Exploring Choas of a Duffy Oscillator with Linear Drag

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Abstract

A mathematical model of duffing equation was modified to add a drag term. Bifurcation diagrams with fixed parameters show characteristic period-doubling cascading into chaos. Time evolution plots, state-space plots and Poincaré of chaotic behavior indicate that the system while periodic under small driving strengths transitions into chaos as driving strength increases.

Introduction

The Duffing equation is a non-linear second order differential equation, it was invented by George Duffing who lived from 1861-1944. it is more complex potential than a simple harmonic oscillator's and can exhibit chaos under certain parameters [1]. Our model in this paper is different because we our modeling a physical mechanical system. The model was inspired by a toy model of the Duffing oscillator in room 110, Physical Sciences building, Chico State University. In our mathematical model a mass in a gravitational potential has a velocity dependent drag. Different initial conditions and parameter spaces are explored to investigate the model for chaos.

The curve the ball will follow through space is such given by: $y = ax^4 - bx^2$

Taking the Lagrangian approach our T and V are:

$T = \frac{1}{2}m(x'(t)^2 + (4Ax(t)^3x'(t) - 2Bx(t)x'(t))^2)$

$V = mg(Ax(t)^4 - Bx(t)^2)$

The Euler Lagrange equation they satisfy is:

$$\frac{\partial L}{\partial x_1} = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x_1}}$$

Inputting T and V into Mathematica's Lagrangian Notebook to turn the machinery, yields a second order non-linear differential equation of motion:

$$\left\{x''(t) = -\frac{2\left(24A^2x(t)^5x'(t)^2 - 16ABx(t)^3x'(t)^2 + 2Agx(t)^3 + 2B^2x(t)x'(t)^2 - Bgx(t)\right)}{16A^2x(t)^6 - 16ABx(t)^4 + 4B^2x(t)^2 + 1}\right\}$$

we add a velocity dependent drag and sinusoidal driving term on the right.

$$\gamma * Sin[\omega_{drive} * time]$$

$$-\beta * velocity$$

We will fix the following parameters and initial conditions as such:
A = 1
B = 1
ω_{drive} = 2π
β = 1
g = 9.8
γ = varied

initial position = 0
initial velocity = 1

The values a, b and omega drive was chosen as 2π for ease of calculation frequency. The damping strength β, initial position and initial velocity were chosen arbitrarily. The gravity near earth surface is g. The driving strength parameter γ is varied to explore if the system is chaotic. The mass of the particle is implicitly set as 1.

Bifurcation

The bifurcation plot in Fig. 1 provides a high level over view of the system's behavior. The plot includes a γ range from 0 to 100 in increments of 1 with all the other parameters are fixed. Regions in the diagram where overlapping dots become lines are chaotic. If the dots appear separate that is not a chaotic region. The plot shows the system is periodic at low γ values, chaotic around 10 to 30 then is intermittently chaotic at higher values.

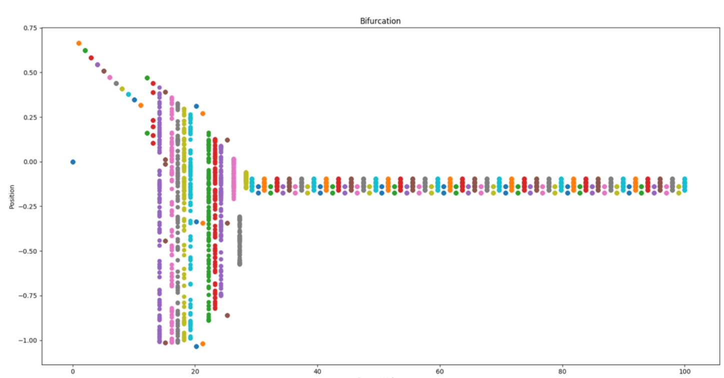

Figure 1: Low gamma values show no chaotic behavior but as gamma increases the system transitions into chaos and then in and out of choas.

To zoom in on the systems features on the road to chaos another bifurcation diagram was done with γ ranging from 0 to 15 in 100 increments. A period doubling cascade was observed in the gamma value range 11-14. This is a typical feature of choatic systems.

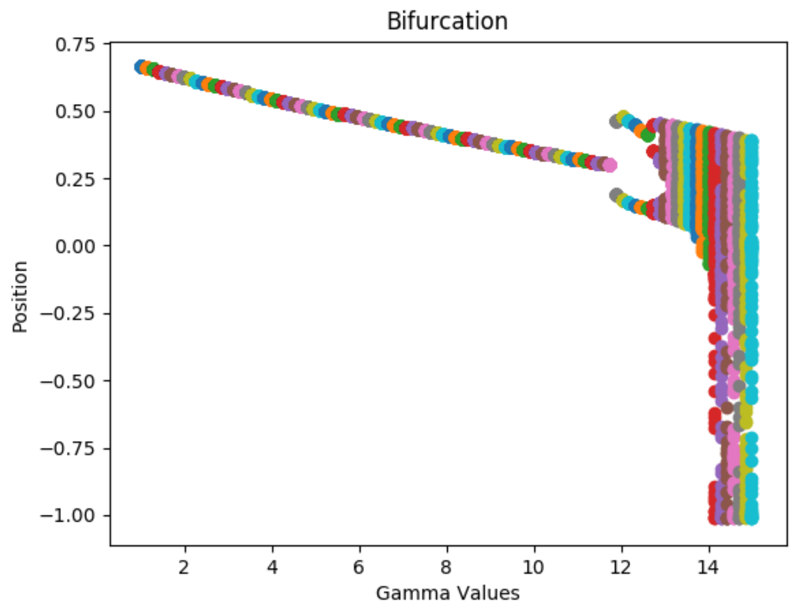

Figure 2: A period-doubling cascade on the ‘road to chaos’ is observed.

A bifurcation diagram with γ values ranging from 11 to 15 shows greater resolution on how the system transitions into chaos. Because of the richness of features such as single period motion, repeated period cascading, and clear evidence of non-periodic chaotic motion this range was choosen as the region of interest to explore.

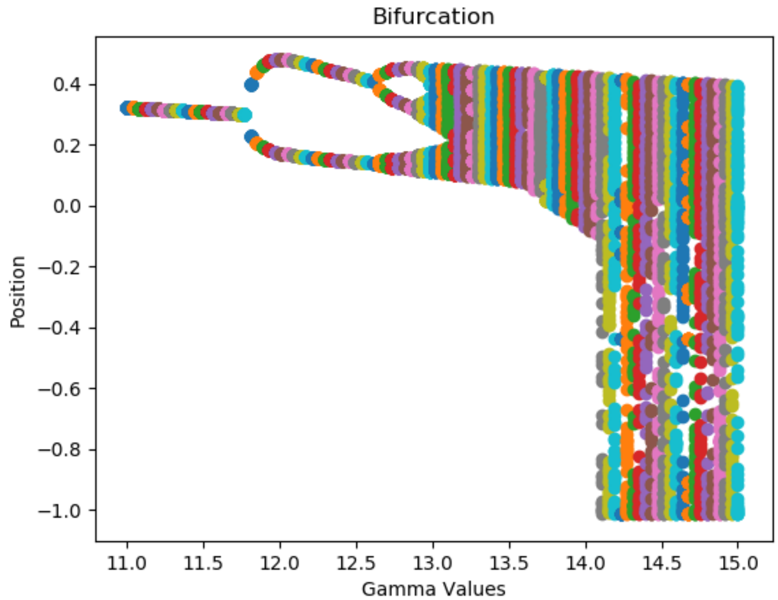

Figure 3: Region of interest

Period Doubling

The driving frequency listed is such that it is the threshold frequency that leads to bifurcation within $\frac{1}{100}th$ precision. The driving values, γ, that exhibit period doubling of 2, 4, and 8 times the driving period are shown below in Fig. 4 through Fig. 6. These series of period doubling cascades leads the system into chaos. It also highlights the naturally cyclic nature of the system at these relatively low driving strengths.

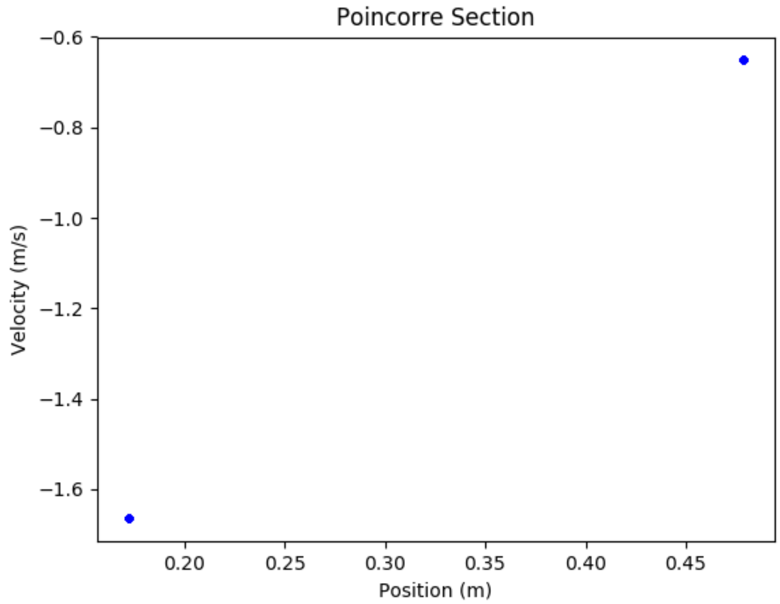

Figure 4: Period 2 driving strength threshold = 12.00

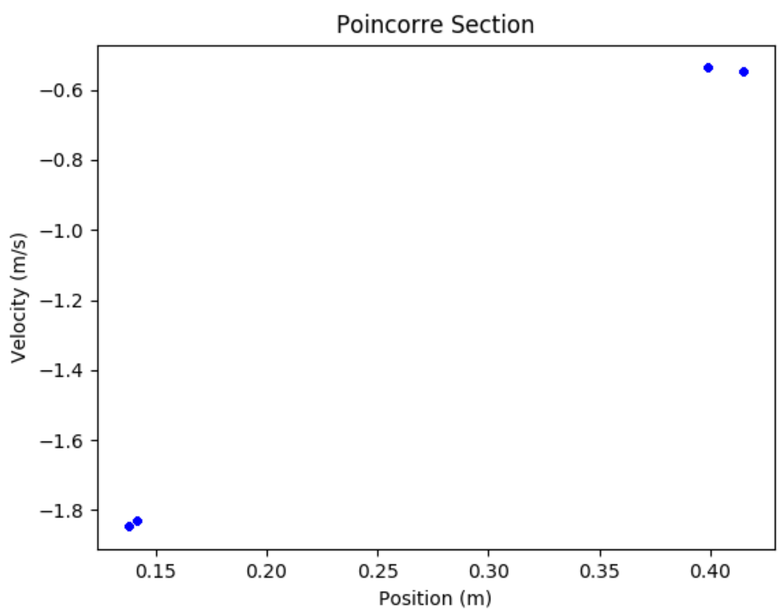

Figure 5: Period 4 driving strength threshold = 12.63

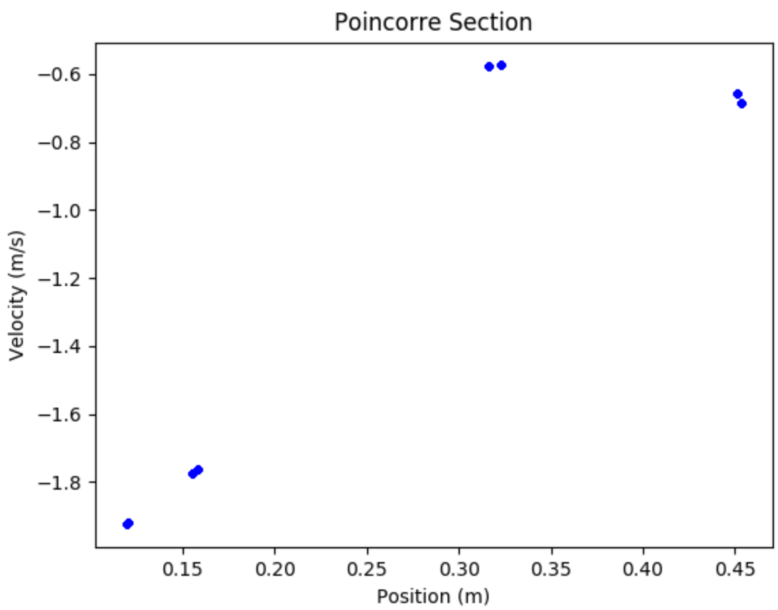

Figure 6: Period 8 driving strength threshold = 12.87

Feigenbaum number

The driving strength parameter ratios do not show the typical Feigenbaum number occurring. For n between 1 and 3, a constant of 2.65 was calculated.

n	period		interval
1	1 to 2	12	
			0.63
2	2 to 4	12.63	
			0.25
3	4 to 8	12.87	

Table 1: Threshold values of driving strength show no evidence of the Feigenbaum number.

Poincorre Sections

Poincorre sections in the region of interest all display non-periodic trajectories characteristic of chaos. The phase plots are no longer a few dots but resemble squid-like shapes, the guardians of chaotic state space between γ values of 11 to 15. The system was is no longer repeating the cyclic motion present during the period doubling region but has transitioned into chaos.

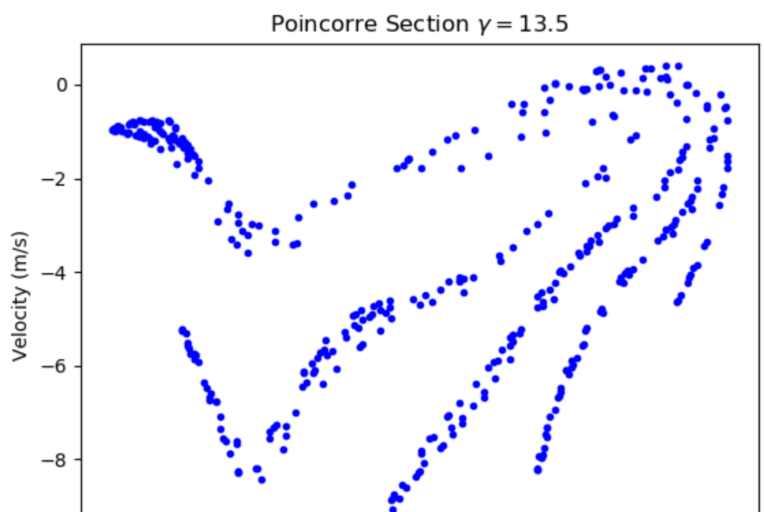

Figure 7: Driving Strength = 13.5

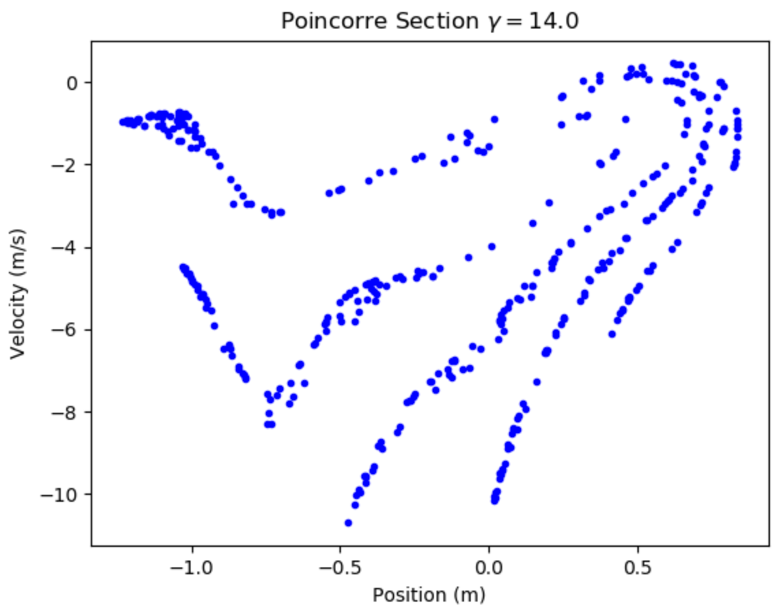

Figure 8: Driving Strength = 14.0

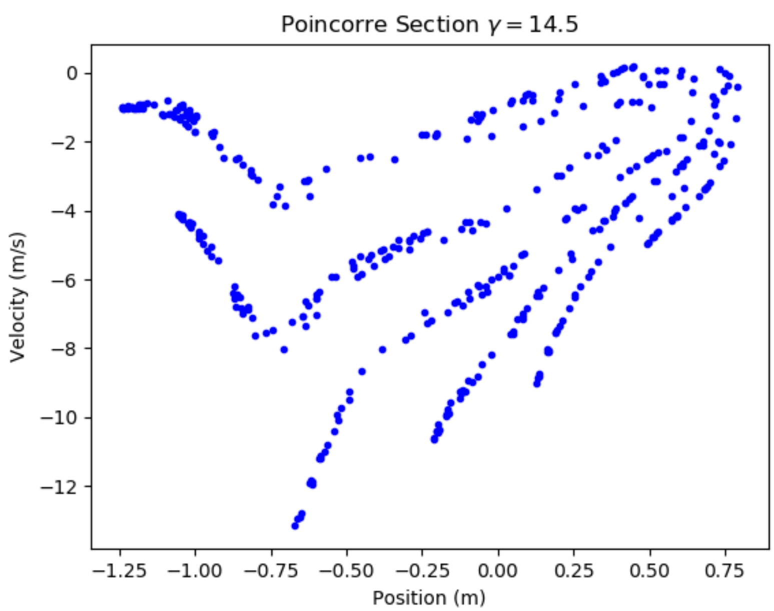

Figure 9: Driving Strength = 14.5

Phase Space Plots

A phase space orbit for γ=13.5 is show below for two different time intervals. The difference in infilling between the phase space plots occurred between $1000 < t < 2000$. After 1000 drive cycles the system is sill tracing new orbits. The infilling is due to the system tracing out new non-repeating trajectories. This is clear evidence of chaotic motion.

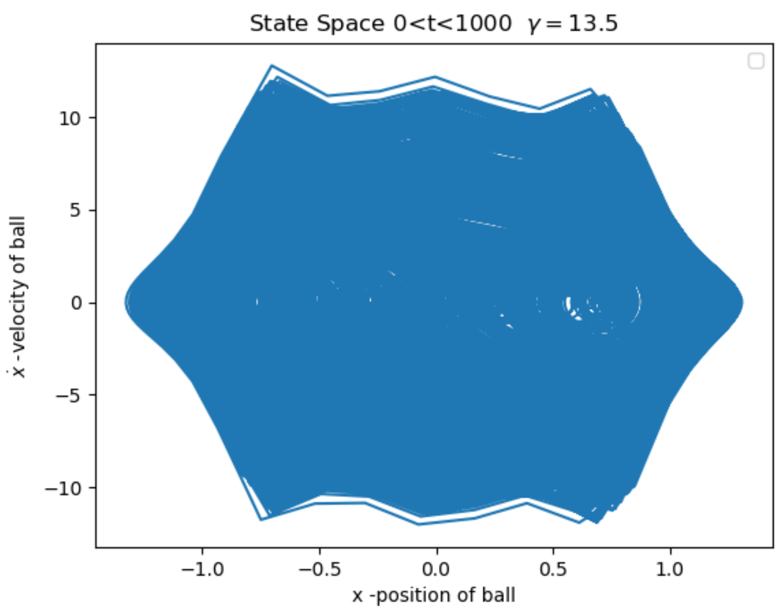

Figure 10: White space is present inside the blue.

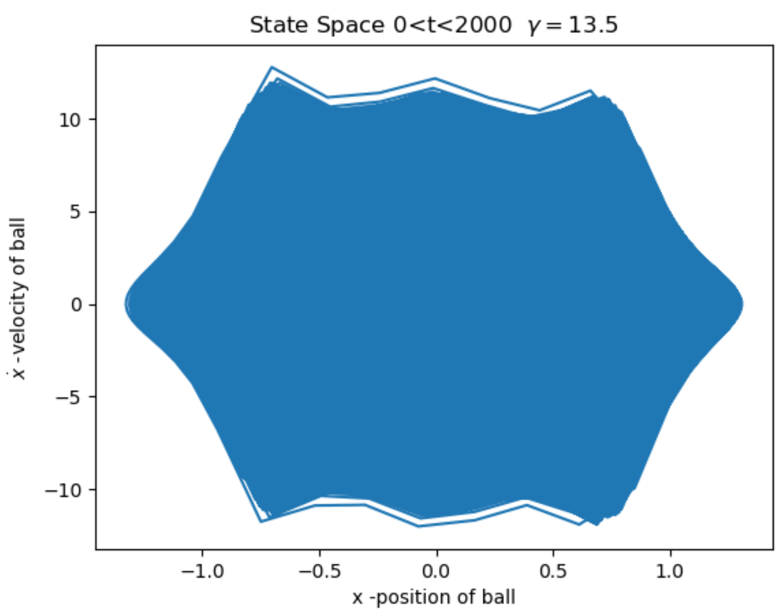

Figure 11: The white space is filled showing the system is still tracing out new trajectories .

Time Evolution

A time evolution plot for γ=13.5 with $1940 < t < 2000$ shows a non-periodic waveform. It's non-repeating trajectory through space and time is characteristic of chaos. If the system was not-chaotic we would expect to see some repeating pattern but this is not the case.

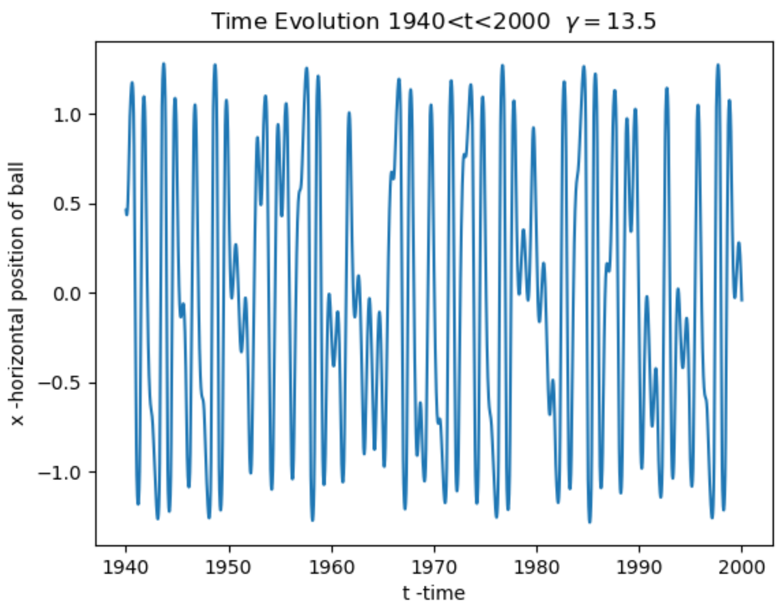

Figure 12: Non-periodic motion through space and time

Conclusion

As gamma increases from null the system exhibits a period doubling cascade that leads into chaotic motion. The Feigenbaum number was not present in the period doubling cascade. The Poincorre sections of γ = 13.0, 13.5 and 14.0 display complex curve's characteristic of chaos. State space plots of the system at γ = 13.5 shows that infilling occurred after 1000 drive cycles. Since the trajectory in phase space is not repeating it gives evidence of chaotic behavior. The time evolution of system with $1940 < t < 2000$ and γ = 13.5 shows a non-periodic waveform giving further evidence of the systems non-repeating chaotic nature. All in all the non-repeating trajectories in the Poincorre sections, phase space and the time evolution plots characterize the duffing oscillator with damping as a chaotic system.

References

[1] Wikipedia. Duffing Equation. *https://en.wikipedia.org/wiki/Duffing*$_{e}quationHistorical.$

[2] Peter Goldreich and Pawan Kumar. Wave generation by turbulent convection. *The Astrophysical Journal*, 363:694, nov 1990.

[3] Pawan Kumar, Peter Goldreich, and Richard Kerswell. Effect of nonlinear interactions on p-mode frequencies and line widths. *The Astrophysical Journal*, 427:483, may 1994.