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ARTICLE TYPE

Decay of solutions for a viscoelastic wave equation with acoustic boundary conditions.

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Summary

In this report we prove that the hypothesis on the memory term g in[1] can be modified to be $g'(t) \leq -\zeta(t) g^p(t)$, $t \geq 0$, $1 \leq p < \frac{3}{2}$ where $\zeta(t)$ provides

$$\zeta(0) > 0,\ \zeta'(t) \leq 0,\ \int_0^\infty \zeta(s)\, ds = +\infty.$$

So the optimal decay results are extended.

KEYWORDS:
Viscoelastic equation, Acoustic boundary conditions, energy decay

1 | INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary Γ and is divided by two closed and disjoint parts Γ_0, Γ_1, Here, $\Gamma_0 \neq \emptyset$, we investigate the solutions to the problem

$$u_{tt}(x,t) - \Delta u(x,t) - \alpha(t) \int_0^t g(t-s)\, \Delta u(x,s)\, ds = 0 \text{ in } \Omega \times (0,+\infty), \tag{1}$$

$$u(x,t) = 0 \text{ on } \Gamma_1 \times (0,+\infty), \tag{2}$$

$$\frac{\partial u}{\partial \nu}(x,t) - \alpha(t) \int_0^t g(t-s) \frac{\partial}{\partial \nu} u(x,s)\, ds = y_t(x,t) \text{ on } \Gamma_0 \times (0,+\infty), \tag{3}$$

$$u_t(x,t) + f(x) y_t(x,t) + m(x) y(x,t) = 0 \text{ on } \Gamma_0 \times (0,+\infty), \tag{4}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{5}$$

where ν is the outward normal to Γ. f and m are essentially bounded nonlinear functions satisfying some general properties. u_0, u_1 are given functions, Ω is a bounded domain of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$. $g, \alpha : \mathbb{R}^+ \to \mathbb{R}^+$ are non-increasing differentiable functions. The equation in consideration reaches from various mathematical models in engineering and physics. We refer the readers to[2,3,4,1] for a motivation and references concerning the subject of problem (1)-(5).

Our main aim in this work is to establish a general decay rate result, depending on the behavior of both α and g, for the energy of problem (1)-(5). Our main novel contribution is an extension and improvement of the previous result from[1] to the time-dependent viscoelastic case with the assumption condition $g'(t) \leq -\zeta(t) g^p(t)$, $t \geq 0$, $1 \leq p < \frac{3}{2}$ where $\zeta(t)$ provides

$$\zeta(0) > 0,\ \zeta'(t) \leq 0,\ \int_0^\infty \zeta(s)\, ds = +\infty.$$

[0]**Abbreviations:** ANA, anti-nuclear antibodies; APC, antigen-presenting cells; IRF, interferon regulatory factor

2 | PRELIMINARIES AND MAIN RESULTS

In this section, we present some material needed in the proof of our result and state the main result. Throughout this paper, we use the notation

$$H^1_{\Gamma_1}(\Omega) = \left\{u \in H^1(\Omega) : u\,|_{\Gamma_1} = 0\right\}$$

endow with the Hilbert structure induced by $H^1(\Omega)$, is a Hilbert space.

we use standard functional spaces and denote that $\|.\|_2$, $\|.\|_{\Gamma_0}$ are $L^2(\Omega)$ norm and $L^2(\Gamma_0)$ norm, respectively, such that:

$$\|u\|_{\Gamma_0} = \int_{\Gamma_0} |u(\eta)|^2 \, d\eta, \ \ \|u\|_2 = \|u\|^2_{L^2(\Omega)} = \int_{\Omega} |u(x)|^2 \, dx.$$

Also, we define the inner products $(u, v) = \int_\Omega u(x)\, v(x)\, dx$ and $(u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)\, v(x)\, d\Gamma$.

The norm in $H^1_0(\Omega)$ is $\|.\|_{H^1_0(\Omega)}$ and is given by:

$$\|u\|^2_{H^1_0(\Omega)} = \int_\Omega |\nabla u|^2 \, dx.$$

Let λ and $\widetilde{\lambda}$ be the smallest positive constants such that

$$\|u\|_2 \leq \sqrt{\lambda}\, \|\nabla u\|_2 \ \text{ and } \ \|u\|_{\Gamma_0} \leq \sqrt{\widetilde{\lambda}}\, \|\nabla u\|_2 \ \ \forall u \in H^1_{\Gamma_1}(\Omega). \tag{6}$$

We wish to use the following hypotheses:

(H1) Hypotheses on g, α $g, \alpha : [0, \infty) \to (0, \infty)$ are a bounded C^1 functions satisfying

$$g(0) > 0, \ \int_0^\infty g(s)\, ds < +\infty, \ \ \alpha(t) > 0, \ 1 - \alpha(t) \int_0^t g(s)\, ds \geq l > 0. \tag{7}$$

(H2) Hypotheses on g There exists a non-increasing differentiable function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\zeta(t) > 0, \ g'(t) \leq -\zeta(t)\, g(t), \ \text{for all } \ t \geq 0, \ \lim_{t\to+\infty} \frac{-\alpha'(t)}{\zeta(t)\, \alpha(t)} = 0 \tag{8}$$

(H3) Hypotheses on f, m For the functions p and q , we assume that $f, m \in C(\Gamma_0)$ and $f(x) > 0$ and $m(x) > 0$ for all $x \in \Gamma_0$.

This assumption implies that there exist positive constant $f_{0,1}, m_{0,1}$ such that

$$f_0 \leq f(x) \leq f_1, \ m_0 \leq m(x) \leq m_1, \ \text{for all } x \in \Gamma_0. \tag{9}$$

Theorem 1. [1, Theorem 2.1.] Assume that (H1) and (H3) hold. For the initial data $(u_0, u_1) \in H^1_{\Gamma_1}(\Omega) \cap H^2(\Omega) \times H^1_{\Gamma_1}(\Omega)$ there exists a unique pair of functions (u, y_t), which is a solution to the problem (1)-(5) in the class

$$u \in L^\infty\left(0, T; H^1_{\Gamma_1}(\Omega) \cap H^2(\Omega)\right), u_t \in L^\infty\left(0, T; H^1_{\Gamma_1}(\Omega)\right),$$
$$u_{tt} \in L^2\left(0, T; L^2(\Omega)\right),$$
$$y, \ y_t \in L^\infty\left(\mathbb{R}^+; L^2\left(\Gamma_0\right)\right),$$

Theorem 2. [1, Theorem 2.2.] Let $(u_0, u_1) \in H^1_{\Gamma_1}(\Omega) \cap H^2(\Omega) \times H^1_{\Gamma_1}(\Omega)$ be given. Assume that (H1)-(H3) hold. Then there exist positive constants K and k such that the solution of (1)-(5) satisfies

$$E(t) \leq K e^{-k \int_0^t \alpha(s)\zeta(s) ds} \ \ t \geq 0, \tag{10}$$

where

$$E(t) = \frac{1}{2}\left\|u_t\right\|^2_2 + \frac{1}{2}\left(1 - \alpha(t) \int_0^t g(s)\, ds\right) \|\nabla u\|^2_2 + \frac{1}{2} \alpha(t)\, (g \circ \nabla u)(t) \tag{11}$$
$$+ \frac{1}{2} \int_{\Gamma_0} m(x)\, |y(x, t)|^2 \, d\Gamma \text{ for } t \in \mathbb{R}^+,$$

A simple differentiation, employing (1), drives to

$$E'(t) = -\int_{\Gamma_0} f\,|y_t(x,y)|^2\,d\Gamma + \frac{1}{2}\alpha(t)\left(g'\circ\nabla u\right)(t) - \frac{1}{2}\alpha(t)\,g(t)\,\|\nabla u\|_2^2 \tag{12}$$
$$+\frac{1}{2}\alpha'(t)\,(g\circ\nabla u)(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2$$
$$\leq \frac{1}{2}\alpha(t)\left(g'\circ\nabla u\right)(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 \quad \text{for } t\in\mathbb{R}^+,$$

Where

$$(g\circ\nabla u)(t) = \int_0^t g(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)|^2\,dx\,ds,$$

In this report, we shall prolong the above exponential rate of decay to the general case. We use the following hypothesis which is weaker than (8).

(H4) Hypotheses on g There exist a fixed $p \in [1, 3/2)$ and a real positive differentiable function ζ so that

$$g'(t) \leq -\zeta(t)\,g^p(t) \quad \text{for all } t \geq 0. \tag{13}$$

and $\zeta(t)$ satisfies

$$\zeta(0) > 0,\ \zeta'(t) \leq 0\ \forall t\in\mathbb{R}^+,\ \int_0^\infty \zeta(s)\,ds = +\infty. \tag{14}$$

Then, we can show the following principal result.

Theorem 3. Suppose that (H3)-(H4) and (7) hold. Then, there exist strictly two positive constants λ_0 and λ_1 such that the energy $E(t)$ of the problem (1)-(5) satisfies, for all $t \in \mathbb{R}^+$, the decay rate

$$E(t) \leq \lambda_0 e^{-\lambda_1\int_0^t \alpha(s)\zeta(s)ds} \quad \text{if } p = 1, \tag{15}$$

$$E(t) \leq \lambda_1\left(1 + \int_0^t \alpha^{2p-1}(s)\,\zeta^{2p-1}(s)\,ds\right)^{\frac{-1}{2p-2}} \quad \text{if } p > 1, \tag{16}$$

Furthermore, if ζ, p in (H4) and α in (H2) satisfy

$$\int_0^\infty \left(1 + \int_0^t \alpha^{2p-1}(s)\,\zeta^{2p-1}(s)\,ds\right)^{\frac{-1}{2p-2}} dt < +\infty, \tag{17}$$

then, for all $t \in \mathbb{R}^+$, we have

$$E(t) \leq \lambda_0\left(1 + \int_0^t \alpha^p(s)\,\zeta^p(s)\,ds\right)^{\frac{-1}{p-1}} \quad \text{if } p > 1, \tag{18}$$

Remark 1.

1. For $p = 1$, (15) retrieve the exponential decay rate in [1, Theorem 2.2.].
2. When $g(t) = a(1+t)^{-d}$, $d > 2$ and $a > 0$. Then hypothesis (H4) holds with $\zeta(t) = b = da^{\frac{-1}{d}}$ and $p = \frac{d+1}{d} \in (1, \frac{3}{2})$. Therefore (17) holds and hence, by (18), we have the following decay rate

   $$E(t) \leq \lambda_0\left(1 + b^p\int_0^t \alpha^p(s)\,ds\right)^{\frac{-1}{p-1}},$$

   which is not addressed in [1].

The following lemma and corollary are essential for the proof of our main result.

Lemma 1. Suppose that g satisfies (7) and (H4) then

$$\int_0^\infty \alpha(t)\zeta(t)g^{1-\sigma}(t)\,dt < +\infty, \quad \forall \sigma < 2-p.$$

Proof. Evoking (7), we gain

$$\zeta(t)g^{1-\sigma}(t) = \zeta(t)g^{1-\sigma}(t)g^{p}(t)g^{-p}(t) \le -g'(t)g^{1-\sigma-p}(t).$$

A simple integration yields

$$\int_0^\infty \alpha(t)\zeta(t)g^{1-\sigma}(t)\,dt \le \int_0^\infty -\alpha(t)g'(t)g^{1-\sigma-p}(t)\,dt$$

$$= -\frac{g^{2-\sigma-p}(t)}{2-\sigma-p}\alpha(t)\,|_0^{+\infty} + \int_0^\infty \alpha'(t)\frac{g^{2-\sigma-p}(t)}{2-\sigma-p}dt < +\infty,$$

since $0 < 2-p-\sigma$ and $\alpha'(t) < 0$.

□

Lemma 2. [3, Lemaa 3.3.] Assume that g satisfies (7) and (H4), and u is the solution of (1)-(5) then, for $0<\sigma<1$, we have

$$(g\circ\nabla u)(t) \le C\left[\left(\int_0^\infty g^{1-\sigma}(t)\,dt\right)E(0)\right]^{\frac{p-1}{p-1+\sigma}}(g^p\circ\nabla u)^{\frac{\sigma}{p-1+\sigma}}(t).$$

Particularly, for $\sigma = \frac{1}{2}$, we have

$$(g\circ\nabla u)(t) \le C\left[\int_0^\infty g^{\frac{1}{2}}(t)\,dt\right]^{\frac{2p-2}{2p-1}}(g^p\circ\nabla u)^{\frac{1}{2p-1}}(t). \tag{19}$$

Corollary 1. Suppose that g satisfies (7) and (H4) and u is the solution of (1)-(5) then

$$\alpha(t)\zeta(t)(g\circ\nabla u)(t) \le C\left[-E'(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2\right]^{\frac{1}{2p-1}}(t).$$

Proof. Multiplying both sides of (19) by $\alpha(t)\zeta(t)$ and evoking Lemma 1 and (12) to get

$$\alpha(t)\zeta(t)(g\circ\nabla u)(t)$$

$$\le C\alpha^{\frac{2p-2}{2p-1}}(t)\zeta(t)^{\frac{2p-2}{2p-1}}\left[\int_0^\infty g^{\frac{1}{2}}(s)\,ds\right]^{\frac{2p-2}{2p-1}}\alpha(t)^{\frac{1}{2p-1}}\zeta(t)^{\frac{1}{2p-1}}(g^p\circ\nabla u)^{\frac{1}{2p-1}}(t)$$

$$\le C\left[\int_0^\infty \alpha(s)\zeta(s)g^{\frac{1}{2}}(s)\,ds\right]^{\frac{2p-2}{2p-1}}\alpha(t)^{\frac{1}{2p-1}}(\zeta g^p\circ\nabla u)^{\frac{1}{2p-1}}(t)$$

$$\le C\left[\int_0^\infty \alpha(s)\zeta(s)g^{\frac{1}{2}}(s)\,ds\right]^{\frac{2p-2}{2p-1}}\left[\alpha(t)\left(-g'\circ\nabla u\right)(t)\right]^{\frac{1}{2p-1}}$$

$$\le C\left[-E'(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2\right]^{\frac{1}{2p-1}}.$$

□

By adopting the idea of[5,2,3,4]. Suppose that hypothesis (H4) holds and determine the modified energy, as in[1]

$$L(t) = ME(t) + \varepsilon\alpha(t) F(t) + \alpha(t) H(t), \ \forall\varepsilon > 0. \tag{20}$$

where

$$F(t) = \int_{\Omega} u_t u dx + \int_{\Gamma 0} u(t) y(t) d\Gamma + \frac{1}{2}\int_{\Gamma_1} m(x) y^2(t) d\Gamma, \tag{21}$$

$$H(t) = \int_{\Omega} u_t \int_0^t g(t-s)(u(s) - u(t)) dsdx, \tag{22}$$

and M, ε is some suitable positive constants to be specified.

$$\begin{aligned} E'(t) &\le \frac{1}{2}\alpha(t)\left(g'\circ\nabla u\right)(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s) ds\right)\|\nabla u\|_2^2 \\ &\le -\frac{1}{2}\zeta(t)\alpha(t)(g^p\circ\nabla u)(t) - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s) ds\right)\|\nabla u\|_2^2 \text{ for } t\in\mathbb{R}^+, \end{aligned} \tag{23}$$

Now, we shall investigate the global decay of the energy function $E(t)$.

First, to achieve the decay result, we use the following lemmas which act of fundamental significance in the proof.

Lemma 3. [1, Lemma 3.2.]There exists $C_1 > 0$ such that

$$|L(t) - E(t)| \le \varepsilon C_1 E(t), \ \forall t \ge 0, \ \forall\varepsilon > 0.$$

Lemma 4. [1, (5.22) in the proof of Theorem 5.5.]There are positive constants C_2, C_3 such that

$$L'(t) \le -C_2\alpha(t) E(t) + C_3\alpha(t)(g\circ\nabla u)(t) \ \forall t \ge t_1 \ge t_0 \tag{24}$$

Now, we conclude the proof of the decay property.

Proof of Theorem 3. Give

$$\varepsilon_0 = \min\left\{\frac{1}{2C_1}, \frac{1}{C_2}\right\}.$$

It results from Lemma 3 that, for $\varepsilon < \varepsilon_0$,

$$\frac{1}{2}E(t) \le L(t) \le \frac{3}{2}E(t), \ \forall t \ge 0. \tag{25}$$

Case when $p = 1$. By the determination of $L(t)$, (23) and (24), we get

$$\begin{aligned} \zeta(t) L'(t) &\le -C_2\zeta(t)\alpha(t) E(t) + C_3\zeta(t)\alpha(t)(g\circ\nabla u)(t) \\ &\le -C_2\zeta(t)\alpha(t) E(t) + C_3\alpha(t)\left(-g'\circ\nabla u\right)(t) \\ &\le -C_2\zeta(t)\alpha(t) E(t) - C_3\alpha'(t)\left(\int_0^t g(s) ds\right)\|\nabla u\|_2^2 - 2C_3 E'(t) \ \forall t \ge t_1 \end{aligned} \tag{26}$$

We fixed

$$K(t) = \zeta(t) L(t) + 2C_3 E(t).$$

Then, $L(t)$ is equivalent to $E(t)$. In fact, we have

$$K(t) \le \zeta(0) L(t) + 2C_3 E(t) \le \left(\frac{3}{2}\zeta(0) + 2C_3\right) E(t)$$

and

$$K(t) \ge \frac{1}{2}\zeta(t) E(t) + 2C_3 E(t) \ge 2C_3 E(t).$$

From (25) and (26), since $\zeta'(t) \le 0$ we get

$$K'(t) = \zeta'(t) L(t) + \zeta(t) L'(t) + 2C_3 E'(t) \tag{27}$$
$$\le -C_2 \zeta(t) \alpha(t) E(t) - C_3 \alpha'(t) \left(\int_0^t g(s)\,ds \right) \|\nabla u\|_2^2$$
$$\le -\zeta(t) \alpha(t) \left[C_2 + \frac{2\alpha'(t)}{l \zeta(t) \alpha(t)} \left(\int_0^t g(s)\,ds \right) \right] E(t) \ \ \forall t \ge t_1$$

By (H2), we can choose $t_2 \ge t_1$ and then (27) gives

$$K'(t) \le -C_2 \zeta(t) \alpha(t) E(t) \quad \forall t \ge t_2 \tag{28}$$

Since $\zeta'(t) \le 0$, we can easily get $K(t) \sim E(t)$ and

$$K'(t) \le -k \zeta(t) \alpha(t) K(t) \ \ \forall t \ge t_2 \tag{29}$$

for some positive constant k. Integrating (29) over $[t_2, t]$, we have

$$K(t) \le K e^{-k \int_0^t \alpha(s)\zeta(s) ds}, \ \ \forall t \ge t_2.$$

Since $K(t) \sim E(t)$, we have

$$E(t) \le K e^{-k \int_0^t \alpha(s)\zeta(s) ds}, \ \ \forall t \ge t_2.$$

By the virtue of the continuity and boundedness of $E(t)$ in the interval $[0, t_2]$, this yields (15).

Case when $p > 1$. To verify (16), once more consider (26) and apply corollary 1 to obtain

$$\zeta(t) L'(t) \le -C_2 \zeta(t) \alpha(t) E(t) + C_3 \zeta(t) \alpha(t) (g \circ \nabla u)(t)$$
$$\le -\lambda \zeta(t) \alpha(t) L(t) + C \left[-E'(t) - \alpha'(t) \left(\int_0^t g(s)\,ds \right) \|\nabla u\|_2^2 \right]^{\frac{1}{2p-1}} \quad \forall t \ge t_1$$

Multiplication of the last inequality by $\alpha(t)^\beta \zeta^\beta L^\beta(t)$, where $\beta = 2p - 2 > 0$, gives

$$\frac{1}{\beta+1} \zeta^{\beta+1} \alpha(t)^\beta \frac{d}{dt} L^{\beta+1}(t) \le -\lambda \zeta^{\beta+1}(t) \alpha^{\beta+1}(t) L^{\beta+1}(t)$$
$$+ C (\zeta \alpha(t) L(t))^\beta \left[-E'(t) - \alpha'(t) \left(\int_0^t g(s)\,ds \right) \|\nabla u\|_2^2 \right]^{\frac{1}{\beta+1}}.$$

Application of Young's inequality, with $q = \beta + 1$ and $q^* = \frac{\beta+1}{\beta}$, yields, for any $\varepsilon > 0$,

$$\frac{1}{\beta+1} \zeta^{\beta+1} \alpha(t)^\beta \frac{d}{dt} L^{\beta+1}(t) \le -\lambda \zeta^{\beta+1}(t) \alpha(t)^{\beta+1} L^{\beta+1}(t) \tag{30}$$
$$+ C \left(\varepsilon \alpha^{\beta+1}(t) \zeta^{\beta+1}(t) L^{\beta+1}(t) - C(\varepsilon) E'(t) \right) - C(\varepsilon) \alpha'(t) \left(\int_0^t g(s)\,ds \right) \|\nabla u\|_2^2$$
$$= -\zeta(t) \alpha(t) \left[(\lambda - \varepsilon C) \alpha(t)^\beta \zeta^\beta(t) L^{\beta+1}(t) + \frac{C(\varepsilon) \alpha'(t)}{l \zeta(t) \alpha(t)} \left(\int_0^t g(s)\,ds \right) \|\nabla u\|_2^2 \right]$$
$$-C(\varepsilon) E'(t) \ \ \forall t \ge t_1$$

By (H2), we can choose $t_2 \geq t_1$ and then (30) gives

$$\begin{aligned}\frac{1}{\beta+1}\zeta^{\beta+1}\alpha(t)^{\beta}\frac{d}{dt}L^{\beta+1}(t) &\leq -\lambda\zeta^{\beta+1}(t)\,\alpha(t)^{\beta+1}L^{\beta+1}(t)\\ +C\left(\varepsilon\alpha^{\beta+1}(t)\,\zeta^{\beta+1}(t)\,L^{\beta+1}(t)-C(\varepsilon)\,E'(t)\right)&-C(\varepsilon)\,\alpha'(t)\left(\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2\\ &=-(\lambda-\varepsilon C)\,\alpha(t)^{\beta+1}\zeta^{\beta+1}(t)\,L^{\beta+1}(t)-C(\varepsilon)\,E'(t)\ \ \forall t\geq t_2\end{aligned}$$

keeping to mind that $\zeta'(t) \leq 0$ and taking $0 < \varepsilon < \frac{\lambda}{C}$, to get

$$\begin{aligned}\frac{d}{dt}\left(\zeta^{\beta+1}\alpha(t)^{\beta}L^{\beta+1}(t)\right) &\leq \zeta^{\beta+1}\alpha(t)^{\beta}\frac{d}{dt}L^{\beta+1}(t)\\ &\leq -C_4\zeta^{\beta+1}(t)\,\alpha(t)^{\beta+1}L^{\beta+1}(t)-C(\varepsilon)\,E'(t);\end{aligned}$$

which means

$$\frac{d}{dt}\left(\zeta^{\beta+1}\alpha(t)^{\beta}L^{\beta+1}(t)+C(\varepsilon)\,E(t)\right)\leq -C_4\zeta^{\beta+1}(t)\,\alpha(t)^{\beta+1}L^{\beta+1}(t) \tag{31}$$

Let $G=\zeta^{\beta+1}\alpha(t)^{\beta}L^{\beta+1}(t)+C(\varepsilon)\,E(t)\sim L\sim E$. Then

$$\frac{d}{dt}G(t)\leq -C_4\zeta^{\beta+1}(t)\,\alpha(t)^{\beta+1}G^{\beta+1}(t)=-C_4\zeta^{2p-1}(t)\,\alpha(t)^{2p-1}G^{2p-1}(t);$$

Integrating over $(0,t)$ and using the fact that $G \sim E$, we get, for some $\lambda_0 > 0$

$$E(t)\leq \lambda_0\left(1+\int_0^t \alpha(s)^{2p-1}\zeta^{2p-1}(s)\,ds\right)^{\frac{-1}{2p-2}},\ \ \forall t\geq t_2.$$

hence by the virtue of the continuity and boundedness of $E(t)$ in the interval $[0,t_2]$, the assertion (16) holds.

To prove (18), we set

$$\varphi(t)=\int_0^t\int_\Omega |\nabla u(s)-\nabla u(t-s)|^2\,dxds$$

From (16) and (17), we have

$$\begin{aligned}\varphi(t)&\leq 2\int_0^t\int_\Omega\left(|\nabla u(s)|^2+|\nabla u(t-s)|^2\right)dxds\\ &\leq \frac{4}{l}\int_0^t(E(t)+E(t-s))\,ds\leq\frac{8}{l}\int_0^t E(t)\,ds<+\infty.\end{aligned}$$

This signifies that

$$\sup_{t\in\mathbb{R}^+}\varphi(t)^{\frac{p-1}{p}}<+\infty \tag{32}$$

Suppose that $\varphi(t) > 0$. Then, from (26), we gain

$$\begin{aligned}\zeta(t)\,L'(t)&\leq -C_2\zeta(t)\,\alpha(t)\,E(t)+C_3\zeta(t)\,\alpha(t)\,(g\circ\nabla u)(t)\\ &\leq -C_2\zeta(t)\,\alpha(t)\,E(t)+C_3\frac{\varphi(t)}{\varphi(t)}\alpha(t)\int_0^t(\zeta^p(s)\,g^p(s))^{\frac{1}{p}}\int_\Omega|\nabla u(s)-\nabla u(t-s)|^2\,dxds\end{aligned} \tag{33}$$

Using Jensen's inequality for the second term of the right-hand side of (32), in the form

$$\begin{aligned}&\frac{1}{\varphi(t)}\int_0^t(\zeta^p(s)\,g^p(s))^{\frac{1}{p}}\int_\Omega|\nabla u(s)-\nabla u(t-s)|^2\,dxds\\ &\leq\left(\frac{1}{\varphi(t)}\int_0^t(\zeta^p(s)\,g^p(s))\int_\Omega|\nabla u(s)-\nabla u(t-s)|^2\,dxds\right)^{\frac{1}{p}},\end{aligned}$$

to get

$$\zeta(t) L'(t) \le -C_2 \zeta(t)\alpha(t) E(t)$$
$$+C_3\varphi(t)\left(\frac{1}{\varphi(t)}\int_0^t \left(\zeta^p(s) g^p(s)\right)\int_\Omega |\nabla u(s) - \nabla u(t-s)|^2 \, dx ds\right)^{\frac{1}{p}}$$

Hence, using (32) we get

$$\zeta(t) L'(t) \le -C_2 \zeta(t)\alpha(t) E(t)$$
$$+C_3\varphi^{\frac{p-1}{p}}(t)\left(\zeta^{p-1}(0)\int_0^t \left(\zeta(s) g^p(s)\right)\int_\Omega |\nabla u(s) - \nabla u(t-s)|^2 \, dx\right)^{\frac{1}{p}} ds$$
$$\le -C_2 \zeta(t)\alpha(t) E(t) + C\left(-g'\circ\nabla u\right)(t)^{\frac{1}{p}}$$

and then

$$\zeta(t) L'(t) \le -C_2 \zeta(t)\alpha(t) E(t) + C\left(-E'(t) - \alpha'(t)\left(\int_0^t g(s)\, ds\right)\|\nabla u\|_2^2\right)^{\frac{1}{p}} \tag{34}$$

If $\varphi(t) = 0$, then $s \to \nabla u(s)$ is a constant function on $[0, t]$. Consequently

$$(g\circ\nabla u)(t) = 0,$$

and so we have, from (26),

$$L'(t) \le -C_2\alpha(t) E(t)$$

which implies (34).

Now, multiplying (34) by $\zeta^\beta(t)\,\alpha(t)^\beta L^\beta(t)$ using again the fact that $E \sim L$, for $\beta = p-1$, and repeating the same estimates as in above, we become at, for suitable $\lambda_0 > 0$,

$$E(t) \le \lambda_0\left(1 + \int_0^t \alpha(s)^p \zeta^p(s)\, ds\right)^{\frac{-1}{p-1}} \quad \text{if } p > 1.$$

□

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The author declare that they have no conflict of interest.

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