ON NEW GENERALIZED NON-INTEGRO-DERIVATIVES AND APPLICATIONS

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Abstract. With respect to the non-integro-fractional derivative, in previous studies, the non-integro-fractional derivative of non-negative real numbers can be calculated. However, by previous definitions, the non-integro-fractional derivative of negative values can not be calculated due to $t^{-\alpha}$, $\alpha \in (0,1)$. For example, $(-2)^{-\frac{1}{2}} \notin \mathbb{R}$ for $t = -2$ and $\alpha = \frac{1}{2}$. So what should we do for the non-integro-fractional derivative of "negative" real numbers? The purpose of this paper is to introduce more general derivative definition and we claim that we will obtain non-integro-fractional derivative of "all" real numbers. Classic derivative, q-derivative, (p,q)-derivative, comformable fractional derivative, Katugampola fractional derivative and backward-forward difference operator in Time Scale are the special cases of these general derivative definitions. These new definitions of ours must give us derivatives on both discrete and continuous calculus.

1. Introduction

Fractional calculus is not a new topic; in reality it has almost the same history as that of the classical calculus. Since the occurrence of fractional or fractional-order derivative, the theories of fractional calculus fractional derivative plus fractional integral has undergone a significant and even heated development, which has been primarily contributed by pure but not applied mathematicians.

There exist many different definitions of fractional derivative, among which we mention the Riemann–Liouville, Caputo, Hadamard, Edrlyi–Kober and Katugampola types [9, 12]. Most of the fractional derivatives are defined via fractional integrals [15]. Due to the same reason, those fractional derivatives inherit some non-local behaviors, which lead them to many interesting applications including memory effects and future dependence. All of these have important applications in several different areas such as mathematics, physics, biology, medicine and engineering. We must recall that to each definition of fractional derivative, there corresponds a specific fractional integral [10, 12].

In 2014, Khalil [11] introduced a new fractional derivative and a corresponding fractional integral with properties similar to the classical (integerorder) derivative and integral. He called the derivative conformable fractional derivative and the integral α-fractional integral. Abdeljawad [1] presented a generalization of the conformable fractional derivative and the α-fractional integral. In the same year, Katugampola [7] introduced the alternative fractional derivative and, from the truncated exponential function, the truncated alternative fractional derivative; to

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both derivatives there corresponds a unique α-fractional integral. Recently, Sousa and Oliveira [18, 19] introduced the M-fractional derivative and the truncated M-fractional derivative, whose properties generalize the properties of integer-order calculus derivatives and integrals.

With respect to the non-integro-fractional derivative, in previous studies, the non-integro-fractional derivative of non-negative real numbers can be calculated. However, by previous definitions, the non-integro-fractional derivative of negative values can not be calculated due to $t^{-\alpha}$, $\alpha \in (0,1)$. For example, $(-2)^{-\frac{1}{2}} \notin \mathbb{R}$ for $t=-2$ and $\alpha = \frac{1}{2}$. So what should we do for the non-integro-fractional derivative of "negative" real numbers? The purpose of this paper is to introduce more general derivative definition and we claim that we will obtain non-integro-fractional derivative of "all" real numbers. For this we will use the absolute value and the modified signal functions. Classic derivative, q-derivative, (p,q)-derivative, comformable fractional derivative, Katugampola fractional derivative and backward-forward difference operator in Time Scale are the special cases of these general derivative definitions. These new definitions of ours must give us derivatives on both discrete and continuous calculus.

We will benefit from the Mittag-Leffler function in this definitions. Too much work has been done on the Mittag-Leffler function. Many updates of Mittag-Leffler function are also available. In this study we will use a Mittag-Leffler function with less parameters.

In third section, we will introduce two generalized derivatives for discrete and continuous analysis and we'll get some special cases of these definitions. In fourth section, we will present the properties and theorems of generalized derivatives. In last section, we will give some examples and simulate these examples with graphics. In these examples, we will see that the fractional derivatives of the negative numbers gives the classical derivative for $\alpha = 1$.

2. Preleminaries

In 1905, Wiman [17] proposed and studied a generalization of the Mittag–Leffler function, the so-called two-parameter Mittag–Leffler function.

Definition 1. *The two-parameter Mittag-Leffler function is given by the series*

$$E_{a,b}(z) = \sum_{i=0}^{\infty} \frac{(z)^i}{\Gamma(ai+b)}$$

with $a, b \in \mathbb{C}$, $Re(a) > 0$ *and* $Re(b) > 0$.

Throughout the study we will accept $\alpha \in (0,1]$ and we need the following definition bounded Mittag–Leffler function:

Definition 2. *Bounded two-parameter Mittag–Leffler function is given by:*

$$E_{a,b}^{m}(z) = \sum_{i=0}^{m} \frac{(z)^i}{\Gamma(ai+b)} \tag{2.1}$$

where $m \in \mathbb{N}$.

Some special cases of (2.1):

i) For $a=b=1$, $t>0$ and $m=\infty$, then

$$E_{1,1}^{\infty}\left(N\varepsilon t^{-\alpha}\right)=\sum_{i=0}^{\infty}\frac{\left(N\varepsilon t^{-\alpha}\right)^{i}}{\Gamma\left(i+1\right)}=\sum_{i=0}^{\infty}\frac{\left(N\varepsilon t^{-\alpha}\right)^{i}}{i!}=e^{N\varepsilon t^{-\alpha}}.$$

ii) For $a=b=m=1$ and $t>0$, then

$$E_{1,1}^{1}\left(\varepsilon t^{-\alpha}\right)=\sum_{i=0}^{1}\frac{\left(\varepsilon t^{-\alpha}\right)^{i}}{\Gamma\left(i+1\right)}=1+\varepsilon t^{-\alpha}.$$

iii) For $a=b=m=1$ and $t>0$, then

$$E_{1,1}^{1}\left(-0\varepsilon t^{-\alpha}\right)=\sum_{i=0}^{1}\frac{\left(-0\varepsilon t^{-\alpha}\right)^{i}}{\Gamma\left(i+1\right)}=1.$$

Also for $n\in\mathbb{R}$ and $M\leq 0\leq N$, the following notation will be used:

$$[n]_{E}=\sum_{i=0}^{n-1}\left(E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)^{n-1-i}\left(E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)^{i}$$

where $\widehat{sgn}\left(t\right)$ is a modified signal function is defined by

$$\widehat{sgn}\left(t\right)=\begin{cases}1, & \text{if } t\geq 0\\ -1, & \text{if } t<0.\end{cases}$$

3. New Generalized Derivatives

In this section, we introduce two generalized derivatives for discrete and continuous analysis. We claim that we will find fractional derivatives of all real numbers by new definition of fractional derivative. Now let establish new definition derivative for discrete analogue as follow:

Definition 3. *Let $\alpha\in(0,1]$, $M\leq 0\leq N$ $(N,M\in\mathbb{R})$ with $N\neq M$, $Re\left(a\right)>0$, $Re\left(b\right)>0$ $(a,b\in\mathbb{C})$, $m\in\mathbb{N}$ and we denote a fuction $f:I\subset\mathbb{R}\to\mathbb{R}$. The generalized discreted derivative of the function f is defined as:*

$$\begin{aligned}&T_{NM}^{\alpha\varepsilon}f\left(t\right)\\ =\;&T_{NM}^{\alpha\varepsilon}\left(m;a,b\right)f\left(t\right)\\ =\;&\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}{\left(N-M\right)\varepsilon}.\end{aligned}\tag{3.1}$$

If f is $T_{NM}^{\alpha\varepsilon}$-differentiable in some $(-t_0,0)\cup(0,t_0)$, $t_0>0$, and $\lim_{t\to0^{\pm}}T_{NM}^{\alpha\varepsilon}f\left(t\right)$ exist. Then, we define $T_{NM}^{\alpha\varepsilon}f\left(0\right)$ such as $T_{NM}^{\alpha\varepsilon}f\left(0\right)=\lim_{t\to0^{\pm}}T_{NM}^{\alpha\varepsilon}f\left(t\right).$

In (3.1) by using limit for $\varepsilon\to 0$, we have defined generalized derivative for continuous analogue as follow:

Definition 4. *Let $\alpha\in(0,1]$, $M\leq 0\leq N$ $(N,M\in\mathbb{R})$ with $N\neq M$, $Re\left(a\right)>0$, $Re\left(b\right)>0$ $(a,b\in\mathbb{C})$, $m\in\mathbb{N}$ and we denote a fuction $f:I\subset\mathbb{R}\to\mathbb{R}$. The* *generalized derivative of the function f is defined as:*

$$
\begin{aligned}
(3.2)\qquad & T_{N,M}^{\alpha} f(t) \\
= & \; T_{N,M}^{\alpha}(m;a,b) f(t) \\
= & \; \lim_{\varepsilon \to 0} \left[T_{NM}^{\alpha\varepsilon} f(t) \right] \\
= & \; \lim_{\varepsilon \to 0} \frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t) N\varepsilon |t|^{-\alpha}\right)\right) - f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t) M\varepsilon |t|^{-\alpha}\right)\right)}{(N-M)\varepsilon}.
\end{aligned}
$$

If f is $T_{N,M}^{\alpha}$-differentiable in some $(-t_0,0)\cup(0,t_0)$, $t_0>0$, and $\lim_{t\to 0^{\pm}} T_{N,M}^{\alpha} f(t)$ exist. Then, we define $T_{N,M}^{\alpha} f(0)$ such as $T_{N,M}^{\alpha} f(0) = \lim_{t\to 0^{\pm}} T_{N,M}^{\alpha} f(t)$.

We will write $T_{NM}^{\alpha\varepsilon} f(t)$ instead of $T_{NM}^{\alpha\varepsilon}(m;a,b) f(t)$ and $T_{N,M}^{\alpha} f(t)$ instead of $T_{N,M}^{\alpha}(m;a,b) f(t)$ throughout the study.

The following results are obtained from (3.1) and (3.2) :

1) If we choose $a=b=m=\alpha=N=1, M=0$ in the definition (3.2), we recaptured classic derivative:

$$T_{1,0}^{1} f(t) = \lim_{\varepsilon\to 0} T_{1,0}^{1\varepsilon} f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon) - f(t)}{\varepsilon} = f'(t).$$

2) If we choose $a=b=m=N=\alpha=1, M=0$, $t>0$ in (3.1) we have

$$T_{1,0}^{1\varepsilon} f(t) = \begin{cases} \dfrac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, & \text{if } t \geq 0 \\ \dfrac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, & \text{if } t < 0 \end{cases}$$

In above by choosing $q = 1+\varepsilon t^{-1}$, we have quantum q-derivative in [6]:

$$T_{1,0}^{1\varepsilon} f(t) = \begin{cases} \dfrac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, & \text{if } t \geq 0 \\ \dfrac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, & \text{if } t < 0 \end{cases} = \frac{f(qt) - f(t)}{(q-1)t} = D_q f(t).$$

3) If we choose $a=b=m=N=-M=\alpha=1$ in (3.1), we have

$$T_{1,-1}^{1\varepsilon} f(t) = \begin{cases} \dfrac{f\left(t+\varepsilon t^{1-\alpha}\right) - f\left(t-\varepsilon t^{1-\alpha}\right)}{2\varepsilon}, & \text{if } t \geq 0 \\ \dfrac{f\left(t+\varepsilon t^{1-\alpha}\right) - f\left(t-\varepsilon t^{1-\alpha}\right)}{2\varepsilon}, & \text{if } t < 0 \end{cases}$$

and by choosing $q = 1+\varepsilon t^{-\alpha}$ with $p = 1-\varepsilon t^{-\alpha}$ replace respectively $\varepsilon = (q-1)t^{-\alpha}$, $\varepsilon = (1-p)t^{-\alpha}$, we have quantum (p,q)-derivative in [14]:

$$T_{1,-1}^{1\varepsilon} f(t) = \begin{cases} \dfrac{f\left(t+\varepsilon t^{1-\alpha}\right) - f\left(t-\varepsilon t^{1-\alpha}\right)}{2\varepsilon}, & \text{if } t \geq 0 \\ \dfrac{f\left(t+\varepsilon t^{1-\alpha}\right) - f\left(t-\varepsilon t^{1-\alpha}\right)}{2\varepsilon}, & \text{if } t < 0 \end{cases} = \frac{f(qt) - f(pt)}{(q-p)t} = D_{p,q} f(t).$$

4) If we choose $a=b=m=N=1$, $M=0$ and $t>0$ with $\alpha\in(0,1)$ in (3.2), we have comformable fractional derivative is obtained as below:

$$T_{1,0}^{\alpha} f(t) = \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} = T_{\alpha}(f)(t)$$

which is defined by Khalil in [11].

5) If we choose $a=b=N=1, M=0, m=\infty$ and $t>0$ with $\alpha\in(0,1)$ in (3.1), we recaptured

$$T_{1,0}^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f(t)}{\varepsilon}=D^{\alpha}(f)(t)$$

which is defined by Katugampola in [7].

6) If we choose $a=\beta\ b=r=1, s=0, m=\infty$ and $t>0$ with $\alpha\in(0,1)$ in (3.2), we recaptured

$$T_{1,0}^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f\left(tE_{\beta}\left(\varepsilon t^{-\alpha}\right)\right)-f(t)}{\varepsilon}=D_{M}^{\alpha,\beta}(f)(t)$$

which is defined by Vanterler in [20].

7) If we choose $a=b=\alpha=N=1, M=0, m=1$ and $\varepsilon=1$ in (3.1) we recaptured

$$E_{a,b}^{m}\left(\widehat{sgn}(t)N\varepsilon|t|^{-\alpha}\right)=1+t^{-1}$$

and

$$T_{1,0}^{1,1}f(t)=f(t+1)-f(t)=f^{\triangle}(t)$$

is the forward difference operator in Time Scale. Also for $a=b=\alpha=-M=1, N=0, m=1$ and $\varepsilon=1$ we recaptured

$$E_{a,b}^{m}\left(\widehat{sgn}(t)M\varepsilon|t|^{-\alpha}\right)=1-t^{-1}.$$

So

$$T_{0,-1}^{1,1}f(t)=f(t)-f(t-1)=f^{\nabla}(t)$$

is the backward difference operator by Hilger in [3].

8) If we choose $a=b=m=N=1$, $M=0$, $\alpha=p$ and $t>0$ with $E_{1,1}^{m}\left(\widehat{sgn}(t)\varepsilon|t|^{-\alpha}\right)=\psi(t,p)$ in (3.2), we have general conformable fractional derivative is obtained as below:

$$T_{1,0}^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon\psi(t,p)\right)-f(t)}{\varepsilon}=D_{\psi}^{p}f(t)$$

which is defined by Zhao in [21].

4. Properties and Theorems of Generalized Derivatives

In this section, we will start the following theorem the properties of $T_{NM}^{\alpha\varepsilon}-$derivative:

Theorem 1. *Let $\alpha\in(0,1]$, $M\leq 0\leq N$ $(N,M\in\mathbb{R})$ with $N\neq M$, $Re(a)>0$, $Re(b)>0$ $(a,b\in\mathbb{C})$, $m\in\mathbb{N}$ and f,g be $T_{NM}^{\alpha\varepsilon}$-differentiable functions. Then,*

(1) $T_{NM}^{\alpha\varepsilon}(cf+dg)=cT_{NM}^{\alpha\varepsilon}(f)+dT_{NM}^{\alpha\varepsilon}(g)$, for all $c,d\in\mathbb{R}$.

(2) $T_{NM}^{\alpha\varepsilon}(C)=0$, for all constant functions, $f(t)=C$.

(3)

$$\begin{aligned}&T_{NM}^{\alpha\varepsilon}(fg)(t)\\=\ &g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)M\varepsilon|t|^{-\alpha}\right)\right)T_{NM}^{\alpha\varepsilon}f(t)+f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)N\varepsilon|t|^{-\alpha}\right)\right)T_{NM}^{\alpha\varepsilon}g(t)\end{aligned}$$

(4)

$$
\begin{aligned}
& T_{NM}^{\alpha\varepsilon}\left(\frac{f}{g}\right)(t) \\
= & \frac{g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)T_{NM}^{\alpha\varepsilon}f(t)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)T_{NM}^{\alpha\varepsilon}g(t)}{g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}
\end{aligned}
$$

where

$$
g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)\neq 0.
$$

(5) $T_{NM}^{\alpha\varepsilon}(f\circ g)(t)=f'(g(t))\,T_{NM}^{\alpha\varepsilon}g(t)$, *for* f *differentiable at* $g(t)$.

Proof. Proof of parts (1) and (2) are clear from definition in (3.1).

Proof part (3):

$$
\begin{aligned}
& T_{NM}^{\alpha\varepsilon}(fg)(t) \\
= & \frac{(fg)\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)-(fg)\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon} \\
= & \frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon} \\
& -\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon} \\
& -\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon} \\
= & g\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)T_{NM}^{\alpha\varepsilon}f(t)+f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)T_{NM}^{\alpha\varepsilon}g(t).
\end{aligned}
$$

This completes the proof of (3). Similarly if $\alpha\in(0,1]$, we have

$$
\begin{aligned}
& T_{NM}^{\alpha\varepsilon}\left(\frac{f}{g}\right)(t) \\
= & \frac{\left(\frac{f}{g}\right)\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)-\left(\frac{f}{g}\right)\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon}
\end{aligned}
$$

$$= \frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}{\left(N-M\right)\varepsilon g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}$$

$$-\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)}{\left(N-M\right)\varepsilon g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}$$

$$+\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)}{\left(N-M\right)\varepsilon g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}$$

$$\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)}{\left(N-M\right)\varepsilon g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}$$

$$= \frac{g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)T_{NM}^{\alpha\varepsilon}f\left(t\right)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)T_{NM}^{\alpha\varepsilon}g\left(t\right)}{g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}.$$

The proof of (4) is completed. Finally let prove the part of (5)

$$T_{NM}^{\alpha\varepsilon}\left(f\circ g\right)\left(t\right)$$

$$= \frac{f\left(g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)\right)-f\left(g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)\right)}{\left(N-M\right)\varepsilon}$$

$$= \frac{f\left(g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)\right)-f\left(g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)\right)}{g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)-g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}$$

$$\times\frac{g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)-g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}{\left(N-M\right)\varepsilon}$$

$$= \frac{f\left(g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)\right)-f\left(g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)\right)}{g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)-g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}T_{NM}^{\alpha\varepsilon}g\left(t\right).$$

Here for $g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)=g\left(t\right)+\varepsilon_0$ and $g\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)=g\left(t\right)+\varepsilon_1$ there exist $\varepsilon_0,\varepsilon_1\in\mathbb{R}$

$$T_{NM}^{\alpha\varepsilon}\left(f\circ g\right)\left(t\right)=\frac{f\left(g\left(t\right)+\varepsilon_0\right)-f\left(g\left(t\right)+\varepsilon_1\right)}{\varepsilon_0-\varepsilon_1}T_{NM}^{\alpha\varepsilon}g\left(t\right).$$

For $\varepsilon_0 \to \varepsilon_1$ we have that

$$\begin{aligned} & T_{NM}^{\alpha\varepsilon}(f \circ g)(t) \\ = & \lim_{\varepsilon_0 \to \varepsilon_1} T_{NM}^{\alpha\varepsilon}(f \circ g)(t) \\ = & \lim_{\varepsilon_0 \to \varepsilon_1} \frac{f(g(t)+\varepsilon_0) - f(g(t)+\varepsilon_1)}{\varepsilon_0 - \varepsilon_1} T_{NM}^{\alpha\varepsilon} g(t). \\ = & f'(g(t)) T_{NM}^{\alpha\varepsilon} g(t) \end{aligned}$$

which is completed the proof of the part of (5). □

The following theorem has been proved as Theorem 1 for the T_{NM}^{α}-derivative:

Theorem 2. *Let* $\alpha \in (0,1]$, $M \le 0 \le N$ $(N, M \in \mathbb{R})$ *with* $N \ne M$, $Re(a) > 0$, $Re(b) > 0$ $(a, b \in \mathbb{C})$, $m \in \mathbb{N}$ *and* f, g *be* T_{NM}^{α}*-differentiable functions. Then,*

(1) $T_{NM}^{\alpha}(cf + dg) = cT_{NM}^{\alpha}(f) + dT_{NM}^{\alpha}(g)$, *for all* $c, d \in \mathbb{R}$.

(2) $T_{NM}^{\alpha}(C) = 0$, *for all constant functions,* $f(t) = C$.

(3)

$$T_{NM}^{\alpha}(fg)(t) = g(t) T_{NM}^{\alpha} f(t) + f(t) T_{NM}^{\alpha} g(t).$$

(4)

$$T_{NM}^{\alpha}\left(\frac{f}{g}\right)(t) = \frac{g(t) T_{NM}^{\alpha} f(t) - f(t) T_{NM}^{\alpha} g(t)}{g^2(t)}, \text{ for } g(t) \ne 0.$$

(5) $T_{NM}^{\alpha}(f \circ g)(t) = f'(g(t)) T_{NM}^{\alpha} g(t)$, *for* f *differentiable at* $g(t)$.

Theorem 3. *For all* $n \in \mathbb{R}$, *the* $T_{NM}^{\alpha\varepsilon}$*-derivative of* t^n *is that*

$$T_{NM}^{\alpha\varepsilon}(t^n) = \left(\frac{1}{\Gamma(a+b)} + H(\varepsilon)\right) [n]_E \, t^n |t|^{-\alpha} \widehat{sgn}(t) \tag{4.1}$$

where

$$\begin{aligned} & H(\varepsilon) \\ = & \frac{\widehat{sgn}(t)(N+M)\varepsilon |t|^{-\alpha}}{\Gamma(2a+b)} + \frac{(\widehat{sgn}(t))^2 \left(N^2 + NM + M^2\right) \varepsilon^2 |t|^{-2\alpha}}{\Gamma(3a+b)} + \cdots \\ & + \frac{(\widehat{sgn}(t))^{m-1}}{\Gamma(ma+b)} \sum_{i=0}^{m-1} N^{m-1-i} M^i \varepsilon^{m-1} |t|^{-(m-1)\alpha} \end{aligned}$$

and

$$[n]_E = \sum_{i=0}^{n-1} \left(E_{a,b}^{m}\left(\widehat{sgn}(t) N\varepsilon |t|^{-\alpha}\right)\right)^{n-1-i} \left(E_{a,b}^{m}\left(\widehat{sgn}(t) M\varepsilon |t|^{-\alpha}\right)\right)^{i}.$$

Proof. From definition of $T_{NM}^{\alpha\varepsilon}$-derivative, we get

$$\begin{aligned} & T_{NM}^{\alpha\varepsilon}(t^n) \\ = & \frac{t^n \left[\left(E_{a,b}^{m}\left(\widehat{sgn}(t) N\varepsilon |t|^{-\alpha}\right)\right)^{n} - \left(E_{a,b}^{m}\left(\widehat{sgn}(t) M\varepsilon |t|^{-\alpha}\right)\right)^{n}\right]}{(N-M)\varepsilon}. \end{aligned}$$

If we take $t > 0$

$$
\begin{aligned}
& T_{NM}^{\alpha\varepsilon}\left(t^{n}\right) \\
= & \frac{t^{n}\left[\left(E_{a,b}^{m}\left(N\varepsilon t^{-\alpha}\right)\right)^{n}-\left(E_{a,b}^{m}\left(M\varepsilon t^{-\alpha}\right)\right)^{n}\right]}{(N-M)\,\varepsilon} \\
= & \frac{t^{n}}{(N-M)\,\varepsilon}\left[E_{a,b}^{m}\left(N\varepsilon t^{-\alpha}\right)-E_{a,b}^{m}\left(M\varepsilon t^{-\alpha}\right)\right] \\
& \times\left[\sum_{i=0}^{n-1}\left(E_{a,b}^{m}\left(N\varepsilon\left|t\right|^{-\alpha}\right)\right)^{n-1-i}\left(E_{a,b}^{m}\left(M\varepsilon\left|t\right|^{-\alpha}\right)\right)^{i}\right] \\
= & \frac{t^{n}}{(N-M)\,\varepsilon}\left[E_{a,b}^{m}\left(N\varepsilon t^{-\alpha}\right)-E_{a,b}^{m}\left(M\varepsilon t^{-\alpha}\right)\right][n]_{E} \\
= & \frac{t^{n}}{(N-M)\,\varepsilon}\sum_{i=0}^{m}\frac{\left(N^{i}-M^{i}\right)\left(\varepsilon t^{-\alpha}\right)^{i}}{\Gamma(ai+b)}[n]_{E} \\
= & \frac{t^{n}}{(N-M)\,\varepsilon}[n]_{E}\left\{\frac{(N-M)\,\varepsilon t^{-\alpha}}{\Gamma(a+b)}+\frac{\left(N^{2}-M^{2}\right)\varepsilon^{2}t^{-2\alpha}}{\Gamma(2a+b)}\right. \\
& \left.+\frac{\left(N^{3}-M^{3}\right)\varepsilon^{3}t^{-3\alpha}}{\Gamma(3a+b)}+\cdots+\frac{\left(N^{m}-M^{m}\right)\varepsilon^{m}t^{-m\alpha}}{\Gamma(ma+b)}\right\} \\
= & \frac{t^{n}}{(N-M)\,\varepsilon}[n]_{E}(N-M)\,\varepsilon t^{-\alpha}\left\{\frac{1}{\Gamma(a+b)}+\frac{(N+M)\,\varepsilon t^{-\alpha}}{\Gamma(2a+b)}\right. \\
& \left.+\frac{\left(N^{2}+NM+M^{2}\right)\varepsilon^{2}t^{-2\alpha}}{\Gamma(3a+b)}+\cdots+\frac{1}{\Gamma(ma+b)}\sum_{i=0}^{m-1}N^{m-1-i}M^{i}\varepsilon^{m-1}t^{-(m-1)\alpha}\right\} \\
= & t^{n-\alpha}[n]_{E}\left\{\frac{1}{\Gamma(a+b)}+\frac{(N+M)\,\varepsilon t^{-\alpha}}{\Gamma(2a+b)}\right. \\
& \left.+\frac{\left(N^{2}+NM+M^{2}\right)\varepsilon^{2}t^{-2\alpha}}{\Gamma(3a+b)}+\cdots+\frac{1}{\Gamma(ma+b)}\sum_{i=0}^{m-1}N^{m-1-i}M^{i}\varepsilon^{m-1}t^{-(m-1)\alpha}\right\}.
\end{aligned}
$$

If we choose

$$
\begin{aligned}
H(\varepsilon) = & \frac{(N+M)\,\varepsilon t^{-\alpha}}{\Gamma(2a+b)}+\frac{\left(N^{2}+NM+M^{2}\right)\varepsilon^{2}t^{-2\alpha}}{\Gamma(3a+b)}+\cdots+\frac{1}{\Gamma(ma+b)}\sum_{i=0}^{m-1}N^{m-1-i}M^{i}\varepsilon^{m-1}t^{-(m-1)\alpha} \\
= & \frac{\widehat{sgn}(t)\,(N+M)\,\varepsilon\left|t\right|^{-\alpha}}{\Gamma(2a+b)}+\frac{\left(\widehat{sgn}(t)\right)^{2}\left(N^{2}+NM+M^{2}\right)\varepsilon^{2}\left|t\right|^{-2\alpha}}{\Gamma(3a+b)}+\cdots \\
& +\frac{\left(\widehat{sgn}(t)\right)^{m-1}}{\Gamma(ma+b)}\sum_{i=0}^{m-1}N^{m-1-i}M^{i}\varepsilon^{m-1}t^{-(m-1)\alpha}
\end{aligned}
$$

then it follows that

$$
T_{NM}^{\alpha\varepsilon}\left(t^{n}\right)=t^{n-\alpha}[n]_{E}\left(\frac{1}{\Gamma(a+b)}+H(\varepsilon)\right). \tag{4.2}
$$

Also if we take $t<0$, then we get

$$T_{NM}^{\alpha\varepsilon}\left(t^{n}\right)=\frac{t^{n}\left[\left(E_{a,b}^{m}\left(-N\varepsilon\left|t\right|^{-\alpha}\right)\right)^{n}-\left(E_{a,b}^{m}\left(-M\varepsilon\left|t\right|^{-\alpha}\right)\right)^{n}\right]}{\left(N-M\right)\varepsilon}$$

$$\begin{aligned}
&= \frac{t^{n}}{\left(N-M\right)\varepsilon}\left[E_{a,b}^{m}\left(-N\varepsilon\left|t\right|^{-\alpha}\right)-E_{a,b}^{m}\left(-M\varepsilon\left|t\right|^{-\alpha}\right)\right] \\
&\quad\times\left[\sum_{i=0}^{n-1}\left(E_{a,b}^{m}\left(-N\varepsilon\left|t\right|^{-\alpha}\right)\right)^{n-1-i}\left(E_{a,b}^{m}\left(-M\varepsilon\left|t\right|^{-\alpha}\right)\right)^{i}\right] \\
&= \frac{t^{n}}{\left(N-M\right)\varepsilon}\left[E_{a,b}^{m}\left(-N\varepsilon\left|t\right|^{-\alpha}\right)-E_{a,b}^{m}\left(-M\varepsilon\left|t\right|^{-\alpha}\right)\right]\left[n\right]_{E}
\end{aligned}$$

$$\begin{aligned}
&= \frac{t^{n}}{\left(N-M\right)\varepsilon}\sum_{i=0}^{m}\frac{\left(\left(-N\right)^{i}-\left(-M\right)^{i}\right)\left(\varepsilon\left|t\right|^{-\alpha}\right)^{i}}{\Gamma\left(ai+b\right)}\left[n\right]_{E} \\
&= \frac{t^{n}}{\left(N-M\right)\varepsilon}\left[n\right]_{E}\left\{\frac{\left(-N+M\right)\varepsilon\left|t\right|^{-\alpha}}{\Gamma\left(a+b\right)}+\frac{\left(N^{2}-M^{2}\right)\varepsilon^{2}\left|t\right|^{-2\alpha}}{\Gamma\left(2a+b\right)}\right. \\
&\quad\left.+\frac{\left(-N^{3}+M^{3}\right)\varepsilon^{3}\left|t\right|^{-3\alpha}}{\Gamma\left(3a+b\right)}+\cdots+\frac{\left(\left(-N\right)^{m}-\left(-M\right)^{m}\right)\varepsilon^{m}\left|t\right|^{-m\alpha}}{\Gamma\left(ma+b\right)}\right\} \\
&= \frac{t^{n}}{\left(N-M\right)\varepsilon}\left[\sum_{i=0}^{n-1}\left(E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)^{n-1-i}\left(E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)^{i}\right] \\
&\quad\times\left(-N+M\right)\varepsilon\left|t\right|^{-\alpha}\left\{\frac{1}{\Gamma\left(a+b\right)}-\frac{\left(N+M\right)\varepsilon\left|t\right|^{-\alpha}}{\Gamma\left(2a+b\right)}\right.
\end{aligned}$$

$$\begin{aligned}
&\quad\left.+\frac{\left(N^{2}+NM+M^{2}\right)\varepsilon^{2}\left|t\right|^{-2\alpha}}{\Gamma\left(3a+b\right)}+\cdots+\frac{\left(-1\right)^{m-1}}{\Gamma\left(ma+b\right)}\sum_{i=0}^{m-1}N^{m-1-i}M^{i}\varepsilon^{m-1}\left|t\right|^{-\left(m-1\right)\alpha}\right\} \\
&= -t^{n}\left|t\right|^{-\alpha}\left[n\right]_{E}\left\{\frac{1}{\Gamma\left(a+b\right)}-\frac{\left(N+M\right)\varepsilon\left|t\right|^{-\alpha}}{\Gamma\left(2a+b\right)}\right. \\
&\quad\left.+\frac{\left(N^{2}+NM+M^{2}\right)\varepsilon^{2}\left|t\right|^{-2\alpha}}{\Gamma\left(3a+b\right)}+\cdots+\frac{\left(-1\right)^{m-1}}{\Gamma\left(ma+b\right)}\sum_{i=0}^{m-1}N^{m-1-i}M^{i}\varepsilon^{m-1}\left|t\right|^{-\left(m-1\right)\alpha}\right\}.
\end{aligned}$$

and

$$\begin{aligned}
H(\varepsilon) &= -\frac{(N+M)\varepsilon|t|^{-\alpha}}{\Gamma(2a+b)}+\frac{\left(N^2+NM+M^2\right)\varepsilon^2|t|^{-2\alpha}}{\Gamma(3a+b)}+\cdots \\
&\quad +\frac{(-1)^{m-1}}{\Gamma(ma+b)}\sum_{i=0}^{m-1}N^{m-1-i}M^i\varepsilon^{m-1}|t|^{-(m-1)\alpha} \\
&= \frac{\widehat{sgn}(t)(N+M)\varepsilon|t|^{-\alpha}}{\Gamma(2a+b)}+\frac{\left(\widehat{sgn}(t)\right)^2\left(N^2+NM+M^2\right)\varepsilon^2|t|^{-2\alpha}}{\Gamma(3a+b)}+\cdots \\
&\quad +\frac{\left(\widehat{sgn}(t)\right)^{m-1}}{\Gamma(ma+b)}\sum_{i=0}^{m-1}N^{m-1-i}M^i\varepsilon^{m-1}|t|^{-(m-1)\alpha}
\end{aligned}$$

such that

$$T_{NM}^{\alpha\varepsilon}(t^n)=-t^n|t|^{-\alpha}[n]_E\left(\frac{1}{\Gamma(a+b)}+H(\varepsilon)\right). \tag{4.3}$$

Therefore, from (4.2) and (4.3) we have reached the desired (4.1). □

Corollary 1. *Let $\alpha\in(0,1]$, $M\leq 0\leq N$ $(N,M\in\mathbb{R})$ with $N\neq M$, $Re(a)>0$, $Re(b)>0$ $(a,b\in\mathbb{C})$, $m\in\mathbb{N}$ and for all $n\in\mathbb{R}$, the T_{NM}^{α}-derivative of t^n is that*

$$T_{NM}^{\alpha}(t^n)=\frac{n}{\Gamma(a+b)}t^n|t|^{-\alpha}\widehat{sgn}(t) \tag{4.4}$$

Proof. Since T_{NM}^{α}-derivative has limit as $\varepsilon\to 0$, if we take the limit of (4.1) and $\alpha\in(0,1]$ we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}T_{NM}^{\alpha\varepsilon}(t^n) &= T_{NM}^{\alpha}(t^n) \\
&= \lim_{\varepsilon\to 0}\left(\frac{1}{\Gamma(a+b)}+H(\varepsilon)\right)[n]_E\,t^n|t|^{-\alpha}\widehat{sgn}(t) \\
&= t^n|t|^{-\alpha}\widehat{sgn}(t)\lim_{\varepsilon\to 0}\left(\frac{1}{\Gamma(a+b)}+H(\varepsilon)\right)\lim_{\varepsilon\to 0}[n]_E.
\end{aligned}$$

Moreover

$$\begin{aligned}
\lim_{\varepsilon\to 0}[n]_E &= \lim_{\varepsilon\to 0}\sum_{i=0}^{n-1}\left(E_{a,b}^m\left(\widehat{sgn}(t)N\varepsilon|t|^{-\alpha}\right)\right)^{n-1-i}\left(E_{a,b}^m\left(\widehat{sgn}(t)M\varepsilon|t|^{-\alpha}\right)\right)^i \\
&= \sum_{i=0}^{n-1}\left(E_{a,b}^m(0)\right)^{n-1-i}\left(E_{a,b}^m(0)\right)^i=\sum_{i=0}^{n-1}1=n
\end{aligned}$$

such that

$$T_{NM}^{\alpha}(t^n)=\frac{n}{\Gamma(a+b)}t^n|t|^{-\alpha}\widehat{sgn}(t).$$

this is completed the proof of (4.4). □

Remark 1. *Under the assumptation of Corollary 1, we have*

$$T_{NM}^{\alpha\varepsilon}(t^{\alpha})=\left(\frac{1}{\Gamma(a+b)}+H(\varepsilon)\right)[\alpha]_E \tag{4.5}$$

where

$$\begin{aligned} & H(\varepsilon) \\ = \; & \frac{\widehat{sgn}(t)(N+M)\varepsilon|t|^{-\alpha}}{\Gamma(2a+b)} + \frac{(\widehat{sgn}(t))^2\left(N^2+NM+M^2\right)\varepsilon^2|t|^{-2\alpha}}{\Gamma(3a+b)} + \cdots \\ & + \frac{(\widehat{sgn}(t))^{m-1}}{\Gamma(ma+b)} \sum_{i=0}^{m-1} N^{m-1-i} M^i \varepsilon^{m-1} |t|^{-(m-1)\alpha} \end{aligned}$$

and

$$T_{NM}^{\alpha}(t^{\alpha}) = \frac{\alpha}{\Gamma(a+b)} \tag{4.6}$$

Proof. In Theorem 3, if $\alpha \in (0,1]$, the domain of t^{α} function must be nonnegative, so

$$\begin{aligned} T_{NM}^{\alpha\varepsilon}(t^{\alpha}) &= \left(\frac{1}{\Gamma(a+b)} + H(\varepsilon)\right)[n]_E\, t^{\alpha} t^{-\alpha} \\ &= \left(\frac{1}{\Gamma(a+b)} + H(\varepsilon)\right)[\alpha]_E \end{aligned}$$

and by limit $\varepsilon \to 0$, we have

$$T_{NM}^{\alpha}(t^{\alpha}) = \frac{\alpha}{\Gamma(a+b)}.$$

□

In (4.4) if $n = 1$ is selected and by using equality $t|t|^{-\alpha}\widehat{sgn}(t) = |t|^{1-\alpha}$ the following result is obtained:

Corollary 2. *For $\alpha \in (0,1]$, the T_{NM}^{α}-derivative of t is that*

$$T_{NM}^{\alpha\varepsilon}(t) = \left(\frac{1}{\Gamma(a+b)} + H(\varepsilon)\right)|t|^{1-\alpha}.$$

and

$$T_{NM}^{\alpha}(t) = \frac{|t|^{1-\alpha}}{\Gamma(a+b)}.$$

Now in the following theorem we will prove the continuity of the $T_{NM}^{\alpha} f$ at point c:

Theorem 4. *Let $\alpha \in (0,1]$, $M \leq 0 \leq N$ $(N, M \in \mathbb{R})$ with $N \neq M$, $Re(a) > 0$, $Re(b) > 0$ $(a, b \in \mathbb{C})$, $m \in \mathbb{N}$. If $f : I \subset \mathbb{R} \to \mathbb{R}$ is T_{NM}^{α}- generalized differentiable at $c \in I$, then, f is continuous at c.*

Proof. Since

$$\begin{aligned} & f\left(cE_{a,b}^m\left(\widehat{sgn}(t)N\varepsilon|c|^{-\alpha}\right)\right) - f\left(cE_{a,b}^m\left(\widehat{sgn}(t)M\varepsilon|c|^{-\alpha}\right)\right) \\ = \; & \frac{f\left(cE_{a,b}^m\left(\widehat{sgn}(t)N\varepsilon|c|^{-\alpha}\right)\right) - f\left(cE_{a,b}^m\left(\widehat{sgn}(t)M\varepsilon|c|^{-\alpha}\right)\right)}{(N-M)\varepsilon}.(N-M)\varepsilon \end{aligned}$$

we get

$$\begin{aligned}
& \lim_{\varepsilon\to 0}\left[f\left(cE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|c\right|^{-\alpha}\right)\right)-f\left(cE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|c\right|^{-\alpha}\right)\right)\right] \\
= & \lim_{\varepsilon\to 0}\frac{f\left(cE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|c\right|^{-\alpha}\right)\right)-f\left(cE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|c\right|^{-\alpha}\right)\right)}{\left(N-M\right)\varepsilon}\lim_{\varepsilon\to 0}\left(N-M\right)\varepsilon \\
= & T_{NM}^{\alpha}f\left(c\right).0=0.
\end{aligned}$$

Let choose $c=\frac{d}{E_{a,b}^{m}\left(\widehat{sgn}(t)M\varepsilon|c|^{-\alpha}\right)}$ then we have

$$\begin{aligned}
& \lim_{\varepsilon\to 0}\left[f\left(d\frac{E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|c\right|^{-\alpha}\right)}{E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|c\right|^{-\alpha}\right)}\right)-f\left(\frac{d}{E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|c\right|^{-\alpha}\right)}E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|c\right|^{-\alpha}\right)\right)\right] \\
= & 0.
\end{aligned}$$

Let assume that

$$\frac{E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|c\right|^{-\alpha}\right)}{E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|c\right|^{-\alpha}\right)}=\frac{\frac{1}{\Gamma(b)}+\frac{\left(\widehat{sgn}(t)N\varepsilon|c|^{-\alpha}\right)}{\Gamma(a+b)}+\frac{\left(\widehat{sgn}(t)N\varepsilon|c|^{-\alpha}\right)^{2}}{\Gamma(2a+b)}...+\frac{\left(\widehat{sgn}(t)N\varepsilon|c|^{-\alpha}\right)^{m}}{\Gamma(ma+b)}}{\frac{1}{\Gamma(b)}+\frac{\left(\widehat{sgn}(t)M\varepsilon|c|^{-\alpha}\right)}{\Gamma(a+b)}+\frac{\left(\widehat{sgn}(t)M\varepsilon|c|^{-\alpha}\right)^{2}}{\Gamma(2a+b)}...+\frac{\left(\widehat{sgn}(t)M\varepsilon|c|^{-\alpha}\right)^{m}}{\Gamma(ma+b)}}=1+h$$

and

$$\begin{aligned}
& \lim_{\varepsilon\to 0}\left[f\left(d\frac{E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|c\right|^{-\alpha}\right)}{E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|c\right|^{-\alpha}\right)}\right)-f\left(d\right)\right] \\
= & \lim_{h\to 0}\left[f\left(d+hd\right)-f\left(d\right)\right] \\
= & \lim_{h\to 0}\left[f\left(d+h\right)-f\left(d\right)\right]=0
\end{aligned}$$

such that

$$\lim_{h\to 0}\left[f\left(d+h\right)-f\left(d\right)\right]=0$$

i.e

$$\lim_{h\to 0}f\left(d+h\right)=f\left(d\right)$$

and replace as $d=cE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|c\right|^{-\alpha}\right)$, by $\lim_{\varepsilon\to 0}\left[E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|c\right|^{-\alpha}\right)\right]=$ 1 then we have

$$\lim_{\varepsilon\to 0}\left[\lim_{h\to 0}f\left(cE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|c\right|^{-\alpha}\right)+h\right)\right]=\lim_{\varepsilon\to 0}f\left(cE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|c\right|^{-\alpha}\right)\right)$$

i.e

$$\lim_{h\to 0}f\left(c+h\right)=f\left(c\right)$$

which implies that f is continuous at c. □

Theorem 5. *Let $\alpha\in(0,1]$, $M\leq 0\leq N$ $(N,M\in\mathbb{R})$ with $N\neq M$, $Re\left(a\right)>0$, $Re\left(b\right)>0$ $(a,b\in\mathbb{C})$, $m\in\mathbb{N}$ and $f:I\subset\mathbb{R}\to\mathbb{R}$ is $T_{NM}^{\alpha\varepsilon}$-generalized differentiable on I. Then, we have*

i) If $T_{NM}^{\alpha\varepsilon}f\left(x\right)>0$, then f is increasing on I.

ii) If $T_{NM}^{\alpha\varepsilon}f(x)<0$, then f is decreasing on I.
iii) If $T_{NM}^{\alpha\varepsilon}f(x)=0$, then f is constant on I.

Proof. Firstly we prove part i). For $\forall t\in I$, let us assume that

$$\lim_{\varepsilon\to 0}\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon}>0$$

(4.7)

$$T_{NM}^{\alpha\varepsilon}f(t)=\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon}>0.$$

In (4.7), if $\varepsilon>0$, then

$$f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)>0$$

is true and so

$$f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)>f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right).$$

On the other hand, for $M\leq 0\leq N$, $t>0$ and $\varepsilon>0$

$$tE_{a,b}^{m}\left(N\varepsilon\,|t|^{-\alpha}\right)>t>tE_{a,b}^{m}\left(M\varepsilon\,|t|^{-\alpha}\right).$$

Also for $M\leq 0\leq N$, $t<0$ and $\varepsilon>0$

$$E_{a,b}^{m}\left(-N\varepsilon\,|t|^{-\alpha}\right)<E_{a,b}^{m}\left(-M\varepsilon\,|t|^{-\alpha}\right).$$

is true and so

$$tE_{a,b}^{m}\left(-N\varepsilon\,|t|^{-\alpha}\right)>tE_{a,b}^{m}\left(-M\varepsilon\,|t|^{-\alpha}\right).$$

Therefore f is increasing on I. Similarly if $\varepsilon<0$, then

$$f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)<0$$

is true and thus

$$f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)<f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right).$$

On the other hand, for $M\leq 0\leq N$, $t>0$ and $\varepsilon<0$,

$$tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)<t<tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right).$$

Similarly for $M\leq 0\leq N$, $t<0$ and $\varepsilon<0$

$$E_{a,b}^{m}\left(-N\varepsilon\,|t|^{-\alpha}\right)<E_{a,b}^{m}\left(-M\varepsilon\,|t|^{-\alpha}\right)$$

i.e.

$$tE_{a,b}^{m}\left(-N\varepsilon\,|t|^{-\alpha}\right)>tE_{a,b}^{m}\left(-M\varepsilon\,|t|^{-\alpha}\right).$$

Therefore f is increasing on I.

Same way ii) can be proved. Finally for $\forall t\in I$ if

$$T_{NM}^{\alpha\varepsilon}f(t)=\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon}=0.$$

we say that

$$f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)=0$$

and

$$f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)=f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right),$$

this means f is constant on I and the proof is completed. □

Theorem 6. *Let $\alpha\in(0,1]$, $M\leq 0\leq N$ $(N,M\in\mathbb{R})$ with $N\neq M$, $Re(a)>0$, $Re(b)>0$ $(a,b\in\mathbb{C})$, $m\in\mathbb{N}$ and $f:I\subset\mathbb{R}\to\mathbb{R}$ is T_{NM}^{α}-generalized differentiable on I, then, we have*

i) If $T_{NM}^{\alpha}f(x)>0$ then f is increasing on I.
ii) If $T_{NM}^{\alpha}f(x)<0$ then f is decreasing on I.
iii) If $T_{NM}^{\alpha}f(x)=0$ then f is constant on I.

This theorem is proved in the same the above theorem.

Theorem 7. *(Rolle's Theorem for $T_{NM}^{\alpha\varepsilon}$ and T_{NM}^{α}-Generalized Differentiable Functions) Let $\alpha\in(0,1]$, $M\leq 0\leq N$ $(N,M\in\mathbb{R})$ with $N\neq M$, $Re(a)>0$, $Re(b)>0$ $(a,b\in\mathbb{C})$, $m\in\mathbb{N}$ and $f:[c,d]\subset\mathbb{R}\to\mathbb{R}$ is a function such that satisfies the following conditions:*

1) f is continuous on $[c,d]$,
2) $f(c)=f(d)$.
Then,
a)if f is $T_{NM}^{\alpha\varepsilon}$-generalized differentiable on (c,d), there exists $x_0\in(c,d)$, such that $T_{NM}^{\alpha\varepsilon}f(x_0)=0$.
b) if f is T_{NM}^{α}-generalized differentiable on (c,d), there exists $x_0\in(c,d)$, such that $T_{NM}^{\alpha}f(x_0)=0$.

Proof. a) Since f is continuous on $[c,d]$ and $f(c)=f(d)$, there is $x_0\in(c,d)$, at which the function has a local extrema. Then,

(4.8)

$$T_{NM}^{\alpha\varepsilon}f(x_0)=\frac{f\left(x_0E_{a,b}^{m}\left(\widehat{sgn}(x_0)N\varepsilon\left|x_0\right|^{-\alpha}\right)\right)-f\left(x_0E_{a,b}^{m}\left(\widehat{sgn}(x_0)M\varepsilon\left|x_0\right|^{-\alpha}\right)\right)}{(N-M)\varepsilon},\ \varepsilon<0$$

and

(4.9)

$$T_{NM}^{\alpha\varepsilon}f(x_0)=\frac{f\left(x_0E_{a,b}^{m}\left(\widehat{sgn}(x_0)N\varepsilon\left|x_0\right|^{-\alpha}\right)\right)-f\left(x_0E_{a,b}^{m}\left(\widehat{sgn}(x_0)M\varepsilon\left|x_0\right|^{-\alpha}\right)\right)}{(N-M)\varepsilon},\ \varepsilon>0$$

In (4.8) and (4.9) the two value have opposite signs. Therefore $T_{NM}^{\alpha\varepsilon}f(x_0)=0$. Moreover $T_{NM}^{\alpha}f(x_0)=\lim_{\varepsilon\to 0}T_{NM}^{\alpha\varepsilon}f(x_0)=0$ so this completes the proof. □

Theorem 8. *(Mean Value Theorem for T_{NM}^{α} and $T_{NM}^{\alpha\varepsilon}$-Generalized Differentiable Functions) Let $\alpha\in(0,1]$, $M\leq 0\leq N$ $(N,M\in\mathbb{R})$ with $N\neq M$, $Re(a)>0$, $Re(b)>0$ $(a,b\in\mathbb{C})$, $m\in\mathbb{N}$ and $f:[c,d]\subset\mathbb{R}\to\mathbb{R}$ is a continuous function. Then,*

1)if f is $T_{NM}^{\alpha\varepsilon}$-generalized differentiable on (c,d), there exists $x_0\in(c,d)$, such that

$$T_{NM}^{\alpha\varepsilon}f(x_0)=\frac{f(d)-f(c)}{(d-c)}\left(\frac{1}{\Gamma(a+b)}+H(\varepsilon)\right)\left|x_0\right|^{1-\alpha}. \tag{4.10}$$

2) if f is T_{NM}^{α}-generalized differentiable on (c,d), there exists $x_0\in(c,d)$, such that

$$T_{NM}^{\alpha}f(x_0)=\frac{f(d)-f(c)}{(d-c)}\frac{\left|x_0\right|^{1-\alpha}}{\Gamma(a+b)}. \tag{4.11}$$

Proof. *1)* Consider the function

$$g(t) = f(t) - f(c) - \frac{f(d) - f(c)}{(d-c)}((t-c)).$$

Then the function g satisfies the conditions of Rolle's theorem. Hence there exists $x_0 \in (c,d)$, such that $T_{NM}^{\alpha\varepsilon} g(x_0) = 0$. Using $T_{NM}^{\alpha\varepsilon}$-derivative

$$\begin{aligned} T_{NM}^{\alpha\varepsilon} g(t) &= T_{NM}^{\alpha\varepsilon}[f(t) - f(c)] \\ &\quad - \frac{f(d) - f(c)}{(d-c)} T_{NM}^{\alpha\varepsilon}(t-c) \end{aligned}$$

and for $t = x_0$

$$\begin{aligned} T_{NM}^{\alpha\varepsilon} f(x_0) &= \frac{f(d) - f(c)}{(d-c)} T_{NM}^{\varepsilon}(t-c) \\ &= \frac{f(d) - f(c)}{(d-c)} \left(\frac{1}{\Gamma(a+b)} + H(\varepsilon) \right) x_0 |x_0|^{-\alpha} \widehat{sgn}(x_0) \\ &= \frac{f(d) - f(c)}{(d-c)} \left(\frac{1}{\Gamma(a+b)} + H(\varepsilon) \right) |x_0|^{1-\alpha}. \end{aligned}$$

2) By taking limit (4.10) the desired is obtained as follow:

$$\begin{aligned} \lim_{\varepsilon \to 0} T_{NM}^{\alpha\varepsilon} f(x_0) &= \lim_{\varepsilon \to 0} \frac{f(d) - f(c)}{(d-c)} \left(\frac{1}{\Gamma(a+b)} + H(\varepsilon) \right) |x_0|^{1-\alpha} \\ T_{NM}^{\alpha} f(x_0) &= \frac{f(d) - f(c)}{(d-c)} \frac{|x_0|^{1-\alpha}}{\Gamma(a+b)} \end{aligned}$$

which completes the proof. □

Theorem 9. *Let $\alpha \in (0,1]$, $M \leq 0 \leq N$ $(N, M \in \mathbb{R})$ with $N \neq M$, $Re(a) > 0$, $Re(b) > 0$ $(a, b \in \mathbb{C})$, $m \in \mathbb{N}$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ is T_{NM}^{α}-generalized differentiable on I. T_{NM}^{α}-generalized derivative and classic derivative have the following relationship:*

$$T_{NM}^{\alpha} f(t) = \frac{|t|^{1-\alpha}}{\Gamma(a+b)} f'(t).$$

Proof. From definition of T_{NM}^{α}, it follows that

$$
\begin{aligned}
(4.12)\qquad & T_{NM}^{\alpha} f(t) \\
&= \lim_{\varepsilon\to 0} \frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right) - f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{(N-M)\,\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right) - f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{E_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right) - E_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)} \\
&\quad \times \frac{E_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right) - E_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)}{(N-M)\,\varepsilon} \\
&= t\lim_{\varepsilon\to 0} \frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)\right) - f\left(tE_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right)}{t\left\{E_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right) - E_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)\right\}} \\
&\quad \times \lim_{\varepsilon\to 0} \frac{E_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right) - E_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)}{(N-M)\,\varepsilon}
\end{aligned}
$$

is obtained. Calculating right limit as follow:

$$
\begin{aligned}
& \frac{E_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right) - E_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)}{(N-M)\,\varepsilon} \\
= \; & \frac{1}{(N-M)\,\varepsilon}\left\{\frac{1}{\Gamma(b)} + \frac{\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)}{\Gamma(a+b)} + \frac{\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)^{2}}{\Gamma(2a+b)} + \cdots + \frac{\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right)^{m}}{\Gamma(ma+b)}\right. \\
& \left. - \left(\frac{1}{\Gamma(b)} + \frac{\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)}{\Gamma(a+b)} + \frac{\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)^{2}}{\Gamma(2a+b)} + \cdots + \frac{\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)^{m}}{\Gamma(ma+b)}\right)\right\} \\
= \; & \frac{1}{(N-M)\,\varepsilon}\left(\frac{\widehat{sgn}(t)\,(N-M)\,\varepsilon\,|t|^{-\alpha}}{\Gamma(a+b)} + \frac{\left(N^{2}-M^{2}\right)\varepsilon^{2}\,|t|^{-2\alpha}}{\Gamma(2a+b)} + \cdots \right. \\
& \left. + \frac{sgn^{m}(t)\,(N^{m}-M^{m})\,\varepsilon^{m}\,|t|^{-m\alpha}}{\Gamma(ma+b)}\right) \\
= \; & \frac{|t|^{-\alpha}\,\widehat{sgn}(t)}{\Gamma(a+b)} + \frac{(N+M)\,\varepsilon^{2}\,|t|^{-2\alpha}}{\Gamma(2a+b)} + \cdots + \sum_{i=0}^{m-1}\frac{N^{m-1-i}M^{i}\varepsilon^{m-1|t|-m\alpha}sgn^{m}(t)}{\Gamma(ma+b)}
\end{aligned}
$$

such that

$$
\begin{aligned}
(4.13)\qquad & \lim_{\varepsilon\to 0}\frac{E_{a,b}^{m}\left(\widehat{sgn}(t)\,N\varepsilon\,|t|^{-\alpha}\right) - E_{a,b}^{m}\left(\widehat{sgn}(t)\,M\varepsilon\,|t|^{-\alpha}\right)}{(N-M)\,\varepsilon} \\
= \; & \lim_{\varepsilon\to 0}\left(\frac{|t|^{-\alpha}\,\widehat{sgn}(t)}{\Gamma(a+b)} + \frac{(N+M)\,\varepsilon^{2}\,|t|^{-2\alpha}}{\Gamma(2a+b)} + \cdots + \sum_{i=0}^{m-1}\frac{N^{m-1-i}M^{i}\varepsilon^{m-1|t|-m\alpha}sgn^{m}(t)}{\Gamma(ma+b)}\right) \\
= \; & \frac{|t|^{-\alpha}\,\widehat{sgn}(t)}{\Gamma(a+b)}.
\end{aligned}
$$

Moreover,

$$\lim_{\varepsilon\to 0}\frac{f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)\right)-f\left(tE_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right)}{t\left\{E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)N\varepsilon\left|t\right|^{-\alpha}\right)-E_{a,b}^{m}\left(\widehat{sgn}\left(t\right)M\varepsilon\left|t\right|^{-\alpha}\right)\right\}}=f'\left(t\right)$$

and from (4.12)-(4.13) we get

$$T_{NM}^{\alpha}f\left(t\right)=\frac{\left|t\right|^{1-\alpha}}{\Gamma\left(a+b\right)}f'\left(t\right)$$

and this gives the proof. □

Theorem 10. *Let $\alpha\in(0,1]$, $M\leq 0\leq N$ $(N,M\in\mathbb{R})$ with $N\neq M$, $Re\left(a\right)>0$, $Re\left(b\right)>0$ $(a,b\in\mathbb{C})$, $m\in\mathbb{N}$, T_{NM}^{α}-derivatives of some functions are as follows:*

(1)

$$T_{NM}^{\alpha}\left(\sin\left(t\right)\right)=\frac{\left|t\right|^{1-\alpha}}{\Gamma\left(a+b\right)}\cos\left(t\right).$$

(2)

$$T_{NM}^{\alpha}\left(\cos\left(t\right)\right)=-\frac{\left|t\right|^{1-\alpha}}{\Gamma\left(a+b\right)}\sin\left(t\right).$$

(3)

$$T_{NM}^{\alpha}\left(e^{t}\right)=\frac{\left|t\right|^{1-\alpha}}{\Gamma\left(a+b\right)}e^{t}.$$

(4) For $t>0$ and $\alpha\in(0,1]$

$$T_{NM}^{\alpha}\log\left(t\right)=\frac{t^{-\alpha}}{\Gamma\left(a+b\right)}.$$

Using Theorem 9, the proof of Theorem 10 can easily be seen.

5. Applications

In this section we claculate some T_{NM}^{α}-generalized derivative for some functions by modified signal function

$$\widehat{sgn}\left(t\right)=\begin{cases}1, & \text{If } t\geq 0\\ -1, & \text{If } t<0.\end{cases}$$

We will give the fractional derivative graphs for some values of α in the FIGURE 1-2. In this figures we chose as $a=b=1$.

Example 1. *1) Let $f:\mathbb{R}\to\mathbb{R}$ and choose the function $f\left(t\right)=t^{2}$ we get*

$$T_{NM}^{\alpha}\left(t^{2}\right)=\frac{2}{\Gamma\left(a+b\right)}t^{2}\left|t\right|^{-\alpha}\widehat{sgn}\left(t\right).$$

If we choose $t=-2$, we have

$$T_{NM}^{\alpha}f\left(-2\right)=-\frac{2^{3-\alpha}}{\Gamma\left(a+b\right)}$$

and let assume $a=b=\alpha=1$ we have classical derivative $T_{NM}^{1}f\left(-2\right)=-4=f'\left(-2\right).$

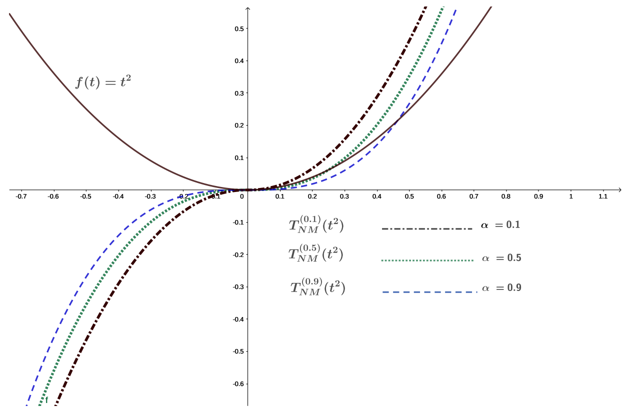

FIGURE 1

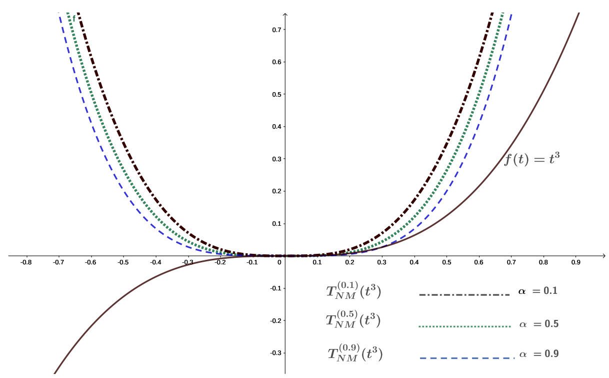

FIGURE 2

2) Let $f:\mathbb{R}\to\mathbb{R}$ *and choose the function* $f(t) = t^3$*. The* T_{NM}^{α}*-generalized fractional derivative of* t^3 *is that:*

$$T_{NM}^{\alpha}\left(t^3\right) = \frac{3}{\Gamma(a+b)} t^3 \left|t\right|^{-\alpha} \widehat{sgn}\left(t\right).$$

For $t = -2$, we have

$$T_{NM}^{\alpha} f(-2) = \frac{3}{\Gamma(a+b)} 2^{3-\alpha}$$

and if we choose $a = b = \alpha = 1$, we have classical derivative $T_{NM}^{1} f(-2) = 12 = f'(-2)$.

3) Let $f:\mathbb{R}\to\mathbb{R}$ and choose the function $f(t) = e^t$, we obtain that

$$T_{NM}^{\alpha}\left(e^t\right) = \frac{\left|t\right|^{1-\alpha}}{\Gamma(a+b)} e^t.$$

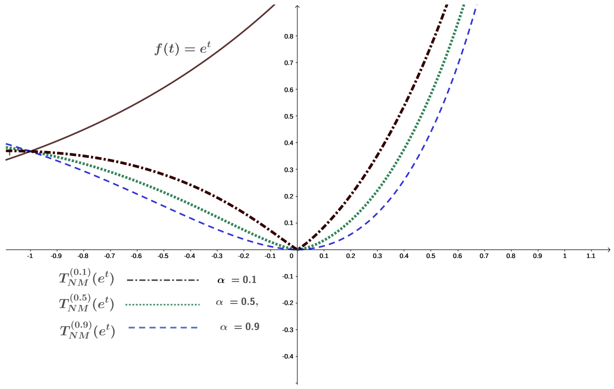

FIGURE 3

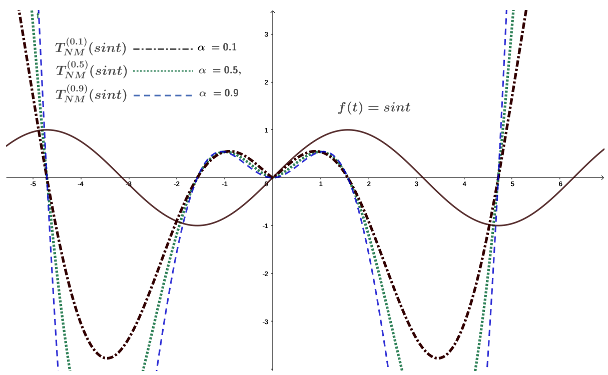

FIGURE 4

For example,

$$\begin{aligned} T_{NM}^{\alpha} f(-3) &= \frac{3^{1-\alpha}}{\Gamma(a+b)} e^{-3}, \\ T_{NM}^{\alpha} f(3) &= \frac{3^{1-\alpha}}{\Gamma(a+b)} e^{3}, \\ T_{NM}^{\alpha} f(0) &= \lim_{t \to 0^{\pm}} \frac{|t|^{1-\alpha}}{\Gamma(a+b)} e^{t} = 0. \end{aligned}$$

If we choose $a=b=\alpha=1$, we have $T_{NM}^{1}(e^t) = |t|^{1-\alpha} e^t$. For example, $T_{NM}^{1} f(-3) = e^{-3} = f'(-3)$, $T_{NM}^{1} f(3) = e^{3} = f'(3)$ and for $t=0$ we get $T_{NM}^{1} f(0) = \lim_{t\to 0^{\pm}} \left(|t|^{1-\alpha} e^{t}\right) = e^{0} = 1 = f'(0)$ these examples turn into classical derivatives.

4) Let $f:\mathbb{R}\to\mathbb{R}$ and choose the function $f(t)=\sin t$ and

$$T_{NM}^{\alpha}(\sin t) = \frac{|t|^{1-\alpha}}{\Gamma(a+b)} \cos(t).$$

For example,

$$\begin{aligned}
T^{\alpha}_{NM} f(-\pi) &= -\frac{\pi^{1-\alpha}}{\Gamma(a+b)}, \\
T^{\alpha}_{NM} f(\pi) &= -\frac{\pi^{1-\alpha}}{\Gamma(a+b)}, \\
T^{\alpha}_{NM} f(0) &= \lim_{t \to 0^{\pm}} \left(\frac{|t|^{1-\alpha}}{\Gamma(a+b)} \cos(t) \right) = 0.
\end{aligned}$$

If we choose $a = b = \alpha = 1$ we have $T^{1}_{NM}(\sin(t)) = |t|^{1-\alpha}\cos(t)$. For example, $T^{1}_{NM}(f(-\pi)) = -1 = f'(-\pi)$, $T^{1}_{NM}(f(\pi)) = -1 = f'(\pi)$ and for $t = 0$ we get $T^{1}_{NM}(f(0)) = \lim_{t \to 0^{\pm}} \left(|t|^{1-\alpha}\cos(t) \right) = \cos(0) = 1 = f'(0)$ these examples turn into classical derivatives.

This new generalized derivative will contribute to the solution of fractional differential equations. We also believe that it will guide us to the next works.

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