Split step Fourier method for non-linear Schroedinger equation

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April 24, 2018

The non-linear Schroedinger equation can be written as:

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) + [V(r) + \alpha|\Psi(\mathbf{r},t)|^2]\Psi(\mathbf{r},t) \tag{1}$$

Where $V(r)$ is the potential energy function and α is the nonlinear parameter. This equation can be divided in two parts. Linear and non-linear components. i.e.

$$i\hbar\frac{\partial\Psi}{\partial t} = \alpha|\Psi|^2 \tag{2}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\Psi(\mathbf{r},t) \tag{3}$$

The solution of first equation is:

$$\overline{\Psi}(\mathbf{r},t_0+\Delta t) = e^{-i\alpha|\Psi(\mathbf{r},t_0)|^2\Delta t}\Psi(\mathbf{r},t_0) \tag{4}$$

and for the second equation

$$\Psi(\mathbf{r},t) = e^{-i[\hat{T}+\hat{V}](t-t_0)}\Psi(\mathbf{r},t_0) \tag{5}$$

By using spectral techniques it is possible to write that (one dimensional):

$$\Psi(x,t_0+\Delta t) = e^{-iV(x)\Delta t}\mathcal{F}^{-1}\{e^{ik^2\Delta t}\mathcal{F}\{e^{-iV(x)\Delta t}\overline{\Psi}(x,t_0+\Delta t)\}\} \tag{6}$$

Replacing (3) on (5), finally we obtain:

$$\Psi(x,t_0+\Delta t) = e^{-iV(x)\Delta t}\mathcal{F}^{-1}\left\{e^{ik^2\Delta t}\mathcal{F}\left\{e^{-iV(x)\Delta t}e^{-i\alpha|\Psi(x,t_0)|^2\Delta t}\Psi(x,t_0)\right\}\right\} \tag{7}$$