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On correct solvability of some non-stationary problems in two-dimensional weighted Stepanov spaces

S. V. Pisareva*

The article deals with the correct solvability of some differential equations in two-dimensional weighted Stepanov spaces. The solution is sought by methods of semigroup theory and related methods using fractional powers of operators. The results of the paper are important for the theory of differential equations in Stepanov's spaces. Also they are important and form a basis for computational methods for differential equations considered based on explicit integral representations for solutions. The results are obtained using ideas of S.G.Krein on fractional powers of operators in Banach spaces.

Keywords: boundary value problem, semigroup, fractional powers of operators, Stepanov weighted space.

1. Introduction

To establish the correct solvability of mathematical problems is one of the most important condition for its numerical implementation. Starting from works of S.G. Krein methods of semigroups theory and related methods using fractional powers of operators became one of the main ones for the study of the correct solvability of initial and boundary- value problems for evolution equations and its applications.

2. Notations, definitions, problem statements and formulations of results

We call the two-dimensional weighted Stepanov spaces the classes of functions $S^+_{p,\alpha,\beta}$ for which the norm is finite

$$\|f\|_{S^+_{p,\alpha,\beta}} = \sup_{(\tau,\xi)\in R^2} \left[\int_0^1 \int_0^1 x^{\alpha-1} y^{\beta-1} |f(x+\tau, y+\tau)|^p dx dy\right]^{\frac{1}{p}}$$

at $0<\alpha<1,\ 0<\beta<1,\ p\geq 1$. Properties of generalized Stepanov spaces are studied un [1].

Our first result is

Theorem 1. *Differential equation*

$$\frac{\partial^2 u(t,x,y)}{\partial t^2} - \frac{\partial u(t,x,y)}{\partial x} - \frac{\partial u(t,x,y)}{\partial y} = 0, \tag{1}$$

satisfying initial boundary conditions

$$u(0,x,y) = g_0(x,y), \tag{2}$$

$$u(t,x,y) = 0 \ \ at \ \ t\to\infty, \tag{3}$$

is uniformly correctly solvable in two-dimensional weighted Stepanov spaces $S^+_{p,\alpha,\beta}$. The solution to the problem (1) - (3) is representable in the form

$$u(t,x,y) = U^+_{\frac{1}{2}}(t) g_0 = \frac{1}{2\sqrt{\pi}} \int_0^\infty t e^{\frac{-t^2}{4s}} s^{\frac{-3}{2}} g_0(x-s, y-s) ds$$

8, Timiryazeva Str., Voronezh, Russia
**Correspondence to: 8, Timiryazeva Str., Voronezh, Russia*

at $s \leq \min\{x, y\}$.

Another result is

Theorem 2. *Cauchy problem*

$$\frac{\partial^2 u(t,x,y)}{\partial t^2} = Au(t,x,y), \quad t \geq 0, \tag{4}$$

with initial boundary conditions

$$u(0,x,y) = u_0(x,y), \tag{5}$$

$$u'_t(0,x,y) = 0 \tag{6}$$

is uniformly correct. The solution to problem (4) - (6) is representable in the form

$$u(t,x,y) = \left(C(t)u_0\right)(x,y) = \frac{1}{2}\left[u_0(x-t,y-t) + u_0(x+t,y+t)\right].$$

Also we state

Theorem 3.

The Cauchy problem

$$\frac{\partial u(t,x,y)}{\partial t} = Au(t,x,y), \quad t \geq 0, \tag{7}$$

$$u(0,x,y) = u_0(x,y), \tag{8}$$

where $u_0 \in S^+_{p,\alpha,\beta}$ *and* $u_0 \geq 0$ *is correctly solvable. The solution to problem (7) - (8) is representable in the form*

$$u(t,x,y) = \left(T(t)u_0\right)(x,y) = \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{\frac{-s^2}{4t}}\left[u_0(x-s,y-s) + u_0(x+s,y+s)\right]ds.$$

3. Proofs of theorems.

Proof of the theorem 1.

Let the operator A_+ being given by the differential expression

$$A^+u(t,x,y) = \frac{\partial u(t,x,y)}{\partial x} + \frac{\partial u(t,x,y)}{\partial y}, \quad t \geq 0, \quad x,y \in R^1$$

and domain $D(A^+) = \left\{u \in S^+_{p,\alpha,\beta},\ A^+u \in C^1(S^+_{p,\alpha,\beta})\right\}$,

We will show that the operator $-A^+$ is a generator of a strongly continuous semigroup $U^+(t)$ defined by the equality

$$U_+(t)f = \begin{cases} f(x-t,y-t), & \text{at } t \leq \min\{x,y\}, \\ 0, & \text{at } t > \min\{x,y\}, \end{cases} \tag{9}$$

with $x, y \in R^1$.

In what follows, we need an operator J_n to be defined for any $n > 0$ (see, for example, [2])

$$J_n = \left(I - n^{-1}\left(-A^+\right)\right)^{-1} = nR\left(n, -A^+\right),$$

for which the next condition is satisfied

$$-A^+J_nf = n(J_n - I)f = nJ_nf - nf. \tag{10}$$

We introduce a family of functions $F_n(x,y)$ by the equality

$$F_n(x,y) = (J_nf)(x,y) = n\int_0^\infty e^{-nt}U^+(t)f\,dt.$$

In accordance with (9), we obtain

$$F_n(x,y) = n\int_0^{\min\{x,y\}} e^{-nt}f(x-t,y-t)dt.$$

In [3] it was shown that

$$\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)F_n(x,y) = nF_n(x,y) - nf(x,y) \tag{11}$$

for any $x, y \in R^1$. Comparing the resulting equality with the general formula (10), we find that

$$-A^+F^n(x,y) = \left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) F_n(x,y).$$

Since $R(J_n) = R(R(n, -A^+)) = D(-A^+)$, it follows that

$$-A^+\phi(x,y) = \left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \phi(x,y)$$

at any $\phi \in D(-A^+)$.

Conversely, now let the functions $\phi(x,y)$, $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial x}$ belong to the generalized Stepanov spaces $S^+_{p,\alpha,\beta}$. We will show that $\phi \in D(-A^+)$ and $-A^+\phi(x,y) = \left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\phi(x,y)$. Further, we define using the relation

$$-\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} - n\phi(x,y) = -nf(x,y)$$

auxiliary function $f(x,y)$. Assuming that $F_n(x,y) = (J_nf)(x,y)$, according to above obtained results, we derive the equality

$$-\frac{\partial}{\partial x}F_n(x,y) - \frac{\partial}{\partial y}F_n(x,y) - nF_n(x,y) = -nf(x,y).$$

Therefore, the function $w(x,y) = \phi(x,y) - F_n(x,y)$ satisfies to the equation

$$-\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} = nw(x,y).$$

The solution of this partial differential equation has the form

$$w(x,y) = Ce^{n(\psi(x-y)-y)},$$

where ψ is an arbitrary function, $C = \text{const}$. Let find the value of the constant C at which $w(x,y) \in S^+_{p,\alpha,\beta}$. As an arbitrary function, we choose $\psi = 0$ and obtain the estimate

$$\|w\|_{S^+_{p,\alpha,\beta}} = \sup_{(\tau,\xi)\in R^2}\left[\int_0^1\int_0^1 x^{\alpha-1}y^{\beta-1}|Ce^{-n(y+\xi)}|^p dxdy\right]^{\frac{1}{p}} = C\sup_{(\tau,\xi)\in R^2}\left[\int_0^1 y^{\beta-1}e^{-np(y+\xi)}\int_0^1 x^{\alpha-1}dxdy\right]^{\frac{1}{p}} =$$

$$= C\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\sup_{(\tau,\xi)\in R^2}\left[\int_0^1 y^{\beta-1}e^{-np(y+\xi)}dy\right]^{\frac{1}{p}} = C\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\sup_{(\tau,\xi)\in R^2}\left(e^{-n\xi}\left[\int_0^1 y^{\beta-1}e^{-npy}dy\right]^{\frac{1}{p}}\right) \geq$$

$$\geq C\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\sup_{(\tau,\xi)\in R^2}\left(e^{-n\xi}\left[\int_0^1 y^{\beta-1}dy\right]^{\frac{1}{p}}\right) = C\left(\frac{1}{\alpha\beta}\right)^{\frac{1}{p}}\sup_{(\tau,\xi)\in R^2}\left(e^{-n\xi}\right).$$

This expression may be valid only for $C = 0$. Consequently, $\phi(x,y) = F_n(x,y) \in D(-A^+)$.

Thus, the domain of definition of the operator $-A^+$ coincides with the set of functions $\phi(x,y)$ bounded in two-dimensional weighted Stepanov spaces $S^+_{p,\alpha,\beta}$, the first-order partial derivatives of which also belong to this space, and it holds for such functions $-A^+\phi(x,y) = \left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\phi(x,y)$. This means that the differential operator $-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ is an infinitesimal generating operator of a semigroup $U^+(t)$.

In accordance with [2], we construct a semigroup $U^+_{\frac{1}{2}}(t)f = U(t, -(A^+)^{\frac{1}{2}})f$ using the formula

$$U(t, -(A^+)^{\frac{1}{2}})f = \frac{1}{2\sqrt{\pi}}\int_0^\infty te^{\frac{-t^2}{4s}}s^{\frac{-3}{2}}U^+(s)fds.$$

Given equality (9), we obtain

$$U^+_{\frac{1}{2}}(t)f = \frac{1}{2\sqrt{\pi}}\int_0^\infty te^{\frac{-t^2}{4s}}s^{\frac{-3}{2}}f(x-s,y-s)ds$$

at $s \leq \min\{x,y\}$.

We show that the semigroup $U^+_{\frac{1}{2}}(t)$ acts from the spaces $S^+_{p,\alpha,\beta}$ in $S^+_{p,\alpha,\beta}$, using the generalized Minkowski integral inequality for estimation

$$\|U^+_{\frac{1}{2}}(t)f\|_{S^+_{p,\alpha,\beta}} = \sup_{(\tau,\xi)\in R^2}\left[\int_0^1\int_0^1 x^{\alpha-1}y^{\beta-1}\left|\frac{1}{2\sqrt{\pi}}\int_0^\infty te^{\frac{-t^2}{4s}}s^{\frac{-3}{2}}f(x+\tau-s,y+\xi-s)ds\right|^p dxdy\right]^{\frac{1}{p}} \leq$$

$$\leq \frac{t}{2\sqrt{\pi}} \sup_{(\tau,\xi)\in R^2} \int_0^\infty t e^{\frac{-t^2}{4s}} s^{\frac{-3}{2}} \left[\int_0^1 \int_0^1 x^{\alpha-1} y^{\beta-1} \left| f(x+\tau-s, y+\xi-s) \right|^p dxdy \right]^{\frac{1}{p}} ds \leq$$

$$\leq \|f\|_{S^+_{p,\alpha,\beta}} \frac{t}{2\sqrt{\pi}} \int_0^\infty t e^{\frac{-t^2}{4s}} s^{\frac{-3}{2}} ds = \|f\|_{S^+_{p,\alpha,\beta}}.$$

In accordance with the method described in the monograph of S. G. Krein (see [4]), we obtain that the differential equation (1) satisfying initial boundary conditions (2) -(3) is uniformly correctly solvable in two-dimensional weighted Stepanov spaces $S^+_{p,\alpha,\beta}$. Moreover, the solution to the problem (1) - (3) is representable in the form

$$u(t,x,y) = U^+_{\frac{1}{2}}(t) g_0 = \frac{1}{2\sqrt{\pi}} \int_0^\infty t e^{\frac{-t^2}{4s}} s^{\frac{-3}{2}} g_0(x-s, y-s) ds$$

at $s \leq \min\{x, y\}$.

So the theorem 1 is valid.

Proof of the theorem 2.

Consider the operator defined by the next differential expression

$$A^- u(t,x,y) = -\frac{\partial u(t,x,y)}{\partial x} - \frac{\partial u(t,x,y)}{\partial y}, \quad t \geq 0, \quad x, y \in R^1$$

and scope $D(A^-) = \left\{ u \in S^+_{p,\alpha,\beta},\ A^- u \in C^1(S^+_{p,\alpha,\beta}) \right\}$.

It can be shown that the operator $-A^-$ is a generator of a strongly continuous semigroup $U^-(t)$ defined by the equality

$$U^-(t) f = \begin{cases} f(x+t, y+t), & \text{at } t \in [0;\infty); \\ 0, & \text{at } t > \min\{x, y\}, \end{cases}$$

where $x, y \in R^1$. Obviously, the semigroup $U^-(t)$ is bounded in two-dimensional weighted Stepanov spaces $S^+_{p,\alpha,\beta}$.

We construct the family of operators $C(t)$ as follows

$$\left(C(t) f\right)(x,y) = \frac{1}{2}\left[U^+(t) f + U^-(t) f\right](x,y) = \frac{1}{2}\left[f(x-t, y-t) + f(x+t, y+t)\right].$$

This family is a strongly continuous operator cosine function, since it satisfies to the next conditions (see [5]):

1) From the following equalities

$$\left(C(t+s) f + C(t-s) f\right)(x,y) =$$

$$= \frac{1}{2}\left[f(x-t-s, y-t-s) + f(x+t+s, y+t+s)\right] + \frac{1}{2}\left[f(x-t+s, y-t+s) + f(x+t-s, y+t-s)\right]$$

and

$$\left(2C(t)C(s) f\right)(x,y) = 2C(t)\left(\frac{1}{2}\left[f(x-s, y-s) + f(x+s, y+s)\right]\right) =$$

$$= \frac{1}{2}\left[f(x-t-s, y-t-s) + f(x+t-s, y+t-s) + f(x+t+s, y+t+s) + f(x-t+s, y-t+s)\right]$$

it follows $C(t+s) + C(t-s) = 2C(t)C(s)$.

2) Since $\left(C(0) f\right)(x,y) = \frac{1}{2}\left[f(x-0, y-0) + f(x+0, y+0)\right] = f(x,y)$, then the equality holds $C(0) = I$.

3) If $f \in S^+_{p,\alpha,\beta}$, then also $\left(C(t) f\right)(x,y) \in S^+_{p,\alpha,\beta}$. We estimate the cosine function by applying the Minkowski integral inequality

$$\|C(t) f\|_{S^+_{p,\alpha,\beta}} =$$

$$= \sup_{(\tau,\xi)\in R^2} \left[\int_0^1 \int_0^1 x^{\alpha-1} y^{\beta-1} \left| \frac{1}{2} f(x+\tau-t, y+\xi-t) + f(x+\tau+t, y+\xi+t) \right|^p dxdy \right]^{\frac{1}{p}} \leq$$

$$\leq \frac{1}{2} \sup_{(\tau,\xi)\in R^2} \left(\left[\int_0^1 \int_0^1 x^{\alpha-1} y^{\beta-1} \left| f(x+\tau-t, y+\xi-t) \right|^p dxdy \right]^{\frac{1}{p}} + \right.$$

$$\left. + \left[\int_0^1 \int_0^1 x^{\alpha-1} y^{\beta-1} \left| f(x+\tau+t, y+\xi+t) \right|^p dxdy \right]^{\frac{1}{p}} \right) \leq$$

$$\leq \frac{1}{2}\left(\|f\|_{S^+_{p,\alpha,\beta}} + \|f\|_{S^+_{p,\alpha,\beta}} \right) = \|f\|_{S^+_{p,\alpha,\beta}}.$$

To find the generator A of the operator cosine function, we use the relation $A = C''$. We carry out calculations

$$\left(C'(t)f\right)(x,y) = \frac{1}{2}\left[-\frac{\partial f(x-t,y-t)}{\partial x} - \frac{\partial f(x-t,y-t)}{\partial y} + \frac{\partial f(x+t,y+t)}{\partial x} + \frac{\partial f(x+t,y+t)}{\partial y}\right],$$

$$\left(C''(t)f\right)(x,y) = \frac{1}{2}\left[\frac{\partial^2 f(x-t,y-t)}{\partial x^2} + 2\frac{\partial^2 f(x-t,y-t)}{\partial x \partial y} + \frac{\partial^2 f(x-t,y-t)}{\partial y^2} + \right.$$
$$\left. + \frac{\partial^2 f(x+t,y+t)}{\partial x^2} + 2\frac{\partial^2 f(x+t,y+t)}{\partial x \partial y} + \frac{\partial^2 f(x+t,y+t)}{\partial y^2}\right],$$

$$\left(C''(0)f\right)(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + 2\frac{\partial^2 f(x,y)}{\partial x \partial y} + \frac{\partial^2 f(x,y)}{\partial y^2}.$$

Thus, the generator of an operator cosine function is an operator A defined by a differential expression

$$Au(t,x,y) = \frac{\partial^2 u(t,x,y)}{\partial x^2} + 2\frac{\partial^2 u(t,x,y)}{\partial x \partial y} + \frac{\partial^2 u(t,x,y)}{\partial y^2}$$

and a domain of definition $D(A) = \left\{u \in S^+_{p,\alpha,\beta},\ Au \in C^2(S^+_{p,\alpha,\beta})\right\}$.

By the correctness theorem (see [5]) it follows that for $u_0 \in S^+_{p,\alpha,\beta}$ the Cauchy problem (4) with initial boundary conditions (5)-(6) is uniformly correct. Moreover, the solution to problem (4) - (6) is representable in the form

$$u(t,x,y) = \left(C(t)u_0\right)(x,y) = \frac{1}{2}\left[u_0(x-t,y-t) + u_0(x+t,y+t)\right].$$

So the theorem 2 is valid.

Proof of the theorem 3.

The operator A generates a semigroup $T(t)$ (see [5]) defined by the formula

$$T(t) = \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{\frac{-s^2}{4t}} C(s)ds.$$

Denote by $\left(C(s)f\right)(x,y) = G(s,x,y)$. It was previously shown that if $f \in S^+_{p,\alpha,\beta}$, then $G(s,x,y) \in S^+_{p,\alpha,\beta}$. We show that the function $\left(T(t)f\right)(x,y)$ also belongs $S^+_{p,\alpha,\beta}$, using the generalized Minkowski integral inequality for estimation,

$$\|T(t)f\|_{S^+_{p,\alpha,\beta}} == \sup_{(\tau,\xi)\in R^2}\left[\int_0^1 \int_0^1 x^{\alpha-1}y^{\beta-1}\left|\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{\frac{-s^2}{4t}} G(s,x+\tau,y+\xi)ds\right|^p dxdy\right]^{\frac{1}{p}} \le$$

$$\le \frac{1}{\sqrt{\pi t}} \sup_{(\tau,\xi)\in R^2}\int_0^\infty e^{\frac{-s^2}{4t}}\left[\int_0^1 \int_0^1 x^{\alpha-1}y^{\beta-1}\left|G(s,x+\tau,y+\xi)\right|^p dxdy\right]^{\frac{1}{p}} ds \le$$
$$\le \|G(s,x,y)\|_{S^+_{p,\alpha,\beta}} \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{\frac{-s^2}{4t}} ds = \|G(s,x,y)\|_{S^+_{p,\alpha,\beta}}$$

(here we used the fact that when $a > 0$ it is also true that $\int_0^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a}$).

So it follows that the Cauchy problem (7) - (8), where $u_0 \in S^+_{p,\alpha,\beta}$ and $u_0 \ge 0$ is correctly solvable. The solution to problem is representable in the form

$$u(t,x,y) = \left(T(t)u_0\right)(x,y) = \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{\frac{-s^2}{4t}}\left[u_0(x-s,y-s) + u_0(x+s,y+s)\right] ds.$$

So the theorem 3 is valid.

4. Conclusions.

In this article we deal with problems of correct solvability of some differential equations in two-dimensional weighted Stepanov spaces. We treat these problems by methods of semigroup theory and related methods using fractional powers of operators.

Exactly, in theorem 1 we prove that the problem (1)–(3) is uniformly correctly solvable in two-dimensional weighted Stepanov spaces $S^+_{p,\alpha,\beta}$ and find the explicit solution representation. In theorem 2 we obtained the same results to the related problem (4)–(6). And finally in theorem 3 we prove correct solvability and representation form for problem (7)–(8). The results of the paper are important for the theory of differential equations in Banach spaces. Also they are important and form a basis for computational methods for differential equations considered based on explicit integral representations for solutions. The results are obtained using ideas of S.G.Krein [4] on fractional powers of operators in Banach spaces.

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