Holonomic quantum control via nonlinear realizations

David Bar Moshe

September 20, 2021

Abstract

Geometric phases appear as holonomies in principal bundles over quantum state spaces. In this work, we consider the case when the principal bundle itself is a Lie group and the quantum space of states a homogeneous space of that group. This structure allows the application of the theory of nonlinear realizations of symmetry for the construction non-Abelian geometric phases corresponding to this bundle structure. When the quantum state space is the complex Grassmann manifold U(N)/(U(N-k) × U(k)), we identify the total non-Abelian Aharonov-Anandan phase as the U(k)-valued cocycle of the U(N) action on the Grassmann manifold . We describe generalizations of this result in two cases: 1) the case of isospectral dynamics of mixed states, 2) the case of non-self adjoint dynamics over the Grassmannian.

Introduction

Holonomic quantum control on Lie group total space principal bundles

In this work, we describe a geometric setup for holonomic quantum control on principal bundles. We treat a particular case in which our control system is a principal bundle whose total space is a Lie group G, and the bundle pojection is the canonical projection onto an appropriately chosen coset space G/H with respect to a closed shubgroup H. The group G and the subgroup H will be chosen such that base space of the bundle becomes a space of quantum states. The control system's role, in this setup, is to generate, by holonomy, a group action on a physical system having a similar structure of a Lie group total space principal bundle.

The Lie group bundle structure allows a special connection: the H*-connection*, whose holonomy along a closed path on the space of quantum states is the non-Abelian geometric phase which will be used to drive the physical system. This setup in which the bundle is a Lie group, allows the use of the theory of nonlinear realizations for the construction of the H-connection and the holonomy.

We put the holonomy into action, through a bundle map to a physical system, having also thethe same structure of a Lie group total space principal bundle space but with different isotropy group $K \subset H$:

$$
\begin{array}{ccccc}
H & & K & = & K \\
i_{HG}\downarrow & & i_{KG}\downarrow & & i_{KH}\downarrow \\
G & \cong & G & & H \\
\pi_{HG}\downarrow & & \pi_{KG}\downarrow & & \pi_{KH}\downarrow \\
G/H & \longleftarrow & G/K & \longleftarrow & H/K
\end{array}
$$

$$\text{Control System} \qquad\qquad \text{Physical System}$$

The control system simulates a closed path on the base space G/H which is a projection of a generally open path on the total space of the principal bundle: $H \overset{i_{HG}}{\to} G \overset{\pi_{HG}}{\to} G/H$ due to the existence of a nontrivial holonomy. This holonomy drives our physical system through a bundle map to a second version of the same Lie group G, this time belonging to our physical system. The initial state of the physical system is chosen so that the isotropy group of the action is now K a subgroup of H. in this geometric setup, the physical state space G/K becomes a fibre bundle over the control system base space G/H with a fibre isomorphic to H/K, when the control system completes one revolution, the physical system will return to the starting point on the base space of this bundle, but in general will end up in a different point on the fibre H/K than the starting point, thus the physical state space is actually encoded in the fibre of this bundle.

This formulation consists of a slightly different approach than [1]. The bundle map from the control system to the physical system is similar

Nonlinear realizations

When a Lie group acts on a manifold without a global linear structure, its action defines a nonlinear representation or a nonlinear realization. Nonlinear realizations can be explicitly constructed when the manifold is a homogeneous space of the Lie group. The name: *Nonlinear realization* was coined (at least in the physics literature) by Wess and Zumino [2] and Weinberg [3] for the case of chiral symmetry groups. It was generalized to arbitrary compact Lie groups in the classic papers of Callan Coleman Wess and Zumino (CCWZ) [4] and[5] for the study of the dynamics of the Nambu-Gooldstone bosons in systems with spontaneous symmetry breaking. The configuration space of the Nambu-Goldstone bosons at very low energies becomes a homogeneous space of the symmetry group G of the theory, with the symmetry group H of the vacuum acting as the isotropy group.

The method of nonlinear realizations was extended to Kählerian homogeneous spaces, which appear as the low energy configuration spaces of spontaneously broken supersymmetric theories, by Bando, Kuratomo, Maskawa and Uhera (BKMU) [6] and [7]. Compact Kählerian homogeneous spaces spaces allow complex non-compact transitive group actions, which can be used to model non-self adjoint dynamics. Non-compact Kählerian homogeneous spaces allow in addition real on-compact transitive group actions. The geometric phases of both cases will be described.

In this work, we use the theory of nonlinear realizations for the construction of non-Abelian geometric phases. We construct the Aharonov-Anandan phase using the CCWZ theory of nonlinear realizations on the Grassmann manifold and identify the total non-Abelian phase (dynamical and geometric) as the U(k)-valued cocycle of the U(N) action on the Grassmann manifold. Next, since, in the BKMU theory, the homogeneous spaces allow transitive actions of non-compact lie groups, we contruct generalizations corresponding to non self-adjoint dynamics, where the geometric phase becomes the group cocycle of the corresponding non-compact group. We make another generalization by replacing the Grassmann manifold by a generalized flag manifold to describe the non-Abelian geometric phase of isospectral dynamics of density matrices. We show how the latter result is related the Uhlmann's phase [8].

1 Control systems on principal bundles

We consider a quantum system living in a Hilbert space $\mathcal{H} \cong \mathbb{C}^k$; we suppose that the system's Hilbert space $\mathcal{H}$ is embedded into a fixed larger Hilbert space $\mathcal{W} \cong \mathbb{C}^N$. In addition, we suppose that we have in our disposal a set of Hamitonians generating unitary transformations in the larger Hilbert space, which do not, in general, leave our system's subspace $\mathcal{H}$ invariant. If we want to use this setting to generate unitary transformations within our initial Hilbert space $\mathcal{H}$, we need to carefully design the evolution such that:

1. The Hilbert space $\mathcal{H}$ returns to its original embedding at the end of an evolution cycle.
2. At the end of the evolution cycle the Hilbert space $\mathcal{H}$ becomes unitarily rotated by a group element $h \in U(\mathcal{H}) \cong U(k)$.

let $P_{\mathcal{H}} \in \mathcal{L}(\mathcal{W})$ be the orthogonal projector onto $\mathcal{H}$. The $U(N)$ orbit of $P_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{W})$ is a Grassmannian $\mathrm{Gr}(k, N)$, which is the base space of a principal $(U(N-k) \times U(k))$ bundle which is in addition isomorphic to the group $U(N)$.

$$\begin{array}{c} U(N-k) \times U(k) \\ i \downarrow \\ U(N) \\ \pi \downarrow \\ \mathrm{Gr}(k, N) \end{array}$$

A cyclic evolution generated by a one parameter group element $g(t) \in \mathrm{Map}([0,1], U(N))$, such that $g(0) = e$ and $g(1) = h \in (U(N-k) \times U(k))$. This transformation is obtained as the holonomy of the path is the geometric phase of our concern.

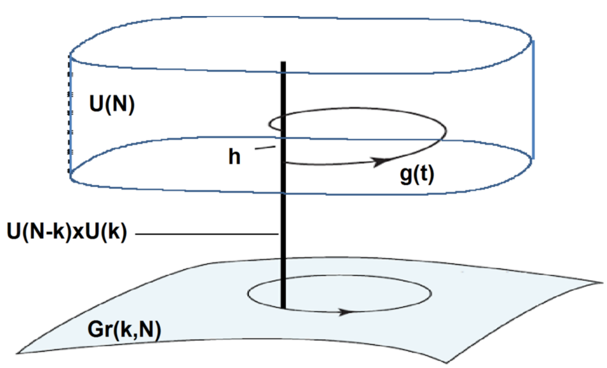

Figure 1: **Geometric phase as a path holonomy**

The base space of the principal bundle can be viewed as a space of quantum states consisting of the unitary orbit of a density matrix of the form:

$$\rho_0 = \mathrm{diag}(\underbrace{\frac{1}{k}, ...\frac{1}{k}}_{\text{k-times}}, \underbrace{0, ...0}_{\text{(N-k)-times}}) \tag{1}$$

Since, by design, the projected path onto the Grassmannian is closed, the above state is left invariant after the evolution cycle. The holonomy belongs to the isotropy group of ρ_0, thus it is left invarinat. However, we can , because the holonomy lies in its isotropy group. Howeverr, this holonomy acts nontrivially on state spaces through bundle maps as follows:

Let

other However, any other state whose density matrix does not commute with ρ_0 would change under the above evolution. We can allow the holonomy to act on a larger class of states by embedding their state space into the Grassmannian. This can be done as follows:

The space of states of a level quantum system living in a Hilbert space $\mathcal{H} \cong \mathcal{C}^N$ is the set of N dimensional density matrices:

$$\mathcal{D} = \{\rho \in \mathcal{L}(\mathbb{C}^N) | \rho \geq 0, \mathrm{Tr}(\rho) = 1\} \tag{2}$$

An evolution of the quantum system can be well described by an automorphism of this space. However, this However, there are cases, where different descriptions of the state space or some of its subsets are favorable. For example, suppose that the Hilbert space of the system $\mathcal{H}$ is a subspace of a fixed larger Hilbert space $W \cong \mathcal{C}^M$. Suppose that we can operate only within the larger Hilbert space, for example when the larger Hilbert space is the computational basis of a quantum computing system. Quantum evolution phases appear in systems with parametrized Hamiltonians, where the system is associated with a family of Hamiltonians,

1.1 Nonlinear realizations (CCWZ)

Let G be a compact Lie group, H a closed subgroup, and $G/H = gH, g \in G$ be the corresponding left coset space. According to the bundle structure theorem [9], G is a principal H bundle over G/H

$$H \xrightarrow{i} G \xrightarrow{\pi} G/H \tag{3}$$

We parametrize the coset space G/H by a section $u : G/H \to G$ according the following commutative diagram:

$$\begin{array}{ccc} G & & \\ \pi \downarrow & u \nwarrow & \\ G/H & \underset{id}{\leftarrow} & G/H \end{array}$$

Thus, for $x \in G/H : \pi(u(x)) = x$.

Choosing G/H to be a left coset space, the left G action: $G \times G/H \to G/H$ takes the form:

$$g.x = \pi(gu(x)) \tag{4}$$

Thus there must exist a function $h : G/H \times G \to H$, such that:

$$gu(x) = u(g.x)h(x, g) \tag{5}$$

By performing two consecutive group actions, we deduce that $h(x, g)$ satisfies the cocycle condition:

$$h(x, g_2 g_1) = h(g_1.x, g_2)h(x, g_1) \tag{6}$$

The infinitesimal version of equation 5 can be obtained by restricting the action to a one parameter subgroup $g(t) = e^{sT}$:

$$\frac{d}{ds} g(t)u(x)|_{s=0} = \frac{d}{ds} \left(u(g(s).x)h(x, g(s)) \right)|_{s=0} \tag{7}$$

Thus, we have

$$Tu(x) = \nabla_T u(x) + u(x)\Omega_T(x) \tag{8}$$

Where $\nabla_T \in \mathrm{Vect}(G/H)$ is the vector field on G/H generating the left group action along T, and $\Omega_T(x) \in H$ is the Lie algebra cocycle corresponding to the group cocycle $h(x, g)$. The vector fields ∇_T satisfies the Lie algebra $\mathfrak{g}$ commutation relations:

$$[\nabla_{T_1}, \nabla_{T_2}] = \nabla_{[T_1, T_2]} \tag{9}$$

while the Lie algebra cocycles satisfy:

$$\nabla_{T_1}\Omega_{T_2}(x) - \nabla_{T_2}\Omega_{T_1}(x) + [\Omega_{T_1}, \Omega_{T_2}] = \Omega_{[T_1, T_2]} \tag{10}$$

We can already see that the Lie algebra cocycles constitute of two parts:

$$\Omega_T(x) = u(x)^{-1}Tu(x) - u(x)^{-1}\nabla_T u(x) \tag{11}$$

The first part is the generator of the dynamical phase, while the second generates the geometric phase. Since the structure group of the principal bundle is H, in the overlap between two charts, the parametrizing section $u(x)$, transforms according to:

$$u'(x) = u(x)h(x) \tag{12}$$

where $h(x)$ a smooth H-valued function in the overlap. The generator of the geometric phase transforms by a gauge transformation:

$$u'(x)^{-1}\nabla_T u'(x) = h(x)^{-1}(u(x)^{-1}\nabla_T u(x))h(x) + h(x)^{-1}\nabla_T h(x) \tag{13}$$

Our purpose is to integrate the Lie algebra cocycles $\Omega_T(x)$ of equation 8, to obtain the group cocycles $h(x, g)$. For this purpose, we choose in 6: $g_2 = e^{sT}$, and $g_1 = g$, and take the derivative with respect to s at $s = 0$.

We obtain:

$$\frac{d}{ds}h(x, e^{sT}g)|_{s=0} = \frac{d}{ds}h(g.x, e^{sT})h(x, g)|_{s=0} \tag{14}$$

Thus, we get

$$L_T h(x, g) = \Omega_T(g.x)h(x, g) \tag{15}$$

Where, $L_T \in \mathrm{Vect}(G)$ is the generator of the left G action on the group manifold.

If we take now an element g given by an anti-time ordered product:

$$g(t) = \mathcal{AT}e^{\int_0^t T(s)ds} \tag{16}$$

We can integrate equation 15 to obtain:

$$h(x, g(t)) = \mathcal{AT}e^{\int_0^t \Omega_{T(s)}(g((s).x)ds} = \mathcal{AT}e^{\int_0^t \Omega_{T(s)}(\mathcal{AT}e^{\int_0^s T(w)dw}.x)ds} \tag{17}$$

Equation 17 is the main result of this work. It constitutes a combination of the non-Abelian geometric and the dynamical phases. The right hand side will be shown to coincide with the Aharonov-Anandan phase when we choose the coset space G/H to be a Grassmann manifold. In addition, it will be used in the non-self adjoint generalizations.

Proof of equation 17:

The anti-time ordered product is given by the following limit:

$$g(t) = \mathcal{AT}e^{\int_0^t T(s)ds} = \lim_{\substack{\epsilon \to 0 \\ N \to \infty \\ \epsilon N = t}} \prod_{j=1}^{N} g_{N-j+1} \tag{18}$$

Where:

$$g_j = 1 + \epsilon T(j\epsilon)$$

Substituting 18 into the left hand side of 17, and consecutively using the cocycle condition, we obtain:

$$h(x, g(t)) = \lim_{\substack{\epsilon \to 0 \\ N \to \infty \\ \epsilon N = t}} \prod_{j=1}^{N} h(x_{N-j}, g_{N-j+1}) \tag{19}$$

Where,

$$x_n = g_n g_{n-1} \ldots g_1 . x$$

Since

$$h(x_{N-j}, g_{N-j+1}) = h(x_{N-j}, 1 + \epsilon T((N-j+1)\epsilon)) = 1 + \epsilon \Omega_{T((N-j+1)\epsilon)}(x_{N-j})$$

We obtain:

$$h(x, g(t)) = \lim_{\substack{\epsilon \to 0 \\ N \to \infty \\ \epsilon N = t}} \prod_{j=1}^{N} (1 + \epsilon \Omega_{T((N-j+1)\epsilon)}(x_{N-j})) \tag{20}$$

The right hand side of equation 17 can also be written as a limit:

$$\mathcal{A}\mathcal{T} e^{\int_0^t \Omega_{T(s)}(g((s).x) ds} = \lim_{\substack{\epsilon \to 0 \\ N \to \infty \\ \epsilon N = t}} \prod_{j=1}^{N} (1 + \epsilon \Omega_{T((N-j)\epsilon)}(x_{N-j})) \tag{21}$$

The right hand sides of equations 20 and 21 coincide in the limit, which completes the proof.

Since G and H are compact, G/H is reductive, i.e., the Lie algebra $\mathfrak{g} = T_e G$ can be decomposed as:

$$\mathfrak{g} = \mathfrak{H} \oplus \mathfrak{M} \tag{22}$$

where $\mathfrak{H}$, is the Lie algebra of H and

$$Ad(H)\mathfrak{M} \subset \mathfrak{M} \tag{23}$$

According to the slice lemma [10], there is a neighborhood $U \subset G/H$ of $0 = \pi(e)$, in which the section $u(x)$, can be chosen as:

$$u(x) = e^X = e^{\sum_{\alpha=0}^{\dim \mathcal{M}} x^\alpha T_\alpha} \tag{24}$$

such that every element in $\pi^{-1}(U)$ can be uniquely written as:

$$g = e^X h, \quad X \in \mathcal{M}, \quad h \in H \tag{25}$$

The left group action of $h \in H$ in the neighborhood U takes the form:

$$he^X = he^X h^{-1} h = e^{\mathrm{ad}(h)X} h$$

Thus the group action and the cocycle have the following forms in the particular case $h \in H$:

$$u(h.x) = hu(x)h^{-1} \tag{26}$$
$$h(x, h) = h \tag{27}$$

The second equality extends to the whole of G/H by continuity.

$$\rho(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho((1\,2)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\rho((2\,3)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \rho((1\,3)) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$\rho((1\,2\,3)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \rho((1\,3\,2)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Nunc a aliquet sem, eget aliquet purus. Vestibulum ac placerat mauris. Proin sed dolor ac justo semper iaculis. Donec varius, nibh sit amet finibus tristique, sapien ante interdum odio, et pretium sapien libero nec massa. In hac habitasse platea dictumst. Donec vel augue ac sapien imperdiet pretium. Maecenas gravida risus id ultricies dignissim. Maecenas Eq. 28 gravida felis quis dolor faucibus, sed maximus lorem tristique

$$\int_a^b u \frac{d^2 v}{dx^2}\, dx = u \frac{dv}{dx} \bigg|_a^b - \int_a^b \frac{du}{dx} \frac{dv}{dx}\, dx. \tag{28}$$

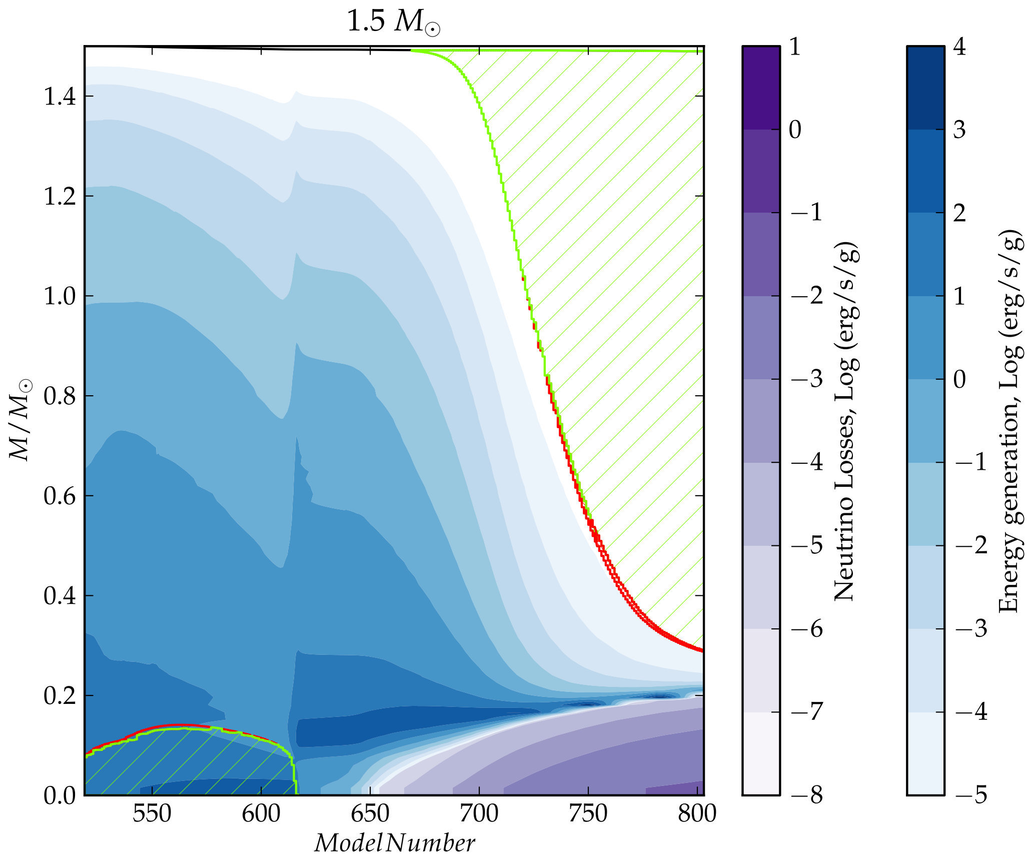

Figure 2: Lorem ipsum dolor sit amet, consectetur adipiscing elit. Cras egestas auctor molestie. In hac habitasse platea dictumst. $\tilde{f}(\omega) = \frac{1}{2\pi}$ Lorem ipsum dolor sit amet, consectetur adipiscing elit. Cras egestas auctor molestie. In hac habitasse platea dictumst. Cras egestas auctor molestie.

Section

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Cras egestas auctor molestie. In hac habitasse platea dictumst. Duis turpis tellus, scelerisque sit amet lectus ut, ultricies cursus enim. Integer fringilla a elit at fringilla. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nulla congue consequat consectetur. Duis ac mi ultricies, mollis ipsum nec, porta est. Aenean augue neque, varius vitae dapibus ac, Fig. 2 dictum ut nisl et Table 1

Table 1: Different quantities and qualities of T_{shell}

Heading	r_c (km)	T_{shell} (s)	t_{waves} (s)	$\mathcal{M}$	ω_{c} (rad/s)	P_{min} (s)	$P_{\text{min,Fe}}$ (s)	$P_{\text{min,NS}}$ (s)
Row	1.6×10^7	4×10^{13}	2×10^5	0.06	3×10^{-6}	2×10^5	40	2×10^{-3}
Row	9.7×10^3	3×10^8	10^6	0.002	4×10^{-3}	2×10^3	50	2.5×10^{-3}
Row	3.6×10^3	4×10^6	10^5	0.004	2×10^{-2}	-	-	-
Row	1.7×10^3	7×10^3	2×10^3	0.02	4×10^{-1}	-	-	-

2 Section

Mauris nec massa leo. Mauris ac diam auctor nisl imperdiet porta. Sed sit amet neque eget nisi dictum placerat. Duis sit amet pellentesque odio. Cras scelerisque sem a consectetur vehicula. Aliquam interdum luctus fringilla. Nunc sollicitudin, lorem in semper viverra, [11], dui nisi sodales sem, ut condimentum erat leo eget arcu [11, 12]. Donec pharetra aliquam metus, non pulvinar tellus interdum a. Mauris a ante pharetra, mollis enim in, eleifend erat. Pellentesque suscipit risus massa, non vestibulum libero euismod feugiat. In hac habitasse platea dictumst. Maecenas rutrum lobortis lobortis. Vestibulum convallis porttitor sem ac ultricies. Mauris volutpat fringilla nisl blandit semper. Proin nec iaculis sem. Aenean neque ipsum, pretium a faucibus non, tincidunt ut sapien.

Non-LaTeX Section

Integer in metus aliquam, cursus dolor eu, **maximus arcu**. Integer vel finibus odio. Maecenas sit amet rhoncus purus. Ut molestie augue vel magna rutrum fermentum. Curabitur eleifend, nisl non rutrum auctor, diam sapien rutrum purus, quis dictum erat leo in leo. Vestibulum semper, velit non malesuada sagittis, tortor dolor sollicitudin enim, sed ullamcorper tellus diam vitae est. Nullam auctor dui ac ultricies porta. Aliquam erat volutpat. Maecenas finibus ultrices felis eu congue. Integer pulvinar, elit sed mollis aliquet, magna turpis molestie nisi, sed auctor justo massa vitae felis. Vivamus dui justo, auctor non magna eget, varius dapibus augue.

References

[1] Dennis Lucarelli. Control aspects of holonomic quantum computation. *Journal of mathematical physics*, 46(5):052103, 2005.

[2] J Wess and Bruno Zumino. Lagrangian method for chiral symmetries. *Physical Review*, 163(5):1727, 1967.

[3] Steven Weinberg. Nonlinear realizations of chiral symmetry. *Physical Review*, 166(5):1568, 1968.

[4] Sidney Coleman, Julius Wess, and Bruno Zumino. Structure of phenomenological Lagrangians. I. *Physical Review*, 177(5):2239, 1969.

[5] Curtis G Callan Jr, Sidney Coleman, Julius Wess, and Bruno Zumino. Structure of phenomenological Lagrangians. II. *Physical Review*, 177(5):2247, 1969.

[6] M Bando, T Kuramoto, T Maskawa, and S Uehara. Structure of non-linear realization in supersymmetric theories. *Physics Letters B*, 138(1-3):94–98, 1984.

[7] Masako Bando, Tetsuji Kuramoto, Toshihide Maskawa, and Shozo Uehara. Non-linear realization in supersymmetric theories. *Progress of Theoretical Physics*, 72(2):313–349, 1984.

[8] Armin Uhlmann. Parallel transport and "quantum holonomy" along density operators. *Reports on Mathematical Physics*, 24(2):229–240, 1986.

[9] Norman Earl Steenrod. *The topology of fibre bundles*, volume 14. Princeton University Press, 1999.

[10] Brian Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*, volume 222. Springer, 2015.

[11] Peter Goldreich and Pawan Kumar. Wave generation by turbulent convection. *The Astrophysical Journal*, 363:694, nov 1990.

[12] Pawan Kumar, Peter Goldreich, and Richard Kerswell. Effect of nonlinear interactions on p-mode frequencies and line widths. *The Astrophysical Journal*, 427:483, may 1994.

[13] Stefan Berceanu. Geometrical phases on hermitian symmetric spaces. *arXiv preprint math/0408233*, 2004.

[14] Bruno Zumino. Supersymmetry and Kähler manifolds. *Phys. Lett. B*, 87(CERN-TH-2734):203–206, 1979.