

A PARTIAL INVERSE PROBLEM FOR QUANTUM GRAPHS WITH A LOOP

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ABSTRACT. We consider the Sturm-Liouville operator on quantum graphs with a loop with the standard matching conditions in the internal vertex and the jump conditions at the boundary vertex. Given the potential on the loop, we try to recover the potential on the boundary edge from the subspectrum. The uniqueness theorem and a constructive algorithm for the solution of this partial inverse problem are provided.

1. INTRODUCTION

Differential operators on quantum graphs model different structures in organic chemistry, acoustics, carbon nanostructure, photonic crystal, and other fields of science and engineering [1]-[3]. In recent years, spectral problems on graphs attract much attention and research in mathematicians. The readers can find the elementary introduction to the theory of quantum graphs in [4]. The classical inverse spectral problems for differential operators, which consist in recovering coefficients of differential equations (especially potentials of Sturm-Liouville operator) from various types of spectral data were studied (see, for example, [5]-[9]).

In this paper, we consider the Sturm-Liouville operator on quantum graphs with a loop. The potential is supposed to be known a priori on a part of the ring, and we try to recover the potential on the remaining part from a part of the spectrum. Such partial inverse problems have been partly studied in [10]-[16] for star-shaped graphs. Here we formulate the partial inverse problem for quantum rings and prove the uniqueness theorem and provide a constructive algorithm for solution of this problem.

Note that there are only a few results on the graph with loops. In [17] authors proved the uniqueness and provided algorithm for the solution

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of this problem without discontinuity, based on the Riesz basis property of some systems of vector functions. [18]-[21] studied the inverse spectral problems of Sturm-Liouville operators with partial potential known on the graph with loops.

Inverse problems with discontinuities inside the interval are related to discontinuous material characters of a intermediary. On a single interval this kind of problem has been studied (see, for instance, [9], [22]-[26]). However, the discontinuous problem on graphs is not investigated.

The paper is organized as follows. Section 2, we state the boundary value problem for discontinuous Sturm-Liouville operator on quantum graphs and study the asymptotic properties of its eigenvalues. Section 3 is devoted to formulate the partial inverse problem and prove the uniqueness of the solution. Constructive algorithm for its solution is given in Section 4. Main results in this work are the uniqueness theorem: Theorem 3.2 and a constructive algorithm: Algorithm I.

2. ASYMPTOTIC FORMULAS FOR EIGENVALUES

Consider the quantum graph G , represented in Figure 1. The edge e_1 is a boundary edge of length $l_1 = 1$, the edge e_2 is a loop of length $l_2 = 2$. Introduce a parameter x_j for each edge $e_j, j = 1, 2, x_j \in [0, l_j]$. The value $x_1 = 0$ corresponds to the boundary vertex, and $x_1 = 1$ corresponds to the interval vertex. For the ring e_2 , both ends $x_2 = 0$ and $x_2 = 2$ correspond to the internal vertex.

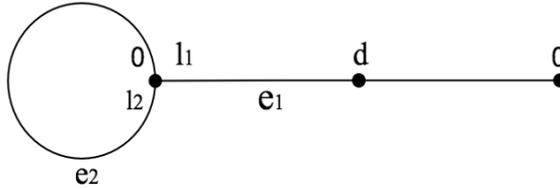


FIGURE 1. Quantum graph

Let $y = [y_j(x_j)]_{j=1,2}$ be a vector function on the quantum graph G . We consider the following boundary value problem L generated by the Sturm-Liouville equation on the edges of G :

$$ly := -y_j'' + q_j(x_j)y_j = \lambda y_j, \quad x \in (0, l_j), j = 1, 2, \quad (2.1)$$

with the standard matching conditions in the internal vertex:

$$y_1(1) = y_2(0) = y_2(2), \quad y_1'(1) + y_2'(2) = y_2'(0), \quad (2.2)$$

the Dirichlet boundary condition $y_1(0) = 0$ in the boundary vertex, and the discontinuous conditions at the point $d \in (0, m)$

$$\begin{cases} y_1(d+0) = \beta y_1(d-0), \\ y_1'(d+0) = \beta^{-1} y_1'(d-0) + \alpha y_1(d-0). \end{cases} \quad (2.3)$$

Here $q_j, j = 1, 2$ are real-valued function from $L_2(0, l_j)$, λ is a spectral parameter, $d \in \mathbb{Q}$, β, α are real, and $\beta > 0$.

For each fixed $j = 1, 2$, let $C_j(x_j, \lambda)$ and $S_j(x_j, \lambda)$ be the solutions of the corresponding equation (2.1) under the initial conditions

$$C_j(0, \lambda) = S_j'(0, \lambda) = 1, \quad C_j'(0, \lambda) = S_j(0, \lambda) = 0. \quad (2.4)$$

Further, we use the following notations. Let $B_{2,a}$ be the class of Paley-Wiener functions of exponential type not greater than a , belong to $L_2(\mathbb{R})$. The symbols $\kappa_{k,odd}(\rho)$ and $\kappa_{k,even}(\rho)$ denote various odd and even functions from $B_{2,k}$, respectively. Note that

$$\kappa_{k,odd}(\rho) = \int_0^k K(t) \sin \rho t dt, \quad \kappa_{k,even}(\rho) = \int_0^k N(t) \cos \rho t dt,$$

where $K, N \in L_2(0, k)$. The following notation κ_n stands for various sequences in l_2 .

Referring [27, 28], we obtain the following relations:

$$\left\{ \begin{array}{l} C_1(d, \rho) = \cos \rho d + \kappa_{d,even}(\rho), \\ C_1(l_1, \rho) = \beta^+ \cos \rho l_1 + \beta^- \cos \rho(2d - l_1) + \kappa_{l_1,even}(\rho), \\ S_1(d, \rho) = \frac{\sin \rho d}{\rho} + \frac{\kappa_{d,odd}(\rho)}{\rho}, \\ S_1(l_1, \rho) = \beta^+ \frac{\sin \rho l_1}{\rho} + \beta^- \frac{\sin \rho(2d - l_1)}{\rho} + \frac{\kappa_{l_1,odd}(\rho)}{\rho}, \\ S_1'(l_1, \rho) = \beta^+ \cos \rho l_1 - \beta^- \cos \rho(2d - l_1) + \kappa_{l_1,even}(\rho), \\ C_2(l_2, \rho) = \cos \rho l_2 + \kappa_{l_2,even}(\rho), \\ S_2(l_2, \rho) = \frac{\sin \rho l_2}{\rho} + \frac{\kappa_{l_2,odd}(\rho)}{\rho}, \\ S_2'(l_2, \rho) = \cos \rho l_2 + \kappa_{l_2,even}(\rho), \end{array} \right. \quad (2.5)$$

where $\beta^+ = \frac{\beta + \beta^{-1}}{2}$, $\beta^- = \frac{\beta - \beta^{-1}}{2}$.

The boundary value problem L is self-adjoint, and has a purely discrete spectrum, consisting of real eigenvalues. The eigenvalues of L coincide with the zeros of the characteristic function:

$$\Delta(\lambda) = S_1'(1, \lambda) S_2(2, \lambda) + S_1(1, \lambda) [S_2'(2, \lambda) + C_2(2, \lambda) - 2]. \quad (2.6)$$

The function $\Delta(\lambda)$ has an at most countable set of zeros with respect to their multiplicities. The asymptotic behavior of the eigenvalues is described by the following lemma.

Lemma 2.1. *The problem L has a countable set of eigenvalues, which can be numbered $\{\lambda_{nk}\}_{n \in \mathbb{Z}, k = \overline{1, 2p}} \cup \{\lambda_{n0}\}_{n \in \mathbb{N}}$ (counting with the multiplicities), satisfying*

$$\begin{cases} \rho_{nk} = \sqrt{\lambda_{nk}} = |2p\pi n + \alpha_k| + \kappa_n, & n \in \mathbb{Z}, k = \overline{1, 2p}, \\ \rho_{n0} = \sqrt{\lambda_{n0}} = n\pi + \kappa_n, & n \in \mathbb{N}. \end{cases} \quad (2.7)$$

where α_k are the simple zeros of equation (2.8) inside an interval $(0, \pi p]$, $|1 - 2d| = \frac{q}{p}$, $(p, q) = 1$, $p, q \in \mathbb{N}$. In particular, when $d = \frac{1}{2}$ one may take $q = 0$.

Proof. In the case $q_j = 0$, $j = 1, 2$, the characteristic function (2.6) takes the form

$$\begin{aligned} \Delta_0(\lambda) = & [\beta^+ \cos \rho - \beta^- \cos \rho(1 - 2d)] \frac{\sin 2\rho}{\rho} \\ & + \frac{1}{\rho} [\beta^+ \sin \rho - \beta^- \sin \rho(1 - 2d)] (2 \cos 2\rho - 2), \end{aligned}$$

where $\rho = \sqrt{\lambda}$, and $|1 - 2d| = \frac{q}{p}$, $(p, q) = 1$, $p, q \in \mathbb{N}$. Note that the function $D(\rho) := \rho \Delta_0(\rho^2)$ is odd and $2\pi p$ -periodic, so it is sufficient to investigate its zeros on $(0, \pi p]$. On the one hand, for $\rho \neq 0$, the equation $D(\rho) = 0$ is equivalent to the following one

$$\frac{1}{2} \cot \rho = \frac{\beta^+ \sin \rho - \beta^- \sin \rho(1 - 2d)}{\beta^+ \cos \rho - \beta^- \cos \rho(1 - 2d)}, \quad (2.8)$$

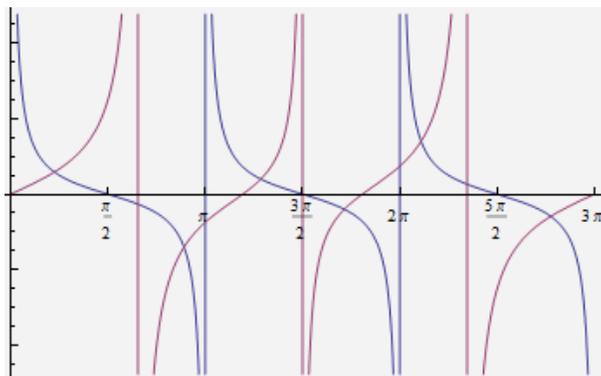
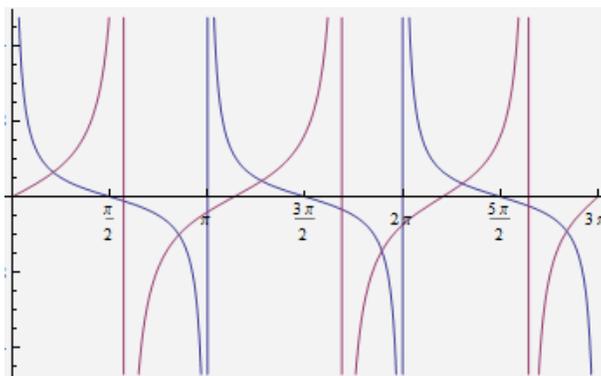
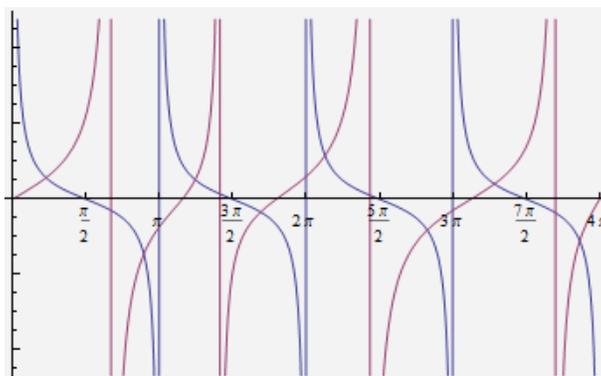
We can check this equation has exactly $2p$ simple roots α_k in a half periodic. For example, $\frac{q}{p} = \frac{1}{3}$ and $\frac{q}{p} = \frac{2}{3}$ there are both 6 points of intersection on $(0, 3\pi]$ (see Fig.2, and Fig.3), when $\frac{q}{p} = \frac{1}{4}$ there are 8 intersections on $(0, 4\pi]$ (see Fig.4). On the other hand, it's easy check $D(\pi) = 0$. Thus the function $\Delta_0(\lambda)$ has the zeros

$$\begin{aligned} \lambda_{nk}^0 &= (2p\pi n + \alpha_k)^2, & n \in \mathbb{Z}, k = \overline{1, 2p}, \\ \lambda_{n0}^0 &= (n\pi)^2, & n \in \mathbb{N}. \end{aligned}$$

Using (2.5), we obtain the relation

$$\Delta(\lambda) = \Delta_0(\lambda) + \frac{\kappa_{3, \text{odd}}(\rho)}{\rho}.$$

Applying the standard argument, based on Rouché's theorem (see, for example, [9], Theorem 1.1.3), we arrive at the asymptotic formulas (2.7) for the eigenvalues of the problem L . \square

FIGURE 2. Plots for equation (2.8), $\frac{q}{p} = \frac{1}{3}$ FIGURE 3. Plots for equation (2.8), $\frac{q}{p} = \frac{2}{3}$ FIGURE 4. Plots for equation (2.8), $\frac{q}{p} = \frac{1}{4}$

3. PARTIAL INVERSE PROBLEM: UNIQUENESS

In this section, we give the statement of the studied partial inverse problem, that is, the uniqueness theorem.

Consider the spectrum $\Lambda := \{\lambda_{nk}\}_{n \in \mathbb{Z}, k = \overline{1, 2p}}$. Here and below we assume that the eigenvalues are numbered with respect to their asymptotics according to Lemma 2.1. Note that this numbering is not unique, so a finite number of first eigenvalues in Λ can be chosen arbitrarily.

The following assumptions are imposed:

(A₁) All the values in Λ are distinct.

(A₂) All the values in Λ are positive.

(A₃) The function $S_2(2, \lambda)$ and $S_2'(2, \lambda) + C_2(2, \lambda) - 2$ don't have common zeros in Λ .

Assumption (A₁) is used for simplicity, the case of multiple eigenvalues requires some technical modification (see discussion in [15]). Assumption (A₂) can be achieved by a shift of the spectrum. From the asymptotics of the λ_{nk} in (2.7), one may see the assumption (A₃) holds for sufficiently n .

Under assumptions (A₁)-(A₃), we study the following partial inverse problem.

Inverse Problem I: Given the potential q_2 , α , β , and the spectrum Λ , find the function q_1 .

Using relations (2.5), we get

$$\begin{cases} S_1(1, \rho) = \beta^+ \frac{\sin \rho}{\rho} + \beta^- \frac{\sin \rho(2d-1)}{\rho} + \frac{1}{\rho} \int_0^1 K(t) \sin \rho t dt, \\ S_1'(1, \rho) = \beta^+ \cos \rho - \beta^- \cos \rho(2d-1) + \int_0^1 N(t) \cos \rho t dt, \end{cases} \quad (3.1)$$

in which K and N are some real-valued functions from $L_2(0, 1)$. Substituting (3.1) into (2.6), we derive the relation

$$\int_0^1 K(t) a_{nj} \sin \rho_{nj} t dt + \int_0^1 N(t) b_{nj} \cos \rho_{nj} t dt = f_{nj}, \quad (n, j) \in \mathcal{I}, \quad (3.2)$$

here $\mathcal{I} = \{(n, j) : \rho_{nj} \in \Lambda\}$,

$$\begin{aligned} a_{nj} &= S_2'(2, \lambda_{nj}) + C_2(2, \lambda_{nj}) - 2, & b_{nj} &= S_2(2, \lambda_{nj}) \rho_{nj}, \\ f_{nj} &= -a_{nj} [\beta^+ \sin \rho_{nj} + \beta^- \sin \rho_{nj} (2d-1)] \\ &\quad - b_{nj} [\beta^+ \cos \rho_{nj} - \beta^- \cos \rho_{nj} (2d-1)]. \end{aligned} \quad (3.3)$$

Introduce the real Hilbert space $\mathcal{H} := L_2(0, 1) \oplus L_2(0, 1)$ with the scalar product

$$(g, h) = \int_0^1 (g_1(t)h_1(t) + g_2(t)h_2(t))dt,$$

where $g, h \in \mathcal{H}$, $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, $h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$.

Obviously, the vector function

$$f(t) = \begin{bmatrix} K(t) \\ N(t) \end{bmatrix}, v_{nj}(t) = \begin{bmatrix} a_{nj} \sin \rho_{nj}t \\ b_{nj} \cos \rho_{nj}t \end{bmatrix}, (n, j) \in \mathcal{I}, \quad (3.4)$$

belong to \mathcal{H} , and relation (3.2) can be written in the form

$$(f, v_{nj})_{\mathcal{H}} = f_{nj}, \quad (n, j) \in \mathcal{I}. \quad (3.5)$$

Lemma 3.1. *The system of vector functions $V = \{v_{nj}\}_{(n,j) \in \mathcal{I}}$ is complete in \mathcal{H} .*

Proof. Suppose $w_1, w_2 \in L_2(0, 1)$ are such functions, that

$$\int_0^1 w_1(t)a_{nj} \sin \rho_{nj}tdt + \int_0^1 w_2(t)b_{nj} \cos \rho_{nj}tdt = 0, \quad (n, j) \in \mathcal{I}. \quad (3.6)$$

When $S_2(2, \lambda_{nj}) \neq 0$ for some $(n, j) \in \mathcal{I}$, in view of (2.6), we have $S_1(1, \lambda_{nj}) \neq 0$. So we get

$$a_{nj} = -\frac{S'_1(1, \lambda_{nj})b_{nj}}{\rho_{nj}S_1(1, \lambda_{nj})}.$$

Substituting this relation into (3.6), we obtain

$$\int_0^1 \left(w_1(t)S'_1(1, \lambda_{nj})\frac{\sin \rho_{nj}t}{\rho_{nj}} - w_2(t)S_1(1, \lambda_{nj}) \cos \rho_{nj}t \right) dt = 0. \quad (3.7)$$

In the other case, $S_2(2, \lambda_{nj}) = 0$. We see that (3.6) is equivalent to

$$\int_0^1 w_1(t)a_{nj} \sin \rho_{nj}tdt = 0, \quad (n, j) \in \mathcal{I}.$$

By the assumption (A_3) , we get $a_{nj} \neq 0$ and $S_1(1, \lambda_{nj}) = 0$, therefore $S'_1(1, \lambda_{nj}) \neq 0$, we also get the formula (3.7). So (3.7) holds for all $(n, j) \in \mathcal{I}$. Thus the entire function

$$W(\lambda) := \int_0^1 \left(w_1(t)S'_1(1, \lambda)\frac{\sin \rho t}{\rho} - w_2(t)S_1(1, \lambda) \cos \rho t \right) dt \quad (3.8)$$

has zeros Λ . Then together with (3.1), we obtain

$$W(\lambda) = O\left(\frac{e^{2|Im\rho|}}{|\rho|}\right), \quad |\rho| \rightarrow \infty. \quad (3.9)$$

Taking the assumption (A_2) into account, one constructs the infinite product

$$D(\lambda) = \prod_{\lambda_{nj} \in \Lambda} \left(1 - \frac{\lambda}{\lambda_{nj}}\right).$$

In view of the assumption (A_1) , the function $\frac{W(\lambda)}{D(\lambda)}$ is entire. According to the asymptotic formula (2.7), the function $D(\lambda)$ can be written in the following form (see [16])

$$D(\lambda) = C \prod_{k=1}^{2p} \left(\cos \frac{1}{p}\rho - \cos \frac{1}{p}\alpha_k\right) + \kappa_{2,even}(\rho), \quad (3.10)$$

where C is a nonzero constant. Moreover, one has the following estimate

$$|D(\rho^2)| \geq C e^{2|Im\rho|}, \quad \varepsilon < \arg \rho < \pi - \varepsilon, \quad |\rho| \geq \rho^*,$$

for some positive ε and ρ^* . Together with (3.9) it yields

$$\frac{W(\lambda)}{D(\lambda)} = o(1), \quad \lambda = \rho^2, \quad \varepsilon < \arg \rho < \pi - \varepsilon, \quad |\rho| \geq \rho^*. \quad (3.11)$$

By Phragmen-Lindelöf's and Liouville's theorems, (3.11) implies that $W(\lambda) \equiv 0$.

Let $\{\mu_n\}_{n \in \mathbb{N}}$ be the zeros of $S_1(1, \lambda)$. Note that $\{\mu_n\}_{n \in \mathbb{N}}$ are the eigenvalues of the boundary values problem

$$l_1 y_1 = \lambda y_1, \quad y_1(0) = 0 = y_1(1).$$

Note that $\{\mu_n\}_{n \in \mathbb{N}}$ is real and simple, $S_1'(1, \mu_n) \neq 0$, one substitutes $\lambda = \mu_n$ into (3.8), we get

$$\int_0^1 w_1(t) \frac{\sin \sqrt{\mu_n} t}{\sqrt{\mu_n}} dt = 0, \quad n \in \mathbb{N}.$$

It follows from [29] that the system $\left\{\frac{\sin \sqrt{\mu_n} t}{\sqrt{\mu_n}}\right\}_{n \in \mathbb{N}}$ is complete in $L_2(0, 1)$. Hence $w_1 = 0$. Then we conclude from (3.9) and $W(\lambda) \equiv 0$, that $w_2 = 0$. Thus, the system V is complete in \mathcal{H} . \square

Relying on Lemma 3.1, we shall prove the uniqueness theorem for the solution of inverse problem. Along with the boundary value problem L , consider the problem \tilde{L} of the same form, but with different potential $\tilde{q}_j \in L_2(0, l_j)$, $j = 1, 2$. We agree that if a certain symbol γ denotes an

object related to L , the corresponding symbol $\tilde{\gamma}$ denotes an analogous object related to \tilde{L} .

Theorem 3.2. *Assume that the boundary value problems L and \tilde{L} together with their subspectrum Λ and $\tilde{\Lambda}$ of the form described above satisfy assumptions (A_1) - (A_3) , and $q_2(x) = \tilde{q}_2(x)$ a.e. on $(0, 2)$, $\Lambda = \tilde{\Lambda}$, then $q_1(x) = \tilde{q}_1(x)$ a.e. on $(0, 1)$. Thus the Inverse Problem I has a unique solution.*

Proof. The relation $q_2(x) = \tilde{q}_2(x)$ a.e. on $(0, 2)$ implies $S_2(1, \lambda) \equiv \tilde{S}_2(2, \lambda)$, $S_2'(2, \lambda) \equiv \tilde{S}_2'(2, \lambda)$. In view of (3.3) and $\Lambda = \tilde{\Lambda}$, we have $v_{nj} = \tilde{v}_{nj}$ in \mathcal{H} and $f_{nj} = \tilde{f}_{nj}$ for $(n, j) \in \mathcal{I}$. Since by Lemma 3.1 the system V is complete in \mathcal{H} , we conclude from (3.5), that $K(t) = \tilde{K}(t)$ and $N(t) = \tilde{N}(t)$ on $(0, 1)$. Then relation (3.1) yield $S_1(1, \lambda) \equiv \tilde{S}_1(1, \lambda)$ and $S_1'(1, \lambda) \equiv \tilde{S}_1'(1, \lambda)$, therefore from [9] we get $q_1(x) = \tilde{q}_1(x)$ a.e. on $(0, 1)$. \square

4. PARTIAL INVERSE PROBLEM: RECONSTRUCTIVE ALGORITHM

In this section we shall provide reconstructive algorithm of the inverse problem in Theorem 3.2.

Lemma 4.1. *The system of vector functions $V = \{v_{nj}\}_{(n,j) \in \mathcal{I}}$ forms a Riesz basis in \mathcal{H} .*

Proof. Using (2.7) and (3.3), we get

$$a_{nj} = 2 \cos 2\rho_{nj} - 2 + \kappa_n, \quad b_{nj} = \sin 2\rho_{nj} + \kappa_n, \quad (n, j) \in \mathcal{I}.$$

Consequently, we have $\{\|v_{nj} - v_{nj}^0\|_{\mathcal{H}}\}_{(n,j) \in \mathcal{I}} \in l_2$, where

$$v_{nk}^0(t) = \begin{bmatrix} (2 \cos 2\alpha_k - 2) \sin |2p\pi n + \alpha_k|t \\ \sin 2\alpha_k \cos |2p\pi n + \alpha_k|t \end{bmatrix}, n \in \mathbb{Z}.$$

Note that $\cos 2\alpha_k \neq 1, \sin 2\alpha_k \neq 0$ from (2.8).

Next we prove the system $V^0 := \{v_{nj}^0\}_{(n,j) \in \mathcal{I}}$ is a Riesz basis in \mathcal{H} .

It follows from the results of [16], that the systems $\bigcup_{k=1}^p \{\sin(2p\pi n +$

$\alpha_k)t\}_{n \in \mathbb{Z}}$ and $\bigcup_{k=p+1}^{2p} \{\cos(2p\pi n + \alpha_k)t\}_{n \in \mathbb{Z}}$ are Riesz basis in $L_2(0, 1)$,

respectively. Consider the two linear operators $A, B : \mathcal{H} \rightarrow \mathcal{H}$, defined as follows:

$$Av = A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 - \frac{v_1}{2 \cos 2\alpha_k - 2} f(v_1) \end{bmatrix}, v \in \mathcal{H},$$

$$Bv = B \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 - \frac{2 \cos 2\alpha_k - 2}{\sin 2\alpha_k} g(v_2) \\ v_2 \end{bmatrix}, v \in \mathcal{H},$$

where

$$f(u_1)(t) = \sum_{j=1}^p \sum_{n \in \mathbb{Z}} c_{n,j}(u_1) \cos |2p\pi n + \alpha_j|t,$$

$$u_1(t) = \sum_{j=1}^p \sum_{n \in \mathbb{Z}} c_{n,j}(u_1) \sin |2p\pi n + \alpha_j|t,$$

$$g(u_2)(t) = \sum_{j=p+1}^{2p} \sum_{n \in \mathbb{Z}} c_{n,j}(u_2) \sin |2p\pi n + \alpha_j|t,$$

$$u_2(t) = \sum_{j=p+1}^{2p} \sum_{n \in \mathbb{Z}} c_{n,j}(u_2) \cos |2p\pi n + \alpha_j|t,$$

i.e. $c_{n,j}(u_1)$ and $c_{n,j}(u_2)$ are the coordinates of the functions $u_1, u_2 \in L_2(0, m)$ with respect to the Riesz basis $\bigcup_{k=1}^p \{\sin(2p\pi n + \alpha_k)t\}_{n \in \mathbb{Z}}$ and $\bigcup_{k=p+1}^{2p} \{\cos(2p\pi n + \alpha_k)t\}_{n \in \mathbb{Z}}$, respectively. It follows from the Riesz-basis property, that there exist positive constants C_1 and C_2 such that

$$C_1 \|u_1\|_{L_2} \leq \|f(u_1)\|_{L_2} \leq C_2 \|u_1\|_{L_2},$$

$$C_1 \|u_1\|_{L_2} \leq \|g(u_2)\|_{L_2} \leq C_2 \|u_1\|_{L_2}.$$

Consequently, the operators A, B and their inverse:

$$A^{-1}v = A^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 + \frac{\sin 2\alpha_k}{2 \cos 2\alpha_k - 2} f(v_1) \end{bmatrix}, v \in \mathcal{H},$$

$$B^{-1}v = B^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + \frac{2 \cos 2\alpha_k - 2}{\sin 2\alpha_k} g(v_2) \\ v_2 \end{bmatrix}, v \in \mathcal{H},$$

are bounded in \mathcal{H} . Note that the operators A and B transforms the sequence V^0 into Riesz basis in \mathcal{H} :

$$Av_{nk}^0(t) = (2 \cos 2\alpha_k - 2) \begin{bmatrix} \sin |2p\pi n + \alpha_k|t \\ 0 \end{bmatrix}, n \in \mathbb{Z}, k = \overline{1, p},$$

$$Bv_{nk}^0(t) = \sin 2\alpha_k \begin{bmatrix} 0 \\ \cos |2p\pi n + \alpha_k|t \end{bmatrix}, n \in \mathbb{Z}, k = \overline{p+1, 2p}.$$

We get the system $\bigcup_{k=1}^p \{Av_{nk}^0 \cup Bv_{n(p+k)}^0\}$ is a Riesz basis in \mathcal{H} , hence the system V^0 is also a Riesz basis in \mathcal{H} . Since the system V is complete in \mathcal{H} by Lemma 3.1 and l_2 -close to the Riesz basis V^0 , so we conclude that V is a Riesz basis in \mathcal{H} . \square

One can uniquely recover the vector function f by its coordinates with respect to the Riesz basis, thus we can obtain the following algorithm for the solution of the inverse problem.

Algorithm I. Let the function q_2, α, β , the eigenvalue set Λ be given.

Step 1. Solving the initial value problems

$$\begin{aligned} l_2 S_2(x_2, \lambda_{nj}) &= \lambda_{nj} S_2(x_2, \lambda_{nj}), & S_2(0, \lambda_{nj}) &= 0, & S_2'(0, \lambda_{nj}) &= 1, \\ l_2 C_2(x_2, \lambda_{nj}) &= \lambda_{nj} C_2(x_2, \lambda_{nj}), & C_2(0, \lambda_{nj}) &= 1, & C_2'(0, \lambda_{nj}) &= 0, \end{aligned}$$

one constructs the functions $S_2(1, \lambda_{nj}), C_2(1, \lambda_{nj})$ and $S_2'(1, \lambda_{nj}), C_2'(1, \lambda_{nj})$ for $(n, j) \in \mathcal{I}$.

Step 2. Find the vector function v_{nj} and the numbers f_{nj} , using (3.3) and (3.4).

Step 3. Construct the vector functions f by its coordinates with respect to the Riesz basis (see (3.5)), i.e. find the functions $K(t)$ and $N(t)$.

Step 4. Find $S_1(1, \rho)$ and $S_1'(1, \rho)$ by (3.1).

Step 5. Recovering the function q_1 from Weyl function theory in [9].

Remark 4.1. In Eq. (2.8), when $l_2 = m \in \mathbb{N}$, and $m \geq 3$, then Theorem 3.2 and Algorithm I are still valid with the same consideration.

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