Singapore Physics Olympiad Training (SPOT) - Mechanics

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January 10, 2018

Newtonian Mechanics

Introduction

Ancient philosophers (such as the Greeks) were interested in the natural world around them, in particular how bodies moved. While the solutions to their pondering would only come much later, they contributed immensely to our understanding of the natural world by starting the initial exploring and definitions related to motion. In this set of notes, we attempt to build up our knowledge from basic axioms. While basic knowledge is assumed (in the form of basic definitions and equations), this set of notes is intended to flow through as we build up to larger concepts, to ensure that by the end of these notes, any loopholes or gaps in understanding have been covered, and the reader is brought to greater awareness about how the methods he or she has learnt may be applied to other situations. Definitions will only be repeated if the author deems them necessary (e.g. due to presence of subtleties to take note of).

1. Statics

To understand why things move, we first need to understand why things do not move. A state in which a stationary body, if left alone, remains stationary or a body in motion remains in motion with the same parameters (See Newton's First Law) is known as equilibrium. Analysis of Equilibrium involves asking four (or more) questions:

1. *What are the forces acting on the body?*
2. *Where are the forces acting towards?*
3. *Where are the forces acting on the body?*
4. *Is it a stable, unstable, or neutral equilibrium?*

Questions 1 and 2 will doubtless be too familiar to you. The third question, related to the moment of a force, should likewise be familiar to you. The ideas involved will be explored in greater depth in later chapters, when we cover the dynamics of rigid-body motion.

Let us discuss question 4 now, by considering a pencil on a table. If the pencil were to be placed vertically on the table with its centre of gravity precisely above the contact point, it would be in a state of *unstable equilibrium*, as any small perturbation of the body away from its equilibrium position results in a greater motion away from (and out of) equilibrium. If the pencil were to be placed horizontally on the table, it would be in a state of *stable equilibrium*, as any small perturbation of the body away from its equilibrium position (i.e. upwards) results in a motion back down towards the equilibrium position. In this case, we also consider the pencil to be in *neutral equilibrium*, as it can roll along the table and find another equilibrium position. Note that this example allows us to understand that it is possible for an object to be in different states of equilibrium in relation to different coordinates simultaneously – equilibrium states are linked to so-called degrees of freedom. Note also that it is possible for an object to be in stable equilibrium along one direction, and in unstable equilibrium along another direction.

Olympiad questions, while relying on the same basic concepts of equilibrium, can still be extremely challenging, e.g. by considering a complicated system, or involving an accelerating frame of reference.

In solving questions, we recommend that you *analyse when it might be necessary to impose force balance on individual components in a system, versus for the system as a whole.* An extreme example may be seen in the following question:

1.1 A chain of mass M hangs between two walls, with its ends at the same height. The chain makes an angle θ with each wall. Find the tension in the chain at the lowest point. Solve it via: (1) Considering the forces on half the chain (Solving as a system), and (2) Using the fact that the height of a hanging chain is given by $y\left(x\right) = \left(\frac{1}{\alpha}\right)\cosh\left(\alpha x\right)$, and considering the vertical forces on an infinitesimal piece at the bottom (Solving per components).

Extension: derive the equation for the shape of a uniform chain suspended by its ends in a uniform gravitational field.

2. Linear Kinematics and Dynamics

An overall imbalance in the forces on an object results in a change in motion (Newton's second law relates this resultant force to the rate of change of linear momentum). In solving Olympiad problems, you should be conscious of the directions in which forces are acting, and translating these directions to mathematical expressions accordingly. A negative sign usually changes the physics profoundly, and you should additionally develop the habit of quickly checking for reasonableness as you work through a problem.

2.1 "String Theory"

In dealing with problems that have a pulley and inextensible string, it is often useful to realise that we are able to obtain an equation (of kinematic constraint) by summing up the lengths of all the components of strings, and noting that it must be a constant, thus the time derivative has to be zero.

2.2 Polar Coordinates

Up till now you might have been entirely comfortable working with Cartesian coordinates. However, for certain problems (e.g. central force motion), it may sometimes be easier to convert to polar coordinates. To do this effectively, we need to comfortable with the form of Newton's law expressed in polar coordinates. We begin with the basic relationships between the two coordinate systems:

$$x = r\cos\theta, y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

In vector form, we write

$$\vec{r} = r\hat{r}$$

Note that $\hat{r}$ is a unit vector that points in the direction from the origin to the point in question, and $\hat{\theta}$ is the unit vector that points in the perpendicular direction, going counter-clockwise.

2.2.1 Write $\hat{r}$ and $\hat{\theta}$ in terms of $\theta, \hat{x}, \hat{y}$.

2.2.2 Take the time derivative of the equations obtained in 2.2.1. Note that $\hat{x}$ and $\hat{y}$ are constant with respect to time. Simplify the equations to remove x and y.

2.2.3 Hence, differentiate $\vec{r}$ by application of the product rule. Check if the answer makes sense.

2.2.4 Hence, differentiate $\dot{\vec{r}}$ by application of the product rule to obtain $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$.

Allow us to analyse the results obtained. The first term is simply the component of the acceleration in the radial direction, and likewise for the third term in the tangential direction. The second term is familiar to us – the centripetal acceleration. The last term isn't as obvious – this is what gives rise to the Coriolis force.

3. Energy

3.1 Potential Energy

We consider Newton's second law:

$$F(x) = ma = mv\frac{dv}{dx}$$

By integration,

$$\frac{1}{2}mv^2 = E + \int_{x_0}^{x} F(x')\, dx'$$

Where E is the constant of integration. If we were to define $V(x) = -\int_{x_0}^{x} F(x')\, dx'$, then we may write:

$$\frac{1}{2}mv^2 + V(x) = E$$

We are now able to examine how different types of forces affect the energy. For any force that can be written solely as a function of x, we are able to define a corresponding $V(x)$. The difference between kinetic energy at two points $x = a$ and $x = b$ can thus be found as a difference of $V(x)$. We call these forces *conservative* *forces*, as they conserve the energy in the system. Note therefore that only conservative forces can have potential energies, and the force is related to the gradient of the potential energy.

Conversely, forces that cannot be written as solely a function of x are known as *non-conservative forces.*

When thinking about force fields that are defined everywhere in space, such a field is conservative if and only if the curl of $\vec{F}$ is zero everywhere. If so, a potential energy function is well-defined (up to a constant).

$$\nabla \times \vec{F} = 0$$

3.2 Work Done

We now look into work done, which will be defined as:

$$W = \int_{x_1}^{x_2} F(x')\, dx' = \frac{1}{2}mv(x_2)^2 - \frac{1}{2}mv(x_1)^2$$

Because potential energy is solely a function of position, we are able to write what we call the Work-Energy Theorem:

$$W = \Delta E$$

Now, before we delve into its application, we must first appreciate its boundaries. The usage of Work-Energy Theorem depends very strongly on the system defined. Consider the case of a ball dropping from a height.

If we define the system as solely the ball, then gravity is an external force, so the Work-Energy theorem says that all the work done by gravity goes into changing the kinetic energy of the ball. Note that as gravity is an external force, there is no gravitational potential energy.

$$W_{earth} = \Delta E$$

$$mgh = \frac{1}{2}mv^2$$

If we define the system as the ball plus earth, then gravity is an internal force, so there is gravitational potential energy and no work done by the earth.

$$0 = \Delta E = \Delta U + \Delta K$$

$$0 = -mgh + \frac{1}{2}mv^2$$

The lesson to be learnt here is that both approaches can work, just remember to never double count the work done!

4. Momentum

4.1 Conservation of Momentum

We recall Newton's Third Law: Every force has an equal and opposite reaction. From this law we are able to determine the *conservation of momentum: the total momentum of a system stays constant.*

4.2 Force as the time derivative of momentum

Now, thus far we have defined force as the product of mass and the acceleration, but we will now swap out to a more precise mathematical definition of force: the time derivative of momentum.

$$F = \frac{dp}{dt}$$

For a practical application of this form of force rather than the simplistic version, consider the following question.

4.2.1 Consider a rocket that has mass m and speed v. In the middle of its journey, the rocket starts to eject fuel to fly faster. Let the fuel be ejected at a velocity u relative to the rocket at a rate of $\frac{dm}{dt}$. If the rocket is made out of 90% fuel, by what factor will the velocity of the rocket increase after all the fuel is dumped?

4.3 Changing Frames

Note that as long as it is an inertial frame, the conservation of momentum must hold provided there are no external forces. A frame that is often convenient is the centre of mass frame. Consider the classic exploratory momentum question:

4.3.1 A mass m with speed v approaches a stationary mass M. The masses bounce off each other without any loss in total energy. What are the final velocities of the particles? Assume all motion take place in 1-D. Use the centre of mass frame. Note that after the collision, both masses must "bounce" with the reverse of their initial velocities.

4.4 Collisions

For 1-D collisions, we have a very elegant theorem: *In a 1-D elastic collision, the relative velocity of two particles after a collision is the negative of the relative velocity before the collision.*

4.4.1 Derive the above theorem.

For collisions in higher dimensions, simply apply the 1-D case in each dimension separately.

5. Lagrangian and Hamiltonian Methods

In a more sophisticated treatment of classical mechanics, you will hear about the Lagrangian and the Hamiltonian. These formulations can be used in place of Newton's laws, and also provided a lot of the foundation for modern physics (e.g. quantum mechanics). Without a proper in-depth treatment, you will probably struggle to see the connection to Newton's laws, and to understand the conceptual and mathematical basis of these formulations.

As these methods are formally beyond the Olympiad syllabus, you might be better off working on the topics that are in the syllabus first. Nonetheless, many of the ideas in the Lagrangian and Hamiltonian formulations are really beautiful, so we'll use this space to sketch out (very roughly!) how they look like.

6. Lagrangian

Note: Usage of the Lagrangian method requires knowledge of partial derivatives.

6.1 Euler-Lagrange Equation

The Lagrangian of a system can be written as:

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

Where $\mathcal{L}$ is the Lagrangian of the system, $\mathcal{T}$ is the total kinetic energy of the system, and $\mathcal{V}$ the total potential energy of the system. Instead of writing $F = ma$ to get the equations of motion, we use the *Euler-Lagrange equation*:

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}}) = \frac{\partial \mathcal{L}}{\partial x}$$

It is important to note that x here refers to a generalised coordinate (i.e. just apply the formula as per your degrees of freedom), and to note that partial differentiation is used rather than the normal differentiation. The beauty of the Lagrangian is in that it allows for direct application no matter which set of coordinates you use, be it polar, spherical, or Cartesian, without further processing.

You must be wondering how the Euler-Lagrange equation come about? Mathematically they are based on Newton's laws (https://en.wikipedia.org/wiki/Lagrangian_mechanics), but they can also be derived from Hamilton's Principle of Least Action. Don't worry too much if the mathematics below seem a bit tough – it's based on the calculus of variations, which deal with functionals (which is sort of like a function of a function!).

6.2 The Principle of Stationary Action (Hamilton's Principle)

Allow us to (for reasons you will see later) define a term Action, S, as the integral of the Lagrangian with respect to time. Consider now two points x_1 andx_2, and a function $x\left(t\right)$ such that $x\left(t_1\right) = x_1$ and $x\left(t_2\right) = x_2$. What function $x\left(t\right)$ will produce a stationary action?

Now assume that a function $x_0\left(t\right)$ satisfies this a stationary S, and consider a function $x_a\left(t\right)$ such that:

$$x_a(t) \equiv x_0(t) + a\beta(t)$$

where $\beta\left(t\right)$ satisfies $\beta\left(t_1\right) = 0$ and $\beta\left(t_2\right) = 0$ but is otherwise allowed to be arbitrary. Note this equation implies:

$$\frac{\partial x_a}{\partial a} = \beta, \frac{\partial \dot{x_a}}{\partial a} = \dot{\beta}$$

Using the chain rule, we have:

$$\begin{aligned}
\frac{d}{da}S[x_a(t)] &= \frac{d}{da}\int_{t_1}^{t_2}\mathcal{L}dt && (1)\\
&= \int_{t_1}^{t_2}\frac{d\mathcal{L}}{da}dt && (2)\\
&= \int_{t_1}^{t_2}(\frac{\partial\mathcal{L}}{\partial x_a}\frac{\partial x_a}{\partial a} + \frac{\partial\mathcal{L}}{\partial\dot{x_a}}\frac{\partial\dot{x_a}}{\partial a})dt && (3)\\
&= \int_{t_1}^{t_2}(\frac{\partial\mathcal{L}}{\partial x_a}\beta + \frac{\partial\mathcal{L}}{\partial\dot{x_a}}\dot{\beta})dt && (4)\\
&= \int_{t_1}^{t_2}(\frac{\partial\mathcal{L}}{\partial x_a} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x_a}})\beta dt + \frac{\partial L}{\partial\dot{x_a}}\beta\bigg|_{t_1}^{t_2} && (5)
\end{aligned}$$

Analysing the result, we note by definition the third term on the right hand side has to be 0, so for the left hand side to be zero (as x_0 results in stationary action) we obtain the Euler-Lagrange equation:

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{x}}) = \frac{\partial\mathcal{L}}{\partial x}$$

Richard Feynman's path integral explanation of quantum mechanics basically generalises this principle, applying it to the calculation of quantum mechanical wavefunctions.

Naturally, calculus of variations goes beyond classical mechanics, and the Euler-Lagrange equation can also be use as a mathematical analogy to solve similar problems. Consider the following question:

6.2.1 A surface of revolution has two rings of radii r1 and r2 as its boundaries at a1 and a2 along a straight line. What should the shape of the surface be such that it has minimum area?

We note that the surface area, A can be found by setting up the following equation:

$$A = \int_{a1}^{a2} 2\pi y \sqrt{1 + y'^2} dx$$

An astute reader will note the remarkable similarity with the derivation of the Euler-Lagrange equation. We can thus take our "Lagrangian" for this case to be:

$$\mathcal{L} = 2\pi y \sqrt{1 + y'^2}$$

The remainder of the solution is left to the reader to work out. To help you out, however,

6.2.1.1 Prove that the function that extremises:

$$\int_{x_1}^{x_2} f(y) \sqrt{1 + y'^2} dx$$

satisfies:

$$f' + f'y'^2 = fy''$$

where

$$f' \equiv \frac{df}{dy}$$

Once the above is done, we note that by multiplying throughout by y' and rearranging, we obtain

$$\frac{y'y''}{1 + y'^2} = \frac{f'y'}{f}$$

Which is easily integratable to obtain

$$1 + y'^2 = Bf(y)^2$$

6.3 Hamiltonian

The Hamiltonian of a system is the Legendre transform of the Lagrangian. In simple cases, this gives

$$\mathcal{H} = \mathcal{T} + \mathcal{V}$$

Where $\mathcal{H}$ is the Hamiltonian of the system, $\mathcal{T}$ is the total kinetic energy of the system, and $\mathcal{V}$ the total potential energy of the system. It is clear, then, that $\mathcal{H}$ can also be thought of as the total energy in the system. Understanding this, and applying the Conservation of Energy, we obtain:

$$\dot{\mathcal{H}} = \frac{d}{dt}(\mathcal{T} + \mathcal{V}) = 0$$

Simple! But actually the Hamiltonian shows the deep connection between symmetries and conservation laws, but let's leave Noether's theorem for another day...

7. Concluding Remarks

Classical Mechanics is one of the oldest topics in the physics olympiad. As such, though problems usually present themselves with a pretty straightforward approach, often times the trick lies in viewing questions with a different perspective (i.e. system versus components, choosing the correct frames, etc), or questioning basic assumptions (see supplementary question 1).

What is important, then, is to build up your mathematical foundations, and slowly expand your repository of skills. With each question solved, ask yourself if you could have done it in another method, if there was a different approach that might have worked. Do this well, do this consistently, and you will see improvements in your skills over time.

8. Supplementary Questions

8.1 Two bars of masses m_1 and m_2 connected by a non-deformed light spring of spring constant k rest on a horizontal plane. The coefficient of friction between the bars and the surface is k. What minimum constant force has to be applied in the horizontal direction to the bar of mass m_1 in order to shift the other bar. $(m_1 + m_2/2)kg$

8.2 A fly is able to provide itself a thrust F to fly in dense air which against an air drag $-kv$. Assuming that it is able to fly at v_1 upwards, v_2 downwards, what is the maximum velocity that it can fly horizontally? $\sqrt{v_1 v_2}$

8.3 A projectile is launched at velocity v_0 at an angle θ above the horizontal. What should θ be such that its trajectory length is maximum? Assume that the start and end points of the trajectory are at the same height. (approx 56.5 degrees)

8.4 A ball is thrown upwards with velocity v_0.

> **8.4.1 Ignoring air drag, what is time taken for the ball to come back down to where it started?**
>
> **8.4.2 Assuming that the air drag takes the form $F = -m\alpha v$, where α is a positive constant, solve for the maximum height of the ball.**
>
> **8.4.3 Obtain an implicit equation for the speed of the ball when it comes back down, v_f.**
>
> **8.4.4 Determine the time taken for the ball to complete the entire journey. Is this longer or shorter as compared to the time taken for it to complete the same journey in the absence of air drag? (shorter)**

8.5 A triangular prism of mass M is placed one side on a frictionless horizontal plane as shown in Figure 1. The other two sides are inclined with respect to the plane at angles and respectively. Two blocks of masses m_1 and m_2, connected by an inextensible thread, can slide without friction on the surface of the prism. The mass of the pulley, which supports the thread, is negligible.

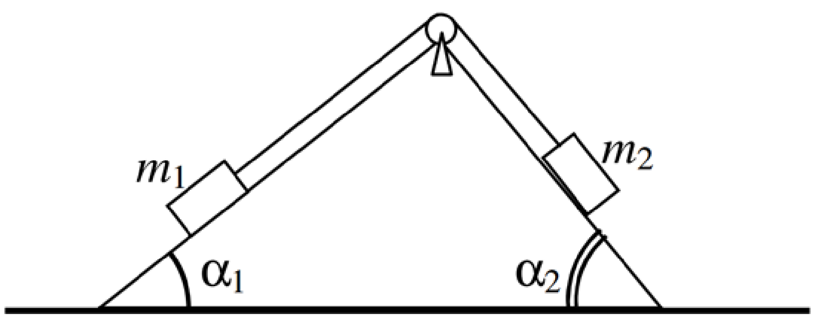

Figure 1: Triangular Prism

> **8.5.1 Express the acceleration of the blocks relative to the prism in terms of the acceleration of the prism.**
>
> **8.5.2 Find the acceleration of the prism in terms of quantities given and the acceleration due to gravity.**
>
> **8.5.3 At what ratio m_1/m_2 will the prism be in equilibrium?**

8.6 Three massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at their ends, as shown in Figure 2. The bottom end of the lower stick is hinged at the ground. They are held such that the two lower sticks are vertical, while the top stick makes an angle with the vertical. They are then released. At this instant, what are the angular accelerations of the three sticks?

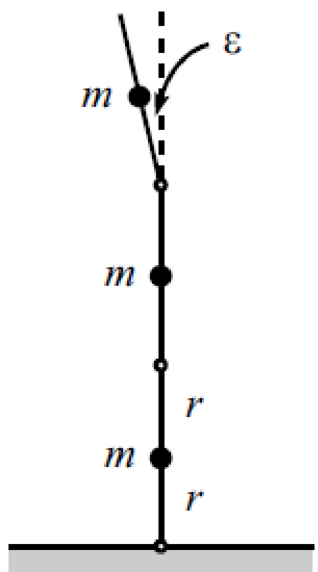

Figure 2: Sticks and Masses

8.7 A thick homogeneous inextensible line of length l and mass M is initially fastened by both ends to two hooks which are close to each other, and hangs freely as shown in Figure 3. Then one end of the line is released and begins to fall. The largest load N which each of the hooks can bear is greater than the weight of the line. What conditions must Mg and N fulfil in order that the second hook is not ripped out? We assume that during the fall each element of the line reaches its final position and later remains at rest. ($N \geq 2Mg$)

8.8 As depicted in Figure 4, a chain of length L and mass M is placed on a table, with $L/2$ hanging off the edge of the table. At $L/4$ from the end of the chain that is on the table, there is a block of mass m . Given that the coefficient of friction between the block and the chain is μ_1 , and the coefficient of friction between the chain and the table is μ_2 , and that both the chain and the block slide off the table at the same time, determine the velocities of the block and chain at the moment they leave the table.

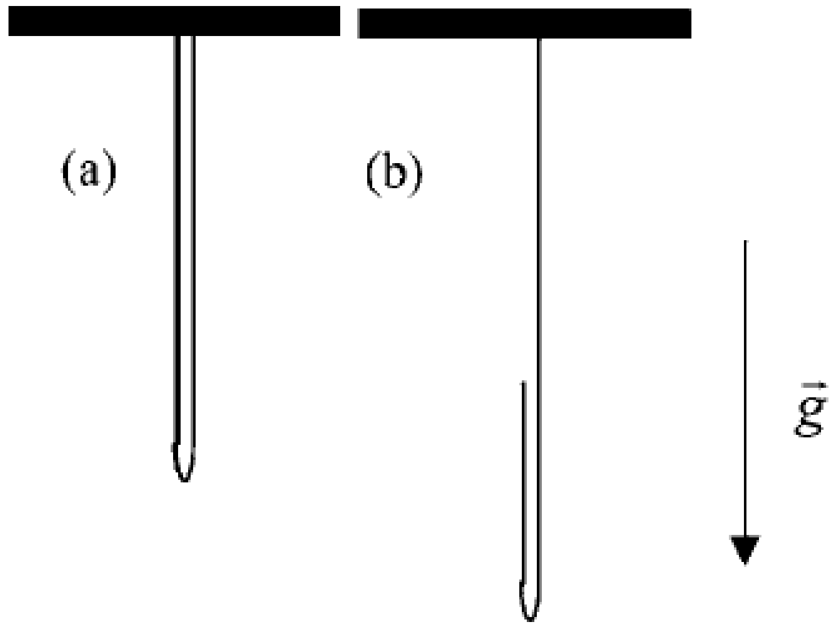

Figure 3: Fishing Line

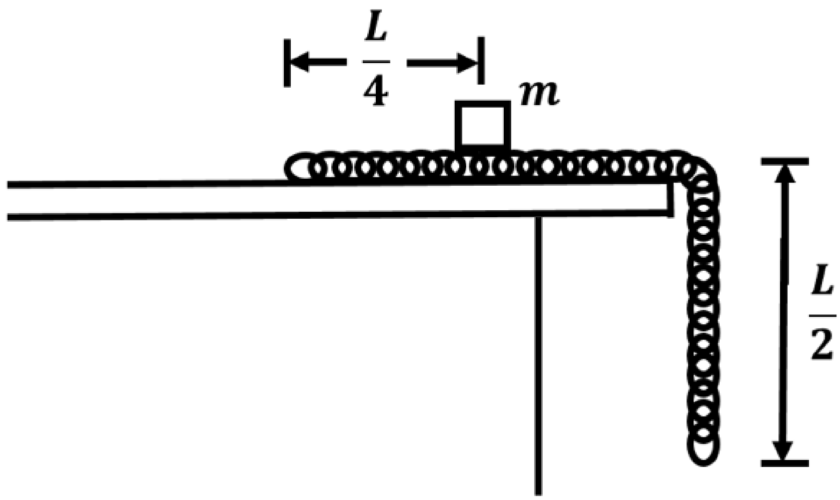

Figure 4: Chain and Mass

8.9 As depicted in figure 5, a point mass m is placed in the centre of a box of mass m . Initially, the mass is given initial velocity v_0 . Assuming that after every collision, the relative velocity of the colliding bodies is decreased by a factor ϵ, determine a) the velocity-time graph of the box. b) After some time has passed, to what value will the velocity of the box tend towards? c) given $\epsilon = \sqrt[4]{0.5}$, what is the maximum number of collisions before the system loses more than 40% of its initial energy?

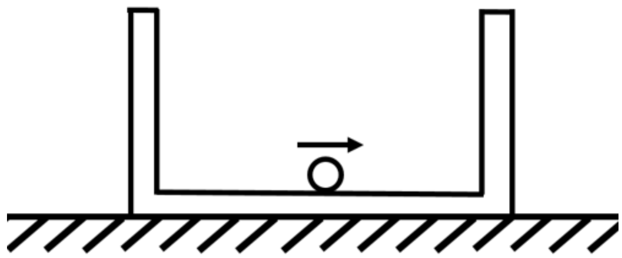

Figure 5: Ball in box

8.10 A fixed pulley carries a weightless thread with masses m_1 and m_2 at its ends. There is friction between the thread and the pulley such that the thread starts slipping when $m_2/m_1 = \eta_0$. Find: a) the friction coefficient; b) the acceleration of the masses when $m_2/m_1 = \eta > \eta_0$.

8.11 A tree-trunk of diameter 20cm lies in a horizontal field. A lazy grasshopper wants to jump over the trunk. Find the minimum take-off speed of the grasshopper that will suffice.

8.12 A tanker full of liquid is parked at rest on a horizontal road as shown in Figure 6. The brake has not been applied, and it may be supposed that the tanker can move without friction. In which direction will the tanker move after the tap on the vertical outlet pipe, which is situated at the rear of the tanker, has been opened? Will the tanker continue to move in this direction?

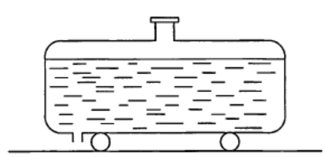

Figure 6: Liquid Tanker

8.13 Beads of equal mass are strung at equal distances on a long, horizontal wire as shown in Figure 7. The beads are initially at rest but can move without friction. One of the beads is continuously accelerated (towards the right) by a constant force F. What are the speeds of the accelerated bead and the front of the 'shock wave', after a long time, if the collisions of the beads are: (i) completely inelastic, (ii) perfectly elastic?

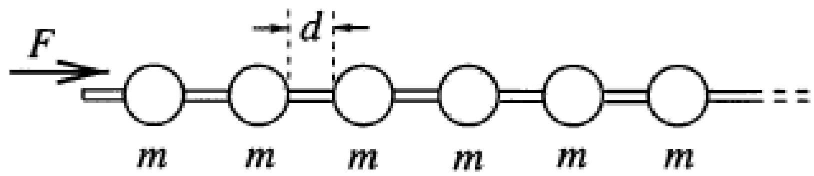

Figure 7: Infinite Beads