A study the effect of the loss functions on E-Bayesian estimation and E-MSE for reliability parameters

Ming Han [*]

School of Science, Ningbo University of Technology,

Ningbo 315211, Zhejiang, China

Abstract

In order to measure the estimated error, the definition of E-MSE(expected mean squared error) is introduced based on the definition of E-Bayesian estimation. Moreover, under different loss functions(include: squared error loss, K-loss function, precautionary loss and entropy loss), the formulas of E-Bayesian estimation and E-MSE for reliability parameters of exponential distribution are given respectively. Monte Carlo simulations are performed to compare the performances of the proposed methods of estimation and a real data set have been analysed for illustrative purposes(also using OpenBUGS), results are compared on the basis of E-MSE. When considering evaluating the E-Bayesian estimations under different loss functions, this paper proposed the E-MSE as evaluation standard.

Key words: E-Bayesian estimation; E-MSE; exponential distribution; loss function; Monte Carlo simulation; reliability parameter.

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1 Introduction

With the development of science and technology, the reliability of industrial products is efficiently improved.The estimation of product reliability has attracted worldwide attention during the past several decades. The estimation procedure usually begins with parameter estimation based on test data.

Lindley and Smith(1972) proposed the thought of hierarchical prior distribution. The hierarchical Bayesian method needs two stages to finish the setting of the prior distribution, hence, it is more robust than Bayesian method. Han (1997) proposed the method to construct hierarchical prior distribution. Yet the hierarchical Bayesian estimation often involves in the computation of complicated integrals and integrals computing is hard work by using hierarchical Bayesian methods in practice, some computing methods such as Markov chain Monte Carlo(MCMC) methods are availablee(see Brooks(1998), Andrieu and Thoms(2008)). In recent years, hierarchical Bayesian methods have been applied to data analysis, for more details, see Ando and Zellner(2010), Osei and Duker(2011),

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Han(2009, 2011, 2017), Yousefzadeh(2017), Kizilaslan(2017), Abdul-Sathar and Krishnan(2019), Shahrastani(2019).

Bayesian methods in statistical inference depend on the choice of the prior distribution and loss function. But prior distribution parameters may depend on the hyper parameters. In this situation, we often use the hierarchical Bayesian method. On the other hand, the loss function is important in Bayesian methods. In the Bayesian inference is the squared error loss the most commonly used. This loss function is symmetrical and gives equal weight to overestimation as well as underestimation. It is well known that the use of symmetric loss functions may be inappropriate in many circumstances, particularly when positive and negative errors have different consequences. A useful asymmetric loss, known as the LINEX(linear exponential) loss function, was introduced by Varian(1975) and was widely used by several authors(see, for example, Zellner(1986), Schabe(1991), and Pandei and Rai(1992)). This function rises approximately exponentially on one side of zero and approxinlately linearly on the other side. And further properties of LINEX loss function have been investigated by Zellner(1986). A suitable alternative to the modified LINEX loss function is the general entropy loss proposed by Calabria and Pulcini(1994). This loss function is a generalization of the entropy loss used. The entropy loss function is another useful asymmetric loss function. Parameter estimation under entropy loss function, see Paorsian and Nematollahi(1996), Yousefzadeh(2017). The balanced loss function was introduced first by Zellner(1994). Ali, Aslam and Kazmi(2013), mathematical properties of Lindley distribution via Bayesian approach are derived under different loss functions(include: squared error loss, weighted squared error loss, precautionary loss, modified squared error loss, logarithmic loss, entropy loss and K-loss). These properties include: Bayes estimators, posterior risks and failure rate function for simulation scheme. Results are compared on the basis of posterior risk. About relevant research of Bayesian estimation(or hierarchical Bayesian estimation) under different loss functions, more details, see Calabria and Pulcini(1996), Kazmi, Aslam and Ali(2012), Ali(2015), Yousefzadeh(2017), Han(2017, 2018), Kizilaslan(2019).

In recent years, the author proposed the E-Bayesian estimation method, which has been cited and developed by some scholars. The hierarchical Bayesian method and E-Bayesian method are two methods to deal with hyper parameters when the prior distribution contains unknown parameters(hyper parameters).

It is well known that in the Bayes framework, the Bayes rule is obtained by considering a specific prior distribution over the parameter of interest but in practice, the use of a specified prior with specific hyper parameters is critical. Specially, when a problem in the Bayes framework is behaved by two or more statisticians, they might agree on a specific prior but might not on the hyper parameter choices. To deal with such an uncertainty issue, E-Bayesian and robust Bayes approaches may be called are derived(Karimnezhad and Moradi, 2016).

When the prior knowledge is vague, the E-Bayesian and robust Bayesian analysis can be used to handle the uncertainty in specifying the prior distribution by considering a class of priors instead of a single prior. The goal of the E-Bayesian method is estimating an unknown parameter or predicting a future value of a sequence of random variables by specifying a prior distribution prior hyper parameter(s)(Kiapour, 2018).

About relevant research of the E-Bayesian estimation method, for more details, see Han(2007, 2009, 2011), Jaheen and Okasha(2011), Okasha (2014), Okasha and Wang(2016), Karimnezhad and Moradi(2016), Yin, Huang and Peng et al.(2016), Kizilaslan (2017),

Gonzalez-Lopez, Gholizadeh and Galarza (2017), Yousefzadeh (2017), Zhang, Zhao and Zhang et al.(2017), Kiapour(2018), Kizilaslan (2019), Londono and Barahona(2019), Rabie and Li(2019). In existing research, we can see, compared with the hierarchical Bayesian method, the E-Bayesian estimation method is so simple that it is easier to perform.

Han(2019)in the case of the two hyper parameters, proposed the definition of E-MSE, and combined exponential distribution under the scaled squared error loss function, the formula of E-Bayesian estimation and the formula of E-MSE are given respectively.

This article is organized as follows: in Section 2, the definition of E-MSE(expected mean square error) is introduced based on the definition of E-Bayesian estimation. In Section 3, introduced the Bayesian estimation under different loss functions. In Section 4, the formulas of E-Bayesian estimation under different loss functions are given, respectively. In Section 5, the formulas of E-MSE under different loss functions are given, respectively. In Section 6, Monte Carlo simulation and comparisons are given. In Section 7, application example is given and Section 8 is the conclusion.

2   Definitions of E-Bayesian estimation and its E-MSE

In this section, the definition of E-Bayesian estimation and the definition of E-MSE will be introduced respectively.

2.1   Definition of E-Bayesian estimation of λ

For exponential distribution with probability density function(pdf)

$$f(t) = \lambda \exp\{-t\lambda\}, \quad t > 0, \tag{1}$$

where $\lambda > 0$ is the failure rate of the exponential distribution.

And the reliability function is given by

$$R(t) = \exp\{-\lambda t\}, t > 0. \tag{2}$$

If we take conjugate prior of λ, namely gamma(a, b), then probability density function(pdf)

$$\pi(\lambda|a, b) = \frac{b^a \lambda^{a-1} \exp(-b\lambda)}{\Gamma(a)}, \lambda > 0, \tag{3}$$

where $\Gamma(a) = \int_0^\infty x^{a-1} \exp(-x)dx$ is the gamma function, and hyper parameters $a > 0$ and $b > 0$.

According to Han (1997) , a and b should be selected to guarantee that $\pi(\lambda|a, b)$ is a decreasing function of λ. The derivative of $\pi(\lambda|a, b)$ with respect to λ is

$$\frac{d[\pi(\lambda|a, b)]}{d\lambda} = \frac{b^a \lambda^{a-2} \exp(-b\lambda)}{\Gamma(a)}[(a - 1) - b\lambda].$$

Note that $a > 0$, $b > 0$, and $\lambda > 0$, it follows $0 < a < 1$, $b > 0$ due to $\frac{d[\pi(\lambda|a,b)]}{d\lambda} < 0$, and therefore $\pi(\lambda|a, b)$ is a decreasing function of λ. Given $0 < a < 1$, the larger b is, the thinner the tail of the Gamma density function will be. Considering the robustness of Bayesian estimate (Berger,1985), the thinner tailed prior distribution often reduces the robustness of Bayesian estimate. Accordingly, b should not be larger then a given upper

bound c, where $c > 0$ is a constant to be determined. Thereby, the hyper parameters a and b should be selected with the restriction of $0 < a < 1$ and $0 < b < c$. How to determine the constant c would be described later in example.

In the case of the one hyper parameter, the definition of E-Bayesian estimation of failure rate was originally addressed by Han(2009). In the case of the two hyper parameters, the E-Bayesian estimation of failure rate is defined as follows.

Definition 1. With $\widehat{\lambda}_B(a,b)$ being continuous

$$\widehat{\lambda}_{EB} = \int\int_D \widehat{\lambda}_B(a,b)\pi(a,b)dadb$$

is called the E-Bayesian estimation of λ (briefly named E-Bayesian estimation, fully name should be expected Bayesian estimation),which is assumed to be finite, where D is the domain of a and b, $\widehat{\lambda}_B(a,b)$ is Bayesian estimation of λ with hyper parameters a and b, and $\pi(a,b)$ is the density function of a and b over D.

Definition 1 indicates that the E-Bayesian estimation of λ

$$\widehat{\lambda}_{EB} = \int\int_D \widehat{\lambda}_B(a,b)\pi(a,b)dadb = E\left[\widehat{\lambda}_B(a,b)\right]$$

is the expectation of Bayesian estimation of λ for all the hyper parameters.

From definition 1 we see, about the definition of E-Bayesian estimation, from the one hyper parameter (Han,2009) generalized to the two hyper parameters.

2.2 Definition of E-MSE of $\widehat{\lambda}_{EB}$

Han(2019b) in the case of the one hyper parameter, proposed the definition of E-MSE(expected mean square error). In the case of the two hyper parameters, Han(2019a)proposed the definition of E-MSE, as shown in definition 2 below.

Definition 2. With $\mathrm{MSE}[\widehat{\lambda}_B(a,b)]$ being continuous,

$$E - MSE(\widehat{\lambda}_{EB}) = \int\int_D MSE[\widehat{\lambda}_B(a,b)]\pi(a,b)dadb$$

is called the E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB}$(briefly named E-MSE, the full name should be expected mean square error),which is assumed to be finite, where D is the domain of a and b, $\mathrm{MSE}[\widehat{\lambda}_B(a,b)]$ is the MSE of Bayesian estimation of λ with hyper parameters a and b, and $\pi(a,b)$ is the density function of a and b over D.

Definition 2 indicates that the E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB}$

$$E - MSE(\widehat{\lambda}_{EB}) = \int\int_D MSE[\widehat{\lambda}_B(a,b)]\pi(a,b)dadb = E\left\{MSE[\widehat{\lambda}_B(a,b)]\right\}$$

is the expectation of MSE of Bayesian estimation for all the hyper parameters.

3 Bayesian estimation under different loss functions

According to Ali, Aslam and Kazmi(2013), we have the following Lemma 1.

Lemma 1. Let $x = (x_1, x_2, \cdots, x_n)$ are the sample observations, for any prior distribution of parameter λ, have the following conclusions:

(i) If the squared error loss function(SELF)$L_1(\lambda, \delta) = (\lambda - \delta)^2$ is used, then the Bayesian estimation of λ is $\widehat{\lambda}_{B1}(x) = E(\lambda|x)$.

(ii) If the K-loss function (KLF)$L_2(\lambda, \delta) = \left(\sqrt{\frac{\lambda}{\delta}} - \sqrt{\frac{\delta}{\lambda}}\right)^2$ is used, then the Bayesian estimation of λ is $\widehat{\lambda}_{B2}(x) = \sqrt{\frac{E(\lambda|x)}{E(\lambda^{-1}|x)}}$.

(iii) If the precautionary loss function (PLF)$L_3(\lambda, \delta) = \frac{(\lambda - \delta)^2}{\delta}$ is used, then the Bayesian estimation of λ is $\widehat{\lambda}_{B3}(x) = \sqrt{E(\lambda^2|x)}$.

(iv) If the entropy loss function(ELF)$L_4(\lambda, \delta) = \frac{\delta}{\lambda} - \ln \frac{\delta}{\lambda} - 1$ is used, then the Bayesian estimation of λ is $\widehat{\lambda}_{B4}(x) = [E(\lambda^{-1}|x)]^{-1}$.

Where δ is an estimate of parameter λ.

4 E-Bayesian estimation of λ

In the following, the complete sample and type I censored sample situations will be introduced respectively.

4.1 In the case of complete sample

Theorem 1. If $x = (x_1, x_2, \cdots, x_n)$ are the sample observations from the exponential distribution(1), $T = \sum\limits_{i=1}^{n} x_i$, the prior density function $\pi(\lambda|a, b)$ of λ is given by (3), the prior density function $\pi(a, b)$ of a and b is given by (4),

$$\pi(a, b) = \frac{1}{c}, \ 0 < a < 1, 0 < b < c, \tag{4}$$

then we have the following conclusions:

(i) If using the squared error loss function(SELF), then the Bayesian estimation of λ is

$$\widehat{\lambda}_{B1}(a, b) = \frac{n + a}{T + b},$$

and the corresponding E-Bayesian estimation of λ is

$$\widehat{\lambda}_{EB1} = \frac{1}{c}\left(n + \frac{1}{2}\right)\ln\left(1 + \frac{c}{T}\right).$$

(ii) If using the K-loss function (KLF), then the Bayesian estimation of λ is

$$\widehat{\lambda}_{B2}(a, b) = \frac{\sqrt{(n + a - 1)(n + a)}}{T + b},$$

and the corresponding E-Bayesian estimation of λ is

$$\widehat{\lambda}_{EB2} = \frac{1}{c}\ln\left(1 + \frac{c}{T}\right)\int_0^1 \sqrt{(n + a - 1)(n + a)}\,da.$$

(iii) If using the precautionary loss function (PLF), then the Bayesian estimation of λ is

$$\widehat{\lambda}_{B3}(a,b) = \frac{\sqrt{(n+a+1)(n+a)}}{T+b},$$

and the corresponding E-Bayesian estimation of λ is

$$\widehat{\lambda}_{EB3} = \frac{1}{c}\ln\left(1+\frac{c}{T}\right)\int_0^1 \sqrt{(n+a+1)(n+a)}da.$$

(iv) If the entropy loss function(ELF), then the Bayesian estimation of λ is

$$\widehat{\lambda}_{B4}(a,b) = \frac{n+a-1}{T+b},$$

and the corresponding E-Bayesian estimation of λ is

$$\widehat{\lambda}_{EB4} = \frac{1}{c}\left(n-\frac{1}{2}\right)\ln\left(1+\frac{c}{T}\right).$$

Proof. If $x = (x_1, x_2, \cdots, x_n)$ are the sample observations from the exponential distribution(1), then the likelihood function can be written as

$$L(x|\lambda) \propto \lambda^n \exp\{-T\lambda\},$$

where $T = \sum\limits_{i=1}^{n} x_i$.

If the prior density function $\pi(\lambda|a,b)$ of λ is given by (3), then the Bayesian theorem leads to the posterior density function of λ is

$$
\begin{aligned}
h(\lambda|x) &= \frac{\pi(\lambda|a,b)L(x|\lambda)}{\int_0^\infty \pi(\lambda|a,b)L(x|\lambda)d\lambda} \\
&= \frac{\lambda^{n+a-1}\exp\{-(T+b)\lambda\}}{\int_0^\infty \lambda^{n+a-1}\exp\{-(T+b)\lambda\}d\lambda} \\
&= \frac{(T+b)^{n+a}}{\Gamma(r+a)}\lambda^{n+a-1}\exp\{-(T+b)\lambda\}, \ \lambda > 0.
\end{aligned}
$$

Therefore the posterior distribution of λ is the gamma distribution: gamma$(n+a, T+b)$.

(i) If using the squared error loss function(SELF), according to (i) of Lemma 1, then the Bayesian estimation of λ is $\widehat{\lambda}_{B1}(a,b) = E(\lambda|x) = \frac{n+a}{T+b}$.

For the prior density function $\pi(a,b)$ of a and b is given by (4), by Definition 1, the E-Bayesian estimation of λ will be

$$
\begin{aligned}
\widehat{\lambda}_{EB1} &= \int\int_D \widehat{\lambda}_{B1}(a,b)\pi(a,b)dadb \\
&= \frac{1}{c}\int_0^1 (n+a)da\int_0^c \frac{1}{T+b}db \\
&= \frac{1}{c}\left(n+\frac{1}{2}\right)\ln\left(1+\frac{c}{T}\right).
\end{aligned}
$$

(ii) According to the posterior density function of λ, we have

$$
\begin{aligned}
E(\lambda^{-1}|x) &= \int_0^\infty \lambda^{-1} h(\lambda|x)d\lambda \\
&= \frac{(T+b)^{n+a}}{\Gamma(n+a)} \int_0^\infty \lambda^{(n+a-1)-1} \exp\{-(T+b)\lambda\}d\lambda \\
&= \frac{\Gamma(n+a-1)(T+b)}{\Gamma(n+a)} \\
&= \frac{T+b}{n+a-1}.
\end{aligned}
$$

If using the K-loss function (KLF), according to (ii) of Lemma 1, then the Bayesian estimation of λ is

$$
\widehat{\lambda}_{B2}(a,b) = \sqrt{\frac{E(\lambda|x)}{E(\lambda^{-1}|x)}} = \frac{\sqrt{(n+a-1)(n+a)}}{T+b}.
$$

For the prior density function $\pi(a,b)$ of a and b is given by (4), by Definition 1, the E-Bayesian estimation of λ will be

$$
\begin{aligned}
\widehat{\lambda}_{EB2} &= \int\!\!\int_D \widehat{\lambda}_{B2}(a,b)\pi(a,b)da\,db \\
&= \frac{1}{c} \int_0^1 \sqrt{(n+a-1)(n+a)}da \int_0^c \frac{1}{T+b}db \\
&= \frac{1}{c} \ln\left(1+\frac{c}{T}\right) \int_0^1 \sqrt{(n+a-1)(n+a)}da.
\end{aligned}
$$

(iii) According to the posterior density function of λ, we have

$$
\begin{aligned}
E(\lambda^2|x) &= \int_0^\infty \lambda^2 h(\lambda|x)d\lambda \\
&= \frac{(T+b)^{n+a}}{\Gamma(n+a)} \int_0^\infty \lambda^{(n+a+2)-1} \exp\{-(T+b)\lambda\}d\lambda \\
&= \frac{\Gamma(n+a+2)}{\Gamma(n+a)(T+b)^2} \\
&= \frac{(n+a+1)(n+a)}{(T+b)^2}.
\end{aligned}
$$

If using the precautionary loss function (PLF), according to (iii) of Lemma 1, then the Bayesian estimation of λ is

$$
\widehat{\lambda}_{B3}(a,b) = \sqrt{E(\lambda^2|x)} = \frac{\sqrt{(n+a+1)(n+a)}}{T+b}.
$$

For the prior density function $\pi(a,b)$ of a and b is given by (4), by Definition 1, the E-Bayesian estimation of λ will be

$$
\begin{aligned}
\widehat{\lambda}_{EB3} &= \int\!\!\int_D \widehat{\lambda}_{B3}(a,b)\pi(a,b)da\,db \\
&= \frac{1}{c} \int_0^1 \sqrt{(n+a+1)(n+a)}da \int_0^c \frac{1}{T+b}db \\
&= \frac{1}{c} \ln\left(1+\frac{c}{T}\right) \int_0^1 \sqrt{(n+a+1)(n+a)}da.
\end{aligned}
$$

(iv) If using the entropy loss function(ELF), according to (iv) of Lemma 1, then the Bayesian estimation of λ is

$$\widehat{\lambda}_{B4}(a,b) = \left[E(\lambda^{-1}|x)\right]^{-1} = \frac{n+a-1}{T+b}.$$

For the prior density function $\pi(a,b)$ of a and b is given by (4), by Definition 1, the E-Bayesian estimation of λ will be

$$
\begin{aligned}
\widehat{\lambda}_{EB4} &= \int\int_D \widehat{\lambda}_{B4}(a,b)\pi(a,b)dadb \\
&= \frac{1}{c}\int_0^1 (n+a-1)da \int_0^c \frac{1}{T+b}db \\
&= \frac{1}{c}\left(n-\frac{1}{2}\right)\ln\left(1+\frac{c}{T}\right).
\end{aligned}
$$

Thus, the proof is completed.

4.2   In the case of type I censored sample

Conduct type I censored life testing m time, denote the censored times as t_i $(i = 1, 2, \cdots, m)$, the corresponding sample numbers as n_i, and the corresponding failure sample numbers observed in the testing process as $r_i(r_i = 0, 1, 2, \cdots, n_i)$, then $\{(n_i, r_i, t_i), i = 1, 2, \cdots, m\}$ is called the testing data set.

Theorem 2 below can be directly deduced from Theorem 1 and Han(2009).

Theorem 2.   For exponential distribution(1), the testing data set $\{(n_i, r_i, t_i), i = 1, 2, \cdots, m\}$ with type I censor, where $r_i = 0, 1, 2, \cdots, n_i$, let $M = \sum\limits_{i=1}^m (n_i - r_i)t_i$ and $r = \sum\limits_{i=1}^m r_i$. If the prior density function $\pi(\lambda|a,b)$ of λ is given by (3), the prior density function $\pi(a,b)$ of a and b is given by (4), then we have the following conclusions:

(i) If using the squared error loss function(SELF), then the Bayesian estimation of λ is

$$\widehat{\lambda}_{B1}(a,b) = \frac{r+a}{M+b},$$

and the corresponding E-Bayesian estimation of λ is

$$\widehat{\lambda}_{EB1} = \frac{1}{c}\left(r+\frac{1}{2}\right)\ln\left(1+\frac{c}{M}\right).$$

(ii) If using the K-loss function (KLF), then the Bayesian estimation of λ is

$$\widehat{\lambda}_{B2}(a,b) = \frac{\sqrt{(r+a-1)(r+a)}}{M+b},$$

and the corresponding E-Bayesian estimation of λ is

$$\widehat{\lambda}_{EB2} = \frac{1}{c}\ln\left(1+\frac{c}{M}\right)\int_0^1 \sqrt{(r+a-1)(r+a)}da.$$

(iii) If using the precautionary loss function (PLF), then the Bayesian estimation of λ is

$$\widehat{\lambda}_{B3}(a,b) = \frac{\sqrt{(r+a+1)(r+a)}}{M+b},$$

and the corresponding E-Bayesian estimation of λ is

$$\widehat{\lambda}_{EB3} = \frac{1}{c} \ln\left(1 + \frac{c}{M}\right) \int_0^1 \sqrt{(r+a+1)(r+a)}\,da.$$

(iv) If the entropy loss function(ELF), then the Bayesian estimation of λ is

$$\widehat{\lambda}_{B4}(a,b) = \frac{r+a-1}{M+b},$$

and the corresponding E-Bayesian estimation of λ is

$$\widehat{\lambda}_{EB4} = \frac{1}{c}\left(r - \frac{1}{2}\right)\ln\left(1 + \frac{c}{M}\right).$$

5　The E-MSE of E-Bayesian estimation

In the following, the complete sample and type I censored sample situations will be introduced respectively.

5.1　In the case of complete sample

Theorem 3. If $x = (x_1, x_2, \cdots, x_n)$ are the sample observations from exponential distribution (1), the prior density function $\pi(\lambda|a,b)$ of λ is given by (3), the prior density function $\pi(a,b)$ of a and b is given by (4), then we have the following conclusions:

(i) If using the squared error loss function(SELF), then the MSE of Bayesian estimation $\widehat{\lambda}_{B1}(a,b)$ is

$$MSE[\widehat{\lambda}_{B1}(a,b)] = \frac{n+a}{(T+b)^2},$$

the corresponding E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB1}$ is

$$E - MSE(\widehat{\lambda}_{EB1}) = \frac{2n+1}{2T(T+c)}.$$

(ii) If using the K-loss function (KLF), then the MSE of Bayesian estimation $\widehat{\lambda}_{B2}(a,b)$ is

$$MSE[\widehat{\lambda}_{B2}(a,b)] = \frac{2(n+a)[(n+a) - \sqrt{(n+a)(n+a-1)}]}{(T+b)^2},$$

the corresponding E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB2}$ is

$$E - MSE(\widehat{\lambda}_{EB2}) = \frac{2}{T(T+c)} \int_0^1 (n+a)[(n+a) - \sqrt{(n+a)(n+a-1)}]\,da.$$

(iii) If using the precautionary loss function (PLF), then the MSE of Bayesian estimation $\widehat{\lambda}_{B3}(a,b)$ is

$$MSE[\widehat{\lambda}_{B3}(a,b)] = \frac{2(n+a)[(n+a+1) - \sqrt{(n+a)(n+a+1)}]}{(T+b)^2},$$

the corresponding E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB3}$ is

$$E - MSE(\widehat{\lambda}_{EB3}) = \frac{2}{T(T+c)} \int_0^1 (n+a)[(n+a+1) - \sqrt{(n+a)(n+a+1)}]da.$$

(iv) If the entropy loss function(ELF), then the MSE of Bayesian estimation $\widehat{\lambda}_{B4}(a,b)$ is

$$MSE[\widehat{\lambda}_{B4}(a,b)] = \frac{n+a+1}{(T+b)^2},$$

the corresponding E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB4}$ is

$$E - MSE(\widehat{\lambda}_{EB4}) = \frac{2n+3}{2T(T+c)}.$$

Proof. (i) If $x = (x_1, x_2, \cdots, x_n)$ are the sample observations from the exponential distribution (1), the prior density function $\pi(\lambda|a,b)$ of λ is given by (3). According to (i) of Theorem 1, then the Bayesian estimation of λ is $\widehat{\lambda}_{B1}(a,b) = E(\lambda|x) = \frac{n+a}{T+b}$. According to proof procedure of Theorem 1, then the posterior distribution of λ is gamma$(n+a, T+b)$, so $Var(\lambda|x) = \frac{n+a}{(T+b)^2}$.

If using the squared error loss function (SELF), then the MSE of Bayesian estimation $\widehat{\lambda}_{B1}(a,b)$ is

$$MSE[\widehat{\lambda}_{B1}(a,b)] = E\left\{\left[\lambda - \widehat{\lambda}_{B1}(a,b)\right]^2 |x\right\} = E\left\{[\lambda - E(\lambda|x)]^2 |x\right\} = Var(\lambda|x) = \frac{n+a}{(T+b)^2}.$$

If the prior density function $\pi(a,b)$ of a and b is given by (4), by Definition 2, then the E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB1}$ will be

$$
\begin{aligned}
E - MSE(\widehat{\lambda}_{EB1}) &= \int\int_D MSE[\widehat{\lambda}_{B1}(a,b)]\pi(a,b)dadb \\
&= \frac{1}{c}\int_0^1 (n+a)da \int_0^c \frac{1}{(T+b)^2}db \\
&= \frac{2n+1}{2T(T+c)}.
\end{aligned}
$$

(ii)If using the K-loss function (KLF), according to (ii) of Theorem 1, the Bayesian estimation of λ is $\widehat{\lambda}_{B2}(a,b) = \frac{\sqrt{(n+a)(n+a-1)}}{T+b}$, then the MSE of Bayesian estimation $\widehat{\lambda}_{B2}(a,b)$ is

$$
\begin{aligned}
MSE[\widehat{\lambda}_{B2}(a,b)] &= E\left\{\left[\lambda - \widehat{\lambda}_{B2}(a,b)\right]^2 |x\right\} \\
&= E(\lambda^2|x) - 2\widehat{\lambda}_{B2}(a,b)E(\lambda|x) + [\widehat{\lambda}_{B2}(a,b)]^2 \\
&= \frac{2(n+a)[(n+a) - \sqrt{(n+a)(n+a-1)}]}{(T+b)^2}.
\end{aligned}
$$

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If the prior density function $\pi(a, b)$ of a and b is given by (4), by Definition 2, then the E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB2}$ will be

$$
\begin{aligned}
E - MSE(\widehat{\lambda}_{EB2}) &= \int\int_D MSE[\widehat{\lambda}_{B2}(a, b)]\pi(a, b)dadb \\
&= \frac{2}{c}\int_0^1 (n+a)[(n+a) - \sqrt{(n+a)(n+a-1)}]da \int_0^c \frac{1}{(T+b)^2}db \\
&= \frac{2}{T(T+c)}\int_0^1 (n+a)[(n+a) - \sqrt{(n+a)(n+a-1)}]da.
\end{aligned}
$$

(iii) If using the precautionary loss function (PLF), according to (iii) of Theorem 1, the Bayesian estimation of λ is $\widehat{\lambda}_{B3}(a, b) = \frac{\sqrt{(n+a+1)(n+a)}}{T+b}$, then the MSE of Bayesian estimation $\widehat{\lambda}_{B3}(a, b)$ is

$$
\begin{aligned}
MSE[\widehat{\lambda}_{B3}(a, b)] &= E\left\{\left[\lambda - \widehat{\lambda}_{B3}(a, b)\right]^2 |x\right\} \\
&= E(\lambda^2|x) - 2\widehat{\lambda}_{B3}(a, b)E(\lambda|x) + [\widehat{\lambda}_{B3}(a, b)]^2 \\
&= \frac{2(n+a)[(n+a+1) - \sqrt{(n+a)(n+a+1)}]}{(T+b)^2}.
\end{aligned}
$$

If the prior density function $\pi(a, b)$ of a and b is given by (4), by Definition 2, then the E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB3}$ will be

$$
\begin{aligned}
E - MSE(\widehat{\lambda}_{EB3}) &= \int\int_D MSE[\widehat{\lambda}_{B3}(a, b)]\pi(a, b)dadb \\
&= \frac{2}{c}\int_0^1 (n+a)[(n+a+1) - \sqrt{(n+a)(n+a+1)}]da \int_0^c \frac{1}{(T+b)^2}db \\
&= \frac{2}{T(T+c)}\int_0^1 (n+a)[(n+a+1) - \sqrt{(n+a)(n+a+1)}]da.
\end{aligned}
$$

(iv) If the entropy loss function(ELF), according to (iv) of Theorem 1, the Bayesian estimation of λ is $\widehat{\lambda}_{B4}(a, b) = \frac{n+a-1}{T+b}$, then the MSE of Bayesian estimation $\widehat{\lambda}_4(a, b)$ is

$$
\begin{aligned}
MSE[\widehat{\lambda}_{B4}(a, b)] &= E\left\{\left[\lambda - \widehat{\lambda}_{B4}(a, b)\right]^2 |x\right\} \\
&= E(\lambda^2|x) - 2\widehat{\lambda}_{B4}(a, b)E(\lambda|x) + [\widehat{\lambda}_{B4}(a, b)]^2 \\
&= \frac{n+a+1}{(T+b)^2}.
\end{aligned}
$$

If the prior density function $\pi(a, b)$ of a and b is given by (4), by Definition 2, then the E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB4}$ will be

$$
\begin{aligned}
E - MSE(\widehat{\lambda}_{EB4}) &= \int\int_D MSE[\widehat{\lambda}_{B4}(a, b)]\pi(a, b)dadb \\
&= \frac{1}{c}\int_0^1 (n+a+1)da \int_0^c \frac{1}{(T+b)^2}db \\
&= \frac{2n+3}{2T(T+c)}.
\end{aligned}
$$

That, the proof is complete.

5.2 In the case of type I censored sample

Following Theorem 4 can be directly deduced from Theorem 3 and Theorem 2.

Theorem 4. For exponential distribution(1), the testing data set $\{(n_i, r_i, t_i), i = 1, 2, \cdots, m\}$ with type I censor, where $r_i = 0, 1, 2, \cdots, n_i$, let $M = \sum\limits_{i=1}^{m}(n_i - r_i)t_i$ and $r = \sum\limits_{i=1}^{m} r_i$. If the prior density function $\pi(\lambda|a, b)$ of λ is given by (3), the prior density function $\pi(a, b)$ of a and b is given by (4), then we have the following conclusions:

(i) If using the squared error loss function(SELF), then the MSE of Bayesian estimation $\widehat{\lambda}_{B1}(a, b)$ is

$$MSE[\widehat{\lambda}_{B1}(a, b)] = \frac{r + a}{(M + b)^2},$$

the corresponding E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB1}$ is

$$E - MSE(\widehat{\lambda}_{EB1}) = \frac{2r + 1}{2M(M + c)}.$$

(ii) If using the K-loss function (KLF), then the MSE of Bayesian estimation $\widehat{\lambda}_{B2}(a, b)$ is

$$MSE[\widehat{\lambda}_{B2}(a, b)] = \frac{2(r + a)[(r + a) - \sqrt{(r + a)(r + a - 1)}]}{(M + b)^2},$$

the corresponding E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB2}$ is

$$E - MSE(\widehat{\lambda}_{EB2}) = \frac{2}{M(M + c)} \int_0^1 (r + a)[(r + a) - \sqrt{(r + a)(r + a - 1)}]da.$$

(iii) If using the precautionary loss function (PLF), then the MSE of Bayesian estimation $\widehat{\lambda}_{B3}(a, b)$ is

$$MSE[\widehat{\lambda}_{B3}(a, b)] = \frac{2(r + a)[(r + a + 1) - \sqrt{(r + a)(r + a + 1)}]}{(M + b)^2},$$

the corresponding E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB3}$ is

$$E - MSE(\widehat{\lambda}_{EB3}) = \frac{2}{M(M + c)} \int_0^1 (r + a)[(r + a + 1) - \sqrt{(r + a)(r + a + 1)}]da.$$

(iv) If the entropy loss function(ELF), then the MSE of Bayesian estimation $\widehat{\lambda}_{B4}(a, b)$ is

$$MSE[\widehat{\lambda}_{B4}(a, b)] = \frac{r + a + 1}{(M + b)^2},$$

the corresponding E-MSE of E-Bayesian estimation $\widehat{\lambda}_{EB4}$ is

$$E - MSE(\widehat{\lambda}_{EB4}) = \frac{2r + 3}{2M(M + c)}.$$

6   Monte Carlo simulation and comparisons

In this section, based on the complete sample, a Monte Carlo simulation is used for a comparison $\widehat{\lambda}_{EBi}(i = 1, 2, 3, 4)$ and E-MSE$(\widehat{\lambda}_{EBi})(i = 1, 2, 3, 4)$.

For known value $\lambda = 0.005$, a sample size n is then generated from the exponential distribution (1). The codes of MATLAB are used to generate from the exponential distribution.

The performance of all estimates has been compared numerically in two terms of their $\widehat{\lambda}_{EBi}(i = 1, 2, 3, 4)$ and E-MSE$(\widehat{\lambda}_{EBi})(i = 1, 2, 3, 4)$. The summary for the repeated 10000 times simulation runs is shown in Tables 1-2.

Table 1   Results of $\widehat{\lambda}_{EBi}(i = 1, 2, 3, 4)$

n	c	$\widehat{\lambda}_{EB1}$	$\widehat{\lambda}_{EB2}$	$\widehat{\lambda}_{EB3}$	$\widehat{\lambda}_{EB4}$
10	10	0.0058	0.0055	0.0061	0.0052
	50	0.0057	0.0054	0.0060	0.0052
	100	0.0056	0.0054	0.0059	0.0051
30	10	0.0053	0.0052	0.0053	0.0051
	50	0.0052	0.0051	0.0053	0.0051
	100	0.0052	0.0051	0.0053	0.0051
50	10	0.0051	0.0051	0.0052	0.0050
	50	0.0051	0.0051	0.0052	0.0050
	100	0.0051	0.0051	0.0052	0.0050
70	10	0.0051	0.0051	0.0051	0.0050
	50	0.0051	0.0051	0.0051	0.0050
	100	0.0051	0.0051	0.0051	0.0050
100	10	0.0051	0.0050	0.0051	0.0050
	50	0.0051	0.0050	0.0051	0.0050
	100	0.0051	0.0050	0.0051	0.0050

Table 2   Results of E-MSE$(\widehat{\lambda}_{EBi})(i = 1, 2, 3, 4)$

n	c	E-MSE$(\widehat{\lambda}_{EB1})$	E-MSE$(\widehat{\lambda}_{EB2})$	E-MSE$((\widehat{\lambda}_{EB3})$	E-MSE$((\widehat{\lambda}_{EB4})$
10	10	3.5803e-006	3.6701e-006	3.6619e-006	3.9213e-006
	50	3.4844e-006	3.5712e-006	3.5632e-006	3.8162e-006
	100	3.3868e-006	3.4718e-006	3.4641e-006	3.7093e-006
30	10	9.3827e-007	9.4610e-007	9.4585e-007	9.6904e-007
	50	9.3160e-007	9.3933e-007	9.3908e-007	9.6214e-007
	100	9.2560e-007	9.3330e-007	9.3305e-007	9.5595e-007
50	10	5.3608e-007	5.3877e-007	5.3871e-007	5.4670e-007
	50	5.3548e-007	5.3816e-007	5.3811e-007	5.4609e-007
	100	5.3084e-007	5.3350e-007	5.3345e-007	5.4136e-007
70	10	3.7576e-007	3.7711e-007	3.7709e-007	3.8109e-007
	50	3.7278e-007	3.7411e-007	3.7409e-007	3.7806e-007
	100	3.7100e-007	3.7232e-007	3.7230e-007	3.7626e-007
100	10	2.5829e-007	2.5894e-007	2.5893e-007	2.6086e-007
	50	2.5868e-007	2.5933e-007	2.5932e-007	2.6126e-007
	100	2.5736e-007	2.5800e-007	2.5799e-007	2.5992e-007

Based on tabulated the values of $\widehat{\lambda}_{EBi}(i = 1, 2, 3, 4)$ and E-MSE$(\widehat{\lambda}_{EBi})(i = 1, 2, 3, 4)$, the following conclusions can be drawn from Tables 1-2.

(1)For the same $n(n = 10, 30, 50, 70, 100)$ and different $c(c = 10, 50, 100)$, we find that the values of $\widehat{\lambda}_{EBi}(i = 1, 2, 3, 4)$ and E-MSE$(\widehat{\lambda}_{EBi})(i = 1, 2, 3, 4)$ are all robust.

(2)For the same $n(n = 10, 30, 50, 70, 100)$ and fixed $c(c = 10, 50, 100)$, we find that the values of E-MSE$(\widehat{\lambda}_{EBi})(i = 1, 2, 3, 4)$ have the following sequential relationship:

E-MSE$(\widehat{\lambda}_{EB1})$ <E-MSE$(\widehat{\lambda}_{EB3})$ <E-MSE$(\widehat{\lambda}_{EB2})$ <E-MSE$(\widehat{\lambda}_{EB4})$.

(3)For fixed $c(c = 10, 50, 100)$, it is observed that by increasing n, the performances of all estimates $\widehat{\lambda}_{EBi}(i = 1, 2, 3, 4)$ and E-MSE$(\widehat{\lambda}_{EBi})(i = 1, 2, 3, 4)$ improve.

7   Application example

In Han(2009) the given testing data for a type of electronic products, which is listed in Table 3(time unit: hour).

Table 3   Testing data of the electronic products

i	1	2	3	4	5	6	7
t_i	480	680	880	1080	1280	1480	1680
n_i	3	3	5	5	8	8	8
r_i	0	0	0	1	0	2	1

According to Han(2009), lifetime of this electronic products obey the exponential distribution.

The following relevant calculation, using Theorems 2, 4 and the MCMC algorithm(by Open BUGS software) respectively.

7.1   Using Theorems 2, 4

By Table 3 and Theorem 2, we can obtain $\widehat{\lambda}_{EBi}(i = 1, 2, 3, 4)$. Some numerical results are listed in Table 4.

Table 4   Results of $\widehat{\lambda}_{EBi}(i = 1, 2, 3, 4)$

c	100	500	1000	2000	Range
$\widehat{\lambda}_{EB1}$	1.0434e-004	1.0386e-004	1.0326e-004	1.0210e-004	1.5000e-006
$\widehat{\lambda}_{EB2}$	9.2013e-005	9.1589e-005	9.1066e-005	9.0045e-005	1.9680e-006
$\widehat{\lambda}_{EB3}$	1.1535e-004	1.1482e-004	1.1416e-004	1.1288e-004	2.4700e-006
$\widehat{\lambda}_{EB4}$	8.1150e-005	8.0776e-005	8.0316e-005	7.9415e-005	1.7350e-006
$\widehat{\lambda}_{-EB}$	3.4200e-005	3.4044e-005	3.3844e-005	3.3465e-005	

Note: In Table 4, $\widehat{\lambda}_{-B} = \widehat{\lambda}_{EB3} - \widehat{\lambda}_{EB4}$.

From Table 4, we find that for the same $c(100, 500, 1000, 2000)$, $\widehat{\lambda}_{EBi}(i = 1, 2, 3)$ are close to each other, and for different $c(100, 500, 1000, 2000)$, $\widehat{\lambda}_{EBi}(i = 1, 2, 3)$ are all robust.

Based on Table 4, we can obtain the corresponding estimate of the reliability functions $\widehat{R}_{EBi}(t) = \exp\{-\widehat{\lambda}_{EBi}t\}(i = 1, 2, 3, 4)$. Some numerical results are listed in Table 5.

Table 5   Results of $\widehat{R}_{EBi}(1000)(i=1,2,3)$

c	100	500	1000	2000	Range
$\widehat{R}_{EB1}(1000)$	0.9009	0.9014	0.9019	0.9029	0.0013
$\widehat{R}_{EB2}(1000)$	0.9121	0.9125	0.9130	0.9139	0.0018
$\widehat{R}_{EB3}(1000)$	0.8911	0.8915	0.8921	0.8933	0.0022
$\widehat{R}_{EB4}(1000)$	0.9221	0.9224	0.9228	0.9237	0.0016
$\widehat{R}_{-EB}(1000)$	0.0310	0.0309	0.0307	0.0208	

Note: In Table 5, $\widehat{R}_{-EB}(1000) = \widehat{R}_{EB4}(1000) - \widehat{R}_{EB3}(1000)$.

From Table 5, we find that for the same $c(100, 500, 1000, 2000)$, $\widehat{R}_{EBi}(1000)(i=1,2,3,4)$ are close to each other, and for different $c(100, 500, 1000, 2000)$, $\widehat{R}_{EBi}(1000)(i=1,2,3)$ are all robust.

By Table 3 and Theorem 4, we can obtain E-MSE$(\widehat{\lambda}_{EBi})(i=1,2,3,4)$. Some numerical results are listed in Table 6 and Figure 1.

Table 6   Results of E-MSE$(\widehat{\lambda}_{EBi})(i=1,2,3,4)$

c	100	500	1000	2000	Range
E-MSE$(\widehat{\lambda}_{EB1})$	2.4191e-009	2.3969e-009	2.3697e-009	2.3171e-009	1.0200e-010
E-MSE$(\widehat{\lambda}_{EB2})$	2.5710e-009	2.5474e-009	2.5185e-009	2.4627e-009	1.0830e-010
E-MSE$(\widehat{\lambda}_{EB3})$	2.5403e-009	2.5170e-009	2.4884e-009	2.4332e-009	1.0710e-010
E-MSE$(\widehat{\lambda}_{EB4})$	2.9567e-009	2.9295e-009	2.8963e-009	2.8321e-009	1.2460e-010
E-MSE$(\widehat{\lambda}_{EB-})$	5.3760e-010	5.3260e-010	4.0790e-010	5.1500e-010	

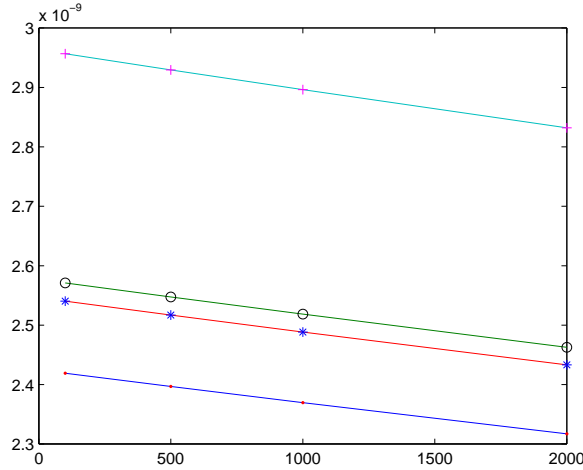Note: In Table 6, E-MSE$(\widehat{\lambda}_{EB-})$ =E-MSE$(\widehat{\lambda}_{EB4})$-E-MSE$(\widehat{\lambda}_{EB1})$.

Figure 1: Relationship between c and E-MSE$(\widehat{\lambda}_{EBi})(i=1,2,3,4)$

Note: In Figure 1, · is the results of E-MSE$(\widehat{\lambda}_{EB1})$, ∘ is the results of E-MSE$(\widehat{\lambda}_{EB2})$, ∗ is the results of E-MSE$(\widehat{\lambda}_{EB3})$, + is the results of E-MSE$(\widehat{\lambda}_{EB4})$.

From Table 6 and Figure 1, we find that for different $c(100, 500, 1000, 2000)$, E-MSE$(\widehat{\lambda}_{EBi})$ $(i=1,2,3,4)$ are all robust; for fixed $c(100, 500, 1000, 2000)$, we have the following sequential relationship: E-MSE$(\widehat{\lambda}_{EB1})$ <E-MSE$(\widehat{\lambda}_{EB3})$ <E-MSE$(\widehat{\lambda}_{EB2})$ <E-MSE$(\widehat{\lambda}_{EB4})$.

If E-MSE as evaluation standard, then we have the following conclusions:

$\widehat{\lambda}_{EB1}$ is superior to $\widehat{\lambda}_{EB3}$, $\widehat{\lambda}_{EB3}$ is superior to $\widehat{\lambda}_{EB2}$ and $\widehat{\lambda}_{EB2}$ is superior to $\widehat{\lambda}_{EB4}$.

According to the above calculation and analysis, to different loss functions(include: squared error loss, K-loss function, precautionary loss and entropy loss), in this paper, the author suggest select squared error loss function.

From Tables 4, 5 and 6, we find that for different $c(100, 500, 1000, 2000)$, $\widehat{\lambda}_{EBi}(i = 1, 2, 3, 4)$, $\widehat{R}_{EBi}(1000)(i = 1, 2, 3, 4)$ and E-MSE$(\widehat{\lambda}_{EBi})$ $(i = 1, 2, 3, 4)$ are all robust. Therefore in application, the author suggest select a value of c in the middle point of interval $(0, 2000]$, that is, $c = 1000$.

7.2 Using MCMC algorithm

In exponential distribution(1), we take the gamma distribution: gamma(0.005, 0.005) as prior distribution of λ.

According to MCMC relevant algorithm, by the OpenBUGS the following results. We obtain summary statistics for 10000 simulations as table 7.

<div align="center">Table 7   Results of λ</div>

node	mean	sd	MC-error	val2.5pc	median	val97.5pc	start	sample
λ	1.01e-4	4.63e-5	1.622e-6	3.295e-5	9.415e-5	2.118e-4	1001	9000

From Tables 4 and 7, we find that the values of $\widehat{\lambda}_{EBi}$ $(i = 1, 2, 3, 4)$ are very close to $\widehat{\lambda}_{B1}$=1.01e-4(mean of λ) or $\widehat{\lambda}_{B2}$=9.415e-5(median of λ).

Remark 1: gamma distribution: gamma(0.005, 0.005) is close to the non-informative prior distribution gamma(0, 0) of λ.

Remark 2: According to MCMC relevant algorithm, by the Open BUGS software are performed to relevant calculate, need to be the lower bound of lifetime on the censoring time.

When $c = 1000$, according to Tables 4 and 7, we can obtain the $\widehat{R}_{EB1}(t) = \exp(-\widehat{\lambda}_{EB1}t)$ and $\widehat{R}_{B1}(t) = \exp(-\widehat{\lambda}_{B1})$. When $t \in [0, 1600]$, some numerical results are listed in Figure 2.
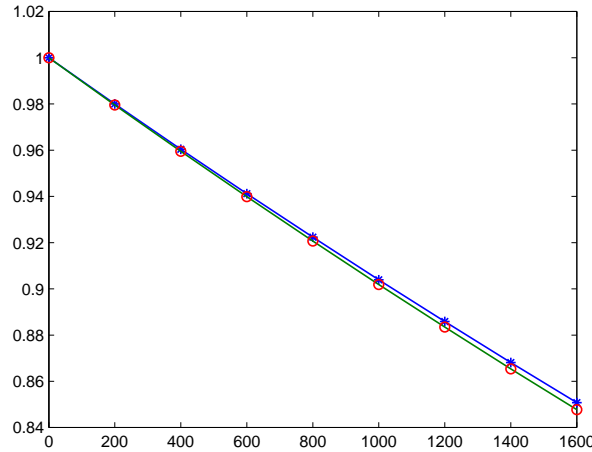

Figure 2: $\widehat{R}_{EB1}(t)$ and $\widehat{R}_{B1}(t)$

Note: In Figure 2, $\circ$ is the results of $\widehat{R}_{EB1}(t)$, $*$ is the results of $\widehat{R}_{B1}(t)$.

From Figure 2, we find that the values of $\widehat{R}_{EB1}(t)$ and $\widehat{R}_{B1}(t)$ is very close.

8 Conclusions

In this paper, we study of the effect of the different loss functions on E-Bayesian estimation and its E-MSE. The definition of E-MSE(expected mean squared error) is introduced based on the definition of E-Bayesian estimation, see Definition 1 and Definition 2. Under the different loss functions, the formulas of E-Bayesian estimation and formulas of E-MSE are given, respectively, see Theorems 1—4.

Need to explain, some results of Theorem 2 and Theorem 4 does not apply to case of $r = 0$(that is case of zero failure data. In this case, will be discussed in a separate paper).

Reviewing the simulation example and application example, we find that for $\mathrm{MSE}(\widehat{\lambda}_{EBi})$ $(i = 1, 2, 3, 4)$, we have the following sequential relationship:

E-MSE$(\widehat{\lambda}_{EB1})$ <E-MSE$(\widehat{\lambda}_{EB3})$ <E-MSE$(\widehat{\lambda}_{EB2})$ <E-MSE$(\widehat{\lambda}_{EB4})$.

It also suggests that, if E-MSE as evaluation standard, then we have the following conclusions:

$\widehat{\lambda}_{EB1}$ is superior to $\widehat{\lambda}_{EB3}$, $\widehat{\lambda}_{EB3}$ is superior to $\widehat{\lambda}_{EB2}$ and $\widehat{\lambda}_{EB2}$ is superior to $\widehat{\lambda}_{EB4}$.

When considering evaluating the E-Bayesian estimations under the different loss functions, this paper proposed the E-MSE(expected mean square error) as evaluation standard.

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