Modal Logic

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Abstract

Algebraic logic started in the XIXth century, with the focus on the algebraic investigation of particular classes of algebras and their possible connection to deductive systems. This introductory work intends to address the algebraization of Modal Logic for the third module of the Applied Mathematics doctoral course in Algebra, Logics and Computation.

Introduction

Modal logic language is an extension of the language of classic propositional logic by a unary modal operator $\Box$. A **logical language** is a set of connectives with fixed arity (greater than 0). Mathematically, this can be expressed as an ordered pair $\langle O, \rho \rangle$ where O is generally a non-empty set called the *labels* or *symbols* of the language and $\rho\colon O \to \mathbb{N}$ is the arity function that assigns a natural number to every symbol of the language. If $\Delta \in O$ then $\rho(\Delta)$ are the number of arguments to which Δ must be applied and $\rho(\Delta) = n$ defines a *n-ary relation* on the terms. The logical connectives can be thought of as the operation symbols of an algebraic similarity type and the formulas as the terms of that algebraic similarity type. Given a logical language $\mathcal{L}$, the definition of formulas over a set of propositional variables Var is defined as follows:

1. If $A \in \mathcal{L}$, then A is a formula over $\mathcal{L}$
2. if β and γ are formulas, then so is the result of applying the elements $A \in \mathcal{L}$ to β and γ, taking into account the degree of the operation defined by A.

The language $\mathcal{L}$ is given by:

$$\mathcal{L} = \{\wedge, \vee, \rightarrow, \Box, \top, \bot\}$$

and the corresponding free-algebra of type $\mathcal{L}$ over the denumerable set of generators Var given by:

$$\mathbf{Fm} = \langle Fm(Var), \wedge^{\textit{Fm(Var)}}, \vee^{\textit{Fm(Var)}}, \rightarrow^{\textit{Fm(Var)}}, \Box^{\textit{Fm(Var)}}, \top^{\textit{Fm(Var)}}, \bot^{\textit{Fm(Var)}} \rangle$$

Sometimes the language $\mathcal{L}$ uses just only one modal functor $(L_{\Box}, L_{\Diamond})$, in that case, the dual modal functor can be determined as follows:

$$\Box\phi = \neg\Diamond\neg\phi$$

$$\Diamond\phi = \neg\Box\neg\phi$$

The symbols $(\Box, \Diamond)$ have particular associated readings. In this sense, $\Box\phi$ is read as "necessarily ϕ" and $\Diamond\phi$ as "it is possible that ϕ". Some simple formulas can be interpreted under these principles, for example:

$$\Box\phi \to \Diamond\phi, \quad \text{what is necessary is possible}$$

$$\phi \to \Box\Diamond\phi, \quad \text{if it is, it is necessarily possible}$$

Models and Frames

Definition 1 (Frame) *A frame is a relational structure bearing a single binary relation $\mathcal{F} = \langle W, R\rangle$ where $W \neq \emptyset$ and R a binary relation on W called accessibility relation.*

Definition 2 (Model) *A model is a structure $\mathcal{M} = \langle \mathcal{F}, V\rangle$ where $\mathcal{F}$ is a frame and V is a valuation on $\mathcal{F}$.*

A valuation V is a map: $Var \to \mathcal{P}(W)$ where $\mathcal{P}(W)$ denotes the powerset of W. One can think informally of $V(p)$ as the points in $\mathcal{M}$ where p is *true*. The valuation V can be extended from proposition letters to every formula with $V(\phi)$ denoting the set of points at which ϕ is true as follows:

$$V(\phi) = \{x \mid \mathcal{M}, x \vDash \phi\}$$

Where $\mathcal{W}, x \vDash \phi$ defines the notion of satisfaction of the formula ϕ in $\mathcal{M}$ at state x. This can be defined inductively as follows:

$\mathcal{M}, x \vDash \phi$	iff $x \in V(p)$
$\mathcal{M}, x \vDash \phi \wedge \psi$	iff $\mathcal{M}, x \vDash \phi$ and $\mathcal{M}, x \vDash \psi$
$\mathcal{M}, x \vDash \phi \vee \psi$	iff $\mathcal{M}, x \vDash \phi$ or $\mathcal{M}, x \vDash \psi$
$\mathcal{M}, x \vDash \neg\phi$	iff $\mathcal{M}, x \nvDash \phi$
$\mathcal{M}, x \vDash \phi \to \psi$	iff $\mathcal{M}, x \nvDash \phi$ or $\mathcal{M}, x \vDash \psi$
$\mathcal{M}, x \vDash \Box\phi$	iff $\mathcal{M}, x' \vDash \phi$ $\forall x' \in W$ such that xRx'
$\mathcal{M}, x \vDash \Diamond\phi$	iff $\mathcal{M}, x' \vDash \phi$ $\exists x' \in W$ such that xRx'
$\mathcal{M}, x \nvDash \bot$	
$\mathcal{M}, x \vDash \top$	

Note that if $\mathcal{M}$ does not satisfy ϕ at x we write $\mathcal{M}, x \nvDash \phi$. The set of all states where ϕ is satisfied in a model $\mathcal{M}$ is written $\|\phi\|_{\mathcal{M}}$ ($\|\phi\|$ when $\mathcal{M}$ is in context). A formula is *satisfiable* in a model $\mathcal{M}$ if there is *some* state in $\mathcal{M}$ in which ϕ is true. Dually, a formula is *refutable* in a model $\mathcal{M}$ if its *complementation* is *satisfiable*. A set of formulas is globally *true*, or globally *satisfiable* if $\mathcal{M} \vDash \phi$ iff $\forall_{x \in W}\ \mathcal{M}, x \vDash \phi$.

Normal Modal Logic

Definition 3 *A normal modal logic is defined as a set of formulas λ that contains all tautologies of classical logic, the formulas $\Box(p \to q) \to (\Box p \to \Box q)$ and $\Diamond p \leftrightarrow \neg\Box\neg p$, and is closed under the rule of Modus Ponens, uniform substitution and generalization (necessity).*

Other modal logics can be obtained by adding extra axioms to the Hilbert formalization. Some examples are presented below:

Name	Formula
D	$\Box\phi \rightarrow \Diamond\phi$
T	$\Box\phi \rightarrow \phi$
5	$\Diamond\phi \rightarrow \Box\Diamond\phi$
M	$\Box\Diamond\phi \rightarrow \Diamond\Box\phi$

Modal Consequence Relations

In order to understand the meaning of a set of formulas Γ to logical entail a modal formula ϕ, it will be introduced two families of consequence relations: the *local consequence relation* $\vDash^l$ and the *global consequence relation* $\vDash^g$ in terms of classes of structures. Before establishing the definitions let's define the set of all models where a formula ϕ is satisfied as $Mod(\phi) = \{\mathcal{M} : \mathcal{M} \vDash \phi\}$, which is easily extended to a set of formulas Γ, and the class of all models built on any frame from $\mathcal{F}$ as $ModF = \{\langle W, R, V\rangle : \langle W, R\rangle \in \mathcal{F}\}$.

Definition 4 (Local consequence relation) $\Gamma \vDash^l_{\mathcal{F}} \phi$ *iff* $\forall_{\mathcal{M}\in MOD(\mathcal{F})}\ \|\Gamma\|_{\mathcal{M}} \subseteq \|\phi\|_{\mathcal{M}}$. *Alternatively, this can be expressed as: if* $\mathcal{M}_x \vDash \Gamma$ *then* $\mathcal{M}_x \vDash \phi\ \forall_{x,\mathcal{M}\in MOD(\mathcal{F})}$

Definition 5 (Global consequence relation) $\Gamma \vDash^g_{\mathcal{F}} \phi$ *iff* $Mod_{\mathcal{F}}(\Gamma) \subseteq Mod_{\mathcal{F}}(\phi)$