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ARTICLE TYPE

Boundary-Domain Integral Equation Systems to the Mixed BVP for Compressible Stokes Equations with Variable Viscosity in 2D †

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Summary

In this paper, the Boundary-Domain Integral Equations (BDIEs) for the mixed boundary value problem(BVP) for a compressible Stokes system of partial differential equation(PDE) with variable coefficient in 2D is considered . An appropriate parametrix is used to reduce this BVP to the BDIEs. Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the spaces to ensure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The properties of corresponding potential operators are investigated. Equivalence of the BDIE systems to the mixed BVP and invertibility of the matrix operators associated with the BDIE systems in appropriate Sobolev spaces are proved.

MSC: 76D07; 35J57; 31A10; 45A05.

Keywords: Stokes equations; Boundary-domain integral equations; Parametrix; Equivalence; Invertibility.

1 | INTRODUCTION

The Stokes system of PDE is derived from the linearised steady-state Navier-Stokes system. Studying this system gives us an opportunity to introduce several tools necessary for a treatment of the full Navier-Stokes equations, see for example ([1, , Chapter I]). In addition to its importance in applications, this system of PDEs has attracted the attention of numerical analysts.

Boundary integral equations and the hydrodynamic potential theory for the Stokes system with constant viscosity have been extensively studied by numerous authors, (see e.g.[2,3,4,5,6,7]). BDIE systems for the incompressible and compressible Stokes system with variable viscosity in three dimensional space have been investigated in[8] and[9] respectively, but BDIE systems in 2D, following a similar approach as in[10] have not yet been studied. In the case of constant viscosity, fundamental solutions for both velocity and pressure are available in analytical form. However, such fundamental solutions are not available for PDEs with variable viscocity. Therefore, the parametrix (Levi function), see, e.g.,[8,9] is used in order to derive and investigate the BDIE systems for the corresponding variable-coefficient BVPs. In[11,10], authors derived and investigated BDIE systems for BVP with variable-coefficient scalar elliptic PDE defined on a bounded domain. In[8,9], authors transformed mixed BVP with variable coefficient for Stokes problem defined on a bounded domain to BDIE systems for their further analysis. In this paper, we shall derive and investigate BDIE systems for variable coefficient Mixed BVP for compressible Stokes equations in appropriate Sobolev-Slobodetski (Bessel potential) spaces.

†BDIE Systems for Mixed Stokes equations in 2D

2 | PRELIMINARIES

Let $\Omega = \Omega^+$ be a bounded and simply-connected open two-dimensional region of $\mathbb{R}^2$ and the boundary $\partial\Omega$ be closed and infinitely smooth curve. Moreover, $\overline{\partial\Omega} = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$ where $\partial\Omega_D$ and $\partial\Omega_N$ are non empty and non-intersecting part of $\partial\Omega$ with infinitely smooth boundary curve $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} \in \mathcal{C}^\infty$.

Let $\mathbf{v}$ be the velocity vector field, p the pressure scalar field and $\mu \in C^\infty(\Omega)$ be the variable kinematic viscosity of the fluid such that $\mu(\mathbf{x}) > c > 0$. For compressible fluid the stress tensor operator, σ_{ij} , for an arbitrary couple $(p, \mathbf{v})$ is defined as

$$\sigma_{ij}(p,\mathbf{v})(\mathbf{x}) := -\delta_i^j p + \mu(\mathbf{x})\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \alpha\delta_i^j \operatorname{div} \mathbf{v}(\mathbf{x})\right),$$

and the Stokes operator is defined as

$$\begin{aligned}\mathcal{A}_j(p,\mathbf{v})(\mathbf{x}) :=& \frac{\partial}{\partial x_i}\sigma_{ij}(p,\mathbf{v})(\mathbf{x}) \\ =& \frac{\partial}{\partial x_i}\left(\mu(\mathbf{x})\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \alpha\delta_i^j \operatorname{div} \mathbf{v}(\mathbf{x})\right)\right) - \frac{\partial p}{\partial x_j}, \quad j, i \in \{1,2\},\end{aligned} \tag{1}$$

where $\alpha = 1$ or $\alpha = \frac{2}{3}$ and δ_i^j is Kronecker symbol. Here and henceforth we assume the Einstein summation in repeated indices from 1 to 2. We denote the Stokes operator as $\mathcal{A} = \{\mathcal{A}_j\}_{j=1}^2$ and $\mathring{\mathcal{A}} := \mathcal{A}|_{\mu=1}$. We will also use the following notation for derivative operators: $\partial_j = \partial_{x_j} := \frac{\partial}{\partial x_j}$ with $j = 1, 2$; $\nabla := (\partial_1, \partial_2)$.

In what follows $H^s(\Omega) = H_2^s(\Omega)$, $H^s(\partial\Omega)$ are the Bessel potential spaces, where s is a real number (see, e.g.[12,13]). We recall that H^s coincide with the Sobolev-Slobodetski spaces W_2^s for any non-negative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^2)$, $\widetilde{H}^s(\Omega) = \{g : g \in H^s(\mathbb{R}^2), \operatorname{supp}(g) \subset \overline{\Omega}\}$; similarly, $\widetilde{H}^s(S_1) = \{g : g \in H^s(\partial\Omega), \operatorname{supp}(g) \subset \overline{S_1}\}$, $L^2_*(\Omega) = L^2(\Omega)/\mathbb{R} = \{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\}$. We will also use the notations $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^2$, $\mathbf{L}^2(\Omega) = \left[L^2(\Omega)\right]^2$, $\boldsymbol{\mathcal{D}}(\Omega) = [\mathscr{D}(\Omega)]^2$ for 2-dimensional vector space. We will also make use of the following space (see, e.g.[14,11,9]).

$$\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}}) := \{(p,\mathbf{v}) \in H^{s-1}(\Omega)\times\mathbf{H}^s(\Omega) : \boldsymbol{\mathcal{A}}(p,\mathbf{v}) \in \mathbf{L}^2(\Omega)\}$$

endowed with the norm

$$\|(p,v)\|^2_{\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})} := \|p\|^2_{H^{s-1}(\Omega)} + \|\ \mathbf{v}\ \|^2_{\mathbf{H}^s(\Omega)} + \|\ \boldsymbol{\mathcal{A}}(p,\mathbf{v})\ \|^2_{\mathbf{L}^2(\Omega)}\ .$$

Let us define a space

$$\mathbf{H}^{1,0}_*(\Omega;\boldsymbol{\mathcal{A}}) := \{(p,\mathbf{v}) \in L^2_*(\Omega)\times\mathbf{H}^1(\Omega) : \boldsymbol{\mathcal{A}}(p,\mathbf{v}) \in \mathbf{L}^2(\Omega)\}$$

endowed with the norm

$$\|(p,\mathbf{v})\|^2_{\mathbf{H}^{1,0}_*(\Omega;\boldsymbol{\mathcal{A}})} := \|p\|^2_{L^2_*(\Omega)} + \|\ \mathbf{v}\ \|^2_{\mathbf{H}^1(\Omega)} + \|\ \boldsymbol{\mathcal{A}}(p,\mathbf{v})\ \|^2_{\mathbf{L}^2(\Omega)}\ .$$

The operator $\boldsymbol{\mathcal{A}}$ acting on $(p, \mathbf{v})$ is well defined in the weak sense provided $\mu(\mathbf{x}) \in L^\infty(\Omega)$ as

$$\langle\boldsymbol{\mathcal{A}}(p,\mathbf{v}),\mathbf{u}\rangle_\Omega := -\mathcal{E}((p,\mathbf{v}),\mathbf{u}), \qquad \forall\mathbf{u} \in \widetilde{\mathbf{H}}^1(\Omega),$$

where the form $\mathcal{E} : \left[L^2(\Omega)\times\mathbf{H}^1(\Omega)\right]\times\widetilde{\mathbf{H}}^1(\Omega) \to \mathbb{R}$ is defined as

$$\mathcal{E}\left((p,\mathbf{v}),\mathbf{u}\right) := \int_\Omega E\left((p,\mathbf{v}),\mathbf{u}\right)(\mathbf{x})\,d\mathbf{x}, \tag{2}$$

and the function $E\left((p, \mathbf{v}), \mathbf{u}\right)$ given by

$$E((p,\mathbf{v}),\mathbf{u})(\mathbf{x}) := \frac{\mu(\mathbf{x})}{2}\left(\frac{\partial u_i(x)}{\partial x_j} + \frac{\partial u_j(x)}{\partial x_i}\right)\left(\frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i}\right) - \alpha\mu(x)\operatorname{div}\mathbf{v}(\mathbf{x})\operatorname{div}\mathbf{u}(\mathbf{x}) - p(\mathbf{x})\operatorname{div}\mathbf{u}(\mathbf{x}).$$

For sufficiently smooth functions $(p, \mathbf{v}) \in H^{s-1}(\Omega^\pm)\times\mathbf{H}^s(\Omega^\pm)$ with $s > 3/2$, we can define the classical traction operators, $\mathbf{T}^{c\pm} = \{T_j^{c\pm}\}_{j=1}^2$ on the boundary $\partial\Omega$ as

$$T_j^{c\pm}(p,\mathbf{v})(\mathbf{x}) := \left[\gamma^\pm\sigma_{ij}(p,\mathbf{v})(\mathbf{x})\right]n_i(\mathbf{x}), \tag{3}$$

where $n_i(\mathbf{x})$ denote components of the unit outward normal vector $\mathbf{n(x)}$ to the boundary $\partial\Omega$ of the domain and $\gamma^\pm$ is the trace operator from inside and outside Ω[8,9].

Traction operator (3) can be continuously extended to the canonical traction operator $\mathbf{T}^{\pm} : \mathbf{H}^{1,0}(\Omega^{\pm};\boldsymbol{\mathcal{A}}) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ defined in the weak form similar to[8,9] as

$$\langle \mathbf{T}^{\pm}(p,\mathbf{v}),\mathbf{w}\rangle_{\partial\Omega} := \pm\int_{\Omega^{\pm}} \left[\boldsymbol{\mathcal{A}}(p,\mathbf{v})(\gamma^{-1}\mathbf{w}) + E((p,\mathbf{v}),\gamma^{-1}\mathbf{w})\right] d\mathbf{x},$$

$$(p,\mathbf{v}) \in \mathbf{H}^{1,0}(\Omega^{\pm};\boldsymbol{\mathcal{A}}), \forall \mathbf{w} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega).$$

Here the operator $\gamma^{-1} : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{1}(\mathbb{R}^2)$ denotes a continuous right inverse of the trace operator $\gamma^{+} : \mathbf{H}^{1}(\mathbb{R}^2) \to \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. In addition, for $(p,\mathbf{v}) \in \mathbf{H}^{1,0}_{*}(\Omega;\boldsymbol{\mathcal{A}})$ the traction operator $\mathbf{T}^{\pm}$ are also defined.

Furthermore, if $(p,\mathbf{v}) \in \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ and $\mathbf{u} \in \mathbf{H}^{1}(\Omega)$, the following first Green identity holds, (see, e.g.,[14,11,15,8] and[9]),

$$\langle \mathbf{T}^{+}(p,\mathbf{v}),\gamma^{+}\mathbf{u}\rangle_{\partial\Omega} := \int_{\Omega} [\boldsymbol{\mathcal{A}}(p,\mathbf{v})\mathbf{u} + E((p,\mathbf{v}),\mathbf{u})(\mathbf{x})]\, d\mathbf{x}. \tag{4}$$

Equation (4) is also defined for $(p,\mathbf{v}) \in \mathbf{H}^{1,0}_{*}(\Omega;\boldsymbol{\mathcal{A}})$ and $\mathbf{u} \in \mathbf{H}^{1}(\Omega)$. Applying the identity (4) to the pairs $(p,\mathbf{v}) \in \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ and $(q,\mathbf{u}) \in \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ with exchanged roles and subtracting the one from the other, we arrive at the second Green identity, (see, e.g.[13,15,8,9]),

$$\int_{\Omega} \left[\mathcal{A}_j(p,\mathbf{v})u_j - \mathcal{A}_j(q,\mathbf{u})v_j + q\,\mathrm{div}\,\mathbf{v} - p\,\mathrm{div}\,\mathbf{u}\right] dx = \int_{\partial\Omega} \left[T_j(p,\mathbf{v})u_j - T_j(q,\mathbf{u})v_j\right] dS_{\mathbf{x}}. \tag{5}$$

Equation (5) is also defined for $(p,\mathbf{v}) \in \mathbf{H}^{1,0}_{*}(\Omega;\boldsymbol{\mathcal{A}})$ and $(q,\mathbf{u}) \in \mathbf{H}^{1,0}_{*}(\Omega;\boldsymbol{\mathcal{A}})$.

3 | FORMULATION OF THE BOUNDARY VALUE PROBLEM

We shall derive and investigate BDIE systems for the following mixed BVP. For $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in L^2(\Omega)$, $\boldsymbol{\varphi}_0 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega_D)$ and $\boldsymbol{\psi}_0 \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N)$, find $(p,\mathbf{v}) \in \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ such that:

$$\boldsymbol{\mathcal{A}}(p,\mathbf{v})(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{6a}$$

$$\mathrm{div}\,\mathbf{v}(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \Omega,, \tag{6b}$$

$$r_{\partial\Omega_D}\gamma^{+}\mathbf{v}(\mathbf{x}) = \boldsymbol{\varphi}_0(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega_D, \tag{6c}$$

$$r_{\partial\Omega_N}\mathbf{T}^{+}(p,\mathbf{v})(\mathbf{x}) = \boldsymbol{\psi}_0(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega_N. \tag{6d}$$

Theorem 1. The BVP (6a)-(6d) has at most one solution in the space $\mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$.

Proof. let $(p_1,\mathbf{v}_1)$ and $(p_2,\mathbf{v}_2)$ are in $\mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ that satisfy the BVP (6a)-(6d).Then $(p,\mathbf{v}) := (p_2,\mathbf{v}_2) - (p_1,\mathbf{v}_1)$ also belongs to $\mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ satisfy the following homogeneous mixed BVP

$$\boldsymbol{\mathcal{A}}(p,\mathbf{v})(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{7a}$$

$$\mathrm{div}\,\mathbf{v}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{7b}$$

$$r_{\partial\Omega_D}\gamma^{+}\mathbf{v}(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega_D, \tag{7c}$$

$$r_{\partial\Omega_N}\mathbf{T}^{+}(p,\mathbf{v})(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega_N. \tag{7d}$$

The first Green identity (4) holds for any $\mathbf{u} \in \mathbf{H}^{1}(\Omega)$and for any pair $(p,\mathbf{v}) \in \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$.
Then due to (7a)-(7d) we have, $\int_{\Omega} E(\mathbf{v},\mathbf{u})(\mathbf{x})d\mathbf{x} = 0$ which implies that $\int_{\Omega} \frac{\mu(\mathbf{x})}{2}\left(\frac{\partial u_i(\mathbf{x})}{\partial x_j} + \frac{\partial u_j(\mathbf{x})}{\partial x_i}\right)\left(\frac{\partial v_i(\mathbf{x})}{\partial x_j} + \frac{\partial v_j(\mathbf{x})}{\partial x_i}\right) d\mathbf{x} = 0$. from (2), $\mathcal{E}(\mathbf{v},\mathbf{u}) = \int_{\Omega} E(\mathbf{v},\mathbf{u})(\mathbf{x})d\mathbf{x}$. In particular, choose $\mathbf{u} = \mathbf{v}$. Then

$$\mathcal{E}(\mathbf{v},\mathbf{v}) = \int_{\Omega} E(\mathbf{v},\mathbf{v})(\mathbf{x})d\mathbf{x} = 0.$$

As $\mu(\mathbf{x}) > 0$, the only possibility is that $\mathbf{v}(\mathbf{x}) = \mathbf{a} + b(-x_2,x_1)^T$, i.e., $\mathbf{v}(\mathbf{x})$ is a rigid movement. Taking into account the Dirichlet condition (7c), we deduce that $\mathbf{v} \equiv 0$. Hence, $\mathbf{v}_1 = \mathbf{v}_2$.

Considering now $\mathbf{v} \equiv 0$ and keeping in mind the Neumann-traction condition (7d), we conclude that $p_1 = p_2$. □

4 | PARAMETRIX AND PARAMETRIX-BASED HYDRODYNAMIC POTENTIALS

4.1 | Parametrix and Remainder

The operator $\mathcal{A}$ becomes the constant-coefficient Stokes operator $\mathring{\mathcal{A}}$ when $\mu = 1$. The fundamental solution defined by the pair of distributions $(\mathring{q}^k, \mathring{\mathbf{u}}^k)$,where $\mathring{u}_j^k$ represent components of the incompressible velocity fundamental solution and $\mathring{q}^k$ represent the components of the pressure fundamental solution, (see, e.g.,[3,2,4,5]). So for $r_0 > 0$, $\mathring{\mathbf{u}}^k$ and $\mathring{q}^k$ will have the form:

$$\begin{aligned}
\mathring{u}_j^k(\mathbf{x},\mathbf{y}) &= \frac{1}{4\pi}\left(\delta_j^k \log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right)\\
\mathring{q}^k(\mathbf{x},\mathbf{y}) &= \frac{-(x_k-y_k)}{2\pi|\mathbf{x}-\mathbf{y}|^2}
\end{aligned}$$

with $(\mathring{q}^k, \mathring{\mathbf{u}}^k)$ satisfying the relations

$$\frac{\partial}{\partial x_k}\mathring{q}^k(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{2}\frac{\partial^2}{\partial x_k^2}\left(-\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|\right) = -\delta(\mathbf{x}-\mathbf{y}) \tag{8}$$

$$\mathring{\mathcal{A}}_j(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{2}\frac{\partial^2\mathring{u}_j^k}{\partial x_i^2} - \frac{\partial\mathring{q}^k}{\partial x_j} = \delta_j^k\delta(\mathbf{x}-\mathbf{y}), \quad \operatorname{div}_{\mathbf{x}}\mathring{\mathbf{u}}^k(\mathbf{x},\mathbf{y}) = 0. \tag{9}$$

Let us denote $\mathring{\sigma}_{ij}(p,\mathbf{v}) := \sigma_{ij}(p,\mathbf{v})|_{\mu=1}$. Then in particular case, for $\mu = 1$ and the fundamental solution $(\mathring{q}^k, \mathring{\mathbf{u}}^k)_{k=1,2}$ of the operator $\mathring{\mathcal{A}}$, the stress tensor $\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})$ reads

$$\mathring{\sigma}_{ij}(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}.$$

Indeed,

$$\begin{aligned}
\mathring{\sigma}_{ij}(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}) =& -\mathring{q}^k\delta_{ij} + \left(\frac{\partial\mathring{u}_i^k}{\partial x_j} + \frac{\partial\mathring{u}_j^k}{\partial x_i}\right)\\
=&\frac{x_k-y_k}{2\pi|\mathbf{x}-\mathbf{y}|^2}\delta_{ij} + \left[\frac{\partial}{\partial x_i}\left(\frac{1}{4\pi}\left(\delta_j^k\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right)\right)\right.\\
&\left.+\frac{\partial}{\partial x_j}\left(\frac{1}{4\pi}\left(\delta_i^k\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_i-y_i)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right)\right)\right].
\end{aligned}$$

Since

$$\mathring{\sigma}_{ij}(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}.$$

the boundary traction becomes

$$\begin{aligned}
\mathring{T}_j^c(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x},\mathbf{y}) &:= \mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(\mathbf{x}-\mathbf{y})n_i(\mathbf{x})\\
&= \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}n_i(\mathbf{x}).
\end{aligned}$$

Let us define a pair of functions $(q^k, \mathbf{u}^k)_{k=1,2}$ similar as in[8,9],

$$u_j^k(\mathbf{x},\mathbf{y}) = \frac{1}{\mu(\mathbf{y})}\mathring{u}_j{}^k(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi\mu(\mathbf{y})}\left(\delta_j^k\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right), \tag{10}$$

$$q^k(\mathbf{x},\mathbf{y}) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{q}^k(\mathbf{x},\mathbf{y}) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\frac{y_k-x_k}{2\pi|\mathbf{x}-\mathbf{y}|^2}, \quad j,k\in\{1,2\}. \tag{11}$$

Then

$$\begin{aligned}
\sigma_{ij}(\mathbf{x};q^k,\mathbf{u}^k)(\mathbf{x}-\mathbf{y}) =& -\delta_i^j q^k + \mu(\mathbf{x})\left(\frac{\partial u_i^k}{\partial x_j} + \frac{\partial u_j^k}{\partial x_i} - \alpha\delta_i^j \operatorname{div} u^k(\mathbf{x})\right)\\
=& -\delta_i^j\frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{q}^k + \mu(\mathbf{x})\left(\frac{\partial(\frac{1}{\mu(\mathbf{y})}\mathring{u}_i^k)}{\partial x_j} + \frac{\partial(\frac{1}{\mu(\mathbf{y})}\mathring{u}_j^k)}{\partial x_i} - \alpha\delta_i^j \operatorname{div}\left(\frac{1}{\mu(y)}\mathring{u}^k(\mathbf{x})\right)\right)
\end{aligned}$$

$$=\frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\left(-\delta_i^j \mathring{q}^k + \left(\frac{\partial \mathring{u}_i^k}{\partial x_j} + \frac{\partial \mathring{u}_j^k}{\partial x_i} - \alpha\delta_i^j \text{ div } \mathring{\mathbf{u}}^k(\mathbf{x})\right)\right) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}).$$

Thus,

$$\sigma_{ij}(x; q^k, \mathbf{u}^k)(\mathbf{x}-\mathbf{y}) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})$$

and

$$T_j(x; q^k, \mathbf{u}^k)(\mathbf{x}, \mathbf{y}) := \sigma_{ij}(\mathbf{x}; q^k, \mathbf{u}^k)(\mathbf{x}-\mathbf{y})n_i(\mathbf{x}) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{T}_j(\mathbf{x}; \mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}, \mathbf{y}) \tag{12}$$

substituting (10)-(11) into Stokes system (1) with variable coefficients, we get

$$\begin{aligned}
\mathcal{A}_j(\mathbf{x}; q^k; \mathbf{u}^k)(\mathbf{x}, \mathbf{y}) &= \frac{\partial}{\partial x_i}\left(\sigma_{ij}(\mathbf{x}; q^k, \mathbf{u}^k)(\mathbf{x}-\mathbf{y})\right) = \frac{\partial}{\partial x_i}\left(\frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})\right)\\
&= \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\frac{\partial}{\partial x_i}\left(\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})\right) + \frac{\partial}{\partial x_i}\left(\frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\right)\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})\\
&= \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{\mathcal{A}}_j(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}) + \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})\\
&= \frac{\mu(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\delta_j^k}{\mu(\mathbf{y})} + \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}\\
&= \frac{\mu(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\delta_j^k}{\mu(\mathbf{y})} + \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})\\
&= \delta_j^k\delta(\mathbf{x}-\mathbf{y} + \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})
\end{aligned}$$

Thus,

$$\mathcal{A}_j(\mathbf{x}; q^k; \mathbf{u}^k)(\mathbf{x}, \mathbf{y}) = \delta_j^k\delta(\mathbf{x}-\mathbf{y}) + R_{kj}(\mathbf{x}, \mathbf{y}), \tag{13}$$

where

$$R_{kj}(\mathbf{x}, \mathbf{y}) = \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}) = \mathcal{O}(|\mathbf{x}-\mathbf{y}|^{-1})$$

is a weakly singular remainder. This implies that $(q^k,\ \mathbf{u}^k)$ is a parametrix of the operator $\boldsymbol{\mathcal{A}}$.

4.2 | Volume and Surface Potentials

Let ρ and $\boldsymbol{\rho}$ be sufficiently smooth scalar and vector function on Ω. The parametrix-based Newton-type and the Remainder vector potential operators are defined as

$$[\boldsymbol{\mathcal{U}}\boldsymbol{\rho}]_k(\mathbf{y}) = \mathcal{U}_{kj}\rho_j(\mathbf{y}) := \int_\Omega u_j^k(\mathbf{x}, \mathbf{y})\rho_j(\mathbf{x})d\mathbf{x}, \qquad [\boldsymbol{\mathcal{R}}\boldsymbol{\rho}]_k(\mathbf{y}) = \mathcal{R}_{kj}\rho_j(\mathbf{y}) := \int_\Omega R_{kj}(\mathbf{x}, \mathbf{y})\rho_j(\mathbf{x})d\mathbf{x}, \ \ \mathbf{y} \in \mathbb{R}^2$$

for the velocity $\mathbf{v}$, and the scalar Newton-type and remainder potentials for the pressure,

$$[\boldsymbol{\mathcal{Q}}\rho]_j(\mathbf{y}) = \mathcal{Q}_j\rho(\mathbf{y}) := -\int_\Omega q^j(\mathbf{x}, \mathbf{y})\rho(\mathbf{x})d\mathbf{x}, \tag{14}$$

$$\mathcal{Q}\boldsymbol{\rho}(\mathbf{y}) = \boldsymbol{\mathcal{Q}}\cdot\boldsymbol{\rho}(\mathbf{y}) = \mathcal{Q}_j\rho_j(\mathbf{y}) := -\int_\Omega q^j(\mathbf{x}, \mathbf{y})\rho_j(\mathbf{x})d\mathbf{x}, \tag{15}$$

$$\boldsymbol{\mathcal{R}}^{\bullet}\boldsymbol{\rho}(\mathbf{y}) = -2\langle\partial_i\mathring{q}^j(., \mathbf{y}), \rho_i\partial_j\mu\rangle_\Omega - 2\rho_i(\mathbf{y})\partial_i\mu(\mathbf{y}) = -2v.p.\int_\Omega \frac{\partial\mathring{q}^j(\mathbf{x}, \mathbf{y})}{\partial x_i}\frac{\partial\mu(\mathbf{x})}{\partial x_i}\rho_j(\mathbf{x})d\mathbf{x} - \rho_j(\mathbf{y})\frac{\partial\mu(\mathbf{y})}{\partial y_j}, \tag{16}$$

for $\mathbf{y} \in \mathbf{R}^2$. The integral in (16) is understood as a 2D strongly singular integral in the Cauchy sense, (see, e.g.,[8,9]).

For the velocity, the parametrix-based single layer and double layer potentials are defined for $\mathbf{y} \notin \partial\Omega$ as :

$$[\mathbf{V}\boldsymbol{\rho}]_k(\mathbf{y}) = V_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} u_j^k(\mathbf{x}, \mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \qquad [\mathbf{W}\boldsymbol{\rho}]_k(\mathbf{y}) = W_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} T_j^+(\mathbf{x}; q^k, \mathbf{u}^k)(\mathbf{x}, \mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}},$$

and for pressure in the variable coefficient Stokes system, the single layer and double layer potentials are defined for $\mathbf{y} \notin \partial\Omega$ as:

$$\Pi^s \boldsymbol{\rho}(\mathbf{y}) = \Pi^s_j \rho_j(\mathbf{y}) := \int_{\partial\Omega} \mathring{q}^j(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \qquad \Pi^d \boldsymbol{\rho}(\mathbf{y}) = \Pi^d_j \rho_j(\mathbf{y}) := 2\int_{\partial\Omega} \frac{\partial \mathring{q}^j(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{x})}\mu(\mathbf{x})\rho_j(\mathbf{x})dS_{\mathbf{x}}.$$

The corresponding boundary integral (pseudo-differential) operators of direct surface values of the single layer potential and the double layer potential, the traction of the single layer potential and the double layer potential are

$$[\boldsymbol{\mathcal{V}}\boldsymbol{\rho}]_k(\mathbf{y}) = \mathcal{V}_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} u^k_j(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \qquad [\boldsymbol{\mathcal{W}}\boldsymbol{\rho}]_k(\mathbf{y}) = \mathcal{W}_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} T^+_j(\mathbf{x}; q^k, \mathbf{u}^k)(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \ \ \mathbf{y}\in\partial\Omega,$$

$$\left[\boldsymbol{\mathcal{W}}'\boldsymbol{\rho}\right]_k(\mathbf{y}) = \mathcal{W}'_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} T^+_j(\mathbf{y}; q^k, \mathbf{u}^k)(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \qquad \boldsymbol{\mathcal{L}}^{\pm}\boldsymbol{\rho}(\mathbf{y}) := \mathbf{T}^{\pm}(\Pi^d\boldsymbol{\rho}, \mathbf{W}\boldsymbol{\rho})(\mathbf{y}), \ \ \mathbf{y}\in\partial\Omega,$$

where $\mathbf{T}^{\pm}$ are the traction operators (see, e.g.,[8,9]).

The parametrix-based integral operators depending on the variable coefficient, $\mu(\mathbf{x})$, can be expressed in terms of the corresponding integral operators for the constant coefficient case, $\mu = 1$, see ([8,9]) for 3D case.

$$\boldsymbol{\mathcal{U}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}, \tag{17}$$

$$[\boldsymbol{\mathcal{R}}\boldsymbol{\rho}]_k = -\frac{1}{\mu}\left[\frac{\partial}{\partial y_j}\mathring{\mathcal{U}}_{ki}(\rho_j\partial_i\mu)(\mathbf{y}) + \frac{\partial}{\partial y_i}\mathring{\mathcal{U}}_{kj}(\rho_j\partial_i\mu) - \mathring{\mathcal{Q}}_k(\rho_j\partial_j\mu)\right], \tag{18}$$

$$\mathcal{Q}\rho = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{Q}}}(\mu\rho), \quad \mathcal{R}^{\bullet}\boldsymbol{\rho} = -2\frac{\partial}{\partial y_i}\mathring{\mathcal{Q}}_j(\rho_j\partial_i\mu) - \rho_j\frac{\partial\mu}{\partial y_i}, \tag{19}$$

$$\mathbf{V}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathbf{V}}\boldsymbol{\rho}, \quad \mathbf{W}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathbf{W}}(\mu\boldsymbol{\rho}), \tag{20}$$

$$\boldsymbol{\mathcal{V}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathcal{V}}\boldsymbol{\rho}, \quad \boldsymbol{\mathcal{W}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathcal{W}}(\mu\boldsymbol{\rho}), \tag{21}$$

$$\Pi^s\boldsymbol{\rho} = \mathring{\Pi}^s\boldsymbol{\rho}, \quad \Pi^d\boldsymbol{\rho} = \mathring{\Pi}^d(\mu\boldsymbol{\rho}), \tag{22}$$

$$\left[\boldsymbol{\mathcal{W}}'\boldsymbol{\rho}\right]_k = \left[\mathring{\boldsymbol{\mathcal{W}}}'\boldsymbol{\rho}\right]_k - \left(\frac{\partial_i\mu}{\mu}\left[\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}\right]_k + \frac{\partial_k\mu}{\mu}\left[\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}\right]_i - \alpha\delta^k_i\frac{\partial_j\mu}{\mu}\left[\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}\right]_j\right)n_i, \tag{23}$$

$$\widehat{\boldsymbol{\mathcal{L}}}(\boldsymbol{\tau}) := \mathring{\boldsymbol{\mathcal{L}}}(\mu\boldsymbol{\tau}). \tag{24}$$

Note that the constant-coefficient velocity potentials $\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}$, $\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}$ and $\mathring{\boldsymbol{\mathcal{W}}}\boldsymbol{\rho}$ are divergence-free in $\Omega^{\pm}$, the corresponding potentials $\boldsymbol{\mathcal{U}}\boldsymbol{\rho}$, $\boldsymbol{\mathcal{V}}\boldsymbol{\rho}$ and $\boldsymbol{\mathcal{W}}\boldsymbol{\rho}$ are not divergence-free for the variable coefficient $\mu(\mathbf{y})$,(see e.g.,[9]). Note also that by 11 and 14,

$$\mathring{\mathcal{Q}}_j\rho = \partial_j P_{\Delta}\rho \tag{25}$$

where

$$P_{\Delta}\rho(\mathbf{y}) = -\frac{1}{2\pi}\int_{\Omega}\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0}\rho(\mathbf{x})d\mathbf{x}$$

is the harmonic Newton potential. Hence

$$\operatorname{div}\,\mathring{\boldsymbol{\mathcal{Q}}}\rho = \partial_j\mathring{\mathcal{Q}}_j\rho = \Delta P_{\Delta}\rho = -\rho. \tag{26}$$

Moreover, for the constant-coefficient potentials we have the following well-known relations,

$$\mathring{\boldsymbol{\mathcal{A}}}(\mathring{\Pi}^s\boldsymbol{\rho}, \mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}) = 0, \quad \mathring{\boldsymbol{\mathcal{A}}}(\mathring{\Pi}^d\boldsymbol{\rho}, \mathring{\boldsymbol{\mathcal{W}}}\boldsymbol{\rho}) = 0, \quad \mathring{\boldsymbol{\mathcal{A}}}(\mathring{\boldsymbol{\mathcal{Q}}}\boldsymbol{\rho}, \mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}) = \boldsymbol{\rho}. \tag{27}$$

In addition, by (25) and (26),

$$\begin{aligned}\mathring{\mathcal{A}}_j((2-\alpha)\rho, -\mathring{\boldsymbol{\mathcal{Q}}}\rho) &= -\partial_i\left(\partial_i\mathring{\mathcal{Q}}_j\rho + \partial_j\mathring{\mathcal{Q}}_i\rho - \alpha\delta^j_i\operatorname{div}\,\mathring{\boldsymbol{\mathcal{Q}}}\rho\right) - (2-\alpha)\partial_j\rho \\ &= -\left(\Delta\mathring{\mathcal{Q}}_j\rho + \partial_j\operatorname{div}\,\mathring{\boldsymbol{\mathcal{Q}}}\rho - \alpha\partial_j\operatorname{div}\,\mathring{\boldsymbol{\mathcal{Q}}}\rho\right) - (2-\alpha)\partial_j\rho = 0\end{aligned} \tag{28}$$

Theorem 2. Let $s \in \mathbb{R}$, the following operators are continuous:

$$\Pi^s : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to L^2(\Omega), \qquad \Pi^d : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to L^2(\Omega) \tag{29}$$

$$\Pi^s : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to L^2_*(\Omega), \qquad \Pi^d : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to L^2_*(\Omega) \tag{30}$$

$$\mathbf{V} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+\frac{3}{2}}(\Omega), \qquad \mathbf{W} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+\frac{1}{2}}(\Omega), \tag{31}$$

$$\mathcal{V} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+1}(\partial\Omega), \qquad \mathcal{W} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+1}(\partial\Omega), \tag{32}$$

$$\mathcal{L}^{\pm} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s-1}(\partial\Omega), \qquad \mathcal{W}' : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+1}(\partial\Omega), \tag{33}$$

$$(\Pi^s, \mathbf{V}) : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{1,0}(\Omega; \mathcal{A}), \qquad (\Pi^d, \mathbf{W}) : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{1,0}(\Omega; \mathcal{A}) \tag{34}$$

$$(\Pi^s, \mathbf{V}) : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{1,0}_*(\Omega; \mathcal{A}), \qquad (\Pi^d, \mathbf{W}) : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{1,0}_*(\Omega; \mathcal{A}). \tag{35}$$

Moreover, the following operators are compact,

$$\mathcal{V} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^s(\partial\Omega), \tag{36}$$

$$\mathcal{W} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^s(\partial\Omega), \tag{37}$$

$$\mathcal{W}' : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^s(\partial\Omega). \tag{38}$$

Proof. The continuity of the operators for the constant coefficient case is proved in[3, section 5.6.4]. Consequently, from the relations (17)-(23) follows the continuity of variable coefficient operators (29) - (33) as well and the continuity of the operators (34) and (35) can be proved similar to[9, Theorem 4.3]. The compactness of the operators (36) - (38) is implied by the Rellich compactness embedding theorem (see,[13, Theorem 3.27]) for scalar case. □

Theorem 3. Let Ω be a bounded open region $\mathbb{R}^2$ with closed, infinitely smooth boundary $\partial\Omega$. The following operators are continuous:

$$\mathcal{U} : \widetilde{\mathbf{H}}^s(\Omega) \to \mathbf{H}^{s+2}(\Omega), \quad s \in \mathbb{R}, \tag{39}$$

$$\mathcal{U} : \mathbf{H}^s(\Omega) \to \mathbf{H}^{s+2}(\Omega), \quad s > -\frac{1}{2}, \tag{40}$$

$$\mathcal{R} : \widetilde{\mathbf{H}}^s(\Omega) \to \mathbf{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{41}$$

$$\mathcal{R} : \mathbf{H}^s(\Omega) \to \mathbf{H}^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{42}$$

$$\mathcal{Q} : \widetilde{H}^s(\Omega) \to \mathbf{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{43}$$

$$\mathcal{Q} : H^s(\Omega) \to \mathbf{H}^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{44}$$

$$\mathcal{Q} : \widetilde{\mathbf{H}}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{45}$$

$$\mathcal{Q} : \mathbf{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{46}$$

$$\mathcal{R}^\bullet : \widetilde{\mathbf{H}}^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2}, \tag{47}$$

$$\mathcal{R}^\bullet : \mathbf{H}^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2}. \tag{48}$$

$$(\mathring{\mathcal{Q}}, \mathcal{U}) : \mathbf{H}^s(\Omega) \to \mathbf{H}^{s+2,0}(\Omega; \mathcal{A}), \quad s \geq 0, \tag{49}$$

$$((2-\alpha)\mu I, -\mathcal{Q}) : H^{s-1}(\Omega) \to \mathbf{H}^{s,0}(\Omega; \mathcal{A}), \quad s \geq 1, \tag{50}$$

$$(\mathcal{R}^\bullet, \mathcal{R}) : \mathbf{H}^s(\Omega) \to \mathbf{H}^{s+1,0}(\Omega; \mathcal{A}), \quad s \geq 1 \tag{51}$$

Proof. We use similar procedure as in[9, Theorem 4.1]. Since the surface $\partial\Omega$ is infinitely differentiable, the operators $\mathcal{U}$ and $\mathcal{Q}$ are respectively pseudodifferential operators of order -2 and -1[[3, , section 9.1.3]]. Then, the continuity of (39) and (43) immediately follows by virtue of the mapping properties of the pseudodifferential operators. Alternatively, these mapping properties are well studied for the constant coefficient case, i.e. operators $\mathring{\mathcal{U}}$ and $\mathring{\mathcal{Q}}$, see, e.g.,[3].Then continuity of operator (45) immediately follows from representation (15) and continuity of operator (43). Consequently, the respective mapping properties for the remainder operators (41) and (47) immediately follow by considering the relation (18).

For the remaining part of the proof, we shall first assume that $s \in (-\frac{1}{2}, \frac{1}{2})$. In this case, $H^s(\Omega)$ is identified with $\widetilde{H}^s(\Omega)$. Hence, the continuity of the operator (40) immediately follows from the continuity of (39).

To prove the case $s \in (\frac{1}{2}, \frac{3}{2})$, we consider $\mathbf{g} = (g_1, g_2)$, $\mathbf{g} \in \mathbf{H}^s(\Omega)$ and by using divergence theorem and the relation $\frac{\partial}{\partial x_i}\mathring{u}_j^k(x,y) = -\frac{\partial}{\partial y_i}\mathring{u}_j^k(x,y)$ we obtain,

$$
\begin{aligned}
\mathring{\mathcal{U}}_{kj}(\partial_i g_j)(\mathbf{y}) &= \int_\Omega \mathring{u}_j^k(\mathbf{x},\mathbf{y})(\frac{\partial}{\partial x_i} g_j)(\mathbf{x})d\mathbf{x} \\
&= \int_{\partial\Omega} \mathring{u}_j^k(\mathbf{x},\mathbf{y})\gamma^+ g_j(\mathbf{x})n_i d\mathbf{x} - \int_\Omega g_j(\mathbf{x})\frac{\partial}{\partial x_i}\mathring{u}_j^k(\mathbf{x},\mathbf{y})d\mathbf{x} = \int_{\partial\Omega} \mathring{u}_j^k(\mathbf{x},\mathbf{y})\gamma^+ g_j(\mathbf{x})n_i d\mathbf{x} + \frac{\partial}{\partial y_i}(\int_\Omega \mathring{u}_j^k(\mathbf{x},\mathbf{y})g_j(\mathbf{x})d\mathbf{x}) \\
&= -\mathring{V}_{kj}(\gamma^+ g_j n_i)(\mathbf{y}) + \frac{\partial}{\partial y_i}(\mathring{\mathcal{U}}_{kj} g_j(\mathbf{y}))
\end{aligned}
$$

that is,

$$\partial_i \mathring{\mathcal{U}}_{kj} g_j = \mathring{\mathcal{U}}_{kj}(\partial_i g_j) + \mathring{V}_{kj}(\gamma^+ g_j n_i), \; i,j,k \in \{1,2\} \tag{52}$$

where n_i denotes the components of the normal vector to the surface $\partial\Omega$ directed outwards the domain. It is well known that $\partial_i g_j \in H^{s-1}(\Omega)$ and $\gamma^+\mathbf{g} \in \mathbf{H}^{s-\frac{1}{2}}(\partial\Omega)$ due to the continuity of the operator ∂_i and the trace theorem.

Due to the mapping properties of $\mathring{\mathbf{V}} : \mathbf{H}^{s-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{s+1}(\Omega)$ in Theorems 2 and $\mathring{\mathcal{U}} : \mathbf{H}^{s-1}(\Omega) \to \mathbf{H}^{s+1}(\Omega)$ in the previous paragraph, we deduce that $\partial_i \mathring{\mathcal{U}}\mathbf{g} \in \mathbf{H}^{s+1}(\Omega)$ is continuous for $i \in \{1,2\}$. Consequently, from relations (17) and (20), for $s \in (\frac{1}{2}, \frac{3}{2})$, immediately follows the continuity of the operator (40). Furthermore, by induction on $k \in \mathbb{N}$, using the representation in identity (52) and one can prove by induction that the operator (40) is also continuous for $s \in (k - \frac{1}{2}, k + \frac{1}{2})$, where k is an arbitrary nonnegative integer. The continuity of the operator (40) for the cases $s = k + \frac{1}{2}$ is proved by applying the theory of interpolation of Bessel potential spaces, (see, e.g.[16, , Chapter 4]). Continuity of the operator (44) and hence (46) can be proved following a similar argument. Continuity of the remainder operators (42) and (48)) immediately follows from the continuity of operators (40) and (44) by relations (18) and (19). Also the Continuity of the operator (49), (50) and (51) can be proved similar as in[9, Theorem 4.1]. □

Theorem 4. Let $\boldsymbol{\tau} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ and $\boldsymbol{\rho} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. Then, the following jump relations hold

$$\gamma^\pm \mathbf{V}\boldsymbol{\rho} = \boldsymbol{\mathcal{V}}\boldsymbol{\rho}, \qquad \gamma^\pm \mathbf{W}\boldsymbol{\tau} = \mp\frac{1}{2}\boldsymbol{\tau} + \boldsymbol{\mathcal{W}}\boldsymbol{\tau} \tag{53}$$

$$\mathbf{T}^\pm(\Pi^s\boldsymbol{\rho}, \mathbf{V}\boldsymbol{\rho}) = \pm\frac{1}{2}\boldsymbol{\rho} + \boldsymbol{\mathcal{W}}'\boldsymbol{\rho}, \tag{54}$$

Proof. For constant coefficient case, $\mu = 1$, the jump relations for the corresponding operators are proved in[3, Lemma 5.6.5]. Due to relations (20) and (23), the theorem holds for (53) and (54) as well. □

Theorem 5. Let $\boldsymbol{\tau} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Then the following jump relation holds

$$(\boldsymbol{\mathcal{L}}_k^\pm - \widehat{\boldsymbol{\mathcal{L}}}_k)\boldsymbol{\tau} = -\gamma^\pm\left[(\partial_i\mu)W_k(\boldsymbol{\tau}) + (\partial_k\mu)W_i(\boldsymbol{\tau}) - \alpha\delta_i^k(\partial_j\mu)W_j(\boldsymbol{\tau})\right] n_i \tag{55}$$

Theorem 6. The proof is similar to the corresponding proof in[9] 3D case.

Proposition 1. Let $s > \frac{1}{2}$. The following operators are compact,

$$
\begin{aligned}
\boldsymbol{\mathcal{R}} &: \mathbf{H}^s(\Omega) \to \mathbf{H}^s(\Omega), \;\; \boldsymbol{\mathcal{R}}^\bullet : \mathbf{H}^s(\Omega) \to H^{s-1}(\Omega), \; s \in \mathbb{R} \\
\gamma^+\boldsymbol{\mathcal{R}} &: \mathbf{H}^s(\Omega) \to \mathbf{H}^{s-\frac{1}{2}}(\partial\Omega), \quad \mathbf{T}^\pm(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) : \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega), \\
\mathbf{T}^\pm(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) &: \mathbf{H}_*^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega).
\end{aligned}
$$

Proof. The proof is similar to the corresponding proof in[9, Theorem 4.2] 3D case. □

5 | INVERTIBILITY OF THE HYDRODYNAMIC SINGLE LAYER POTENTIAL OPERATOR IN 2D

Suppose that $\boldsymbol{\rho} = \mathbf{T}^+(p, \mathbf{v})$ where $(p, \mathbf{v}) \in \mathbf{H}_*^{1,0}(\Omega)$. The single layer potential operator is a Fredholm of index zero. In 3D case, for $\boldsymbol{\rho} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$, if $\mathbf{V}\boldsymbol{\rho}(\mathbf{y}) = 0$, $\mathbf{y} \in \Omega$, then $\boldsymbol{\rho} = 0$. But this is not generally true for 2D case.

It is well known[7, p.696] that in $\mathbb{R}^2$ the single layer operator fail to be invertible. So that for some 2D domains the kernel of the operator $\mathring{\boldsymbol{\mathcal{V}}} : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ is non-zero, which is by the first relation in (21) implies that also ker$\boldsymbol{\mathcal{V}} \neq \{0\}$ for some domains. The following example is from[17, Lemma 1] and illustrates this fact.

Example 1. Take the density function $\rho_j^m = \delta_{jm}$ and $\Omega = B(0,R)$ to be a disc of radius R centered at the origin and $\partial\Omega = \partial B(0,R)$ be the circular boundary of the disc. We want to show that

$$\mu(\mathbf{y})\mathcal{V}_{kj}\rho_j^m(\mathbf{y}) = \mathring{\mathcal{V}}_{kj}\rho_j^m(\mathbf{y}) = -\frac{R}{2}\delta_{km}(2\log\frac{R}{r_0} - 1), \quad |\mathbf{y}| \le R, \quad k,j,m \in \{1,2\}.$$

Remark 1. If we set $r_0 = Re^{-\frac{1}{2}}$ in Example 1, with $\mu(\mathbf{y}) \neq 0$, we get, $[\mathbf{V}\boldsymbol{\rho}]_k(\mathbf{y}) = 0$ in $\overline{\Omega}$.

In order to have invertibility for the single layer potential operator in 2D, we define the subspace $\mathbf{H}^s_{**}(\partial\Omega)$ of the space $\mathbf{H}^s(\partial\Omega)$, see for example[7, (Appendix A, in particular $s=-\frac{1}{2}$ and $\frac{1}{2}$)],

$$\mathbf{H}^s_{**}(\partial\Omega) := \{\boldsymbol{\rho} \in \mathbf{H}^s(\partial\Omega) : \langle \rho_i, 1\rangle_{\partial\Omega} = 0 \quad \text{for} \quad i = 1,2\}, \tag{56}$$

where the norm in $\mathbf{H}^s_{**}(\partial\Omega)$ is induced norm of $\mathbf{H}^s(\partial\Omega)$.

The boundary integral operator, $\mathring{\boldsymbol{\mathcal{V}}}$ is a Fredholm operator of index zero on $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ as in[7, Lemma A.2] and also $\boldsymbol{\mathcal{V}} : \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega) \to \mathbf{H}^{\frac{1}{2}}_{**}(\partial\Omega)$ by the relation (21).

Theorem 7. If $\boldsymbol{\Psi} \in \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega)$ satisfies $\boldsymbol{\mathcal{V}}\boldsymbol{\Psi} = 0$ on $\partial\Omega$, then $\boldsymbol{\Psi} = 0$.

Proof. Let us proof by using similar procedure as in[13, Corollary 8.11]. The single layer potential $(\mathring{p}, \mathring{\mathbf{v}}) = (\mathring{\Pi}^s\boldsymbol{\Psi}, \mathring{\mathbf{V}}\boldsymbol{\Psi})$ satisfies

$$\Delta\mathring{\mathbf{v}} - \nabla\mathring{p} = 0 \quad \text{in} \quad \Omega^{\pm}, \tag{57}$$

$$\text{div}\,(\mathring{\mathbf{v}}) = 0 \quad \text{in} \quad \Omega^{\pm}, \tag{58}$$

$$\gamma^{\pm}\mathring{\mathbf{v}} = 0 \quad \text{on} \quad \partial\Omega. \tag{59}$$

For the exterior problem , we use the following growth conditions at infinity,

$$\mathring{\mathbf{v}}(\mathbf{x}) = A\log\frac{|\mathbf{x}|}{r_0} + \mathcal{O}(1), \quad \mathring{p} = \mathcal{O}(|\mathbf{x}|^{-1}) \quad \text{as} \quad |\mathbf{x}| \to \infty,$$

where $A = \int_{\partial\Omega}\boldsymbol{\Psi}dS_{\mathbf{x}}$, see e.g.[3, section 2.3.1]. Since $\boldsymbol{\Psi} \in \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega)$, i.e., $\int_{\partial\Omega}\boldsymbol{\Psi}dS_{\mathbf{x}} = 0$, it follows that $\mathring{\mathbf{v}} = 0$ and $\mathring{p} = 0$ in Ω^-. For the interior problem, using first Green identity and Dirichlet condition, we get, $\mathring{\mathbf{v}} = 0$ and using interior part of (57), we have that $\nabla\mathring{p} = 0$ in Ω. Since $p \in L^2_*(\Omega)$, then $p = 0$. Consequently, $\boldsymbol{\Psi} = \mathring{\mathbf{T}}^+(\mathring{\Pi}^s\boldsymbol{\Psi}, \mathring{\mathbf{V}}\boldsymbol{\Psi}) - \mathring{\mathbf{T}}^-(\mathring{\Pi}^s\boldsymbol{\Psi}, \mathring{\mathbf{V}}\boldsymbol{\Psi}) = 0$. Thus, $\boldsymbol{\Psi} = 0$. That is, from $\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\Psi} = 0$ follows that $\boldsymbol{\Psi} = 0$ and relation (21) implies for the operator $\boldsymbol{\mathcal{V}}$ as well. □

Theorem 8. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then the single layer potential $\boldsymbol{\mathcal{V}} : \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega) \to \mathbf{H}^{\frac{1}{2}}_{**}(\partial\Omega)$ is invertible.

Proof. Due to[7, Lemma A.2] the operator $\mathring{\boldsymbol{\mathcal{V}}}$ is Fredholm of index zero and the first relation in (21) implies that so is operator $\boldsymbol{\mathcal{V}}$. Theorem 7 implies the injectivity of operetor $\boldsymbol{\mathcal{V}}$ and hence the invertibility of operator $\boldsymbol{\mathcal{V}}$. □

To prove the $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$- ellipticity of the single-layer potential operator for the Stokes system by setting the condition on the domain, for $r_0 > 0$, consider the fundamental solution

$$\mathring{u}_j^k(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\left(\delta_j^k\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right)$$

.

$$\mathring{\mathcal{V}}_j^k w_j(\mathbf{x},\mathbf{y}) = -\int_{\partial\Omega}\frac{1}{4\pi}\left(\delta_j^k\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right)w_j(\mathbf{x})dS_{\mathbf{x}}.$$

Due to[18, Appendix], the single layer potential operator $\mathring{\boldsymbol{\mathcal{V}}}$ is positive , that is,

$$\langle\mathring{\boldsymbol{\mathcal{V}}}\widetilde{\mathbf{w}}, \widetilde{\mathbf{w}}\rangle_S > 0 \tag{60}$$

for a non-zero $\widetilde{\mathbf{w}}$ that satisfy $\int_S\widetilde{\mathbf{w}}dS = 0$ where S is the boundary of the domain and follows the theorem.

Consider the following basis of the space of rigid body translations in plane: $\mathbf{e}^1 = [1,0]^T$, $\mathbf{e}^2 = [0,1]^T$.

Theorem 9. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\partial\Omega$ is contained in the interior of a circular disk with a radius R. If $r_0 \geq Re^{-\frac{1}{2}}$, then $\mathcal{V}$ is $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$- elliptic.

Proof. First we show the positivity of $\mathring{\mathcal{V}}$ by a similar procedure as in[19, Proposition 2]. Let ∂B denote the boundary of the disk with radius R containing $\partial\Omega$. The operator $\mathring{\mathcal{V}}$ is positive by (60). So that

$$\langle [\mathring{\mathcal{V}}\widetilde{\mathbf{w}}]_j, \widetilde{w}_j\rangle_{(\partial\Omega\cup\partial B)} > 0 \tag{61}$$

for non-zero $\widetilde{\mathbf{w}} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega \cup \partial B)$ satisfying

$$\int\limits_{\partial\Omega\cup\partial B} \widetilde{w}_j(\mathbf{x}) dS_{\mathbf{x}} = 0. \tag{62}$$

Let us take $\widetilde{\mathbf{w}}$ in the form $\widetilde{\mathbf{w}} = \begin{cases} \mathbf{w} & \text{on} \quad \partial\Omega, \\ \sum_{k=1}^2 \omega_k \mathbf{e}^k & \text{on} \quad \partial B \end{cases}$, with ω_k chosen so that (62) is satisfied. Let $c_j = \int_{\partial\Omega} w_j(\mathbf{x}) dS_{\mathbf{x}}$. Condition (62) gives $0 = \int_{\partial\Omega\cup\partial B} \widetilde{w}_j(\mathbf{x}) dS_{\mathbf{x}} = \int_{\partial\Omega} w_j(\mathbf{x}) dS_{\mathbf{x}} + \int_{\partial B} \sum_{k=1}^2 \omega_k e_j^k dS_{\mathbf{x}} = c_j + 2\pi R\omega_j$. But

$$\begin{aligned}
\langle [\mathring{\mathcal{V}}\mathbf{w}]_j, w_j\rangle_{(\partial\Omega\cup\partial B)} &= \langle -\int\limits_{\partial\Omega\cup\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{(\partial\Omega\cup\partial B)} \\
&= \langle -\int\limits_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{(\partial\Omega\cup\partial B)} + \langle -\int\limits_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{(\partial\Omega\cup\partial B)} \\
&= \langle -\int\limits_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{\partial\Omega} + \langle -\int\limits_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{\partial\Omega} \\
&+ \langle -\int\limits_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{\partial B} + \langle -\int\limits_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{\partial B} \\
&= \langle [\mathring{\mathcal{V}}\mathbf{w}]_j, w_j\rangle_{\partial\Omega} + \langle -\int\limits_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{\partial\Omega} + \langle -\int\limits_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{\partial B} \\
&+ \langle -\int\limits_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{(\partial B)}
\end{aligned}$$

and

$$\begin{aligned}
\langle -\int\limits_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k(y) dS_{\mathbf{y}}, w_j\rangle_{\partial B} &= \sum_{j,k=1}^{2} \int\limits_{\partial\Omega} w_j(\mathbf{x}) \left[-\int\limits_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y}) w_k dS_{\mathbf{y}} \right] dS_{\mathbf{x}} \\
&= \sum_{j,k=1}^{2} (-\int\limits_{\partial\Omega} w_j(\mathbf{x}) \int\limits_{\partial B} \frac{1}{4\pi}(\delta_j^k \log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}) w_k dS_{\mathbf{y}} dS_{\mathbf{x}}) \\
&= -\sum_{j,k=1}^{2} \int\limits_{\partial\Omega} w_j(\mathbf{x}) [\int\limits_{\partial B} \frac{1}{4\pi}(\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0}) w_j dS_{\mathbf{y}}] dS_{\mathbf{x}} - \sum_{j,k=1}^{2} \int\limits_{\partial\Omega} w_j(\mathbf{x}) [\int\limits_{\partial B} \frac{1}{4\pi}(-\frac{(x_k-y_k)^2}{|\mathbf{x}-\mathbf{y}|^2}) w_j dS_{\mathbf{y}}] dS_{\mathbf{x}} \\
&= \sum_{j=1}^{2} (-\int\limits_{\partial\Omega} w_j(\mathbf{x}) [\int\limits_{\partial B} \frac{1}{4\pi}(2\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - 1) w_j dS_{\mathbf{y}}] dS_{\mathbf{x}}) \\
&= \sum_{j=1}^{2} (-\frac{1}{4\pi}(2\log\frac{R}{r_0} - 1) \int\limits_{\partial\Omega} w_j(\mathbf{x}) dS_{\mathbf{x}} \int\limits_{\partial B} w_j(\mathbf{y}) dS_{\mathbf{y}}) \\
&= -\sum_{j=1}^{2} \frac{1}{4\pi}(2\log\frac{R}{r_0} - 1)(-c_j^2) \\
&= -\frac{1}{4\pi}(-2\log\frac{R}{r_0} + 1)(c_1^2 + c_2^2).
\end{aligned}$$

Similarly,

$$\langle -\int_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial B} = -\frac{1}{4\pi}\left[-2\log\frac{R}{r_0}+1\right](c_1^2+c_2^2),$$
$$\langle -\int_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{(\partial B)} = \frac{1}{4\pi}\left[-2\log\frac{R}{r_0}+1\right](c_1^2+c_2^2).$$

Therefore, the integral (61) yields

$$0 < \langle[\mathring{\mathcal{V}}\mathbf{w}]_j, w_j\rangle_{\partial\Omega} + \int_{\partial\Omega} w_j(\mathbf{x})\left[-\int_{\partial B}\mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k dS_{\mathbf{y}}\right]dS_{\mathbf{x}} + \int_{\partial B} w_j\left[-\int_{\partial\Omega}\mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}\right]dS_{\mathbf{x}}$$
$$+\int_{\partial B} w_j(\mathbf{x})\left[-\int_{\partial B}\mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(\mathbf{y})dS_{\mathbf{y}}\right]dS_{\mathbf{x}}. \quad (63)$$

Hence, equation (63) becomes

$$0 < [\langle\mathring{\mathcal{V}}\mathbf{w}]_j, w_j\rangle_{\partial\Omega} - \frac{1}{4\pi}\left[-2\log\frac{R}{r_0}+1\right](c_1^2+c_2^2). \quad (64)$$

Also equation (64) can be written as

$$\frac{1}{4c\pi}\left[-2\log\frac{R}{r_0}+1\right](c_1^2+c_2^2) < \langle[\mathcal{V}\mathbf{w}]_j, w_j\rangle_{\partial\Omega}. \quad (65)$$

Then the relation $\langle[\mathcal{V}\mathbf{w}]_j, w_j\rangle_{\partial\Omega} > 0$ is always true for $r_0 \geq Re^{-\frac{1}{2}}$, therefore (65) must be positive for any non-zero $\mathbf{w}$. From[3, Theorem 5.6.13, eq.5.6.50] and[18, Eq.(A.15)] satisfy Gårding inequality. Thus from positivity and Gårding inequality implies that $\mathcal{V}$ is $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$- elliptic that is due to Lemma 5.2.5 in[3]. □

Theorem 10. Let $\Omega \subset \mathbb{R}^2$. If $r_0 > \frac{1}{2}e^{-\frac{1}{2}}\mathrm{diam}(\Omega)$, then the operator $\mathcal{V}$ has a bounded inverse on $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$.

Proof. By Theorem 9 the operator $\mathcal{V}$ is $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$- elliptic and due to Theorem 2 it is also continuous, that is, bounded. Hence the Lax-Milgram Lemma implies $\mathcal{V}$ has a bounded inverse. □

Theorem 11. Let S_1 and S_2 be non empty, non-intersecting $\partial\Omega = \overline{S}_1 \cup \overline{S}_2$. Then for $s \in \mathbb{R}$, the following operators are compact,

$$r_{S_2}\mathcal{V} : \widetilde{\mathbf{H}}^s(S_1) \to \mathbf{H}^s(S_2), \qquad r_{S_2}\mathcal{W} : \widetilde{\mathbf{H}}^s(S_1) \to \mathbf{H}^s(S_2), \qquad r_{S_2}\mathcal{W}' : \widetilde{\mathbf{H}}^s(S_1) \to \mathbf{H}^s(S_2)$$

Proof. From Theorem 2, the following operators are continuous:

$$r_{S_2}\mathcal{V} : \widetilde{\mathbf{H}}^s(S_1) \to \mathbf{H}^{s+1}(S_2), \qquad r_{S_2}\mathcal{W} : \widetilde{\mathbf{H}}^s(S_1) \to \mathbf{H}^{s+1}(S_2), \qquad r_{S_2}\mathcal{W}' : \widetilde{\mathbf{H}}^s(S_1) \to \mathbf{H}^{s+1}(S_2).$$

Since $\mathbf{H}^{s+1}(S_2) \subset \mathbf{H}^s(S_2)$ is compact, the theorem follows. □

Theorem 12. Let S_2 be a non-empty open smooth part of $\partial\Omega$ with smooth boundary. Then the operator

$$r_{S_2}\widehat{\mathcal{L}} : \widetilde{\mathbf{H}}^{\frac{1}{2}}(S_2) \to \mathbf{H}^{-\frac{1}{2}}(S_2)$$

is invertible and the operator

$$r_{S_2}(\mathcal{L}^+ - \widehat{\mathcal{L}}) : \widetilde{\mathbf{H}}^{\frac{1}{2}}(S_2) \to \mathbf{H}^{-\frac{1}{2}}(S_2)$$

is bounded and compact.

Proof. Similar to lamé system as in[4, Lemma 1.18] $\langle\mathring{\mathcal{L}}\boldsymbol{\tau},\boldsymbol{\tau}\rangle_{\partial\Omega} \geq c \parallel \boldsymbol{\tau} \parallel^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}$ for all $\boldsymbol{\tau} \in \mathbf{H}^{\frac{1}{2}}_{\mathcal{R}}(\partial\Omega) = \{\mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega) : \langle\mathbf{v},\mathbf{w}\rangle_{\partial\Omega} = 0 \quad \text{for all} \quad \mathbf{w} \in \mathcal{R}\}$. As in the norm equivalence sobolev[5, Theorem 2.6], we define,

$$\parallel \boldsymbol{\tau} \parallel_{\mathbf{H}^{\frac{1}{2}}_{\mathcal{R}}(\partial\Omega)} = \{[\langle\boldsymbol{\tau},\mathbf{w}\rangle_{\partial\Omega}]^2 + |\boldsymbol{\tau}|^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}\}$$

and then we get,

$$\langle\mathring{\mathcal{L}}\boldsymbol{\tau},\boldsymbol{\tau}\rangle_{\partial\Omega} \geq \widetilde{c}|\boldsymbol{\tau}|^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)} \quad \text{for all } \boldsymbol{\tau} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega).$$

For a given $\widetilde{\boldsymbol{\tau}} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(S_2)$. let $\boldsymbol{\tau} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ denote the extension defined by

$$\boldsymbol{\tau} = \begin{cases} \widetilde{\boldsymbol{\tau}} & \text{for } x \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(S_2) \\ \mathbf{0} & otherwise. \end{cases}$$

As in the norm equivalence sobolev[5, Theorem 2.6],

$$\| \boldsymbol{\tau} \|_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega),S_2} := \{ \| \boldsymbol{\tau} \|_{\mathbf{L}^2(\partial\Omega/S_2)} + |\boldsymbol{\tau}|^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)} \}$$

to be equivalent norm in $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. So that,

$$\begin{aligned} \langle \mathring{\mathcal{L}}\widetilde{\boldsymbol{\tau}}, \widetilde{\boldsymbol{\tau}} \rangle_{S_2} \geq & \langle \mathring{\mathcal{L}}\boldsymbol{\tau}, \boldsymbol{\tau} \rangle_{\partial\Omega} \geq C|\boldsymbol{\tau}|^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)} \\ =& C\{ \| \boldsymbol{\tau} \|_{\mathbf{L}^2(\partial\Omega/S_2)} + |\boldsymbol{\tau}|^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)} \} = C \| \boldsymbol{\tau} \|_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega),S_2} \\ \geq & \widetilde{C} \| \boldsymbol{\tau} \|_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)} = \widetilde{C} \| \widetilde{\boldsymbol{\tau}} \|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(S_2)} . \end{aligned}$$

The continuity of this operator and the Lax-Milgram lemma then imply its invertibility. The operator $\mathcal{W}_k, \mathcal{W}_i, \mathcal{W}_j$ are continuous and since $H^{\frac{3}{2}}(S_2)$ is continuously embedded in $H^{\frac{1}{2}}(S_2)$, using the relation

$$\mathcal{L}_k^+\boldsymbol{\tau} - \widehat{\mathcal{L}}_k\boldsymbol{\tau} = -\frac{\partial\mu}{\partial n_i}\left((-\frac{1}{2}I + \mathcal{W}_k) + \delta_i^k(-\frac{1}{2}I + \mathcal{W}_i) + \delta_j^k(-\frac{1}{2}I + \mathcal{W}_j) \right)\boldsymbol{\tau}$$

,we obtain continuity of the operator $\boldsymbol{\mathcal{L}}^+ - \widehat{\boldsymbol{\mathcal{L}}}$. The embedding $\mathbf{H}^{\frac{1}{2}}(S_2) \subset \mathbf{H}^{-\frac{1}{2}}(S_2)$ is compact ,which implies that the operator $\boldsymbol{\mathcal{L}}^+ - \widehat{\boldsymbol{\mathcal{L}}} : \mathbf{H}^{\frac{1}{2}}(S_2) \to \mathbf{H}^{-\frac{1}{2}}(S_2)$ is compact . □

Theorem 13. Let S_1 be a non-empty part of the boundary curve $\partial\Omega$.

i) The operator

$$r_{S_1}\boldsymbol{\mathcal{V}} : \widetilde{\mathbf{H}}^{-\frac{1}{2}}(S_1) \to \mathbf{H}^{\frac{1}{2}}(S_1) \tag{66}$$

is bounded and fredholm of index zero.

ii) If $\widetilde{\boldsymbol{\psi}} \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}_{**}(S_1)$ satisfies $r_{S_1}\boldsymbol{\mathcal{V}}\widetilde{\boldsymbol{\psi}} = 0$ on S_1, then $\widetilde{\boldsymbol{\psi}} = 0$.

Proof. i) Since the operator $\boldsymbol{\mathcal{V}} : \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ is bounded so that (66) also bounded. The operators $r_{S_1}\mathring{\boldsymbol{\mathcal{V}}}$ admits the decomposition $r_{S_1}\mathring{\boldsymbol{\mathcal{V}}} = r_{S_1}\boldsymbol{\mathcal{V}}_\Delta + r_{S_1}\mathbf{K}$, see,[7].

$$\boldsymbol{\mathcal{V}}_\Delta = \begin{bmatrix} \mathcal{V}_\Delta & 0 \\ 0 & \mathcal{V}_\Delta \end{bmatrix}, \quad \mathcal{V}_\Delta\widetilde{\boldsymbol{\psi}} = -\frac{1}{4\pi}\int_{S_1} \log\frac{|x-y|}{r_0}\widetilde{\boldsymbol{\psi}}\, dS_{\mathbf{x}}.$$

The operator $r_{S_1}\boldsymbol{\mathcal{V}}_\Delta$ is a Fredholm of index zero because each of the components are Fredholm of index zero as in[10, corollary 2.7(i)] and $r_{S_1}\mathbf{K}$ is a compact operator as in[7, lemma A.2]. Thus by relation $\boldsymbol{\mathcal{V}} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{V}}}$, we obtain that operator (66) is Fredholm of index zero as well.

ii) Suppose $\widetilde{\boldsymbol{\psi}} \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}_{**}(S_1)$, i.e. $\langle \widetilde{\psi}_i, 1 \rangle_{S_1} = \langle \widetilde{\psi}_i, 1 \rangle_{\partial\Omega} = 0$, which implies $\widetilde{\boldsymbol{\psi}} \in \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega)$. For $\widetilde{\boldsymbol{\psi}} \in \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega)$, we have $\langle \mathring{\boldsymbol{\mathcal{V}}}\widetilde{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\psi}} \rangle_{\partial\Omega} \geq 0$, moreover, if $\langle \mathring{\boldsymbol{\mathcal{V}}}\widetilde{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\psi}} \rangle_{\partial\Omega} = 0$, then $\widetilde{\boldsymbol{\psi}} = 0$ on $\partial\Omega$, see,[18, Appendix]. Hence, if $r_{S_1}\boldsymbol{\mathcal{V}}\widetilde{\boldsymbol{\psi}} = 0$, then $r_{S_1}\mathring{\boldsymbol{\mathcal{V}}}\widetilde{\boldsymbol{\psi}} = 0$ and $\langle \mathring{\boldsymbol{\mathcal{V}}}\widetilde{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\psi}} \rangle_{\partial\Omega} = \langle r_{S_1}\mathring{\boldsymbol{\mathcal{V}}}\widetilde{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\psi}} \rangle_{S_1} = 0$ implies $\widetilde{\boldsymbol{\psi}} = 0$.

□

Lemma 1.

(i) Let either $\boldsymbol{\Psi}^* \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \frac{1}{2}e^{-\frac{1}{2}}\text{diam}(\Omega)$ or $\boldsymbol{\Psi}^* \in \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega)$. If

$$\mathbf{V}\boldsymbol{\Psi}^*(\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega, \tag{67}$$

then $\boldsymbol{\Psi}^* = 0$

(ii) Let $\mathbf{\Phi}^* \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. If

$$\mathbf{W\Phi}^*(\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega, \tag{68}$$

then $\mathbf{\Phi}^* = 0$.

Proof. We will use similar procedures as in [20].

(i) Taking the trace of (67) on $\partial\Omega$ and using jump relation (53). Then we have $\mathcal{V}\mathbf{\Psi}^* = 0$ on $\partial\Omega$. If $\mathbf{\Psi}^* \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \frac{1}{2}e^{-\frac{1}{2}}\text{diam}(\Omega)$, then the result follows from the invertability of the single layer potential given by Theorem 10. On the other hand, if $\mathbf{\Psi}^* \in \mathbf{H}_{**}^{-\frac{1}{2}}(\partial\Omega)$, then the result is implied by Theorem 8.

(ii) Taking the trace of (68) and then by (53) gives $-\frac{1}{2}\mathbf{\Phi}^* + \mathcal{W}\mathbf{\Phi}^* = 0$ on $\partial\Omega$, due to (21), $-\frac{1}{2}\widehat{\mathbf{\Phi}}^* + \mathring{\mathcal{W}}\widehat{\mathbf{\Phi}}^* = 0$ *on* $\partial\Omega$, where $\widehat{\mathbf{\Phi}}^* = \mu\mathbf{\Phi}^*$. Due to the contraction property of the operator $-\frac{1}{2}\mathbf{I} + \mathring{\mathcal{W}}$, then $\widehat{\mathbf{\Phi}}^*$ is uniquely solvable and $\mu(\mathbf{y}) \neq 0$, $\widehat{\mathbf{\Phi}}^* = 0$ implies $\mathbf{\Phi}^* = 0$.

□

Lemma 2. Let $\partial\Omega = \overline{S_1} \cup \overline{S_2}$, where S_1 ad S_2 are open non-empty non-intersecting. Let $\mathbf{\Psi}^* \in \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(S_1)$, $\mathbf{\Phi}^* \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(S_2)$. If

$$\mathbf{V\Psi}^* - \mathbf{W\Phi}^* = \mathbf{0}, \qquad \Pi^s\mathbf{\Psi}^* - \Pi^d\mathbf{\Phi}^* = 0, \ \ in \ \ \Omega, \tag{69}$$

then $\mathbf{\Psi}^* = \mathbf{0}$ and $\mathbf{\Phi}^* = \mathbf{0}$ on $\partial\Omega$.

Proof. Multiply the first equation in (69) by μ and applying the relation (21), we have

$$\mathring{\mathbf{V}}\mathbf{\Psi}^* - \mathring{\mathbf{W}}(\mu\mathbf{\Phi}^*) = \mathbf{0} \quad \text{in } \Omega.$$

Taking the trace of this equation on S_1

$$r_{S_1}[\gamma^+\mathring{\mathbf{V}}\mathbf{\Psi}^* - \gamma^+\mathring{\mathbf{W}}(\mu\mathbf{\Phi}^*)] = \mathbf{0} \quad \text{on } S_1$$
$$r_{S_1}\mathring{\mathcal{V}}\mathbf{\Psi}^* - r_{S_1}\mathring{\mathcal{W}}(\mu\mathbf{\Phi}^*) + \frac{1}{2}r_{S_1}(\mu\mathbf{\Phi}^*) = \mathbf{0} \quad \text{on } S_1$$
$$r_{S_1}\mathring{\mathcal{V}}\mathbf{\Psi}^* - r_{S_1}\mathring{\mathcal{W}}(\mu\mathbf{\Phi}^*) = \mathbf{0} \quad \text{on } S_1.$$

Taking the traction on S_2

$$r_{S_2}[\mathbf{T}^+(\mathring{\Pi}^s\mathbf{\Psi}^*, \mathring{\mathbf{V}}\mathbf{\Psi}^*) - \mathbf{T}^+(\mathring{\Pi}^d(\mu\mathbf{\Phi}^*), \mathring{\mathbf{W}}(\mu\mathbf{\Phi}^*))] = \mathbf{0} \quad \text{on } S_2$$

which implies $r_{S_2}\mathring{\mathcal{W}}'\mathbf{\Psi}^* - r_{S_2}\mathring{\mathcal{L}}(\mu\mathbf{\Phi}^*) = \mathbf{0}$ on S_2. Thus we obtain

$$\begin{cases} r_{S_1}\mathring{\mathcal{V}}\mathbf{\Psi}^* - r_{S_1}\mathring{\mathcal{W}}\widehat{\mathbf{\Phi}}^* = \mathbf{0} & \text{on } S_1, \\ r_{S_2}\mathring{\mathcal{W}}'\mathbf{\Psi}^* - r_{S_2}\mathring{\mathcal{L}}\widehat{\mathbf{\Phi}}^* = 0 & \text{on } S_2 \end{cases}$$

where $\widehat{\mathbf{\Phi}}^* = \mu\mathbf{\Phi}^*$. The above system of equation can be written in matrix form as

$$\mathring{\mathcal{M}}\mathring{\mathcal{X}} = 0, \tag{70}$$

where

$$\mathring{\mathcal{M}} := \begin{bmatrix} r_{S_1}\mathring{\mathcal{V}} & -r_{S_1}\mathring{\mathcal{W}} \\ r_{S_2}\mathring{\mathcal{W}}' & -r_{S_2}\mathring{\mathcal{L}} \end{bmatrix}, \quad \mathring{\mathcal{X}} = \begin{bmatrix} \mathbf{\Psi}^* \\ \widehat{\mathbf{\Phi}}^* \end{bmatrix}.$$

From [3, Theorem 5.6.13, eq.5.6.50], we have $\langle r_{S_1}\mathring{\mathcal{V}}\mathbf{\Psi}^*, \Psi^*\rangle_{S_1} + \langle\mathbf{\Psi}^*, \mathbf{n}\rangle_{\partial\Omega} \geq c\|\mathbf{\Psi}^*\|^2_{\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)}$. We know that $\mathbf{\Psi}^* = T^+(p, \mathbf{v})$ with $(p, \mathbf{v}) \in \mathbf{H}_*^{1,0}(\Omega, \mathcal{A})$ and $\mathbf{\Psi}^* \in \mathbf{H}_{**}^{-\frac{1}{2}}(\partial\Omega)$. So that $\langle\mathbf{\Psi}^*, \mathbf{n}\rangle_{\partial\Omega} = 0$.
Then $\langle r_{S_1}\mathring{\mathcal{V}}\mathbf{\Psi}^*, \Psi^*\rangle_{S_1} \geq c\|\mathbf{\Psi}^*\|^2_{\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)}$ for $\mathbf{\Psi}^* \in \mathbf{H}_{**}^{-\frac{1}{2}}(S_1)$. In the proof of Theorem 12 , $\langle -r_{S_2}\mathring{\mathcal{L}}\widehat{\mathbf{\Phi}}^*, \widehat{\mathbf{\Phi}}^*\rangle_{S_2} \geq c\|\widehat{\mathbf{\Phi}}^*\|^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}$. In addition, the operators

$$r_{S_1}\mathring{\mathcal{W}} : \widetilde{\mathbf{H}}^{\frac{1}{2}}(S_2) \longrightarrow \mathbf{H}^{\frac{1}{2}}(S_1) \text{ and } r_{S_2}\mathring{\mathcal{W}}' : \widetilde{\mathbf{H}}^{-\frac{1}{2}}(S_1) \longrightarrow \mathbf{H}^{-\frac{1}{2}}(S_2)$$

are mutually adjoint, i.e.,$\langle r_{S_1}\mathring{\mathcal{W}}\widehat{\boldsymbol{\Phi}}^*, \boldsymbol{\Psi}^*\rangle_{S_1} = \langle \widehat{\boldsymbol{\Phi}}^*, r_{S_2}\mathring{\mathcal{W}}'\boldsymbol{\Psi}^*\rangle_{S_2}$ for arbitrary $\boldsymbol{\Psi}^* \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}_{**}(S_1)$ and arbitrary $\widehat{\boldsymbol{\Phi}}^* \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(S_2)$. Consequently, we derive the inequality

$$\langle \mathring{\mathcal{M}}\mathring{\mathcal{X}}, \mathring{\mathcal{X}}\rangle \geq c(\|\boldsymbol{\Psi}^*\|^2_{\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)} + \|\widehat{\boldsymbol{\Phi}}^*\|^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}) = c\|\mathcal{X}\|^2_{\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}.$$

Due to (70), this implies $\boldsymbol{\Psi}^* = 0$ and $\widehat{\boldsymbol{\Phi}}^* = 0$. Keeping in mind that $\mu(\mathbf{y}) \neq 0$, we have $\boldsymbol{\Phi}^* = 0$ on $\partial\Omega$. □

6 | THE THIRD GREEN IDENTITIES

Theorem 14. For any $(p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ the following third Green identities hold

$$\mathbf{v} + \mathcal{R}\mathbf{v} - \mathbf{V}\mathbf{T}^+(p, \mathbf{v}) + \mathbf{W}\gamma^+\mathbf{v} = \boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{A}}(p, \mathbf{v}) - \boldsymbol{\mathcal{Q}}\,\mathrm{div}\,\mathbf{v} \;\text{ in } \Omega, \tag{71}$$

$$p + \mathcal{R}^{\bullet}\mathbf{v} - \Pi^s\mathbf{T}^+(p, \mathbf{v}) + \Pi^d\gamma^+\mathbf{v} = \mathring{\mathcal{Q}}\boldsymbol{\mathcal{A}}(p, \mathbf{v}) + (2-\alpha)\mu\,\mathrm{div}\,\mathbf{v} \;\text{ in } \Omega. \tag{72}$$

Proof. The proof is similar to the corresponding proof in[9] 3D case . □

If the couple $(p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ is a solution of the Stokes PDE (6a) with variable coefficient, then (71) and (72) give

$$\mathbf{v} + \boldsymbol{\mathcal{R}}\mathbf{v} - \mathbf{V}\mathbf{T}^+(p, \mathbf{v}) + \mathbf{W}\gamma^+\mathbf{v} = \boldsymbol{\mathcal{U}}\mathbf{f} - \boldsymbol{\mathcal{Q}}g, \;\; \text{in } \Omega \tag{73}$$

$$p + \mathcal{R}^{\bullet}\mathbf{v} - \Pi^s\mathbf{T}^+(p, \mathbf{v}) + \Pi^d\gamma^+\mathbf{v} = \mathring{\mathcal{Q}}\mathbf{f} + (2-\alpha)\mu g, \;\; \text{in } \Omega \tag{74}$$

We will also need the trace and traction of the third Green identities (73) and (74) on $\partial\Omega$.

$$\frac{1}{2}\gamma^+\mathbf{v} + \boldsymbol{\mathcal{R}}^+\mathbf{v} - \boldsymbol{\mathcal{V}}\mathbf{T}^+(p, \mathbf{v}) + \boldsymbol{\mathcal{W}}\gamma^+\mathbf{v} = \gamma^+\boldsymbol{\mathcal{U}}\mathbf{f} - \gamma^+\boldsymbol{\mathcal{Q}}g \tag{75}$$

$$\frac{1}{2}\mathbf{T}^+(p, \mathbf{v}) + \mathbf{T}^+(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})\mathbf{v} - \boldsymbol{\mathcal{W}}'\mathbf{T}^+(p, \mathbf{v}) + \boldsymbol{\mathcal{L}}^+\gamma^+\mathbf{v} = \mathbf{T}^+(\mathring{\mathcal{Q}}\mathbf{f} + (2-\alpha)\mu g, \, \boldsymbol{\mathcal{U}}\mathbf{f} - \boldsymbol{\mathcal{Q}}g) \tag{76}$$

One can prove the following two assertions that are instrumental for proof of equivalence of the BDIEs and the original PDE.

Lemma 3. Let $\mathbf{v} \in \mathbf{H}^1(\Omega), p \in L^2(\Omega), g \in L^2(\Omega), \mathbf{f} \in \mathbf{L}^2(\Omega), \boldsymbol{\Psi} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega), \boldsymbol{\Phi} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ satisfy equations.

$$\mathbf{v} + \boldsymbol{\mathcal{R}}\mathbf{v} - \mathbf{V}\boldsymbol{\Psi} + \mathbf{W}\boldsymbol{\Phi} = \boldsymbol{\mathcal{U}}\mathbf{f} - \boldsymbol{\mathcal{Q}}g, \;\; \text{in } \Omega, \tag{77}$$

$$p + \mathcal{R}^{\bullet}\mathbf{v} - \Pi^s\boldsymbol{\Psi} + \Pi^d\boldsymbol{\Phi} = \mathring{\mathcal{Q}}\mathbf{f} + (2-\alpha)\mu g, \;\; \text{in } \Omega. \tag{78}$$

Then $(p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ and solve the equations

$$\boldsymbol{\mathcal{A}}(\mathbf{y}; p, \mathbf{v}) = \mathbf{f}, \;\; \mathrm{div}\,\mathbf{v} = g. \tag{79}$$

Moreover, the following relations hold true:

$$\mathbf{V}(\boldsymbol{\Psi} - \mathbf{T}^+(p, \mathbf{v}))(\mathbf{y}) - \mathbf{W}(\boldsymbol{\Phi} - \gamma^+\mathbf{v})(\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega, \tag{80}$$

$$\Pi^s(\boldsymbol{\Psi} - \mathbf{T}^+(p, \mathbf{v}))(\mathbf{y}) - \Pi^d(\boldsymbol{\Phi} - \gamma^+\mathbf{v})(\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega. \tag{81}$$

Proof. The proof is similar to the corresponding proof in[9] 3D case . □

7 | BDIES FOR MIXED BVP

We aim to obtain a segregated boundary-domain integral equation system for mixed BVP (6a)-(6d). We will use similar procedures as in[11,8,9].

To this end, let $\boldsymbol{\Phi}_0 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ be a fixed extension of the given data $\boldsymbol{\varphi}_0$ from $\partial\Omega_D$ to $\partial\Omega$. An arbitrary extension $\boldsymbol{\Phi} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ preserving the function space can then be represented as $\boldsymbol{\Phi} = \boldsymbol{\Phi}_0 + \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$. Analogously, let $\boldsymbol{\Psi}_0 \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ be a fixed extension of the given data $\boldsymbol{\psi}_0$ from $\partial\Omega_N$ to $\partial\Omega$. An arbitrary extension $\boldsymbol{\Psi} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ preserving the function space can then be represented as $\boldsymbol{\Psi} = \boldsymbol{\Psi}_0 + \boldsymbol{\psi}$ with $\boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D)$. Let us now represent

$$\gamma^+\mathbf{v} = \boldsymbol{\Phi}_0 + \boldsymbol{\varphi}, \quad \mathbf{T}^+(p,\mathbf{v}) = \boldsymbol{\Psi}_0 + \boldsymbol{\psi} \ \ on \ \ \partial\Omega \tag{82}$$

where $\boldsymbol{\varphi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ and $\boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D)$ are unknown boundary functions.

BDIE system(M11). Let us now take equations (73) and (74) in Ω and restrictions of equations (75) and (76) to the boundary parts $\partial\Omega_D$ and $\partial\Omega_N$, respectively. Substituting there representations (82) and considering further the unknown boundary functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as formally independent of (segregated from) the unknown domain functions p and $\mathbf{v}$, we obtain the following BDIE system (**M11**) consisting of four BDIEs for four unknowns, $(p,\mathbf{v},\boldsymbol{\psi},\boldsymbol{\varphi}) \in \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$:

$$p + \mathcal{R}^{\bullet}\mathbf{v} - \Pi^s\boldsymbol{\psi} + \Pi^d\boldsymbol{\varphi} = F_0 \quad \text{in } \ \Omega, \tag{83a}$$

$$\mathbf{v} + \boldsymbol{\mathcal{R}}\mathbf{v} - \mathbf{V}\boldsymbol{\psi} + \mathbf{W}\boldsymbol{\varphi} = \mathbf{F} \quad \text{in } \ \Omega, \tag{83b}$$

$$r_{\partial\Omega_D}\gamma^+\boldsymbol{\mathcal{R}}\mathbf{v} - r_{\partial\Omega_D}\boldsymbol{\mathcal{V}}\boldsymbol{\psi} + r_{\partial\Omega_D}\boldsymbol{\mathcal{W}}\boldsymbol{\varphi} = r_{\partial\Omega_D}\gamma^+\mathbf{F} - \boldsymbol{\varphi}_0 \quad \text{on } \ \partial\Omega_D, \tag{83c}$$

$$r_{\partial\Omega_N}\mathbf{T}^+(\mathcal{R}^{\bullet},\boldsymbol{\mathcal{R}})\mathbf{v} - r_{\partial\Omega_N}\boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + r_{\partial\Omega_N}\boldsymbol{\mathcal{L}}^+\boldsymbol{\varphi} = r_{\partial\Omega_N}\mathbf{T}^+(F_0,\mathbf{F}) - \boldsymbol{\psi}_0 \quad \text{on } \ \partial\Omega_N. \tag{83d}$$

where

$$F_0 := \mathring{\mathcal{Q}}\mathbf{f} - (2-\alpha)\mu g + \Pi^s\boldsymbol{\Psi}_0 - \Pi^d\boldsymbol{\Phi}_0, \quad \mathbf{F} := \boldsymbol{\mathcal{U}}\mathbf{f} - \boldsymbol{\mathcal{Q}}g + \mathbf{V}\boldsymbol{\Psi}_0 - \mathbf{W}\boldsymbol{\Phi}_0 \tag{84}$$

Applying Theorems 2 and 3 to (84) implies $(F_0,\mathbf{F}) \in \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$.

We denote the right hand side of BDIE system (83a)- (83d) as

$$\mathcal{F}^{11} := [F_0, \mathbf{F}, \ r_{\partial\Omega_D}\gamma^+\mathbf{F} - \boldsymbol{\varphi}_0, \ r_{\partial\Omega_N}T^+(F_0,\mathbf{F}) - \boldsymbol{\psi}_0]^T, \tag{85}$$

which implies $\mathcal{F}^{11} \in \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N)$

Note that BDIE system (83a)- (83d) can be split into three vector equations (83b), (83c), (83d) for three vector unknowns, $\mathbf{v}$, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, and the scalar equation (83a) that can be used, after solving the system, to obtain the pressure, p. The system (**M11**) given by equations (83a)- (83d) can be written using matrix notation as

$$\mathcal{M}^{11}\mathcal{X} = \mathcal{F}^{11} \tag{86}$$

where $\mathcal{X}$ represents the vector containing the unknowns of the system

$$\mathcal{X} = (p,\mathbf{v},\boldsymbol{\psi},\boldsymbol{\varphi})^T \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N).$$

The matrix operator $\mathcal{M}^{11}$ is defined by

$$\mathcal{M}^{11} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^s & \Pi^d \\ 0 & \mathbf{I} + \boldsymbol{\mathcal{R}} & -\mathbf{V} & \mathbf{W} \\ 0 & r_{\partial\Omega_D}\gamma^+\boldsymbol{\mathcal{R}} & -r_{\partial\Omega_D}\boldsymbol{\mathcal{V}} & r_{\partial\Omega_D}\boldsymbol{\mathcal{W}} \\ 0 & r_{\partial\Omega_N}T^+(\mathcal{R}^{\bullet},\boldsymbol{\mathcal{R}}) & -r_{\partial\Omega_N}\boldsymbol{\mathcal{W}}' & r_{\partial\Omega_N}\boldsymbol{\mathcal{L}}^+ \end{bmatrix}.$$

Remark 2. The term $\mathcal{F}^{11} = \mathbf{0}$ if and only if $(\mathbf{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = \mathbf{0}$.

Suppose $\mathcal{F}^{11} = \mathbf{0}$, then $[F_0, \ \mathbf{F}, \ r_{\partial\Omega_D}\gamma^+\mathbf{F} - \boldsymbol{\varphi}_0, \ r_{\partial\Omega_N}T^+(F_0,\mathbf{F}) - \boldsymbol{\psi}_0]^T = \mathbf{0}$. Taking into account how the terms $\mathbf{F}$ and F_0 are defined, see (84), considering that $F_0 = 0$ for p and $\mathbf{F} = 0$ for $\mathbf{v}$, we can deduce by applying Lemma 3 to equations (84) we obtain that $\mathbf{f} = \mathbf{0}$, $g = 0$ and we have,

$$\Pi^s\boldsymbol{\Psi}_0 - \Pi^d\boldsymbol{\Phi}_0 = 0, \quad \mathbf{V}\boldsymbol{\Psi}_0 - \mathbf{W}\boldsymbol{\Phi}_0 = 0$$

In addition, as $F_0 = 0$ and $\mathbf{F} = \mathbf{0}$, we get

$$r_{\partial\Omega_D}\gamma^+\mathbf{F} - r_{\partial\Omega_D}\boldsymbol{\Phi}_0 = \mathbf{0} \ \text{ implies } \ r_{\partial\Omega_D}\boldsymbol{\Phi}_0 = \mathbf{0}, \quad r_{\partial\Omega_N}\mathbf{T}^+(F_0,\mathbf{F}) - r_{\partial\Omega_N}\boldsymbol{\Psi}_0 = \mathbf{0} \ \text{ implies } \ r_{\partial\Omega_N}\boldsymbol{\Psi}_0 = \mathbf{0}.$$

Consequently, $\boldsymbol{\Psi}_0 \in \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\boldsymbol{\Phi}_0 \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$. Therefore, the hypotheses of Lemma 2 are satisfied, we thus obtain that $\boldsymbol{\Psi}_0 = 0$ and $\boldsymbol{\Phi}_0 = 0$ on $\partial\Omega$. On the other hand assume that $(\mathbf{f}, \ g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$. Then it is evidently $\mathcal{F}^{11} = 0$.

BDIE system(M12). Let us now take equations (73) and (74) in Ω and equation (75) on the whole boundary $\partial\Omega$. Substituting there representations (82), we arrive at the following system of BDIEs,

$$p + \mathcal{R}^{\bullet}\mathbf{v} - \Pi^s\boldsymbol{\psi} + \Pi^d\boldsymbol{\varphi} = F_0 \quad \text{in } \Omega, \tag{87a}$$

$$\mathbf{v} + \boldsymbol{\mathcal{R}}\mathbf{v} - \mathbf{V}\boldsymbol{\psi} + \mathbf{W}\boldsymbol{\varphi} = \mathbf{F} \quad \text{in } \Omega, \tag{87b}$$

$$\frac{1}{2}\boldsymbol{\varphi} + \gamma^+\boldsymbol{\mathcal{R}}\mathbf{v} - \boldsymbol{\mathcal{V}}\boldsymbol{\psi} + \boldsymbol{\mathcal{W}}\boldsymbol{\varphi} = \gamma^+\mathbf{F} - \boldsymbol{\Phi}_0 \quad \text{on } \partial\Omega. \tag{87c}$$

where F_0 and $\mathbf{F}$ are given by (84).Denote the system in matrix form as $\mathcal{M}^{12}\mathcal{X} = \mathcal{F}^{12}$ where,

$$\mathcal{M}^{12} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^s & \Pi^d \\ \mathbf{0} & \mathbf{I}+\boldsymbol{\mathcal{R}} & -\mathbf{V} & \mathbf{W} \\ \mathbf{0} & \gamma^+\boldsymbol{\mathcal{R}} & -\boldsymbol{\mathcal{V}} & (\frac{1}{2}\mathbf{I}+\boldsymbol{\mathcal{W}}) \end{bmatrix}, \quad \mathcal{F}^{12} = \begin{bmatrix} F_0 \\ \mathbf{F} \\ \gamma^+\mathbf{F} - \boldsymbol{\Phi}_0 \end{bmatrix}.$$

Remark 3. Let $\boldsymbol{\Psi}_0 \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}_{**}(\partial\Omega_D)$, Then $\mathcal{F}^{12} = 0$ if and only if $(\mathbf{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$.
Indeed, from (84) we immediately obtain that $(\mathbf{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$ implies $\mathcal{F}^{12} = 0$. Let us now prove that if $\mathcal{F}^{12} = 0$ then $(\mathbf{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$. Taking into account how the terms $\mathbf{F}$ and F_0 are defined, considering that $F_0 = 0$ for p and $\mathbf{F} = 0$ for $\mathbf{v}$, we can deduce by applying Lemma 3 to equations (84) we obtain $\mathbf{f} = 0$, $g = 0$ and

$$\Pi^s\boldsymbol{\Psi}_0 - \Pi^d\boldsymbol{\Phi}_0 = 0, \quad \mathbf{V}\boldsymbol{\Psi}_0 - \mathbf{W}\boldsymbol{\Phi}_0 = \mathbf{0}.$$

The equality $\gamma^+\mathbf{F} - \boldsymbol{\Phi}_0 = \mathbf{0}$ implies $\boldsymbol{\Phi}_0 = \mathbf{0}$ on $\partial\Omega$. Thus $\mathbf{V}\boldsymbol{\Psi}_0 = 0$, hence by Lemma 1 (i) it follows $\boldsymbol{\Psi}_0 = 0$.

BDIE system(M21). Let us now take equations (73) and (74) in Ω and equation (76) on the whole boundary $\partial\Omega$. Substituting there representations (82), we arrive at the following system of BDIEs,

$$p + \mathcal{R}^{\bullet}\mathbf{v} - \Pi^s\boldsymbol{\psi} + \Pi^d\boldsymbol{\varphi} = F_0 \quad \text{in } \Omega, \tag{88a}$$

$$\mathbf{v} + \boldsymbol{\mathcal{R}}\mathbf{v} - \mathbf{V}\boldsymbol{\psi} + \mathbf{W}\boldsymbol{\varphi} = \mathbf{F} \quad \text{in } \Omega, \tag{88b}$$

$$\frac{1}{2}\boldsymbol{\psi} + \mathbf{T}^+(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})\mathbf{v} - \boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + \boldsymbol{\mathcal{L}}^+\boldsymbol{\varphi} = \mathbf{T}^+(F_0, \mathbf{F}) - \boldsymbol{\Psi}_0 \quad \text{on } \partial\Omega \tag{88c}$$

where F_0 and $\mathbf{F}$ are given by (84). Denote the system in matrix form as $\mathcal{M}^{21}\mathcal{X} = \mathcal{F}^{21}$ where,

$$\mathcal{M}^{21} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^s & \Pi^d \\ \mathbf{0} & \mathbf{I}+\boldsymbol{\mathcal{R}} & -\mathbf{V} & \mathbf{W} \\ \mathbf{0} & \mathbf{T}^+(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}}) & (\frac{1}{2}\mathbf{I}-\boldsymbol{\mathcal{W}}') & \boldsymbol{\mathcal{L}}^+ \end{bmatrix}, \quad \mathcal{F}^{21} = \begin{bmatrix} F_0 \\ \mathbf{F} \\ \mathbf{T}^+(F_0, \mathbf{F}) - \boldsymbol{\Psi}_0 \end{bmatrix}.$$

Remark 4. $\mathcal{F}^{21} = \mathbf{0}$ if and only if $(\mathbf{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = \mathbf{0}$. We can show this similarly as in Remark 3.

BDIE system(M22). Let us now take equations (73) and (74) in Ω and restrictions of equations (76) and (75) to the boundary parts $\partial\Omega_D$ and $\partial\Omega_N$, respectively. Substituting there representations (82) and considering further the unknown boundary functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as formally independent of (segregated from) the unknown domain functions p and $\mathbf{v}$, we obtain the following BDIE system (**M22**) consisting of four BDIEs for four unknowns, $(p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$, $\boldsymbol{\varphi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ and $\boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D)$:

$$p + \mathcal{R}^{\bullet}\mathbf{v} - \Pi^s\boldsymbol{\psi} + \Pi^d\boldsymbol{\varphi} = F_0 \quad \text{in } \Omega, \tag{89a}$$

$$\mathbf{v} + \boldsymbol{\mathcal{R}}\mathbf{v} - \mathbf{V}\boldsymbol{\psi} + \mathbf{W}\boldsymbol{\varphi} = \mathbf{F} \quad \text{in } \Omega, \tag{89b}$$

$$\frac{1}{2}\boldsymbol{\psi} + r_{_{\partial\Omega_D}}[\mathbf{T}^+(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})\mathbf{v} - \boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + \boldsymbol{\mathcal{L}}^+\boldsymbol{\varphi}] = r_{_{\partial\Omega_D}}[\mathbf{T}^+(F_0, \mathbf{F}) - \boldsymbol{\Psi}_0] \quad \text{on } \partial\Omega_D, \tag{89c}$$

$$\frac{1}{2}\boldsymbol{\varphi} + r_{_{\partial\Omega_N}}[\gamma^+\boldsymbol{\mathcal{R}}v - \boldsymbol{\mathcal{V}}\boldsymbol{\psi} + \boldsymbol{\mathcal{W}}\boldsymbol{\varphi}] = r_{_{\partial\Omega_N}}[\gamma^+\mathbf{F} - \boldsymbol{\Phi}_0] \quad \text{on } \partial\Omega_N. \tag{89d}$$

where the terms in the right hand side F_0 and $\mathbf{F}$ are given by (84).

Note that the BDIE system (89a)-(89d) can be split into three vector equations, (89b)-(89d), for three vector unknowns, $\mathbf{v}$, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, and the separate equation (89a) that can be used, after solving the system, to obtain the pressure, p. However, since the couple $(p, \mathbf{v})$ shares the space $\mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$, equations (89b), (89c) and (89d) are not completely separate from equation (89a).

The system (89a)-(89d) can be written using matrix notation as follows $\mathcal{M}^{22}\mathcal{X} = \mathcal{F}^{22}$, where,

$$\mathcal{M}^{22} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^{s} & \Pi^{d} \\ \mathbf{0} & \mathbf{I}+\boldsymbol{\mathcal{R}} & -\mathbf{V} & \mathbf{W} \\ \mathbf{0} & r_{\partial\Omega_D}\mathbf{T}^{+}(\mathcal{R}^{\bullet},\boldsymbol{\mathcal{R}}) & r_{\partial\Omega_D}\left(\frac{1}{2}\mathbf{I}-\boldsymbol{\mathcal{W}}'\right) & r_{\partial\Omega_D}\boldsymbol{\mathcal{L}}^{+} \\ \mathbf{0} & r_{\partial\Omega_N}\gamma^{+}\boldsymbol{\mathcal{R}} & -r_{\partial\Omega_N}\boldsymbol{\mathcal{V}} & r_{\partial\Omega_N}\left(\frac{1}{2}\mathbf{I}+\boldsymbol{\mathcal{W}}\right) \end{bmatrix}, \quad \mathcal{F}^{22} = \begin{bmatrix} F_0 \\ \mathbf{F} \\ r_{\partial\Omega_D}T^{+}(F_0,\mathbf{F}) - r_{\partial\Omega_D}\boldsymbol{\Psi}_0 \\ r_{\partial\Omega_N}\gamma^{+}\mathbf{F} - r_{\partial\Omega_N}\boldsymbol{\Phi}_0 \end{bmatrix}.$$

Remark 5. The term $\mathcal{F}^{22} = \mathbf{0}$ if and only if $(\mathbf{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = \mathbf{0}$.
From (84) it follows that $(\mathbf{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = \mathbf{0}$ which implies $\mathcal{F}^{22} = \mathbf{0}$. Conversely, if $\mathcal{F}^{22} = \mathbf{0}$, then $(\mathbf{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = \mathbf{0}$. Taking into account how the terms $\mathbf{F}$ and F_0 are defined and considering that $F_0 = 0$ for p and $\mathbf{F} = \mathbf{0}$ for $\mathbf{v}$ by applying Lemma 3 to (84) we obtain $\mathbf{f} = \mathbf{0}$, $g = 0$ and

$$\Pi^{s}\boldsymbol{\Psi}_0 - \Pi^{d}\boldsymbol{\Phi}_0 = 0, \quad \mathbf{V}\boldsymbol{\Psi}_0 - \mathbf{W}\boldsymbol{\Phi}_0 = \mathbf{0}.$$

In addition since $F_0 = 0$ and $\mathbf{F} = \mathbf{0}$, one can easily see that

$$r_{\partial\Omega_D}T^{+}(F_0,\mathbf{F}) - r_{\partial\Omega_D}\boldsymbol{\Psi}_0 = \mathbf{0} \text{ implies } r_{\partial\Omega_D}\boldsymbol{\Psi}_0 = \mathbf{0}$$
$$r_{\partial\Omega_N}\gamma^{+}\mathbf{F} - r_{\partial\Omega_N}\boldsymbol{\Phi}_0 = \mathbf{0} \text{ implies } r_{\partial\Omega_N}\boldsymbol{\Phi}_0 = \mathbf{0}$$

Consequently we see that $\boldsymbol{\Psi}_0 \in \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_N)$ and $\boldsymbol{\Phi}_0 \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_D)$. Therefore, by Lemma 2 with $S_1 = \partial\Omega_N$ and $S_2 = \partial\Omega_D$, we thus obtain $\boldsymbol{\Psi}_0 = \mathbf{0}$ and $\boldsymbol{\Phi}_0 = \mathbf{0}$ on $\partial\Omega$.

8 | EQUIVALENCE AND INVERTIBILITY

Theorem 15. Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in L^2(\Omega)$, and $\boldsymbol{\Phi}_0 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$, and $\boldsymbol{\Psi}_0 \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\boldsymbol{\varphi}_0 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega_D)$ and $\boldsymbol{\psi}_0 \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N)$ respectively.

(i) If some $(p, \mathbf{v}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega)$ solves mixed BVP (6a) - (6d), then the solution is unique and the set $(p, \mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$, where

$$\boldsymbol{\varphi} = \gamma^{+}\mathbf{v} - \boldsymbol{\Phi}_0, \;\; \boldsymbol{\psi} = \mathbf{T}^{+}(p, \mathbf{v}) - \boldsymbol{\Psi}_0 \quad \text{on } \partial\Omega \tag{90}$$

solves BDIE systems (**M11**), (**M12**), (**M21**) and (**M22**).

(ii) If $(p, \mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ solves one of the BDIE systems (**M11**) or (**M12**)or (**M21**) or (**M22**), then the solution is unique and solves the BDIE systems, while $(p, \mathbf{v})$ belongs to $\mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ and solve mixed BVP (6a) - (6d) and the relations (90).

Proof. (i) Let $(p, \mathbf{v}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega)$ be a solution of the BVP (6a) - (6d). Since $\mathbf{f} \in \mathbf{L}^2(\Omega)$ then $(p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$. Due to Theorem 1 it is unique. Let us define the functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ by (90). By the BVP boundary conditions, $\gamma^{+}\mathbf{v} = \boldsymbol{\varphi}_0 = \boldsymbol{\Phi}_0$ on $\partial\Omega_D$ and $\mathbf{T}^{+}(p, \mathbf{v}) = \boldsymbol{\psi}_0 = \boldsymbol{\Psi}_0$ on $\partial\Omega_N$. This implies that $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ and recalling how BDIE systems (**M11**), (**M12**), (**M21**) and (**M22**) were constructed, we obtain that $(p, \mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})$ solves systems (**M11**), (**M12**), (**M21**) and (**M22**).

(ii) let $(p, \mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ solve BDIE system(**M11**) or (**M12**) or (**M21**) or (**M22**) . The first two equations in BDIE system and Theorems 2 and 3 imply that $(p, \mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$. The hypotheses of Lemma 3 are satisfied for the first two equations in BDIE system, implying that $(p, \mathbf{v})$ solves PDEs (6a)-(6b) in Ω, while the following equations holds:

$$\Pi^{s}\boldsymbol{\Psi}^{*} - \Pi^{d}\boldsymbol{\Phi}^{*} = 0, \;\; \mathbf{V}\boldsymbol{\Psi}^{*} - \mathbf{W}\boldsymbol{\Phi}^{*} = \mathbf{0} \text{ in } \Omega, \tag{91}$$

where $\boldsymbol{\Psi}^{*} := \boldsymbol{\psi} + \boldsymbol{\Psi}_0 - \mathbf{T}^{+}(p, \mathbf{v})$ and $\boldsymbol{\Phi}^{*} := \boldsymbol{\varphi} + \boldsymbol{\Phi}_0 - \gamma^{+}\mathbf{v}$ on $\partial\Omega$.

Suppose first that the tuple $(p, \mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ solves BDIE system (**M11**). Taking the trace of equation (83b) on $\partial\Omega_D$ using the jump relations (53) , and subtracting equation (83c) from it, we obtain

$$\gamma^+\mathbf{v} = \boldsymbol{\varphi}_0 \quad \text{on} \quad \partial\Omega_D, \tag{92}$$

that is $(p, \mathbf{v})$ satisfies the Dirichlet condition (6c). Taking the traction of equations (83a) and (83b) on $\partial\Omega_N$, using the jump relation (54) and subtracting equation (83d) from it, we obtain

$$\mathbf{T}^+(p, \mathbf{v}) = \boldsymbol{\psi}_0 \ \text{ on } \ \partial\Omega_N, \tag{93}$$

that is $(p, \mathbf{v})$ satisfies the Neumann condition (6d). Hence $(p, \mathbf{v})$ solves the mixed BVP (6a) - (6d).

Taking into account $\boldsymbol{\varphi} = 0$, $\boldsymbol{\Phi}_0 = \boldsymbol{\varphi}_0$ on $\partial\Omega_D$ and $\boldsymbol{\psi} = 0$, $\boldsymbol{\Psi} = 0$, $\boldsymbol{\Psi}_0 = \boldsymbol{\psi}_0$ on $\partial\Omega_N$, equations (92) and (93) imply that the second equation in (90) is satisfied on $\partial\Omega_N$ and the first equation in (90) is satisfied on $\partial\Omega_D$. Thus we have $\boldsymbol{\Psi}^* \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\boldsymbol{\Phi}^* \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ in (91). Let $S_1 = \partial\Omega_D$ and $S_2 = \partial\Omega_N$. Then for $\boldsymbol{\Psi}^* \in \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$ and Lemma 2 implies $\boldsymbol{\Psi}^* = \boldsymbol{\Phi}^* = 0$, which completes the proof of conditions in (90). Uniqueness of the solution to BDIE system (**M11**) follows from (90) along with Remark 2 and Theorem 1.

Similar arguments work if we suppose that instead of the BDIE systems (**M11**), $(p, \mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ solves BDIE systems (**M12**) or (**M21**) or (**M22**).

□

Theorem 16. The following operators are invertible

$$\mathcal{M}^{11} : L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}_{**}^{\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N), \tag{94}$$

$$\mathcal{M}^{11} : \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}_{**}^{\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N). \tag{95}$$

Proof. Remark 2 implies that the operators (94) and (95) are injective. Let us denote

$$\widetilde{\mathcal{M}}^{11} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^s & \Pi^d \\ \mathbf{0} & \mathbf{I} & -\mathbf{V} & \mathbf{W} \\ \mathbf{0} & \mathbf{0} & -r_{\partial\Omega_D}\boldsymbol{\mathcal{V}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & r_{\partial\Omega_N}\hat{\boldsymbol{\mathcal{L}}} \end{bmatrix}$$

where $\hat{\boldsymbol{\mathcal{L}}}$ is given by (55). Then $\widetilde{\mathcal{M}}^{11} : L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}_{**}^{\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N)$ is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators I, $\mathbf{I}$, $\boldsymbol{\mathcal{V}}$ from Theorem 13 and $\hat{\mathcal{L}}$ from Theorem 12,

$$I : L^2(\Omega) \to L^2(\Omega), \quad \mathbf{I} : \mathbf{H}^1(\Omega) \to \mathbf{H}^1(\Omega), r_{\partial\Omega_D}\boldsymbol{\mathcal{V}} : \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \to \mathbf{H}_{**}^{\frac{1}{2}}(\partial\Omega_D), \quad r_{\partial\Omega_N}\hat{\boldsymbol{\mathcal{L}}} : \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N).$$

By Proposition 1, Theorem 11 and 12 the operator

$$\mathcal{M}^{11} - \widetilde{\mathcal{M}}^{11} := \begin{bmatrix} 0 & \mathcal{R}^{\bullet} & 0 & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{R}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{\partial\Omega_D}\gamma^+\boldsymbol{\mathcal{R}} & \mathbf{0} & r_{\partial\Omega_D}\boldsymbol{\mathcal{W}} \\ \mathbf{0} & r_{\partial\Omega_N}T^+(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}}) & \mathbf{0} & r_{\partial\Omega_N}(\boldsymbol{\mathcal{L}}^+ - \hat{\boldsymbol{\mathcal{L}}}) \end{bmatrix}$$

is compact. Note that, we can write the operator $\mathcal{M}^{11}$ as a sum of compact and inveritible operator, $\mathcal{M}^{11} = (\mathcal{M}^{11} - \widetilde{\mathcal{M}}^{11}) + \widetilde{\mathcal{M}}^{11}$, like as for scalar case implying that it is a Fredholm operator with zero index, see e.g.,[13, Theorem 2.26]. Then the injectivity of operator (94) implies its invertibility, see e.g.,[13, Theorem 2.27].

To prove invertibility of operator (95), we remark that for any $\mathcal{F}^{11} \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}_{**}^{\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N)$ a solution of system

$$\mathcal{M}^{11}\mathcal{X} = \mathcal{F}^{11}, \tag{96}$$

can be written as $\mathcal{X} = [\mathcal{M}^{11}]^{-1}\mathcal{F}^{11}$, where $[\mathcal{M}^{11}]^{-1} : L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}_{**}^{\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N) \to L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ is the continuous inverse operator to operator (94). Applying Lemma 3 the first two equations of system (96) implies that $\mathcal{X} = [\mathcal{M}^{11}]^{-1}\mathcal{F}^{11} \in \mathbf{H}^{1,0}(\Omega;\mathcal{A}) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ and the operator $[\mathcal{M}^{11}]^{-1}$ is a continuous inverse to (95) as well. □

Theorem 17. The operator

$$\mathcal{M}^{12} : L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega), \tag{97}$$

is invertible.

Proof. The operator is injective, i.e., $\ker \mathcal{M}^{12} = \{0\}$. To see this, let $\mathcal{M}^{12}\mathcal{X} = \mathbf{0}$, which implies $\mathcal{F}^{12} = \mathbf{0}$ or $(F_0, \mathbf{F}, \gamma^+\mathbf{F} - \mathbf{\Phi}_0)^T = \mathbf{0}$. By Remark 3, $(\mathbf{f}, g, \mathbf{\Phi}_0, \mathbf{\Psi}_0) = \mathbf{0}$. This means $\mathbf{f} = \mathbf{0}, g = 0, \mathbf{\Phi}_0 = \mathbf{0}, \mathbf{\Psi}_0 = \mathbf{0}$, hence $\mathcal{A}(p, \mathbf{v}) = \mathbf{0}$, $\operatorname{div} \mathbf{v} = 0$ in Ω, $\gamma^+\mathbf{v} = \mathbf{0}$ on $\partial\Omega_D$, and $\mathbf{T}^+(p, \mathbf{v}) = \mathbf{0}$ on $\partial\Omega_N$. Theorem 1 implies $p = 0$, $\mathbf{v} = \mathbf{0}$ and then by Theorem 15, $\boldsymbol{\varphi} = \mathbf{0}, \boldsymbol{\psi} = \mathbf{0}$. Therefore, $\mathcal{X} = \mathbf{0}$. Let us denote

$$\widetilde{\mathcal{M}}^{12} := \begin{bmatrix} I & 0 & -\Pi^s & \Pi^d \\ \mathbf{0} & \mathbf{I} & -\mathbf{V} & \mathbf{W} \\ \mathbf{0} & \mathbf{0} & -\mathcal{V} & \frac{1}{2}\mathbf{I} \end{bmatrix}.$$

It is not hard to see that $\widetilde{\mathcal{M}}^{12}$ is bounded. Due to the mapping properties of the operators involved in the matrix $\mathcal{M}^{12} - \widetilde{\mathcal{M}}^{12}$, by Theorem 2 and Proposition 1, the operator

$$\mathcal{M}^{12} - \widetilde{\mathcal{M}}^{12} := \begin{bmatrix} 0 & \mathcal{R}^{\bullet} & 0 & 0 \\ \mathbf{0} & \mathcal{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^+\mathcal{R} & \mathbf{0} & \mathcal{W} \end{bmatrix}$$

is compact. To show the invertibility of $\widetilde{\mathcal{M}}^{12}$, consider the equation

$$\widetilde{\mathcal{M}}^{12}\mathcal{X} = \widetilde{\mathcal{F}}, \tag{98}$$

with an unknown vector $\mathcal{X} = (p, \mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})^T \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ and a given vector $\widetilde{\mathcal{F}} = (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3) \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Rewrite (98) componentwise

$$p - \Pi^s\boldsymbol{\psi} + \Pi^d\boldsymbol{\varphi} = \widetilde{F}_1 \quad \text{in } \Omega, \tag{99}$$

$$\mathbf{v} - \mathbf{V}\boldsymbol{\psi} + \mathbf{W}\boldsymbol{\varphi} = \widetilde{F}_2 \quad \text{in } \Omega, \tag{100}$$

$$\frac{1}{2}\boldsymbol{\varphi} - \mathcal{V}\boldsymbol{\psi} = \widetilde{F}_3 \quad \text{on } \partial\Omega, \tag{101}$$

The restriction of equation (101) on $\partial\Omega_D$ gives

$$-r_{\partial\Omega_D}\mathcal{V}\boldsymbol{\psi} = r_{\partial\Omega_D}\widetilde{F}_3. \tag{102}$$

Due to Theorem 13, equation (102) is uniquely solvable. i.e., for arbitrary $\widetilde{F}_3 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ there exist a unique $\boldsymbol{\psi} \in \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$ satisfying (102). Note that in accordance with (102)

$$[\mathcal{V}\boldsymbol{\psi} + \widetilde{F}_3] \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N). \tag{103}$$

Then (101) along with (103) yield that $\boldsymbol{\varphi}$ is defined also uniquely as

$$\boldsymbol{\varphi} = 2[\mathcal{V}\psi + \widetilde{F}_3] \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N).$$

Thus, equation (101) with arbitrary $\widetilde{F}_3 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ defines $\boldsymbol{\varphi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ and $\boldsymbol{\psi} \in \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$ uniquely. Remark that we then have $\Pi^s\boldsymbol{\psi}, \Pi^d\boldsymbol{\varphi} \in L^2(\Omega)$, $\mathbf{V}\boldsymbol{\psi}$, $\mathbf{W}\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)$ and from equation (99) and (100) we get

$$p = \Pi^s\boldsymbol{\psi} - \Pi^d\boldsymbol{\varphi} + \widetilde{F}_1 \quad \text{in } \Omega, \ \mathbf{v} = \mathbf{V}\boldsymbol{\psi} - \mathbf{W}\boldsymbol{\varphi} + \widetilde{F}_2 \quad \text{in } \Omega.$$

That is, the functions $(p, \mathbf{v}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega)$ is defined also uniquely. We conclude that $\widetilde{\mathcal{M}}^{12}$ is invertible. Note that, we can write the operator $\mathcal{M}^{12}$ as a sum of compact and inveritible operator, $\mathcal{M}^{12} = (\mathcal{M}^{12} - \widetilde{\mathcal{M}}^{12}) + \widetilde{\mathcal{M}}^{12}$, implying that it is a Fredholm operator with zero index. Then the injectivity of operator (97) implies its invertibility. □

Theorem 18. The operator

$$\mathcal{M}^{21} : L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega), \tag{104}$$

is invertible.

Proof. It is straight forward to show that the operator $\mathcal{M}^{21}$ is injective. To see this, let $\mathcal{M}^{21}\mathcal{X} = \mathbf{0}$. We show that $\mathcal{X} = \mathbf{0}$. Since $\mathcal{F}^{21} = \mathbf{0}$, we have:

$$\left(F_0,\ \mathbf{F},\ \mathbf{T}^{+}(F_0, \mathbf{F}) - \mathbf{\Psi}_0\right)^T = \mathbf{0}$$

which implies $(\mathbf{f}, g, \mathbf{\Phi}_0, \mathbf{\Psi}_0) = \mathbf{0}$, see Remark 4. Hence $\boldsymbol{\mathcal{A}}(p, \mathbf{v}) = \mathbf{0}$ and $\operatorname{div} \mathbf{v} = 0$ in Ω, $\gamma^{+}\mathbf{v} = \mathbf{0}$ on $\partial\Omega_D$, and $\mathbf{T}^{+}(p, \mathbf{v}) = \mathbf{0}$ on $\partial\Omega_N$. Furthermore, Theorem 1 implies that $p = 0$, $\mathbf{v} = \mathbf{0}$. We thus have $\boldsymbol{\varphi} = \mathbf{0}$ and $\boldsymbol{\psi} = \mathbf{0}$ by Theorem 15. Then we get $\mathcal{X} = \mathbf{0}$ as desired.

Let us set

$$\widetilde{\mathcal{M}}^{21} := \begin{bmatrix} I & 0 & -\Pi^s & \Pi^d \\ \mathbf{0} & \mathbf{I} & -\mathbf{V} & \mathbf{W} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{I} & \widehat{\mathcal{L}} \end{bmatrix}.$$

It is not hard to see that $\widetilde{\mathcal{M}}^{21}$ is bounded. By Theorem 2 and Proposition 1, the operator

$$\mathcal{M}^{21} - \widetilde{\mathcal{M}}^{21} = \begin{bmatrix} 0 & \mathcal{R}^{\bullet} & 0 & 0 \\ \mathbf{0} & \mathcal{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{+}(\mathcal{R}^{\bullet}, \mathcal{R}) & -\mathcal{W}' & \mathcal{L}^{+} - \widehat{\mathcal{L}} \end{bmatrix}$$

is compact. Since the operators $\widehat{\mathcal{L}}$, I, and $\mathbf{I}$ are invertible and by arguments similar to those in the proof of Theorem 17 and then $\widetilde{\mathcal{M}}^{21}$ is invertible . Note that, we can write the operator $\mathcal{M}^{21}$ as a sum of compact and inveritible operator, that is, $\mathcal{M}^{21} = (\mathcal{M}^{21} - \widetilde{\mathcal{M}}^{21}) + \widetilde{\mathcal{M}}^{21}$. This implies that $\mathcal{M}^{21}$ is a Fredholm operator with zero index. Then the injectivity of this operator implies its invertibilit and, hence, the theorem. □

To prove the invertibility of the operator $\mathcal{M}^{22}$ we need some auxiliary assertions.

Lemma 4. Let $\partial\Omega = \overline{S_1} \cup \overline{S_2}$, where S_1 and S_2 are two non-intersecting non-empty of $\partial\Omega$ with infinitely smooth boundaries. For any vector

$$\mathcal{F} = (F_0, \mathbf{F}, \mathbf{\Psi}, \mathbf{\Phi})^T \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(S_1) \times \mathbf{H}^{\frac{1}{2}}(S_2)$$

there exists another vector

$$(g_*, \mathbf{f}_*, \mathbf{\Psi}_*, \mathbf{\Phi}_*)^T = \widetilde{C}_{S_1,S_2}\mathcal{F} \in L^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$$

which is uniquely determined by $\mathcal{F}$ and such that

$$\begin{aligned} \mathring{\mathcal{Q}}\mathbf{f}_* + (2-\alpha)\mu g_* + \Pi^s\mathbf{\Psi}_* - \Pi^d\mathbf{\Phi}_* &= F_0, \text{ in } \Omega, \\ \boldsymbol{\mathcal{U}}\mathbf{f}_* - \boldsymbol{\mathcal{Q}}g_* + \mathbf{V}\mathbf{\Psi}_* - \mathbf{W}\mathbf{\Phi}_* &= \mathbf{F}, \text{ in } \Omega, \\ r_{\partial\Omega_{S_1}}\mathbf{\Psi}_* &= \mathbf{\Psi}, \text{ on } S_1, \\ r_{\partial\Omega_{S_2}}\mathbf{\Phi}_* &= \mathbf{\Phi}, \text{ on } S_2 \end{aligned}$$

Furthermore, the operator

$$\widetilde{C}_{S_1,S_2} : \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(S_1) \times \mathbf{H}^{\frac{1}{2}}(S_2) \to L^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$$

is continuous.

Proof. The proof is the same with the corresponding Lemma for 3D case[9, Lemma 7.5] by including further assumptions $\mathbf{\Psi}^* \in \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega)$. □

Corollary 1. For any $\mathcal{F} = (F_0, \mathbf{F}, \mathcal{F}_2, \mathcal{F}_3)^T \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(S_1) \times \mathbf{H}^{\frac{1}{2}}(S_2)$, there exists a unique vector

$$(g_*, \mathbf{f}_*, \mathbf{\Psi}_*, \mathbf{\Phi}_*)^T = C_{S_1,S_2}\mathcal{F} \in L_2(\Omega) \times \mathbf{L}_2(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$$

which is uniquely determined by $\mathcal{F}$ and such that

$$\begin{aligned} \mathring{\mathcal{Q}}\mathbf{f}_* + (2-\alpha)\mu g_* + \Pi^s\mathbf{\Psi}_* - \Pi^d\mathbf{\Phi}_* &= F_0, \quad \text{in } \Omega, \\ \boldsymbol{\mathcal{U}}\mathbf{f}_* - \boldsymbol{\mathcal{Q}}g_* + \mathbf{V}\mathbf{\Psi}_* - \mathbf{W}\mathbf{\Phi}_* &= \mathbf{F}, \text{ in } \Omega, \\ r_{S_1}(\mathbf{T}^{+}(\mathbf{F}_0, \mathbf{F}) - \mathbf{\Psi}_*) &= \mathcal{F}_2, \text{ on } S_1, \\ r_{S_2}(\gamma^{+}\mathbf{F} - \mathbf{\Phi}_*) &= \mathcal{F}_3, \text{ on } S_2. \end{aligned}$$

Furthermore, the operator

$$\mathcal{C}_{S_1,S_2}\ :\ \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{-\frac{1}{2}}(S_1)\times\mathbf{H}^{\frac{1}{2}}(S_2)\to L^2(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$$

is continuous.

Proof. This corollary follows from applying Lemma 4 with $\boldsymbol{\Psi}=r_{S_1}\mathbf{T}^{+}(F_0,\mathbf{F})-\mathcal{F}_2$ and $\boldsymbol{\Phi}=r_{S_2}\gamma^{+}\mathbf{F}-\mathcal{F}_3$. □

Theorem 19. The operator

$$\mathcal{M}^{22}\ :\ \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})\times\widetilde{\mathbf{H}}^{-\frac{1}{2}}_{**}(\partial\Omega_D)\times\widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)\longrightarrow\mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega_D)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega_N)\tag{105}$$

is continuously invertible

Proof. Let us consider an arbitrary right hand side to the system $\mathcal{M}^{22}\mathcal{X}=\mathcal{F}^{22}$, $\mathcal{F}^{22}\in\mathbf{H}^{1,0}(\Omega)\times\mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega_D)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega_N)$. By Corollary 1, the right hand side $\mathcal{F}^{22}$ can be written in the form

$$\begin{aligned}
\mathring{\mathcal{Q}}\mathbf{f}_*+(2-\alpha)\mu g_*+\Pi^s\boldsymbol{\Psi}_*-\Pi^d\boldsymbol{\Phi}_*\quad &=F_0,\quad\text{in }\Omega,\\
\boldsymbol{\mathcal{U}}\mathbf{f}_*-\boldsymbol{\mathcal{Q}}g_*+\mathbf{V}\boldsymbol{\Psi}_*-\mathbf{W}\boldsymbol{\Phi}_*\quad &=\mathbf{F},\ \text{in }\Omega,\\
r_{_{\partial\Omega_D}}(\mathbf{T}^{+}(\mathbf{F}_0,\mathbf{F})-\boldsymbol{\Psi}_*)\ &=\mathcal{F}^{22}_2,\ \text{on }\partial\Omega_D,\\
r_{_{\partial\Omega_N}}(\gamma^{+}\mathbf{F}-\boldsymbol{\Phi}_*)\ &=\mathcal{F}^{22}_3,\ \text{on }\partial\Omega_N
\end{aligned}$$

where, $(g_*,\mathbf{f}_*,\boldsymbol{\Psi}_*,\boldsymbol{\Phi}_*)^T=\mathcal{C}_{\partial\Omega_D,\partial\Omega_N}\mathcal{F}^{22}$ where the operator $\mathcal{C}_{\partial\Omega_D,\partial\Omega_N}$ is bounded and has the following mapping property

$$\mathcal{C}_{\partial\Omega_D,\partial\Omega_N}\ :\ \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega_D)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega_N)\to L^2(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$$

By Corollary 2 and the equivalence theorem of the system (**M22**), Theorem 15, there exists a solution of the equation $\mathcal{M}^{22}\mathcal{X}=\mathcal{F}^{22}$. This solution can be represented as

$$\mathcal{X}=[p,\mathbf{v},\boldsymbol{\psi},\boldsymbol{\varphi}]^T=(\mathcal{M}^{22})^{-1}\mathcal{F}^{22},$$

where

$$(\mathcal{M}^{22})^{-1}\ :\ \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega_D)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega_N)\longrightarrow\mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})\times\widetilde{\mathbf{H}}^{-\frac{1}{2}}_{**}(\partial\Omega_D)\times\widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$$

is given by

$$(p,\mathbf{v})\ =\ \mathcal{A}_M^{-1}[g_*,\mathbf{f}_*,r_{\partial\Omega_D}\boldsymbol{\Psi}_*,r_{\partial\Omega_N}\boldsymbol{\Phi}_*]^T,\tag{106}$$

$$\boldsymbol{\psi}\ =\ \mathbf{T}^{+}(p,\mathbf{v})-\boldsymbol{\Psi}_*=\mathbf{T}^{+}(p,\mathbf{v})-(\mathcal{C}_{\partial\Omega_D,\partial\Omega_N}\mathcal{F}^{22})_2,\tag{107}$$

$$\boldsymbol{\varphi}\ =\ \gamma^{+}\mathbf{v}-\boldsymbol{\Phi}_*=\gamma^{+}\mathbf{v}-(\mathcal{C}_{\partial\Omega_D,\partial\Omega_N}\mathcal{F}^{22})_3.\tag{108}$$

Consequently, the operator $(\mathcal{M}^{22})^{-1}$ is continuous by continuity of the operators in (106)-(108) . □

The original BVP (6a)-(6d) can be written in the form

$$\boldsymbol{\mathcal{A}}_M\mathcal{X}=\boldsymbol{\mathcal{F}}_M$$

where

$$\boldsymbol{\mathcal{A}}_M=\begin{bmatrix}\boldsymbol{\mathcal{A}}\\ \mathrm{div}\\ r_{\Omega_D}\gamma^{+}\\ r_{\Omega_N}\mathbf{T}^{+}\end{bmatrix},\quad \mathcal{F}_M=\begin{bmatrix}\mathbf{f}\\ g\\ \boldsymbol{\varphi}_0\\ \boldsymbol{\psi}_0\end{bmatrix}.$$

The operator $\boldsymbol{\mathcal{A}}_M\ :\ \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})\longrightarrow\mathbf{L}^2(\Omega)\times L^2(\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega_D)\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N)$ is continuous and due to the uniqueness theorem for the BVP is also injective. The invertibility of the operator $\mathcal{M}^{11}$ from Theorem 16 and equivalence Theorem 15 lead to the following

Corollary 2. The operator

$$\boldsymbol{\mathcal{A}}_M\ :\ \mathbf{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})\longrightarrow\mathbf{L}^2(\Omega)\times L^2(\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega_D)\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N)$$

is continuously invertible.

Particularly, when $\mu = 1$, the operator $\mathcal{A}$ becomes $\mathring{\mathcal{A}}$ and $\mathcal{R} = \mathcal{R}^\bullet = 0$. Consequently, the boundary domain integral equations system (89a)- (89d) can be reduced to a BIE system consisting of two vector equations

$$r_{\partial\Omega_D}\left(\frac{1}{2}\boldsymbol{\psi} - \mathring{\mathcal{W}}'\boldsymbol{\psi} + \mathring{\mathcal{L}}\boldsymbol{\varphi}\right) = r_{\partial\Omega_D}\mathbf{T}^+(F_0, \mathbf{F}) - r_{\partial\Omega_D}\boldsymbol{\Psi}_0 \text{ on } \partial\Omega_D, \tag{109}$$

$$r_{\partial\Omega_N}\left(\frac{1}{2}\boldsymbol{\varphi} - \mathring{\mathcal{V}}\psi + \mathring{\mathcal{W}}\boldsymbol{\varphi}\right) = r_{\partial\Omega_N}\gamma^+\mathbf{F} - r_{\partial\Omega_N}\boldsymbol{\Phi}_0 \text{ on } \partial\Omega_N \tag{110}$$

and a BDIE system consisting of a scalar equation and a vector equation

$$p = F_0 + \mathring{\Pi}^s\boldsymbol{\psi} - \mathring{\Pi}^d\boldsymbol{\varphi} \text{ in } \Omega \tag{111}$$

$$\mathbf{v} = \mathbf{F} + \mathring{\mathbf{V}}\boldsymbol{\psi} - \mathring{\mathbf{W}}\boldsymbol{\varphi} \text{ in } \Omega \tag{112}$$

where the terms F_0 and $\mathbf{F}$ are given by (84). The theorem of equivalence between the BVP and BDIE system, Theorem 15 leads to the following result of equivalence for the constant coefficient case.

Theorem 20. Let $\mu = 1$ in Ω, $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $g \in L^2(\Omega)$. Moreover, Let $\boldsymbol{\Phi}_0 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ and $\boldsymbol{\Psi}_0 \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\boldsymbol{\varphi}_0 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega_D)$ and $\boldsymbol{\psi}_0 \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_N)$ respectively.

(i) If some $(p, v) \in L^2(\Omega)\times\mathbf{H}^1(\Omega)$ solve mixed BVP (6a) - (6d), then the solution is unique, the couple $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D)\times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ given by

$$\boldsymbol{\varphi} = \gamma^+\mathbf{v} - \boldsymbol{\Phi}_0, \quad \boldsymbol{\psi} = \mathbf{T}^+(p, \mathbf{v}) - \boldsymbol{\Psi}_0 \text{ on } \partial\Omega \tag{113}$$

solves the BIE system (109)-(110) and $(p, \mathbf{v})$ satisfies (111)-(112).

(ii) If $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$ solve the BIE system (109)-(110), then $(p, \mathbf{v})$ given by (111)-(112) solves mixed BVP (6a) - (6d) and equations (113) hold. Moreover, the BIE system (109)-(110) is uniquely solvable in $\widetilde{\mathbf{H}}_{**}^{-\frac{1}{2}}(\partial\Omega_D)\times\widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$.

The system (109)-(110) can be expressed using matrix notation as follows

$$\mathring{\mathcal{M}}^{22}\mathring{\mathcal{X}} = \mathring{\mathcal{F}}^{22} \tag{114}$$

where $\mathring{\mathcal{X}} = (\boldsymbol{\psi}, \boldsymbol{\varphi})^T \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N)$, the operator

$$\mathring{\mathcal{M}}^{22} := \begin{bmatrix} r_{\partial\Omega_D}\left(\frac{1}{2}\mathbf{I} - \mathring{\mathcal{W}}'\right) & r_{\partial\Omega_D}\mathring{\mathcal{L}}^+ \\ -r_{\partial\Omega_N}\mathring{\mathcal{V}} & r_{\partial\Omega_N}\left(\frac{1}{2}\mathbf{I} + \mathring{\mathcal{W}}\right) \end{bmatrix}, \quad \mathring{\mathcal{F}}^{22} := \begin{bmatrix} r_{\partial\Omega_D}\mathbf{T}^+F_0 - r_{\partial\Omega_D}\boldsymbol{\Psi}_0 \\ r_{\partial\Omega_N}F_0^+ - r_{\partial\Omega_N}\boldsymbol{\Phi}_0 \end{bmatrix}$$

$\mathring{\mathcal{F}}^{22} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega_N)$. Moreover, the operator $\mathring{\mathcal{M}}^{22} : \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega_N)$ is bounded and injective.

Theorem 21. The operator $\mathring{\mathcal{M}}^{22} : \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega_N)$ is invertible.

Proof. A solution of system (114) with an arbitrary $(\mathring{\mathcal{F}}^{22})^T = (\mathring{\mathcal{F}}_2{}^{22}, \mathring{\mathcal{F}}_3{}^{22}) \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega_N)$ is delivered by the couple $(\boldsymbol{\psi}, \boldsymbol{\varphi})$ satisfying the extended system

$$\widehat{\mathcal{M}}^{22}\mathcal{X} = \widehat{\mathcal{F}}^{22},$$

where $\mathcal{X} = (p, \mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})^T$, $\widehat{\mathcal{F}}^{22} = (0, \mathbf{0}, \widehat{\mathcal{F}}_2{}^{22}, \widehat{\mathcal{F}}_3{}^{22})^T$, and

$$\widehat{\mathcal{M}}^{22} := \begin{bmatrix} I & 0 & -\mathring{\Pi}^s & \mathring{\Pi}^d \\ \mathbf{0} & \mathbf{I} & -\mathring{\mathbf{V}} & \mathring{\mathbf{W}} \\ \mathbf{0} & \mathbf{0} & r_{\partial\Omega_D}\left(\frac{1}{2}\mathbf{I} - \mathring{\mathcal{W}}'\right) & r_{\partial\Omega_D}\mathring{\mathcal{L}}^+ \\ \mathbf{0} & \mathbf{0} & -r_{\partial\Omega_N}\mathring{\mathcal{V}} & r_{\partial\Omega_N}\left(\frac{1}{2}\mathbf{I} + \mathring{\mathcal{W}}\right) \end{bmatrix}, \tag{115}$$

The operator $\widehat{\mathcal{M}}^{22}$ has a continuous inverse due to Theorem 19 for $\mu = 1$. Consequently, the operator $\widehat{\mathcal{M}}^{22}$ has a bounded right inverse, which is also a two-side inverse due to injectivity of the operator $\widehat{\mathcal{M}}^{22}$, this implies that operator $\mathring{\mathcal{M}}^{22}$ is surjective. Theorem 20 implies that operator $\mathring{\mathcal{M}}^{22}$ is also injective and thus an isomorphism. $\square$

Theorem 22. The operator

$$\mathcal{M}^{22} : L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega_N),$$

is invertible.

Proof. Let us consider the following operator,

$$\widetilde{\mathcal{M}}^{22} := \begin{bmatrix} I & 0 & -\Pi^s & \Pi^d \\ \mathbf{0} & \mathbf{I} & -\mathbf{V} & \mathring{\mathbf{W}} \\ \mathbf{0} & \mathbf{0} & r_{\partial\Omega_D}\left(\frac{1}{2}\mathbf{I} - \mathring{\mathcal{W}}'\right) & r_{\partial\Omega_D}\mathcal{L}^+ \\ \mathbf{0} & \mathbf{0} & -r_{\partial\Omega_N}\mathcal{V} & r_{\partial\Omega_N}\left(\frac{1}{2}\mathbf{I} + \mathcal{W}\right) \end{bmatrix}, \tag{116}$$

By Theorem 2 ,3,11 and 12 , the operator $\widetilde{\mathcal{M}}^{22}$ is a compact perturbation of the operator $\mathcal{M}^{22}$. Taking into account relations (20) and (23), the above operator can be represented as

$$\widetilde{\mathcal{M}}^{22} = \operatorname{diag}(1, \frac{1}{\mu}\mathbf{I}, \mathbf{I}, \frac{1}{\mu}\mathbf{I})\widehat{\mathcal{M}}^{22}\operatorname{diag}(1, \mu\mathbf{I}, \mathbf{I}, \mu\mathbf{I}),$$

where $\operatorname{diag}(1, \frac{1}{\mu}\mathbf{I}, \mathbf{I}, \frac{1}{\mu}\mathbf{I})$ and $\operatorname{diag}(1, \mu\mathbf{I}, \mathbf{I}, \mu\mathbf{I})$ are diagonal 7×7 matrices.

The operator $\widehat{\mathcal{M}}^{22}$ is given by (115), is a triangular block matrix operator with the following diagonal operators

$$I : L^2(\Omega) \to L^2(\Omega), \quad \mathbf{I} : \mathbf{H}^1(\Omega) \to \mathbf{H}^1(\Omega), \quad \mathring{\mathcal{M}}^{22} : \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\partial\Omega_N) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega_N).$$

The operator $\mathring{\mathcal{M}}^{22}$ is invertible due to Theorem 21. Consequently (116) is an invertible operator as well. Taking into account that $\mu > constant > 0$ and is bounded, this implies the diagonal matrices $\operatorname{diag}(1, \frac{1}{\mu}\mathbf{I}, \mathbf{I}, \frac{1}{\mu}\mathbf{I})$ and $\operatorname{diag}(1, \mu\mathbf{I}, \mathbf{I}, \mu\mathbf{I})$ are invertible and the operator $\widetilde{\mathcal{M}}^{22}$ is invertible. This implies the operator $\mathcal{M}^{22}$ possesses the Fredholm property and its index is zero. The invertibility of the operator simply follows from the injectivity of the operator $\mathcal{M}^{22}$ derived from Theorem 15 (iii). $\square$

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