$\oplus-$ESSENTIAL SUPPLEMENTED MODULES

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Abstract

In this work $\oplus-e-$supplemented modules are defined and some properties of these modules are investigated. It is proved that the finite direct sum of $\oplus-e-$supplemented modules is also $\oplus-e-$supplemented. Let M be a distributive and $\oplus-e-$supplemented $R-$module. Then every factor module and homomorphic image of M are $\oplus-e-$supplemented. Let M be a $\oplus-e-$supplemented $R-$module with SSP property. Then every direct summand of M is $\oplus-e-$supplemented.

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1 INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an $R-$module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any $R-$module M, we have $M = M \oplus 0$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $RadM$. If M has no maximal submodules, then it is defined $RadM = M$. A submodule N of an R -module M is called an *essential submodule* and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $K = 0$ for every $K \leq M$ with $N \cap K = 0$. Let M be an $R-$module. M is called a *hollow* module if every proper submodule of M is small in M. M is called a *local module* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented module* if every submodule of M has a supplement in M. If every submodule of M has a supplement that is a direct summand in M, then M is called a $\oplus-$*supplemented module*. M is said to be *essential supplemented* (briefly, *e-supplemented*), if every essential submodule of M has a supplement in M. A module M is said to have the *Summand Sum Property* (SSP) if the sum of two direct summands of M is again a direct summand of M (see [2]). We say that a module M has $(D3)$ property if $M_1 \cap M_2$ is a direct summand of M for every direct summands M_1 and M_2 of M with $M = M_1 + M_2$ (see [8]). The intersection of all essential maximal submodules of an $R-$module M is called the *generalized radical* of M and denoted by $Rad_g M$. If M have no essential maximal submodules, then we denote $Rad_g M = M$.

More informations about supplemented modules are in [1], [8], [11] and [12]. More results about $\oplus-$supplemented modules are in [3], [4], [7] and [8]. More details about essential supplemented modules are in [9] and [10]. More informations about generalized radical are in [5] and [6].

Lemma 1.1 *Let M be an $R-$module.*

(1) *If $K \leq L \leq M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.*

(2) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq M$.*

(3) *For $N \leq K \leq M$, if $K/N \trianglelefteq M/N$, then $K \trianglelefteq M$.*

(4) *If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.*

(5) *If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.*

Proof. See [12, 17.3]. ■

Lemma 1.2 *Let M be an $R-$module.*

(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*

(2) *Let $K_i \ll L_i \leq M$ for $i = 1, 2, ..., n$. Then $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.*

(3) *Let $K_i \ll M$ for $i = 1, 2, ..., n$. Then $K_1 + K_2 + ... + K_n \ll M$.*

(4) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$.*

(5) *If $K \ll L \leq M$, then $K \ll M$.*

Proof. See [12, 19.3]. ■

2 $\oplus-$ESSENTIAL SUPPLEMENTED MODULES

Definition 2.1 *Let M be an $R-$module. If every essential submodule of M has a supplement that is a direct summand of M, then M is called a $\oplus-$essential supplemented (briefly, $\oplus-e-$ supplemented) module.*

Clearly we can see that every $\oplus-$supplemented module is $\oplus-e-$supplemented. Hollow and local modules are $\oplus-e-$supplemented.

Definition 2.2 *Let M be an $R-$module and $X \leq M$. If X is a supplement of an essential submodule of M, then X is called an e-supplement submodule in M. (See [9, Definition 2.2])*

Lemma 2.3 *Let $M = M_1 \oplus M_2$. If M_1 and M_2 are $\oplus-e-$supplemented, then M is also $\oplus-e-$supplemented.*

Proof. Let U be any essential submodule of M. Since $U \trianglelefteq M$, by Lemma 1.1, $M_1 + U \trianglelefteq M$ and $(M_1 + U) \cap M_2 \trianglelefteq M_2$. Since M_2 is $\oplus-e-$supplemented, $(M_1 + U) \cap M_2$ has a supplement X that is a direct summand of M_2. Since X is a supplement of $(M_1 + U) \cap M_2$ in M_2, $M_2 = (M_1 + U) \cap M_2 + X$ and $(M_1 + U) \cap X = (M_1 + U) \cap M_2 \cap X \ll X$. By $M_2 = (M_1 + U) \cap M_2 + X$, $M = M_1 \oplus M_2 = M_1 + (M_1 + U) \cap M_2 + X = M_1 + U + X$. Since $U \trianglelefteq M$, by Lemma 1.1, $U + X \trianglelefteq M$ and $(U + X) \cap M_1 \trianglelefteq M_1$. Since M_1 is $\oplus-e-$supplemented, $(U + X) \cap M_1$ has a supplement Y that is a direct summand of M_1. Since Y is a supplement of $(U + X) \cap M_1$ in M_1, $M_1 = (U + X) \cap M_1 + Y$ and $(U + X) \cap Y = (U + X) \cap M_1 \cap Y \ll Y$. By $M_1 = (U + X) \cap M_1 + Y$, $M = M_1 + U + X = (U + X) \cap M_1 + Y + U + X = U + X + Y$. Since $(M_1 + U) \cap X \ll X$ and $(U + X) \cap Y \ll Y$, by Lemma 1.2, $U \cap (X + Y) \leq (U + Y) \cap X + (U + X) \cap Y \leq (M_1 + U) \cap X + (U + X) \cap Y \ll X + Y$. Hence $X + Y$ is a supplement of U in M. Since X is a direct summand of M_2 and Y is a direct summand of M_1, $X + Y$ is a direct summand of $M = M_1 \oplus M_2$. Hence M is $\oplus-e-$supplemented. ■

Corollary 2.4 *Let M be an $R-$module and $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. If M_i $\oplus-e-$supplemented for every $i = 1, 2, ..., n$, then M is also $\oplus-e-$supplemented.*

Proof. Clear from Lemma 2.3. ■

Proposition 2.5 *Let M be a $\oplus-e-$supplemented module. If every e-supplement submodule in M is a direct summand of M, then every direct summand of M is $\oplus-e-$supplemented.*

Proof. Let N be a direct summand of M and $M = N \oplus K$ with $K \leq M$. Since M is e-supplemented, by [9, Lemma 2.9], M/K is e-supplemented. By $\frac{M}{K} = \frac{N \oplus K}{K} \cong \frac{N}{N \cap K} = \frac{N}{0} \cong N$, N is also e-supplemented. Let $X \trianglelefteq N$ and Y be a supplement of X in N. Since $M = N \oplus K$, by [1, Exercise 20.39 (1)], Y is a supplement of $K + X$ in M. Since $X \trianglelefteq N$ and $M = N \oplus K$, by using the canonical projection $\pi : M \longrightarrow N$ we can see by Lemma 1.1, $K + X = \pi^{-1}(X) \trianglelefteq$

M. Hence Y is an e-supplement submodule in M. Since every e-supplement submodule in M is a direct summand of M, Y is a direct summand of M. By $Y \leq N$, Y is also a direct summand of N. Hence N is $\oplus - e-$supplemented. ■

Lemma 2.6 *Let M be a $\oplus - e-$supplemented $R-$module and $K \leq M$. If $\frac{X+K}{K}$ is a direct summand of $\frac{M}{K}$ for every direct summand X of M, then $\frac{M}{K}$ is $\oplus - e-$supplemented.*

Proof. Let U/K be any essential submodule of M/K. Since $U/K \trianglelefteq M/K$, by Lemma 1.1, $U \trianglelefteq M$. Since M is $\oplus - e-$supplemented, U has a supplement X in M that is a direct summand in M. Since X is a supplement of U in M and $K \leq U$, by [12, 41.1 (7)], $\frac{X+K}{K}$ is a supplement of U/K in M/K. Since X is a direct summand of M, by hypothesis, $\frac{X+K}{K}$ is a direct summand of M/K. Hence M/K is $\oplus - e-$supplemented. ■

Lemma 2.7 *Let M be a distributive and $\oplus - e-$supplemented $R-$module. Then every factor module of M is $\oplus - e-$supplemented.*

Proof. Let $K \leq M$ and X be a direct summand of M. Since X is a direct summand of M, there exists $Y \leq M$ such that $M = X \oplus Y$. Since $M = X \oplus Y$, $\frac{M}{K} = \frac{X+K}{K} + \frac{Y+K}{K}$. Since M is distributive, $(X+K) \cap (Y+K) = K$ and $\frac{X+K}{K} \cap \frac{Y+K}{K} = \frac{(X+K)\cap(Y+K)}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{X+K}{K} \oplus \frac{Y+K}{K}$ and by Lemma 2.6, M/K is $\oplus - e-$supplemented. ■

Corollary 2.8 *Let M be a distributive and $\oplus - e-$supplemented $R-$module. Then every homomorphic image of M is $\oplus - e-$supplemented.*

Proof. Clear from Lemma 2.7. ■

Lemma 2.9 *Let M be an $R-$module with $(D3)$ property and N be a direct summand of M. If every essential submodule of N has a supplement that is a direct summand in M, then N is $\oplus - e-$supplemented.*

Proof. Since N is a direct summand of M, there exists $K \leq M$ such that $M = N \oplus K$. Let $U \trianglelefteq N$. By hypothesis, U has a supplement X that is a direct summand in M. Here $M = U + X$ and $U \cap X \ll X$. Since $U \leq N$, $M = U + X = N + X$ and since M has $(D3)$ property, $N \cap X$ is a direct summand of M. Then there exists $Y \leq M$ such that $M = (N \cap X) \oplus Y$. Here $N = (N \cap X) \oplus (N \cap Y)$. Since $M = U + X$ and $U \leq N$, by Modular Law, $N = U + (N \cap X)$. Let $\pi : M \longrightarrow N \cap X$ be a canonical projection. Since $U \cap X \ll X \leq M$, by Lemma 1.2, $U \cap N \cap X = U \cap X = \pi(U \cap X) \ll N \cap X$. Hence $N \cap X$ is a supplement of U that is a direct summand in N. Therefore, N is $\oplus - e-$supplemented. ■

Corollary 2.10 *Let M be an $R-$module with $(D3)$ property and $M = X \oplus Y$. If every essential submodule of Y has a supplement that is a direct summand in M,then M/X is $\oplus - e-$supplemented.*

Proof. By Lemma 2.9, Y is $\oplus - e-$supplemented. Then by $\frac{M}{X} = \frac{X+Y}{X} \cong \frac{Y}{X \cap Y} = \frac{Y}{0} \cong Y$, M/X is also $\oplus - e-$supplemented. ■

Corollary 2.11 *Let $f : M \longrightarrow N$ be an $R-$module epimorphism and $M = Ker\,(f) \oplus K$. If every essential submodule of K has a supplement that is a direct summand in M, then N is $\oplus - e-$supplemented.*

Proof. Clear from Corollary 2.10, since $M/Ker\,(f) \cong \mathrm{Im}\,(f) = N$. ■

Lemma 2.12 *Let M be a $\oplus - e-$supplemented $R-$module, $K \leq M$ and $K = (K \cap M_1) \oplus (K \cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then M/K is $\oplus - e-$supplemented.*

Proof. Let $U/K \trianglelefteq M/K$. Then by Lemma 1.1, $U \trianglelefteq M$. Since M is $\oplus - e-$supplemented, U has a supplement V that is a direct summand in M. Here there exists $X \leq M$ such that $M = V \oplus X$. By hypothesis, $K = (K \cap V) \oplus (K \cap X)$. Since V is a supplement of U in M and $K \leq U$, by [12, 41.1 (7)], $\frac{V+K}{K}$ is a supplement of U/K in M/K. Since $M = V \oplus X$, $\frac{M}{K} = \frac{V+K}{K} + \frac{X+K}{K}$. Here $\frac{V+K}{K} \cap \frac{X+K}{K} = \frac{(V+K)\cap(X+K)}{K} = \frac{(V+K)\cap X+K}{K} = \frac{(V+K\cap V+K\cap X)\cap X+K}{K} = \frac{(V+K\cap X)\cap X+K}{K} = \frac{V\cap X+K\cap X+K}{K} = \frac{0+K}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{X+K}{K}$. Therefore, M/K is $\oplus - e-$supplemented. ■

Corollary 2.13 *Let M be a $\oplus - e-$supplemented $R-$module, $f : M \longrightarrow N$ be an $R-$module epimorphism with N be an $R-$module and $Ker\,(f) = (Ker\,(f) \cap M_1) \oplus (Ker\,(f) \cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then N is $\oplus - e-$supplemented.*

Proof. Clear from Lemma 2.12, since $M/Ker\,(f) \cong \mathrm{Im}\,(f) = N$. ■

Proposition 2.14 *Let M be a $\oplus - e-$supplemented $R-$module and $RadM \trianglelefteq M$. Then there exist $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$, $RadM_1 \ll M_1$ and $RadM_2 = M_2$.*

Proof. Since M is $\oplus - e-$supplemented and $RadM \trianglelefteq M$, $RadM$ has a supplement M_1 in M such that M_1 is a direct summand of M. Since M_1 is a direct summand of M, there exists $M_2 \leq M$ such that $M = M_1 \oplus M_2$. Since M_1 is a supplement of $RadM$ in M, $M = RadM + M_1$ and $M_1 \cap RadM \ll M_1$. By [12, 41.1 (5)], $RadM_1 = M_1 \cap RadM$. Hence $RadM_1 \ll M_1$. Since $M = M_1 \oplus M_2$, by [12, 21.6 (5)], $RadM = RadM_1 \oplus RadM_2$ and $M = RadM + M_1 = RadM_1 + RadM_2 + M_1 = M_1 \oplus RadM_2$. Hence $M_2 = M_2 \cap M = M_2 \cap (M_1 \oplus RadM_2) = (M_2 \cap M_1) \oplus RadM_2 = 0 \oplus RadM_2 = RadM_2$. ■

Proposition 2.15 *Let M be a $\oplus - e-$supplemented $R-$module and $Rad_g M \trianglelefteq M$. Then there exist $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$, $Rad_g M_1 \ll M_1$ and $Rad_g M_2 = M_2$.*

Proof. We can also prove this similar to proof of the previous Proposition. But we prove by different way. Since M is $\oplus-e-$supplemented and $Rad_g M \trianglelefteq M$, $Rad_g M$ has a supplement M_1 in M such that M_1 is a direct summand of M. Since M_1 is a direct summand of M, there exists $M_2 \leq M$ such that $M = M_1 \oplus M_2$. Since M_1 is a supplement of $Rad_g M$ in M, $M = Rad_g M + M_1$ and $M_1 \cap Rad_g M \ll M_1$. By [9, Corollary 2.4], $Rad_g M_1 = M_1 \cap Rad_g M$ holds. Then we have $Rad_g M_1 \ll M_1$. Assume that X be an essential maximal submodule of M_2. Since $X \trianglelefteq M_2$ and $M = M_1 \oplus M_2$, by using the canonical projection $\pi : M \longrightarrow M_2$ we can see by Lemma 1.1, $M_1 + X = \pi^{-1}(X) \trianglelefteq M$. Since $\frac{M}{M_1+X} = \frac{M_1+X+M_2}{M_1+X} \cong \frac{M_2}{M_2\cap(M_1+X)} = \frac{M_2}{M_2\cap M_1+X} = \frac{M_2}{X}$, $M_1 + X$ is a maximal submodule of M. Then $M_1 + X$ is an essential maximal submodule of M and $M = Rad_g M + M_1 \leq M_1 + X$. This is a contradiction. Hence $Rad_g M_2 = M_2$. ∎

Lemma 2.16 *Let M be a $\oplus-e-$supplemented $R-$module with SSP property. Then M/K is $\oplus-e-$supplemented for every direct summand K of M.*

Proof. Let K be any direct summand of M and $U/K \trianglelefteq M/K$. By Lemma 1.1, $U \trianglelefteq M$. Since M is $\oplus-e-$supplemented, U has a supplement V in M such that V is a direct summand of M. By [12, 41.1 (7)], $\frac{V+K}{K}$ is a supplement of U/K in M/K. Since K and V are direct summands of M and M has SSP property, $K+V$ is also a direct summand of M. Hence there exists $T \leq M$ such that $M = (K+V) \oplus T$. Since $M = (K+V) \oplus T$, $\frac{M}{K} = \frac{K+V+T}{K} = \frac{V+K}{K} + \frac{T+K}{K}$. Since $(V+K) \cap T = 0$, $\frac{V+K}{K} \cap \frac{T+K}{K} = \frac{(V+K)\cap(T+K)}{K} = \frac{(V+K)\cap T+K}{K} = \frac{0+K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{T+K}{K}$ and M/K is $\oplus-e-$supplemented. ∎

Corollary 2.17 *Let M be a $\oplus-e-$supplemented $R-$module with SSP property. Then every direct summand of M is $\oplus-e-$supplemented.*

Proof. Let T be any direct summand of M. Then there exists a submodule K of M such that $M = T \oplus K$. By Lemma 2.16, M/K is $\oplus-e-$supplemented. Since $\frac{M}{K} = \frac{T+K}{K} \cong \frac{T}{T\cap K} = \frac{T}{0} \cong T$, T is also $\oplus-e-$supplemented. ∎

Remark 2.18 *Let M be an $R-$module which has only four proper submodules 0, A, B, C with $C \leq A$, $C \leq B$, $A \nleq B$ and $B \nleq A$. Then M is e-supplemented but not $\oplus-e-$supplemented.*

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