aligned} \tag{28}$$

Using $\|W\|_{L^\infty(I^2)} = 1$, the triangle inequality, the Cauchy-Schwarz inequality and continuity of $sin(\cdot)$, we estimate the first term on the right-hand side of (28),

$$
\begin{aligned}
&\left|\int_I^2 W_n(x,y)[\sin(u_n(y,t)-u_n(x,t))-\sin(u(y,t)-u(x,t))]\times\xi_n(x,t)dxdy\right|\\
\leq&\int_{I^2}|[u_n(y,t)-u_n(x,t)-u(y,t)+u(x,t)]\,\xi_n(x,t)|dxdy\\
\leq&\int_{I^2}|[(u_n(y,t)-u(y,t))+(u_n(x,t)-u(x,t))]\,\xi_n(x,t)|dxdy\\
\leq&\int_{I^2}|(\xi_n(y,t)-\xi_n(x,t))\xi_n(x,t)|dxdy\\
=&\int_{I^2}|\xi_n(y,t)\xi_n(x,t)-[\xi_n(x,t)]^2|dxdy\\
\leq&\int_{I^2}|\xi_n(y,t)\xi_n(x,t)|+|\xi_n(x,t)|^2dxdy\\
\leq&\left[\int_I|\xi_n(y,t)|^2dy\right]^{\frac{1}{2}}\left[\int_I|\xi_n(x,t)|^2dx\right]^{\frac{1}{2}}+\int_I|\xi_n(x,t)|^2dx\\
=&\int_I|\xi(\cdot,t)|^2dx+\int_I|\xi(\cdot,t)|^2dx\\
=&2\int_I|\xi(\cdot,t)|^2dx=2[(\int_I|\xi(\cdot,t)|^2dx)^{\frac{1}{2}}]^2=2\|\xi_{\mathbf{n}}\|^2_{L^2(I)}.
\end{aligned}
\tag{29}
$$

Using the Cauchy-Schwarz inequality and the bound on $sin(\cdot)$, $|sin(\cdot)|\leq 1$, we estimate the second term on the right-hand side of (28),

$$
\begin{aligned}
&\left|\int_{I^2}(W_n(x,y)-W(x,y))\sin(u(y,t)-u(x,t))\xi_n(x,t)dxdy\right|\\
\leq&\operatorname*{ess\,sup}_{(x,y,t)\in I^2\times[0,T]}|\sin(u(y,t)-u(x,t))|\times\left|\int_{I^2}\Big(W_n(x,y)-W(x,y)\Big)\xi_n(x,t)dxdy\right|\\
\leq&\left[\int_{I^2}\Big(W_n(x,y)-W(x,y)\Big)^2dxdy\right]^{\frac{1}{2}}\left[\int_I(\xi_n(x,t))^2dx\right]^{\frac{1}{2}}\\
\leq&\|W-W_n\|_{L^2(I^2)}\|\xi_{\mathbf{n}}\|_{L^2(I^2)}.
\end{aligned}
\tag{30}
$$

Using continuity of $sin(\cdot)$, we estimate the third term on the right-hand side of (28)

$$
\begin{aligned}
&\left|\int_I h[\sin(u_n(x,t))-\sin(u(x,t))]\xi_n(x,t)dx\right|\\
\leq&h\left|\int_I[u_n(x,t)-u(x,t)]\xi_n(x,t)dx\right|\\
=&h\left|\int_I(\xi_n(x,t))^2dx\right|\\
=&h\|\xi_{\mathbf{n}}\|^2_{L^2(I)}.
\end{aligned}
\tag{31}
$$

Combining (29), (30) and (31), we obtain

$$\begin{aligned}\frac{d}{dt}\|\xi_{\mathbf{n}}\|^2_{L^2(I)} \leq& 4\|\xi_{\mathbf{n}}\|^2_{L^2(I)} + 2\|W - W_n\|_{L^2(I^2)}\|\xi_{\mathbf{n}}\|_{L^2(I)} + h\|\xi_{\mathbf{n}}\|^2_{L^2(I)} \\ =&(4+h)\|\xi_{\mathbf{n}}\|^2_{L^2(I)} + 2\|W - W_n\|_{L^2(I^2)}\|\xi_{\mathbf{n}}\|_{L^2(I)}.\end{aligned} \tag{32}$$

For the sake of national simplicity, we set

$$\phi_\varepsilon(t) = \sqrt{\|\xi_{\mathbf{n}}\|^2_{L^2(I)} + \varepsilon},$$

where $\varepsilon > 0$ is arbitrary but fixed.

By (32), we have

$$\frac{d}{dt}\phi_\varepsilon(t)^2 \leq (4+h)\phi_\varepsilon(t)^2 + 2\|W - W_n\|_{L^2(I^2)}\phi_\varepsilon(t). \tag{33}$$

Because $\phi_\varepsilon(t)$ is positive $[0, T]$, from (33), we obtain

$$\frac{d}{dt}\phi_\varepsilon(t) \leq \frac{(4+h)}{2}\phi_\varepsilon(t) + \|W - W_n\|_{L^2(I^2)}, t \in [0, T].$$

We now use Gronwall's inequality to have

$$\sup_{t\in[0,T]} \phi_\varepsilon(t) \leq \left(\phi_\varepsilon(0) + \frac{\|W - W_n\|_{L^2(I^2)}}{2 + \frac{h}{2}}\right)\exp\{(2 + \frac{h}{2})T\}. \tag{34}$$

Due to $\varepsilon > 0$ is arbitrary, from (34), we get

$$\sup_{t\in[0,T]} \|\xi_n(t)\|_{L^2(I)} \leq \left(\|g - g_n\|_{L^2(I)} + \frac{C_2\|W - W_n\|_{L^2(I^2)}}{2 + \frac{h}{2}}\right)\exp\{(2 + \frac{h}{2})T\}. \tag{35}$$

Note that

$$W_n \to W \text{ and } g_n \to g, \text{ as } n \to \infty,$$

almost everywhere on I^2 and I respectively. Hence, by the dominated convergence theorem, we have

$$\|W - W_n\|_{L^2(I^2)} \to 0 \text{ as } n \to \infty.$$

The statement of the theorem follows from (35). The proof is completed. □

4 | ASYMPTOTIC BEHAVIOR OF THE DISCRETE KURAMOTO MODEL ON W-RANDOM GRAPHS BY RANDOM SEQUENCES

Definition 4.1. ($W-$random graph) Denote $X = (x_1, x_2, x_3 \ldots)$ and $X_n = (x_1, x_2, \ldots, x_n)$, where $x_i, i \in N$ are independent identically distributed (IID) random variables(RVs). RV x_1 has uniform on $I = [0, 1]$ distribution, that is $\mathcal{L}(x_1) = U(I)$. Let $W \in \mathcal{W}_0$ be a class of symmetric functions on I^2 with values in I. Define $G_n = \langle [n]), E(G_n)\rangle$ such that for each $(i, j) \in [n]^2$

$$\mathbb{P}\{(i, j) \in E(G_n)\} = \begin{cases} W(x_i, x_j), & i \neq j, \\ 0, & \text{otherwise,}\end{cases}$$

where $\mathbb{P}\{\cdot\}$ stands for the probability of an event. Graph G_n is called a $W-$random graph generated by the random sequence X_n and denoted by $G_n = G(X_n, W)$.

For convenience, (2) can be rewritten as

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = \omega + \displaystyle\int_I W(x,y)\sin(u(y,t) - u(x,t))\,dy + h\sin(u(x,t)), \\ u(x,0) = g(x), x \in I. \end{cases} \tag{36}$$

On $\{G_n\}$, the corresponding discrete Kuramoto model of (36) is shown below:

$$\begin{cases} \dfrac{d}{dt}u_{ni}(t) = \omega + \dfrac{1}{n}\displaystyle\sum_{j:(i,j)\in E(G_n)} \sin(u_{nj} - u_{ni}) + h\sin(u_{ni}), \\ u_{ni}(0) = g(x_i), i \in [n]. \end{cases} \tag{37}$$

Define the projection of the solution for (36) $u(x,t)$ onto X_n by

$$\mathbf{P}_{X_n} u(x,t) = \left(u(x_1,t), u(x_2,t), \cdots, u(x_n,t)\right).$$

Both $u_n(t)$ and $\mathbf{P}_{X_n} u(x,t)$ are defined on the discrete set X_n. For such functions, we will use the weighted Euclidean inner product

$$(u,v)_n = \frac{1}{n}\sum_{i=1}^{n} u_i v_i,$$

where $u=(u_1,u_2,\cdots u_n)^T$, $v=(v_1,v_2,\cdots v_n)^T$, and the corresponding norm $\|u\|_{2,n}=\sqrt{(u,u)_n}$. We are going to use $\|\cdot\|_{2,n}$ to figure out the relationship between the solutions of (1) and (2) in this and the next section.

Theorem 4.2. Let $T>0$ and assume that the solution of the IVP for (36) $u(x,t)$ satisfies the following inequality

$$\min_{t\in[0,T]} \int_I \left\{ \int_I W(x,y)\sin\left(u(y,t)-u(x,y)\right)^2 dy - \left(\int_I W(x,y)\sin(u(y,t)-u(x,t))dy\right)^2 \right\} dx \geq C_1, \tag{38}$$

for some constant $C_1>0$. Then the solutions of the IVPs for the discrete and continuum models (37) and (36) have the following relation

$$\lim_{n\to\infty} \mathbb{P}\left\{ n^{\frac{1}{2}} \sup_{t\in[0,T]} \|u_n(t)-\mathbf{P}_{X_n} u(x,t)\|_{2,n} \leq C \right\} = 1,$$

for some constant $C>0$.

Because $\mathbf{u}\in C(0,T;L^\infty(I))$, $\|W\|_{L^\infty(I^2)}=1$ and continuity of $sin(\cdot)$, the use of min in (38) can be proved. To prove the theorem 4.2, we need to apply the Lyapunov Central Limit Theorem (cf. [19]). The following lemma will be useful.

Lemma 4.3. Let $f\in L^\infty(I^2)$. Define RVs $\{\varsigma_{ij}\}, (i,j)\in\mathbb{N}^2$, such that $\mathcal{L}(\varsigma_{ij}\,|X) = Bin(W(x_i,x_j))$, here $\mathcal{L}(\cdot)$ denotes the probability distribution of a RV, $Bin(W(x_i,x_j))$ is the binomial distribution with parameter $W(x_i,x_j)$. In particular,

$$\mathbb{P}(\varsigma_{ij}=1\,|X) = W(x_i,x_j), \mathbb{P}(\varsigma_{ij}=0\,|X) = 1-W(x_i,x_j).$$

Denote

$$\eta_{ij} = \varsigma_{ij} f(x_i,x_j), (i,j)\in\mathbb{N}^2,$$
$$z_{ni} = \frac{1}{n}\sum_{j=1}^{n}\eta_{ij} - \int_I f(x_i,y)W(x_i,y)\,dy, \text{ and } S_n = \sum_{i=1}^{n} z_{ni}^2.$$

We assume

$$\sigma^2 = \int_{I^2} f(x,y)^2 W(x,y)dxdy - \int_I \left(\int_I f(x,y)W(x,y)\,dy\right)^2 dx > 0.$$

Then

$$\frac{S_n-\sigma^2}{n^{-1/2}\sqrt{5\sigma^4+\mathrm{O}(n^{-1})}} \to_d \mathcal{N}(0,1),$$

where $\to_d$ stands for convergence in distribution, and $\mathcal{N}(0,1)$ is the standard normal distribution.

Proof. $\{\eta_{ij}\}$ are IID RVs, and

$$u(x_i) = \mathbb{E}(\eta_{ij}\,|x_i) = \int_I f(x_i,y)W(x_i,y)\,dy.$$

Thus,

$$u = \mathbb{E}\eta_{ij} = \mathbb{E}\mathbb{E}(\eta_{ij}|x_i) = \int_I\int_I f(x,y)W(x,y)dxdy.$$

$$
\begin{aligned}
V_{\eta_{ij}} =&\mathbb{E}\mathbb{E}((\eta_{ij}-u(x_i))^2|x_i)\\
=&\mathbb{E}\mathbb{E}((\eta_{ij}^2-2\eta_{ij}u(x_i)+u(x_i)^2)|x_i)\\
=&\mathbb{E}\mathbb{E}((\eta_{ij}^2-2\eta_{ij}\mathbb{E}(\eta_{ij}|x_i)+\mathbb{E}(\eta_{ij}|x_i)^2)|x_i)\\
=&\mathbb{E}[\mathbb{E}(\eta_{ij}^2|x_i)-2\mathbb{E}(\eta_{ij}|x_i)\mathbb{E}(\eta_{ij}|x_i)+\mathbb{E}(\eta_{ij}|x_i)^2]\\
=&\mathbb{E}\mathbb{E}(\eta_{ij}^2|x_i)-\mathbb{E}[\mathbb{E}(\eta_{ij}|x_i)^2]\\
=&\int_{I^2} f(x,y)^2 W(x,y)dxdy-\int_I\left(\int_I f(x,y)W(x,y)dy\right)^2 dx=\sigma^2.
\end{aligned}
$$

Let $y_{ni}=\sqrt{n}z_{ni}$. Then we will apply the Lyapunov CLT to $\sum_{i=1}^{n} y_{ni}^2$. So let's do the next calculation,

$$
\begin{aligned}
\mathbb{E}y_{ni}^2 =&\frac{1}{n}\mathbb{E}\mathbb{E}\left(\sum_{1\le j,k\le n}(\eta_{ij}-u(x_i))(\eta_{ik}-u(x_i))|x_i\right)\\
=&\frac{2}{n}\mathbb{E}\mathbb{E}\left(\sum_{1\le j<k\le n}(\eta_{ij}-u(x_i))(\eta_{ik}-u(x_i))|x_i\right)+\frac{1}{n}\mathbb{E}\mathbb{E}\left(\sum_{1\le j\le n}(\eta_{ij}-u(x_i))^2|x_i\right)\\
=&\sigma^2.
\end{aligned}
$$

Because when $j\neq k$, $\eta_{ij}-u\left(x_i\right)$ and $\eta_{ik}-u\left(x_i\right)$ are independent of each other, and $\mathbb{E}\left(\left(\eta_{ij}-u\left(x_i\right)\right)|\, x_i\right)=0$.

Once again, we calculate

$$
\begin{aligned}
\mathbb{E}y_{ni}^4 =&\frac{1}{n^2}\mathbb{E}\mathbb{E}\left(\sum_{1\le j_1,j_2,j_3,j_4\le n}(\eta_{ij_1}-u(x_i))\cdots(\eta_{ij_4}-u(x_i))|x_i\right)\\
=&\frac{6}{n^2}\mathbb{E}\mathbb{E}\left(\sum_{1\le j<k\le n}(\eta_{ij}-u(x_i))^2(\eta_{ik}-u(x_i))^2|x_i\right)+\frac{1}{n^2}\mathbb{E}\left(\sum_{1\le j\le n}\mathbb{E}((\eta_{ij}-u(x_i))|x_i)^4\right)\\
=&\frac{6n(n-1)}{n^2}\sigma^4+\mathrm{O}(n^{-1})\\
\le&6\sigma^4+\mathrm{O}(n^{-1}).
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}y_{ni}^6 =&\frac{1}{n^3}\mathbb{E}\mathbb{E}\left(\sum_{1\le j_1,j_2,j_3,j_4,j_5,j_6\le n}\left(\eta_{ij_1}-u\left(x_i\right)\right)\cdots\left(\eta_{ij_6}-u\left(x_i\right)\right)|x_i\right)\\
=&\binom{6}{2}\binom{4}{2}\frac{1}{n^3}\mathbb{E}\left(\sum_{1\le j<k<l\le n}\mathbb{E}\left(\eta_{ij}-u(x_i)\left|x_i\right.\right)^2\mathbb{E}(\eta_{ik}-u(x_i)\left|x_i\right.)^2\right.\\
&\left.\times\ \mathbb{E}(\eta_{il}-u(x_i)\left|x_i\right.)^2\right)+\mathrm{O}\left(n^{-1}\right)\\
=&\frac{90n\left(n-1\right)\left(n-2\right)}{n^3}\sigma^6+\mathrm{O}\left(n^{-1}\right)\\
\le&90\sigma^6+\mathrm{O}\left(n^{-1}\right).
\end{aligned}
$$

For $n\in\mathbb{N}$, let

$$
\zeta_{ni}=\frac{y_{ni}^2-\mathbb{E}y_{ni}^2}{\sqrt{nVar\left(y_{ni}^2\right)}}=\frac{y_{ni}^2-\sigma^2}{n^{\frac{1}{2}}\sqrt{5\sigma^4+\mathrm{O}\left(n^{-1}\right)}},i\in[n].
$$

Thus, $\zeta_{ni},i\in[n]$ are IID RVs. Further, $\mathbb{E}\zeta_{ni}=0,V\left(\sum_{i=1}^{n}\zeta_{ni}\right)=1$.

Now let's verify the Lyapunov conditions,

$$
\sum_{i=1}^{n}\mathbb{E}|\zeta_{ni}|^3\le\frac{\sum_{i=1}^{n}\mathbb{E}\left(y_{ni}^6+3y_{ni}^4\sigma^2+3y_{ni}^2\sigma^4+\sigma^6\right)}{n^{\frac{3}{2}}\left(5\sigma^4+\mathrm{O}\left(n^{-1}\right)\right)}=\mathrm{O}\left(n^{\frac{-1}{2}}\right)\to0\text{ as }n\to+\infty.
$$

Thus by the CLT, we come to the conclusion that

$$
\frac{\sum_{i=1}^{n}\left(y_{ni}^2-\sigma^2\right)}{\sqrt{n\left(5\sigma^4+\mathrm{O}\left(n^{-1}\right)\right)}}=\frac{n^{-1}\sum_{i=1}^{n}y_{ni}^2-\sigma^2}{n^{\frac{-1}{2}}\sqrt{5\sigma^4+\mathrm{O}\left(n^{-1}\right)}}\to_d\mathcal{N}\left(0,1\right)\ \text{ as }n\to+\infty.
$$

that is,

$$\frac{S_n-\sigma^2}{n^{\frac{-1}{2}}\sqrt{5\sigma^4+\mathrm{O}\left(n^{-1}\right)}}\rightarrow_d \mathcal{N}\left(0,1\right).$$

This completes the proof. □

In order to prove the Theorem 4.2, we need to extend Lemma 4.3.

Corollary 4.4. Suppose that $\mathbf{f}$ in Lemma 4.3 also depends on $t\in[0,T]$, and $\mathbf{f}\in C\left(0,T;L^\infty\left(I^2\right)\right)$, $\mathbf{f}(t)=f(\cdot,t)$. All variables defined in terms of $\mathbf{f}$ in Lemma 4.3 are related to t. In addition, preserve the notation of the Lemma 4.3, we assume that

$$\min_{t\in[0,T]}\sigma^2\left(t\right)\geq C_1>0.$$

Then for every $t\in[0,T]$, we have the following relation:

$$\frac{S_n\left(t\right)-\sigma^2\left(t\right)}{n^{\frac{-1}{2}}\sqrt{5\sigma^4\left(t\right)+\mathrm{O}\left(n^{-1}\right)}}\rightarrow_d \mathcal{N}\left(0,1\right)\ \text{ as } n\rightarrow+\infty.$$

Proof. Because $\mathbf{f}\in C\left(0,T;L^\infty\left(I^2\right)\right)$ and

$$\sigma^2\left(t\right)=\int\limits_I\int\limits_I f(x,y,t)^2W\left(x,y\right)dxdy-\int\limits_I\left(\int\limits_I f\left(x,y,t\right)W\left(x,y\right)dy\right)^2dx,$$

we have

$$0<C_1\leq\sigma^2\left(t\right)\leq 2\left\|\mathbf{f}\right\|^2_{C(0,T;L^\infty(I^2))}.$$

We show that t-dependent moments of $y^2_{ni}\left(t\right)$ are bounded uniformly in $t\in[0,T]$. Therefore, according to the proof process of the Lemma 4.3, the corollary can be proved to be true for every $t\in[0,T]$. □

Now we are able to prove the Theorem 4.2 of this section.

Proof of Theorem 4.2. Denote $\xi_{ni}\left(t\right)=u\left(x_i,t\right)-u_{ni}\left(t\right)$, $i\in[n]$ and let

$$\xi_n\left(t\right)=\left(\xi_{n1}\left(t\right),\xi_{n2}\left(t\right),\cdots\xi_{nn}\left(t\right)\right).$$

At $x=x_i$, we have

$$\begin{aligned}\frac{d}{dt}\xi_{ni}(t)=&z_{ni}(t)+\frac{1}{n}\sum_{j=1}^n\varsigma_{ij}[\sin(u(x_j,t)-u(x_i,t))-\sin(u_{nj}(t)-u_{ni}(t))]\\&+h[\sin(u(x_i,t))-\sin(u_{ni}(t))],\end{aligned}\tag{39}$$

where

$$z_{ni}\left(t\right)=\int\limits_I W\left(x_i,y\right)\sin\left(u\left(y,t\right)-u\left(x_i,t\right)\right)dy-\frac{1}{n}\sum_{j=1}^n\varsigma_{ij}\sin\left(u\left(x_j,t\right)-u\left(x_i,t\right)\right),$$

and ς_{ij} are defined in the Lemma 4.3.

Next, we multiply $\frac{1}{n}\xi_{ni}$ on both sides of (39) and sum over i to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\xi_n(t)\|^2_{2,n}=(z_n(t),\xi_n(t))_n+\frac{h}{n}\sum_{i=1}^n[\sin(u(x_i,t))-\sin(u_{ni}(t))]\xi_{ni}\\&+\frac{1}{n^2}\sum_{i,j=1}^n\varsigma_{ij}\left[\sin(u(x_j,t)-u(x_i,t))\ -\sin(u_{nj}(t)-u_{ni}(t))\right]\xi_{ni},\end{aligned}\tag{40}$$

where $z_n=(z_{n1},z_{n2},\cdots,z_{nn})$.

We use the Cauchy-Schwarz inequality to estimate the first term on the right-hand side of (40) as follows:

$$\left|\left(z_n,\xi_n\right)_n\right|\leq\|z_n\|_{2,n}\|\xi_n\|_{2,n}\leq\frac{1}{2}\left(\|z_n\|^2_{2,n}+\|\xi_n\|^2_{2,n}\right).\tag{41}$$

By using $|\varsigma_{ij}| \leq 1$, the Cauchy-Schwarz inequality, the triangle inequality and the bound on $sin(\cdot)$, $|sin(\cdot)| \leq 1$, we estimate the second term on the right-hand side of (40) as follows:

$$
\begin{aligned}
&\left|\frac{1}{n^2}\sum_{i,j=1}^{n}\varsigma_{ij}\left[\sin\left(u\left(x_j,t\right)-u\left(x_i,t\right)\right)-\sin\left(u_{nj}\left(t\right)-u_{ni}\left(t\right)\right)\right]\xi_{ni}\right| \\
\leq&\frac{1}{n^2}\sum_{i,j=1}^{n}\left|u\left(x_j,t\right)-u\left(x_i,t\right)-u_{nj}\left(t\right)+u_{ni}\left(t\right)\right|\left|\xi_{ni}\left(t\right)\right| \\
=&\frac{1}{n^2}\sum_{i,j=1}^{n}\left|\xi_{nj}\left(t\right)-\xi_{ni}\left(t\right)\right|\left|\xi_{ni}\left(t\right)\right| \\
\leq&\frac{1}{n^2}\sum_{i\neq j}^{n}\left(\left|\xi_{nj}\left(t\right)\right|+\left|\xi_{ni}\left(t\right)\right|\right)\left|\xi_{ni}\left(t\right)\right| \\
\leq&2\left\|\xi_n\left(t\right)\right\|_{2,n}^2.
\end{aligned}
\tag{42}
$$

We use the Cauchy-Schwarz inequality and continuity of $sin(\cdot)$ to estimate the third term on the right-hand side of (40) as follows:

$$
\left|\frac{h}{n}\sum_{i=1}^{n}[\sin\left(u\left(x_i,t\right)\right)-\sin u_{ni}\left(t\right)]\xi_{ni}\right| \leq\frac{h}{n}\sum_{i=1}^{n}\left|u\left(x_i,t\right)-u_{ni}\left(t\right)\right|\left|\xi_{ni}\right| = \frac{h}{n}\sum_{i=1}^{n}\left|\xi_{ni}\right|^2 = h\left\|\xi_n\right\|_{2,n}^2. \tag{43}
$$

Combining (41), (42) and (43), we obtain

$$
\frac{d}{dt}\left\|\xi\right\|_{2,n}^2 \leq \left(5+2h\right)\left\|\xi\right\|_{2,n}^2+\left\|z_n\right\|_{2,n}^2.
$$

We now use Gronwall's inequality to obtain

$$
\sup_{t\in[0,T]}\left\|\xi_n\right\|_{2.n}^2 \leq \frac{\sup\limits_{t\in[0,T]}\left\|z_n\left(t\right)\right\|_{2,n}^2}{5+2h}\exp\left\{\left(5+2h\right)T\right\}.
$$

To estimate the bound $\sup\limits_{t\in[0,T]}\left\|\xi_n\left(t\right)\right\|_{2,n}^2$, we need to estimate the bound $\sup\limits_{t\in[0,T]}\left\|z_n\left(t\right)\right\|_{2,n}^2$. Let

$$
f\left(x,y,t\right)=\sin\left(u\left(y,t\right)-u\left(x,t\right)\right).
$$

We use $\mathbf{u}\in C\left(0,T;L^\infty\left(I\right)\right)$ and the triangle inequality to have the following relation:

$$
\|\mathbf{f}\|_{C(0,T;L^\infty(I^2))} \leq \max_{t\in[0,T]} \operatorname*{\mathit{ess}\,sup}_{(x,y)\in I^2}\left|u\left(y,t\right)-u\left(x,t\right)\right| \leq 2\|\mathbf{u}\|_{C(0,T;L^\infty(I))}.
$$

This implies that

$$
C_1 \leq \sigma^2\left(t\right) \leq 4\left\|\mathbf{u}\right\|_{C(0,T;L^\infty(I))}^2 \triangleq C_2.
$$

By using Corollary 4.4, we have

$$
\frac{n\left\|z_n\left(t\right)\right\|_{2,n}^2-\sigma^2\left(t\right)}{n^{\frac{-1}{2}}\sqrt{5\sigma^4\left(t\right)+\mathrm{O}\left(n^{-1}\right)}} \to_d \mathcal{N}\left(0,1\right).
$$

As $n\to\infty$, we have

$$
\begin{aligned}
&\mathbb{P}\left(\left|n\left\|z_n\left(t\right)\right\|_{2,n}^2-\sigma^2\left(t\right)\right|>1\right) \\
=&\mathbb{P}\left(\frac{\left|n\left\|z_n\left(t\right)_{2,n}^2\right\|-\sigma^2\left(t\right)\right|}{n^{\frac{-1}{2}}\sqrt{5\sigma^4\left(t\right)+\mathrm{O}\left(n^{-1}\right)}} > \frac{n^{\frac{1}{2}}}{\sqrt{5\sigma^4\left(t\right)+\mathrm{O}\left(n^{-1}\right)}}\right) \\
\leq&\mathbb{P}\left(\left|\frac{n\left\|z_n\left(t\right)\right\|_{2,n}^2-\sigma^2\left(t\right)}{n^{\frac{-1}{2}}\sqrt{5\sigma^4\left(t\right)+\mathrm{O}\left(n^{-1}\right)}}\right| > \frac{n^{\frac{1}{2}}}{\sqrt{5C_2^2+\mathrm{O}\left(n^{-1}\right)}}\right) \\
\leq&\mathbb{P}\left(\left|\frac{n\left\|z_n\left(t\right)\right\|_{2,n}^2-\sigma^2\left(t\right)}{n^{\frac{-1}{2}}\sqrt{5\sigma^4\left(t\right)+\mathrm{O}\left(n^{-1}\right)}}\right| > \frac{n^{\frac{1}{2}}}{C_2}\right) \to 0.
\end{aligned}
\tag{44}
$$

Convergence in (44) is uniform for $t \in [0,T]$. Thus, $\|z_n(t)\|_{2,n}^2$ tends to zero in probability uniformly in t. As $n \to \infty$,

$$\mathbb{P}\left(\|z_n(t)\|_{2,n}^2 > (C_2+1)\, n^{-1}\right) \le \mathbb{P}\left(\left|n\, \|z_n(t)\|_{2,n}^2 - \sigma^2(t)\right| > 1\right) \to 0.$$

uniformly for $t \in [0,T]$. For arbitrary $\varepsilon > 0$, some $N \in \mathbb{N}$, we have

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|z_n(t)\|_{2,n}^2 > C_3 n^{-1}\right) < \varepsilon.$$

where $C_3 := C_2 + 1$. This completes the proof.

5 | ASYMPTOTIC BEHAVIOR OF THE DISCRETE KURAMOTO MODEL ON SMALL-WORLD GRAPHS

In this section, we consider the Kuramoto model on SW graphs. Because complex systems in real life are not regular and symmetry, neither the regular network nor the random network mentioned above can be used to simulate the complex systems in the real world. In 1998, SW graphs proposed by Watts and Strogatz, interpolate between graphs with regular local connections and completely random graphs, they exhibit the combination of properties that are characteristic to both regular and random graphs, it is used to abstract and describe various complex networks in real life. such as social networks, etc.

Definition 5.1. (SW graph) Let

$$X_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\right\},$$

and

$$W(x,y) = \begin{cases} 1, & d(x,y) \le r, \\ 0, & \text{otherwise}, \end{cases}$$

where $d(x,y) = \min\{|x-y|, 1-|x-y|\}$, and parameter $r \in (0,1)$ is fixed. Next, define

$$W_p(x,y) = (1-p)\, W(x,y) + p\,(1 - W(x,y)), p \in [0, 0.5].$$

With the above definitions, $G_{n,p} = G(W_p, X_n)$ is called a SW graph.

Replacing W by W_p in (2) yields the continuum Kuramoto model on SW graphs as follows:

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \omega + \displaystyle\int_I W_p(x,y) \sin(u(y,t) - u(x,t))\, dy + h \sin(u(x,t)), \\ u(x,0) = g(x), x \in I. \end{cases} \tag{45}$$

A discrete counterpart of (45) on $G_{n,p}$ is given by

$$\begin{cases} \dfrac{d}{dt} u_{ni}(t) = \omega + \dfrac{1}{n} \displaystyle\sum_{(i,j)\in E(G_n,p)} \sin(u_{nj} - u_{ni}) + h \sin(u_{ni}(t)), \\ u_{ni}(0) = g(x_i), i \in [n]. \end{cases} \tag{46}$$

We define a step-function $u_n : I \times \mathbb{R} \to \mathbb{R}$ as follows

$$u_n(x,t) = u_{ni}(t), x \in \left[(i-1)\, n^{-1}, in^{-1}\right],\ i \in [n],\ t \in \mathbb{R},$$

where $u_n(t) = (u_{n1}(t), u_{n2}(t), \cdots u_{nn}(t))$ is the solution of the IVP (46).

Theorem 5.2. Let $W_p \in W_0$ is almost everywhere continuous on I^2, and $g \in L^\infty(I)$. Assume that the solution of the IVP (46) $u(x,t)$ satisfies the following inequality

$$\min_{t\in[0,T]} \int_{I^2} \sin(u(y,t) - u(x,t))^2 W_p(x,y)\,(1 - W_p(x,y))\, dxdy > 0, \tag{47}$$

for some $T > 0$. Then as $n \to \infty$ we have

$$\|\mathbf{u_n} - \mathbf{u}\|_{C(0,T;L^2(I))} \to_P 0. \tag{48}$$

The convergence in (48) is in probability.

Because $\mathbf{u} \in C(\mathbb{R}, L^\infty(I))$ and W_p is bounded, the integral in 46 denotes a continuous function of t. Hence, the use min in 47 can be proved. In order to prove Theorem 5.2, we need to borrow some of auxiliary results below. The first result is similar to Lemma 4.3 of the previous section.

Lemma 5.3. Let $\{W_{nij}\}$ be a real array and $\{f_{nij}\}$ be a bounded real array, these two arrays are denoted for $n \in \mathbb{N}$ and $i, j \in [n]$. Suppose $\{\xi_{nij}\}$, $n \in \mathbb{N}$, $(i,j) \in [n]^2$ is independent binomial RVs $\mathcal{L}(\xi_{nij}) = Bin(W_{nij})$ with parameter $0 \le W_{nij} \le 1$. Denote

$$
\begin{aligned}
&\sigma_{ni}^2 = n^{-1} \sum_{j=1}^n f_{nij}^2 W_{nij}(1 - W_{nij}),\ i \in [n],\ \sigma_n^2 = n^{-1} \sum_{i=1}^n \sigma_{ni}^2, \\
&\lim_{n\to\infty} \inf \sigma_n^2 > 0,\ \eta_{nij} = \xi_{nij} f_{nij}, (i,j) \in [n]^2, \\
&z_{ni} = \frac{1}{n} \sum_{j=1}^n (\eta_{nij} - f_{nij} W_{nij}),\ S_n = \sum_{i=1}^n z_{ni}^2.
\end{aligned}
$$

Then

$$
\frac{S_n - {\sigma_n}^2}{n^{\frac{-1}{2}} \sqrt{5{\sigma_n}^4 + \mathrm{O}(n^{-1})}} \to_d \mathcal{N}(0,1) \text{ as } n \to +\infty.
$$

Proof. According to the definition of the independent RVs $\{\eta_{nij}\}$, $n \in \mathbb{N}$, $(i,j) \in [n]^2$, we have

$$
\mathbb{E}\eta_{nij}^k = f_{nij}^k W_{nij},\ k \in \mathbb{N}.
$$

Hence, for $y_{ni} = \sqrt{n} z_{ni}$, $i \in [n]$, we have

$$
\begin{aligned}
\mathbb{E} y_{ni} =& \mathbb{E}\left(\sqrt{n} z_{ni}\right) = \mathbb{E}\left(\sqrt{n} \cdot \frac{1}{n} \sum_{j=1}^n (\eta_{nij} - f_{nij} W_{nij})\right) \\
=& \frac{\sqrt{n}}{n} \mathbb{E}\left(\sum_{j=1}^n (\eta_{nij} - f_{nij} W_{nij})\right) = \frac{\sqrt{n}}{n} \sum_{j=1}^n \mathbb{E}(\eta_{nij} - f_{nij} W_{nij}) \\
=& \frac{\sqrt{n}}{n} \sum_{j=1}^n [\mathbb{E}(\eta_{nij}) - f_{nij} W_{nij}] = \frac{\sqrt{n}}{n} \sum_{j=1}^n (f_{nij} W_{nij} - f_{nij} W_{nij}) = 0.
\end{aligned}
$$

Then we will apply the Lyapunov CLT to $\sum\limits_{i=1}^n y_{ni}^2$. So let's do the next calculation,

$$
\begin{aligned}
&\mathbb{E} y_{ni}^2 = \mathbb{E}(n z_{ni}^2) = \mathbb{E}\left[n \left(\frac{1}{n} \sum_{j=1}^n \eta_{nij} - f_{nij} W_{nij}\right)^2\right] \\
=& \mathbb{E}\left[\frac{1}{n} \left(\sum_{j=1}^n \eta_{nij} - n f_{nij} W_{nij}\right)^2\right] \\
=& n^{-1} \mathbb{E}\left(\sum_{1 \le j,k \le n} (\eta_{nij} - f_{nij} W_{nij})(\eta_{nik} - f_{nik} W_{nik})\right) \\
=& n^{-1} \mathbb{E}\left(\sum_{1 \le j \le n} (\eta_{nij} - f_{nij} W_{nij})^2\right) \\
&+ 2n^{-1} \mathbb{E}\left(\sum_{1 \le j < k \le n} (\eta_{nij} - f_{nij} W_{nij})(\eta_{nik} - f_{nik} W_{nik})\right) \\
=& \sigma_{ni}^2 + 2n^{-1} \sum_{1 \le j < k \le n} \mathbb{E}(\eta_{nij} - f_{nij} W_{nij}) \mathbb{E}(\eta_{nik} - f_{nik} W_{nik}) \\
=& \sigma_{ni}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{ni}^2 =& n^{-1}\mathbb{E}\Big(\sum_{1\le j\le n}\big(\eta_{nij}-f_{nij}W_{nij}\big)^2\Big)\\
=& n^{-1}\sum_{1\le j\le n}\mathbb{E}(\eta_{nij}-f_{nij}W_{nij})^2\\
=& n^{-1}\sum_{1\le j\le n}\mathbb{E}\left(\eta_{nij}^2-2\eta_{_{nij}}f_{nij}W_{nij}+{f_{nij}}^2{W_{nij}}^2\right)\\
=& n^{-1}\sum_{1\le j\le n}\left[\mathbb{E}\left(\eta_{nij}^2\right)-2\mathbb{E}\left(\eta_{_{nij}}f_{nij}W_{nij}\right)+\mathbb{E}\left({f_{nij}}^2{W_{nij}}^2\right)\right]\\
=& n^{-1}\sum_{1\le j\le n}\left[\mathbb{E}\left(\eta_{nij}^2\right)-\mathbb{E}\left({f_{nij}}^2{W_{nij}}^2\right)\right]\\
=& n^{-1}\sum_{1\le j\le n}\mathbb{E}\left(\eta_{nij}^2-{f_{nij}}^2{W_{nij}}^2\right)\\
=& n^{-1}\sum_{1\le j\le n}\left({f_{nij}}^2W_{nij}-{f_{nij}}^2{W_{nij}}^2\right)\\
=& n^{-1}\sum_{1\le j\le n}{f_{nij}}^2W_{nij}\left(1-W_{nij}\right).
\end{aligned}
$$

Once again, we calculate

$$
\begin{aligned}
\mathbb{E}y_{ni}^4 =& n^{-2}\mathbb{E}\Bigg(\sum_{1\le j_1,j_2,j_3,j_4\le n}(\eta_{nij_1}-f_{nij_1}W_{nij_1})\cdots(\eta_{nij_4}-f_{nij_4}W_{nij_4})\Bigg)\\
=& 6n^{-2}\sum_{1\le j<k\le n}\mathbb{E}(\eta_{nij}-f_{nij}W_{nij})^2\mathbb{E}(\eta_{nik}-f_{nik}W_{nik})^2+n^{-2}\sum_{1\le j\le n}E(\eta_{nij}-f_{nij}W_{nij})^4\\
=& \frac{6n(n-1)}{n^2}\sigma_{ni}^4+\mathrm{O}(n^{-1})\\
\le& 6\sigma_{ni}^4+\mathrm{O}(n^{-1}).
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}y_{ni}^6 =& n^{-3}\mathbb{E}\Bigg(\sum_{1\le j_1,j_2,j_3,j_4,j_5,j_6\le n}(\eta_{nij_1}-f_{nij_1}W_{nij_1})\cdots(\eta_{nij_6}-f_{nij_6}W_{nij_6})\Bigg)\\
=& (\begin{matrix}6\\2\end{matrix})(\begin{matrix}4\\2\end{matrix})n^{-2}\sum_{1\le j<k<l\le n}\mathbb{E}(\eta_{nij}-f_{nij}W_{nij})^2\times\mathbb{E}(\eta_{nik}-f_{nik}W_{nik})^2\mathbb{E}(\eta_{nil}-f_{nil}W_{nil})^2+\mathrm{O}(n^{-1})\\
=& \frac{90n(n-1)(n-2)}{n^3}\sigma_{ni}^6+\mathrm{O}(n^{-1})\\
\le& 90\sigma_{ni}^6+\mathrm{O}(n^{-1}).
\end{aligned}
$$

For $n\in\mathbb{N}$, let

$$
\zeta_{ni}=\frac{y_{ni}^2-\mathbb{E}y_{ni}^2}{\sqrt{nVar\left(y_{ni}^2\right)}}=\frac{y_{ni}^2-{\sigma_{ni}}^2}{n^{\frac{1}{2}}\sqrt{5\sigma_{ni}^4+\mathrm{O}\left(n^{-1}\right)}},\quad i\in[n]\,.
$$

Consider

$$
\zeta_{n1},\zeta_{n2},\cdots,\zeta_{nn}.
$$

Thus $\zeta_{ni}, i\in[n]$ are IID RVs. Further, $E\zeta_{ni}=0, V\left(\sum_{i=1}^{n}\zeta_{ni}\right)=1$.

Now let's verify the Lyapunov conditions,

$$
\sum_{i=1}^{n}\mathbb{E}|\zeta_{ni}|^3\le\frac{\sum_{i=1}^{n}\mathbb{E}\left(y_{ni}^6+3y_{ni}^4{\sigma_{ni}}^2+3y_{ni}^2{\sigma_{ni}}^4+{\sigma_{ni}}^6\right)}{n^{\frac{3}{2}}\left(5{\sigma_{ni}}^4+\mathrm{O}\left(n^{-1}\right)\right)}=\mathrm{O}\left(n^{\frac{-1}{2}}\right)\to 0\text{ as }n\to+\infty.
$$

Thus by the CLT, we come to the conclusion that, as $n \to \infty$

$$\frac{\sum\limits_{i=1}^{n}\left(y_{ni}^2 - {\sigma_{ni}}^2\right)}{\sqrt{n\left(5{\sigma_n}^4 + \mathrm{O}\left(n^{-1}\right)\right)}} = \frac{n^{-1}\sum\limits_{i=1}^{n} y_{ni}^2 - {\sigma_n}^2}{n^{\frac{-1}{2}}\sqrt{5{\sigma_n}^4 + \mathrm{O}\left(n^{-1}\right)}} \to_d \mathcal{N}\left(0,1\right),$$

that is,

$$\frac{S_n - {\sigma_n}^2}{n^{\frac{-1}{2}}\sqrt{5{\sigma_n}^4 + \mathrm{O}\left(n^{-1}\right)}} \to_d \mathcal{N}\left(0,1\right).$$

This completes the proof. □

In order to prove the Theorem 5.2, we need to extend Lemma 5.3.

Corollary 5.4. Suppose f_{nij} in Lemma 5.3 depend on real parameter $t \in [0,T]$ for some T and the functions $f_{nij}\left(t\right)$, $n \in \mathbb{N}$, $i,j \in [n]$, are uniformly bounded for $t \in [0,T]$. We add t-dependence to all variables defined using $f_{nij}\left(t\right)$, keeping the notation of Lemma 5.3 and assume that

$$\lim_{n\to\infty}\inf \sigma_n^2\left(t\right) = \lim_{n\to\infty}\inf n^{-2}\sum_{i,j=1}^{n} f_{nij}^2\left(t\right)W_{nij}\left(1 - W_{nij}\right) \geq C_1 > 0, \tag{49}$$

for every $t \in [0,T]$.

Then for every $t \in [0,T]$, we have the following relation:

$$\frac{S_n\left(t\right) - {\sigma_n}^2\left(t\right)}{n^{\frac{-1}{2}}\sqrt{5{\sigma_n}^4\left(t\right) + \mathrm{O}\left(n^{-1}\right)}} \to_d N\left(0,1\right) \text{ as } n \to \infty.$$

The proof of Theorem 5.2 in addition to the CLT used above, we need to introduce an auxiliary IVP for the Kuramoto model on a weighted graph $\tilde{G}_n = \mathbb{H}\left(W, X_n\right)$. The latter is a complete graph with the node set $V\left(\tilde{G}_n\right) = [n]$. Each edge of $\tilde{G}_n$ is supplied with the weight

$$W_{nij} = W\left(x_{ni}, x_{nj}\right), \quad (i,j) \in [n]^2, \quad i \neq j.$$

The IVP for the discrete Kuramoto model on the weighted graph $\tilde{G}_n$

$$\begin{cases} \dfrac{d}{dt}v_{ni}(t) = \omega + \dfrac{1}{n}\displaystyle\sum_{j:(i,j)\in E\left(\tilde{G}_n\right)} W_{nij}\sin\left(v_{nj}\left(t\right) - v_{ni}\left(t\right)\right) + h\sin\left(v_{ni}(t)\right), \\ v_{ni}(0) = g\left(x_i\right), i \in [n] \end{cases} \tag{50}$$

Let $v_n\left(t\right) = \left(v_{n1}\left(t\right), v_{n2}\left(t\right), \cdots, v_{nn}\left(t\right)\right)$ be the solution of the IVP 50. Define a function $v_n\left(x,t\right)$ on $I \times \mathbb{R}$

$$v_n\left(x,t\right) = v_n\left(t\right), \quad t \in \mathbb{R}, \quad x \in I_{ni}, \quad i \in [n].$$

Next, denote a step-function W_n on I^2

$$W_n\left(x,y\right) = W_{nij}, \quad (x,y) \in I_{ni} \times I_{nj}, \quad i,j \in [n].$$

Then by construction $v_n\left(x,t\right)$ satisfies the following IVP

$$\begin{cases} \dfrac{\partial}{\partial t}v_n\left(x,t\right) = \omega + \displaystyle\int_I W_n\left(x,y\right)\sin\left(v_n\left(y,t\right) - v_n\left(x,t\right)\right)dy + h\sin\left(v_n\left(x,t\right)\right), \\ v_n\left(x,0\right) = g\left(x_{ni}\right), x \in I_{ni}, i \in [n]. \end{cases} \tag{51}$$

Theorem 3.3 shows that for large n, $v_n\left(x,t\right)$ is approximately the solution of IVP 45. Therefore, we have the lemma shown below.

Lemma 5.5. Assume $W \in L^\infty\left(I^2\right)$ is almost everywhere continuous on I^2, and $g \in L^\infty\left(I\right)$. Then for $T > 0$ is arbitrary,

$$\|\mathbf{u} - \mathbf{v_n}\|_{C(0,T;L^2(I))} \to 0 \text{ as } n \to \infty.$$

We need to use Lemma 5.5 to show the following result.

Lemma 5.6. Assume $W \in W_0$ is continuous almost everywhere on I^2, and $g \in L^\infty(I)$. Let $u(x,t)$ and $v_n(x,t)$ be the solutions of the IVPs 45 and 51 respectively, and let

$$\sigma^2(t) = \int_{I^2} \sin(u(y,t) - u(x,t))^2 W_p(x,y)(1 - W_p(x,y))\,dxdy,$$

$$\sigma_n^2(t) = \int_{I^2} \sin(v_n(y,t) - v_n(x,t))^2 W_n(x,y)(1 - W_n(x,y))\,dxdy.$$

Then

$$\sup_{t\in[0,T]} \left|\sigma_n^2(t) - \sigma^2(t)\right| \le C_2\left[\|\mathbf{v_n} - \mathbf{u}\|_{C(0,T;L^2(I))} + \|W_n - W_p\|_{L^2(I^2)}\right],$$

for some $C_2 > 0$. In particular, $\sigma_n^2 \to \sigma^2$ uniformly in $t \in [0,T]$.

Proof. The proof is divided into three steps.

Step 1. We now use triangle inequality, continuity of $sin(\cdot)$ and the bound on $sin(\cdot)$, $|sin(\cdot)| \le 1$ to obtain that for any $t \in [0,T]$,

$$\begin{aligned}
&\left|\int_{I^2} \sin(v_n(y,t) - v_n(x,t))^2 - \sin(u(y,t) - u(x,t))^2 dxdy\right| \\
\le&\left|\int_{I^2} [\sin(v_n(y,t) - v_n(x,t)) - \sin(u(y,t) - u(x,t))] \times [\sin(v_n(y,t) - v_n(x,t)) + \sin(u(y,t) - u(x,t))]dxdy\right| \\
\le&2\left|\int_{I^2} [\sin(v_n(y,t) - v_n(x,t)) - \sin(u(y,t) - u(x,t))]dxdy\right| \\
\le&2\left|\int_{I^2} [v_n(y,t) - v_n(x,t) - u(y,t) + u(x,t)]dxdy\right| \\
\le&2\int_{I^2} |v_n(y,t) - u(y,t)| + |v_n(x,t) - u(x,t)|dxdy \\
\le&4\|\mathbf{v_n} - \mathbf{u}\|_{C(0,T;L^2(I))} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Thus,

$$\max_{t\in[0,T]} \left|\int_{I^2} \sin(v_n(y,t) - v_n(x,t))^2 dxdy\right| \le C_3, n \in \mathbb{N}, \tag{52}$$

for some C_3 independent of n.

Step 2. Denote $q(x) = x(1-x)$. For $x, y \in [0,1]$, $|q(x) - q(y)| \le |x - y|$. Hence,

$$|q(x) - q(y)| \le |W_p - W_n|. \tag{53}$$

Step 3. Finally, we estimate $\left|\sigma_n^2(t) - \sigma^2(t)\right|$. For any $t \in [0,T]$, we have

$$\begin{aligned}
&\left|\int_{I^2} \sin(v_n(y,t) - v_n(x,t))^2 q(W_n(x,y))dxdy - \int_{I^2} \sin(u(y,t) - u(x,t))^2 q(W_p(x,y))dxdy\right| \\
\le&\left|\int_{I^2} \sin(v_n(y,t) - v_n(x,t))^2 [q(W_n(x,y)) - q(W_p(x,y))]dxdy\right| \\
&+\left|\int_{I^2} [\sin(v_n(y,t) - v_n(x,t))^2 - \sin(u(y,t) - u(x,t))^2] \times \mathrm{q(W_p(x,y))dxdy}\right|.
\end{aligned} \tag{54}$$

We now use the Cauchy-Schwarz inequality, $|q(W_p)| \le 1$, 52 and 53 from 54 to obtain

$$\sup_{t\in[0,T]} |\sigma_n^2(t) - \sigma^2(t)| \le C_2\|W_p - W_n\|_{L^2(I^2)} + 4\|\mathbf{v_n} - \mathbf{u}\|_{C(0,T;L^2(I))}. \tag{55}$$

As $n \to \infty$, $W_n \to W_p$ at every point of continuity W_p, that is almost continuous everywhere on I^2. So we have the following relationship by the the dominated convergence theorem,

$$\|W_p - W_n\|_{L^2(I^2)} \to 0 \text{ as } n \to +\infty. \tag{56}$$

We combine 55, 56 with Lemma 5.5 to prove the Lemma 5.6. □

Proof of Theorem 5.2. Denote $\varsigma_{ni}(t) = u_{ni}(t) - v_{ni}(t)$, $i \in [n]$, and

$$\varsigma_n(t) = (\varsigma_{n1}(t), \varsigma_{n2}(t), \cdots, \varsigma_{nn}(t)).$$

By subtracting Equation i in 50 from the corresponding equation in 46, we obtain

$$\begin{aligned}\frac{d}{dt}\varsigma_{ni} =& \frac{1}{n}\left(\sum_{j=1}^{n}\xi_{nij}\sin(u_{nj}-u_{ni}) - \sum_{j=1}^{n}W_{nij}\sin(v_{nj}-v_{ni})\right) + h[\sin(u_{ni}) - \sin(v_{ni})]\\ =& \frac{1}{n}\sum_{j=1}^{n}\xi_{nij}\left[\sin(u_{nj}-u_{ni}) - \sin(v_{nj}-v_{ni})\right] + z_{ni} + h[\sin(u_{ni}) - \sin(v_{ni})],\end{aligned} \tag{57}$$

where $z_{ni} = \frac{1}{n}\sum\limits_{j=1}^{n}\xi_{nij}\sin(v_{nj}-v_{ni}) - \frac{1}{n}\sum\limits_{j=1}^{n}W_{nij}\sin(v_{nj}-v_{ni})$.

We multiply $n^{-1}\varsigma_{ni}$ on both sides of 57 and sum it over i to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varsigma_n\|_{2,n}^2 =& \frac{1}{n^2}\sum_{i,j=1}^{n}\xi_{nij}[\sin(u_{nj}-u_{ni}) - \sin(v_{nj}-v_{ni})]\varsigma_{ni}\\ &+ (z_n, \varsigma_n)_n + \frac{h}{n}\sum_{i=1}^{n}[\sin(u_{ni}) - \sin(v_{ni})]\varsigma_{ni}.\end{aligned} \tag{58}$$

We use $|\xi_{nij}| \le 1$, the Cauchy-Schwarz inequality, continuity of $sin(\cdot)$ and the triangle inequality to estimate the first term on the right-hand side of (58) as follows:

$$\begin{aligned}&\left|\frac{1}{n^2}\sum_{i,j=1}^{n}\xi_{nij}[\sin(u_{nj}-u_{ni}) - \sin(v_{nj}-v_{ni})]\varsigma_{ni}\right|\\ =& \frac{1}{n^2}\sum_{i,j=1}^{n}|\xi_{nij}||u_{nj}-u_{ni}-v_{nj}+v_{ni}||\varsigma_{ni}|\\ =& \frac{1}{n^2}\sum_{i,j=1}^{n}|\xi_{nij}||\varsigma_{nj}-\varsigma_{ni}||\varsigma_{ni}|\\ \le& \frac{1}{n^2}\sum_{i,j=1}^{n}(|\varsigma_{nj}| + |\varsigma_{ni}|)|\varsigma_{ni}|\\ \le& 2\|\varsigma_n\|_{2,n}^2.\end{aligned} \tag{59}$$

We use the Cauchy-Schwarz inequality to estimate the second term on the right hand side of (58)

$$|(z_n, \varsigma_n)_n| \le \|z_n\|_{2,n}\|\varsigma_n\|_{2,n} \le \frac{1}{2}\left(\|z_n\|_{2,n}^2 + \|\varsigma_n\|_{2,n}^2\right), \tag{60}$$

where $z_n = (z_{n1}, z_{n2}, \cdots z_{nn})$.

We use the Cauchy-Schwarz inequality and continuity of $sin(\cdot)$ to estimate the third term on the right-hand side of (58) as follows:

$$\left|\frac{h}{n}\sum_{i=1}^{n}[\sin(u_{ni}) - \sin(v_{ni})]\varsigma_{ni}\right| \le \frac{h}{n}\sum_{i=1}^{n}|u_{ni}-v_{ni}||\varsigma_{ni}| = \frac{h}{n}\sum_{i=1}^{n}|\varsigma_{ni}^2| = h\|\varsigma_n\|_{2,n}^2. \tag{61}$$

Combining (58), (59), (60) and (61), we obtain

$$\frac{d}{dt}\|\varsigma_n\|_{2,n}^2 \le (5+2h)\|\varsigma_n\|_{2,n}^2 + \|z_n\|_{2,n}^2.$$

We use Gronwall's inequality to yield

$$\max_{t\in[0,T]} \|\varsigma_n\|_{2,n}^2 \leq \frac{\max\limits_{t\in[0,T]} \|z_n\|_{2,n}^2}{(5+2h)} \exp\left\{(5+2h)\,T\right\}. \tag{62}$$

Hence,

$$\max_{t\in[0,T]} \|\varsigma_n\|_{2,n} \leq \frac{\max\limits_{t\in[0,T]} \|z_n\|_{2,n}}{\sqrt{(5+2h)}} \exp\left\{(5+h)\,T\right\}. \tag{63}$$

In order to estimate $\|z_n\|_{2,n}$, we use Corollary (5.4) with

$$f_{nij}(t) = \sin\left(v_{nj}(t) - v_{ni}(t)\right) \text{ and } W_{nij} = W\left(x_{ni}, x_{nj}\right).$$

By Lemma 5.6 and (47), we have

$$\min_{t\in[0,T]} \sigma_n^2(t) \geq C_4 > 0, \tag{64}$$

for sufficiently large n. In particular, (49) holds. Hence, by Lemma (5.6), we obtain

$$\max_{t\in[0,T]} \sigma_n^2(t) \leq C_5, \quad n \in \mathbb{N}. \tag{65}$$

For arbitrary $t \in [0,T]$, we use Corollary (5.4) to have

$$\begin{aligned}
&\mathbb{P}\left(\left|n\,\|z_n(t)\|_{2,n}^2 - \sigma_n{}^2(t)\right| > 1\right)\\
=&\mathbb{P}\left(\left|\frac{n\left\|z_n(t)_{2,n}^2\right\| - \sigma_n^2(t)}{n^{\frac{-1}{2}}\sqrt{5\sigma_n{}^4(t) + \mathrm{O}(n^{-1})}}\right| > \frac{n^{\frac{1}{2}}}{\sqrt{5\sigma_n{}^4(t) + \mathrm{O}(n^{-1})}}\right)\\
\leq&\mathbb{P}\left(\left|\frac{n\left\|z_n(t)_{2,n}^2\right\| - \sigma_n^2(t)}{n^{\frac{-1}{2}}\sqrt{5\sigma_n{}^4(t) + \mathrm{O}(n^{-1})}}\right| > \frac{n^{\frac{1}{2}}}{\sqrt{5C_5^2 + \mathrm{O}(n^{-1})}}\right) \to 0,
\end{aligned} \tag{66}$$

as $n \to \infty$.

Combining (65) with (66), we have

$$\mathbb{P}\left(\|z_n(t)\|_{2,n}^2 \leq (C_5+1)\,n^{-1}\right) \leq \mathbb{P}\left(\left|n\,\|z_n(t)\|_{2,n}^2 - \sigma_n^2(t) > 1\right|\right) \to 0 \text{ as } n \to \infty. \tag{67}$$

Due to $t \in [0,T]$ is arbitrary in (67), we have

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{t\in[0,T]} \|z_n(t)\|_{2,n} \leq C_6 n^{\frac{-1}{2}}\right) = 0. \tag{68}$$

We combine (62) with (68) to find that $\|\varsigma_n\|_{2,n}$ tends to 0 in probability.

With the definitions of $\|\varsigma_n\|$ and $\mathbf{u}_n$ in hand, we have

$$\|\mathbf{u_n} - \mathbf{u}\|_{C(0,T;L^2(I))} \leq \max_{t\in[0,T]} \|\varsigma_n(t)\|_{2,n} + \|\mathbf{v_n} - \mathbf{u}\|_{C(0,T;L^2(I))}.$$

The Lemma 5.2 combines with (68), we show that $\|\mathbf{u_n} - \mathbf{u}\|_{C(0,T;L^2(I))}$ tends to 0 in probability as $n \to \infty$.

6 | CONCLUSION

This paper improves on the previous ones and adds external drive of a certain strength to the Kuramoto model oscillators. We study coupled Kuramoto model with external drive and show that the solutions of the IVPs for discrete Kuramoto model converge to the solution of the IVP for its continuum limit on deterministic graphs, W-random graphs and SW graphs respectively. Our results also reveal that properties of graphs affect the convergence rate of discrete problem solutions and the accuracy of the continuous limit. For deterministic graphs, we show that the rate of convergence depends on the regularity of the boundary of support of the graph limit. For random graphs, the rate of convergence depends on the Central Limit Theorem and the regularity of the graph on W. However, studies on the generalized Kuramoto model under different network structures are limited and the complexity of complex networks makes the study of complex networks challenging, it is hoped that further studies can be carried out in the future. For example: We can study asymptotic behavior of generalized Kuramoto model on scale-free networks.

ACKNOWLEDGMENTS

The authors are grateful to the editor and the anonymous reviewers for their valuable comments and suggestions. This work does not have any conflicts of interest. This work is supported by the National Natural Science Foundation of China (No.11201095), the Fundamental Research Funds for the Central Universities (No. 2020), the Postdoctoral research startup foundation of Heilongjiang (No. LBH-Q14044), the Science Research Funds for Overseas Returned Chinese Scholars of Heilongjiang Province (No. LC201502).

References

1. Strogatz SH, Stewat I. Coupled oscillators and biological synchronization. *Sci. Amer.* 1993;269:102–109.
2. Medvedev GS, Zhuravytska S. The geometry of spontaneous spiking in neuronal networks. *J. Nonlinear Sci.* 2012;22:689–725.
3. Ermentrout GB, Kopell N. Multiple pulse interactions and averaging in systems of coupled neural oscillators. *J. Math. Biol.* 1991;29:195–217.
4. Li RD, Erneux T. Preferential instability in arrays of coupled lasers. *Phys. Rev.* 1992;46:4252–4260.
5. Acerbi E, Mingione G. Regularity results for electrorheological fluids: the stationary case. *C. R. Math. Acad. Sci.* 2002;334:817–822.
6. Watanabe S, Strogatz SH. Constants of motion for superconducting Josephson arrays. *Phys. D.* 1994;74:197–253.
7. Pluchino A, Latora V, Rapisarda A. Changing opinions in a changing world: a new perspective in sociophysics. *Int. J. Mod. Phys. C.* 2005;16:515–531.
8. Kuramoto Y. Self-entrainment of a population of coupled non-linear oscillators. In: pp. 420-422. Lecture Notes in Phys., vol. 39: Berlin: Springer 1975.
9. Lovász L. Large networks and graph limits. In: American Mathematical Society, vol. 60: Providence, RI: American Mathematical Society Colloquium Publications 2012.
10. Ermentrout GB. Synchronization in a pool of mutually coupled oscillators with random frequencies. *J. Math. Biol.* 1985;22:1–9.
11. Girnyk T, Hasler M, Maistrenko Y. Multistability of twisted states in non-locally coupled Kuramoto-type models. *Chaos.* 2012;22:013114.
12. Wiley DA, Strogatz SH, Girvan M. The size of the sync basin. *Chaos.* 2006;16:015103.
13. Medvedev GS. The nonlinear heat equation on dense graphs and graph limits. *SIAM J. Math Anal.* 2014;546:2743–2766.
14. Medvedev GS. The nonlinear heat equation on W-random graphs. *Arch. Ration. Mech. Anal.* 2014;212:781–803.
15. Medvedev GS. Small-world networks of Kuramoto oscillators. *Phys. D.* 2014;266:13–22.
16. Medvedev GS. The continuum limit of the Kuramoto model on sparse random graphs http://arxiv.org/abs/1802.03787, 2018; .
17. Borgs C, Chayes J, L Lovász, Sós V VK. Limits of randomly grown graph sequences. *Eur. J. Comb.* 2011;32:985–999.
18. Lovász L, Szegedy B. Limits of dense graph sequences. *J. Combin. Theorey Ser. B.* 2006;96:933–957.
19. Billingsley P. *Probability and Measure.* Willey: New York; 1995.

20. Daleckii JL, Krein MG. Stability of solutions of differential equations in Banach space. In: American Mathematical Society. Providence, RI: American Mathematical Society Colloquium Publications 1974.

21. Falconer K. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons; 1997.

22. Kaliuzhnyi-Verbovetskyi D, Medvedev GS. The semilinear heat equation on sparse random graphs. *SIAM J. Math Anal.* 2017;49:1333–1355.

23. Hoppensteadt FC. *Izhikevich E.M. Weakly connected neural networks*. Springer; 1997.

$\square$