U3_AOS1_Topic04_Higher power polynomials

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February 12, 2019

Question Group information

Name: Higher Power Polynomials

Summary:

Linear and quadratic functions are not the only polynomials we need to know. Cubics and quartics are the next functions in the chain, having higher degrees than the others. We'll look at what these types of polynomials look like and how we can deal with them. Not only this, we'll examine the basic forms of polynomials with degrees higher than a quartic.

Videos:

Video1 url:

Video 1 title:

Video 1 thumb url:

Start at:

End at:

Tutorial number: 1

Prompt: How do we sketch cubic functions?

Title: Cubic functions

Cubic functions are the next polynomial after quadratics as it has a degree of 3 (the highest power is 3). Some of the characteristics are similar to quadratics, yet others can be very different.

Shape

Unlike quadratics, cubic graphs have various shapes depending on the equation. In general, **positive cubic graphs** look something like these:

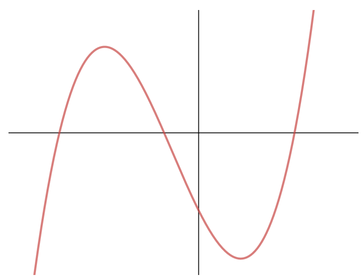

Figure 1: {{s3_url}}/tutorial-01/positive_cubic1.png

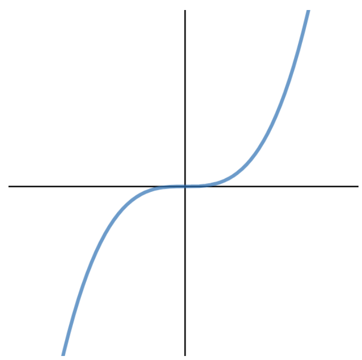

Figure 2: {{s3_url}}/tutorial-01/positive_cubic2.png

Keep in mind that a positive graph **ends at the top right**.

Negative cubic graphs can have the following shapes:

Keep in mind that a negative graph **ends on the bottom right**.

Expanded form

The expanded form is given by:

$$y = ax^3 + bx^2 + cx + d$$

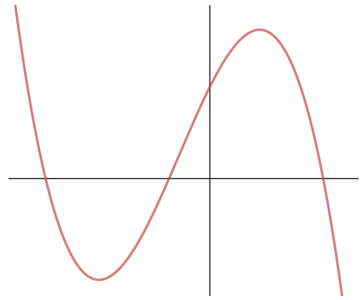

Figure 3: {{s3_url}}/tutorial-01/negative_cubic1.png

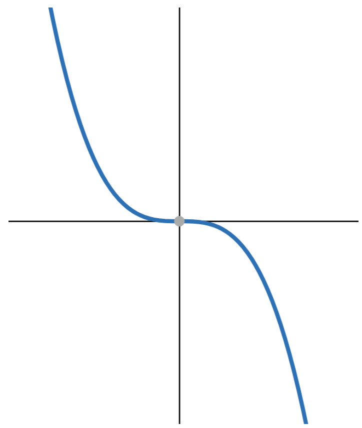

Figure 4: {{s3_url}}/tutorial-01/negative_cubic2.png

The y-intercept can be easily obtained from this form.

$$y - intercept = (0, d)$$

We can only tell if the graph is **positive or negative** from the sign of a: if the sign of a is positive, the graph is positive; and if the sign of a is negative, the graph is negative.

Other than this, we can't really tell what shape it has. How many intercepts there are can only be found after we convert it to factorised form. Refer to Topic 2: Algebric techniques to learn how to factorise cubic functions.

Factorised form

$$y = a(x - x_1)(x - x_2)(x - x_3)$$

The positive/negative shape system in factorised form works in the same way as the expanded form. If a is positive, the graph is positive, and if a is negative, the graph is negative.

Three x-intercepts

The factorised form with three x-intercepts is given by:

$$y = a(x - x_1)(x - x_2)(x - x_3)$$

where x_1, x_2 and x_3 are the x-intercepts of the cubic graph.

There are **three linear factors** in this form.

Let's look at an example: $y = (x+1)(x-1)(x-2)$.

This is a positive graph, as there is no negative sign in front.

From the equation, we can see that there are three linear factors. Each of them represent one x-intercept. To locate all the x-intercepts, we can simply make $y = 0$ and solve for x using null factor law.

$$(x+1)(x-1)(x-2) = 0 \Rightarrow x+1 = 0 \quad o \quad r \quad x-1 = 0 \quad or \quad x-2 = 0 \therefore x = -1 \quad or \quad x = 1 \quad or \quad x = 2$$

Hence, the x-intercepts are located at $(-1, 0)$, $(1, 0)$ and $(2, 0)$.

Now, we can find the y-intercept. To do this, we can make $x = 0$.

$$y = (0+1)(0-1)(0-2)y \qquad = (1)(-1)(-2)y = 2$$

Hence, the y-intercept is at $(0, 2)$.

With all this information, we can sketch the graph.

Figure 5: {{s3_url}}/tutorial-01/cubic_3_x_int.png

Two x-intercepts

It's possible for a cubic graph to have two x-intercepts. This means we have **one linear factor and one quadratic factor**.

$$y = a(x - x_1)(x - x_2)^2$$

where x_1 is an x-intercept, and x_2 is an x-intercept that's also a turning point.

One of the turning points is located at $(x_2, 0)$ because $(x - x_2)^2$ is a quadratic factor. So the graph acts like a **quadratic** around the x-intercept at $(x_2, 0)$. What this means is that instead of crossing over the graph at the x-intercept, the graph actually smooths out and **just touches** the graph at that point, before coming back up on the same side just like a quadratic.

Let's look at an example: $y = -2(x-3)(x+2)^2$

From the equation, we can say this is the negative graph because of the negative sign in front. So the graph should end at the bottom right.

Now, let's find the x-intercepts by making $y = 0$ and then use null factor law.

$$-2(x-3)(x+2)^2 = 0 \Rightarrow x-3 = 0 \quad or \quad x+2 = 0 \therefore x = 3 \quad o \qquad r \quad x = -2$$

Therefore, the x-intercepts are located at $(3, 0)$ and $(-2, 0)$.

To find the y-intercept, we need to make $x = 0$.

$$y = -2(0-3)(0+2)^2 y \qquad = -2(-3)(2)^2 y = 24$$

Hence, the y-intercept is at $(0, 24)$.

Keep in mind that we have an x-intercept and a turning point at $(-2, 0)$. With all the information we gathered, we can sketch the graph!

Figure 6: {{s3_url}}/tutorial-01/cubic_2_x_int.png

We have seen cubic graphs with three x-intercepts and two x-intercepts. Is one x-intercept cubic graph possible?

The answer is yes! And that will bring us to the **power form** of cubic.

Power form

$$y = a(x-h)^3 + k$$

This looks very similar to the quadratic turning point form. In fact, it's almost the exact same! The reason why we don't say that cubics come in turning point form is because we name the so-called *turning point* of a cubic a **stationary point of inflection**. We will go into types of stationary points later in the course, though for now the following diagrams show the general shape of stationary points of inflection.

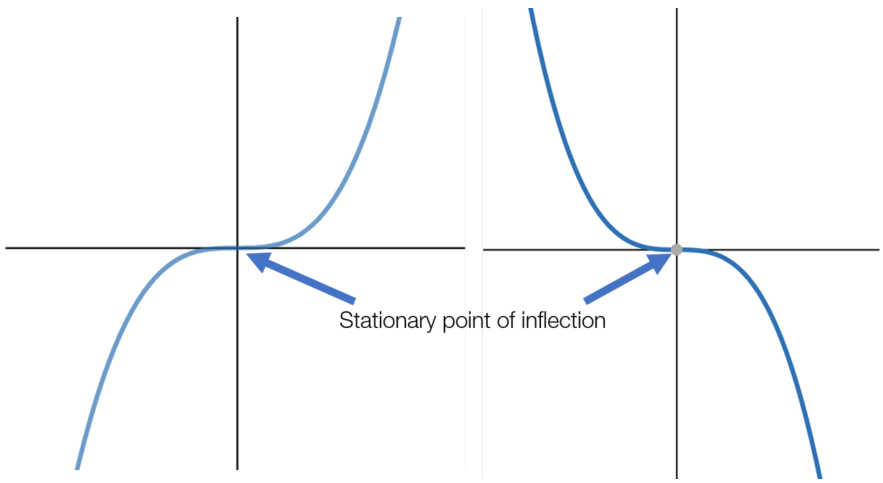

Figure 7: {{s3_url}}/tutorial-01/ cubic_power_form.png

The meaning of a, h and k is the same as quadratics' turning point form.

Translation summary					
Letters	Types of transformation	Definitions			
Magnitude of a	Dilation	$\|a\| > 1$	Narrower	$\|a\| < 1$	Wider
Sign of a	Reflection	$a > 0$	Upright	$a < 0$	Upside dow
h	Horizontal translation	$h > 0$	Moved to the right	$h < 0$	Moved to
k	Vertical translation	$k > 0$	Moved up	$k < 0$	Moved dow

Hence, the stationary point of inflection is located at (h, k).

Let's have a look at a few examples:

Sketch $y = 3(x-1)^3 + 4$.

From the equation, we can locate the stationary point straight away. It's at $(1, 4)$ because we know (h, k) is the location of stationary point.

We can also see that this graph is a positive graph because there is no negative sign in front of the function.

Then, we can find the x-intercept by making $y = 0$.

$$0 = 3(x-1)^3 + 4 - 4 \qquad = 3(x-1)^3(x-1)^3 = -\frac{4}{3}(x-1)^3 \qquad = -\frac{4}{3}x - 1 = \sqrt[3]{-\frac{4}{3}}x \qquad = \sqrt[3]{-\frac{4}{3}} + 1$$

So the x-intercept is at $(\sqrt[3]{-\frac{4}{3}} + 1, 0)$.

Now, let's find the y-intercept by making $x = 0$.

$$y = 3(0-1)^3 + 4y \qquad = 3(-1)^3 + 4y = 3(-1) + 4y \qquad = -3 + 4y = 1$$

The y-intercept is at $(0, 1)$.

With this information, we can now sketch the graph.

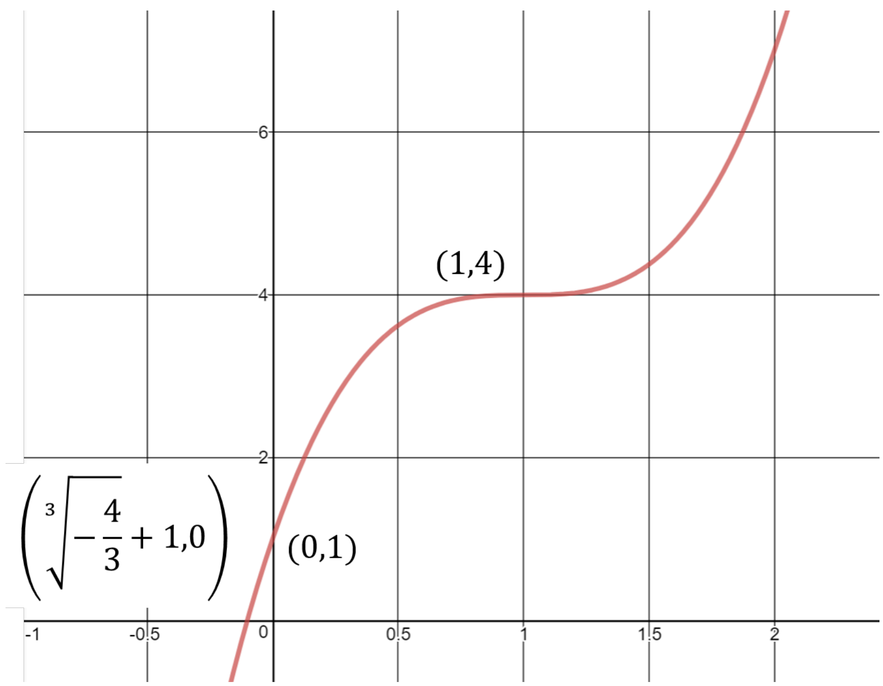

Figure 8: {{s3_url}}/tutorial-01/ cubic_power_form_e1.png

How about $y = -(x+3)^3 - 7$?

From the equation, we can quickly tell that this is a negative graph. So the graph should end at the bottom right.

Another thing we can see straight away is the stationary point: it's at $(-3, -7)$.

Let's find the x-intercept by making $y = 0$ and solve for x.

$$-(x+3)^3 - 7 = 0(x+3)^3 \qquad = -7x + 3 = \sqrt[3]{-7}x \qquad = \sqrt[3]{-7} - 3$$

So the x-intercept is at $(\sqrt[3]{-7} - 3, 0)$.

Next, we need to find the y-intercept by making $x = 0$ and solve for y.

$$y = -(0+3)^3 - 7y = -27 - 7y = -34$$

The y-intercept $= (0, -34)$

So the graph of $y = -(x+3)^3 - 7$ will look like this.

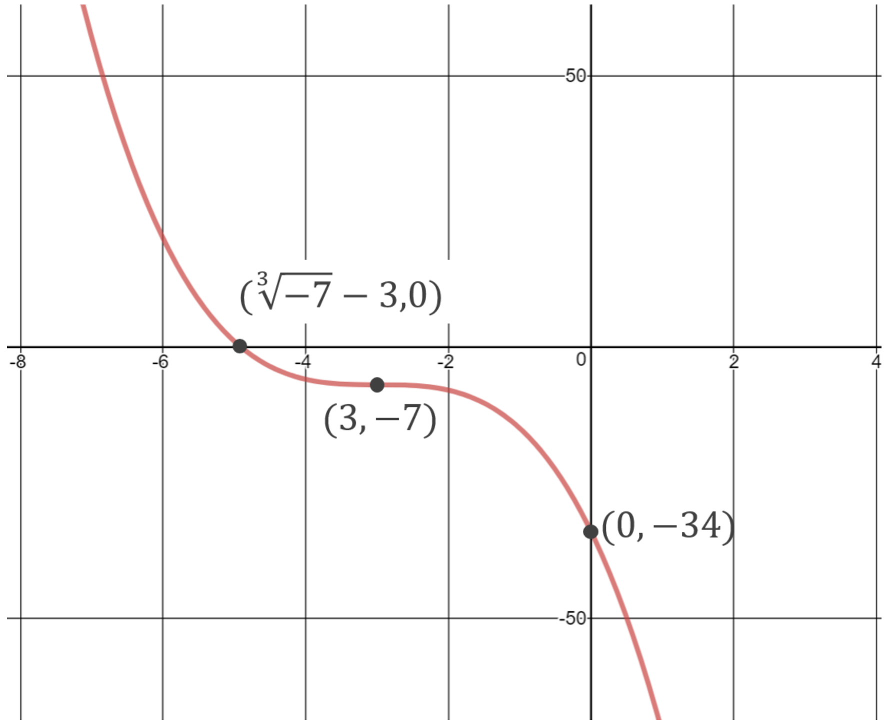

Figure 9: {{s3_url}}/tutorial-01/ cubic_power_form_e2.png

Tutorial number: 2

Prompt: How do we sketch quartic functions?

Title: Quartic functions

Quartic functions are polynomials with a degree of 4. Although these have a higher power than quadratics, some quartics look quite similar to parabolas. On the other hand, other quartic functions can produce some rather wacky looking graphs!

Shape

Just like quadratics and cubics, quartic functions can also have positive and negative shapes. So what can these shapes look like?

Positive quartics are easy to recognize as they always end up in the **top right** of the graph.

Figure 10: {{s3_url}}/tutorial-02/quartics_tute02_graph1.png

This is an example of a basic quartic, and while it looks similar to quadratic it's shape is slightly different. Quartics are generally described as a **fatter** version of a quadratic.

Figure 11: {{s3_url}}/tutorial-02/quartics_tute02_graph2.png

Here's the more wacky type of quartic that we may see more often. The degree of 4 allows quartics to have more dips and turns, even more than a cubic and quadratic.

Negative quartics have the opposite shape to the ones above, especially since these end up in the **bottom right** of the graph.

Figure 12: {{s3_url}}/tutorial-02/quartics_tute02_graph3.png

Figure 13: {{s3_url}}/tutorial-02/quartics_tute02_graph4.png

Expanded form

Just like quadratics and cubics, the expanded form of a quartic follows a pattern:

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

y-intercept

The easiest thing to spot on a quartic in expanded form is its y-intercept, which is given as:

$$(0, e)$$

In other words, the constant on the end of the equation is the y-intercept.

Shape

The sign of a tells us whether the graph is positive and upright, or negative and upside-down:

- $a > 0 \Rightarrow$ The quartic is positive.
- $a < 0 \Rightarrow$ The quartic is negative.

Factorised form

The factorised form tells us the location of the quartic's x-intercepts.

$$y = a(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

If a function is in this form, then its x-intercepts are located at:

$$(x_1, 0), (x_2, 0), (x_3, 0), (x_4, 0)$$

Don't forget the minus signs in front of the factors, and that we have to swap the sign to get the x-intercept!

Shape

The sign of a tells us whether the graph is positive and upright, or negative and upside-down, just like the expanded form:

- $a > 0 \Rightarrow$ The quartic is positive.
- $a < 0 \Rightarrow$ The quartic is negative.

Four x-intercepts

The form from above tells us that there are four distinct x-intercepts, assuming that x_1, x_2, x_3 and x_4 are all different.

$$y = a(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

This quartic in factorised form is made up of four individual **linear factors**. These linear factors tell us that the graph **crosses over** the x-axis at these points, just like a line would cut through the axis.

Let's take a look at the quartic function $y = x(x - 1)(x - 2)(x - 3)$. To find the x-intercepts, we'd have to let $y = 0$ and use the null factor law:

$$x(x-1)(x-2)(x-3)=0 \Rightarrow x \qquad = 0 \textit{ or } x-1=0 \textit{ or } x-2=0 \textit{ or } x-3=0 \therefore x=0,1,2 \textit{ or } 3$$

Based on this, we have our four x-intercepts at $(0,0)$, $(1,0)$, $(2,0)$ and $(3,0)$.

We can see that there's no negative sign in front of the equation, so we can sketch out a positive, upright quartic.

Figure 14: {{s3_url}}/tutorial-02/quartics_tute02_graph5.png

Three x-intercepts

We can also have three x-intercepts instead of four. This is in the following form:

$$y = a(x-x_1)(x-x_2)(x-x_3)^2$$

This form says we have two linear factors as well as one quadratic factor. The two linear factors tell us that the graph crosses over the x-axis at those two points. The quadratic factor tells us that the graph **acts like a quadratic** and smoothly touches the x-axis at that point.

Note that we can switch the order of the linear and quadratic factors, though we'll still have three x-intercepts.

Let's take a look at the following quartic:

$$y = -(x+1)^2(x-1)(x-2)$$

We can see a negative sign at the front, meaning the quartic is upside-down.

The x-intercepts can be solved by letting $y=0$ and solving for x with the null factor law.

$$-(x+1)^2(x-1)(x-2)=0 \Rightarrow x+1 \qquad = 0 \textit{ or } x-1=0 \textit{ or } x-2=0 \therefore x=-1,1,2$$

We have x-intercepts at $(-1,0)$, $(1,0)$ and $(2,0)$. Keep in mind the intercept at $(-1,0)$ will behave like a quadratic, since it came from a quadratic factor.

The y-intercept exists when $x=0$.

$$y = -(0+1)(0-1)(0-2)y \qquad = -(1)(-1)(-2) \Rightarrow y=-2$$

Thus, we have our y-intercept at $(0,-2)$.

With all this information, we can sketch the graph like this:

Figure 15: {s3_url}}/tutorial-02/quartics_tute02_graph6.png

Two x-intercepts

We can have quartics that have fewer x-intercepts, such as only two intercepts. This can come in two forms:

1. $y = a(x - x_1)^2(x - x_2)^2$
2. $y = a(x - x_1)(x - x_2)(x^2 + bx + c)$

The first form tells us that there are two quadratic factors, meaning that both the graph's x-intercepts will behave like quadratics.

Let's look at the function with the rule $y = (x + 1)^2(x - 2)^2$. This graph is a positive quartic, since there's no negative sign at the front.

The x-intercepts are given by the two numbers in the brackets: $(-1, 0)$ and $(2, 0)$. Both of these will behave like quadratics.

The y-intercept is found when $x = 0$.

$$y = (0 + 1)^2(0 - 2)^2 y \qquad = (1)^2(-2)^2 y = (1)(4) \Rightarrow y \qquad = 4$$

The y-intercept occurs at $(0, 4)$. With these pieces of information, we can sketch the graph like so:

Figure 16: {{s3_url}}/tutorial-02/quartics_tute02_graph7.png

The second form, $y = a(x - x_1)(x - x_2)(bx^2 + cx + d)$, tells us that there are two linear factors, and a quadratic factor that has **no solution**. In other words, we get two x-intercepts that act like lines crossing the axis, but the other quadratic factor has no x-intercepts when solved.

When we attempt to solve the equation for x, we'll notice that one of the solutions using the null factor law will be $bx^2 + cx + d = 0$. Taking this further, this equation will have $\Delta < 0$ or a negative discriminant. Don't forget that a negative discriminant implies that the quadratic has no solutions.

For example, the graph of $y = (x + 2)(x^2 + 1)(x - 1)$ has two linear factors that give us x-intercepts at $(-2, 0)$ and $(1, 0)$ using the null factor law, though we recognise that $x^2 + 1 = 0$ has no solution.

$$x^2 + 1 = 0 x^2 \qquad = -1 x = \pm\sqrt{-1}$$

We there is no real solution for $\sqrt{-1}$ since we can't square root negative numbers. Hence, we end up with two x-intercepts.

The y-intercept is at $(0, -2)$ when we sub in $x = 0$. With this, we can roughly sketch it's graph.

Figure 17: {{s3_url}}/tutorial-02/quartics_tute02_graph8.png

One x-intercept

It's also possible for a quartic to have only one x-intercept. Once again, there are two forms for this, though the most common one is in the following form:

$$y = a(x - x_1)^2(bx^2 + cx + d)$$

This is composed of two quadratic factors, one that yields one x-intercept, and the other having no solutions at all.

For example, $y = (x^2 + 2x + 2)(x - 1)^2$ has one solution for x when solving for the x-intercept. It has a quadratic x-intercept at $(1, 0)$.

The other quadratic factor, $(x^2 + 2x + 2)$, has a discriminant of $\Delta = 2^2 - 4(1)(2) = -4 < 0$, meaning there are no solutions for it.

With a y-intercept at $(0, 2)$, the graph looks as follows.

Figure 18: {{s3_url}}/tutorial-02/quartics_tute02_graph9.png

The second form with one x-intercept is a special case of the quartic power form.

Power form

The power form of a quartic is almost identical to that of a quadratic and a cubic. The only difference is the power is now 4.

$$y = a(x - h)^4 + k$$

The constants are given as follows.

Translation summary					
Letters	Types of transformation	Definitions			
Magnitude of a	Dilation	$\|a\| > 1$	Narrower	$\|a\| < 1$	Wider
Sign of a	Reflection	$a > 0$	Upright	$a < 0$	Upside do
h	Horizontal translation	$h > 0$	Moved to the right	$h < 0$	Moved to
k	Vertical translation	$k > 0$	Moved up	$k < 0$	Moved dov

The coordinate of the turning point is given by (h, k), as with all other power forms.

Note that if $k = 0$, we end up with a factorised form with a quartic factor with one x-intercept.

Let's take an example of the function $y = -(x - 2)^4 - 1$. We can see that the sign of a is a negative, so the graph is reflected upside-down. Additionally, the coordinate of the turning point is given by $(2, -1)$.

The x-intercepts are given when $y = 0$.

$$-(x-2)^4 - 1 = 0 - (x-2)^4 \qquad = 1(x-2)^4 = -1x - 2 \qquad = \pm\sqrt[4]{-1}$$

Notice that we'd have to take the fourth root of -1. We don't have any real solution for this, so there are no x-intercepts for this graph.

The y-intercept is given when $x = 0$.

$$y = -(0-2)^4 - 1y \qquad = -(-2)^4 - 1y = -16 - 1 \Rightarrow y \qquad = -17$$

Thus, we have a y-intercept at $(0, -17)$. With all this, we can sketch the graph!

Figure 19: {{s3_url}}/tutorial-02/quartics_tute02_graph10.png

Tutorial number: 3

Prompt: How do we sketch polynomials with a degree greater than four?

Title: Polynomials with degree higher than four

From the previous tutorial, we can see that quartic graphs look very similar to quadratic graphs. In fact, there is a trend with all the polynomials.

Power form

Positive, even power

These graphs have an equation in the form of:

$$y = x^n$$

where n is a positive even number.

For example, $y = x^2$, $y = x^4$and $y = x^6$ etc. These graph have very similar shapes as they all look like a quadratic's parabola.

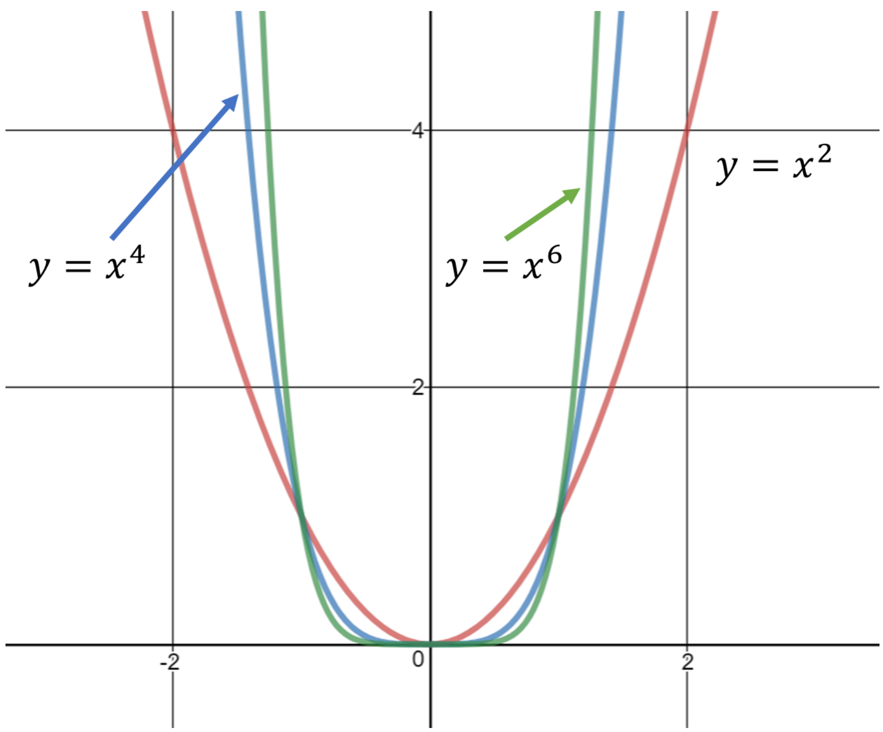

Figure 20: {{s3_url}}/tutorial-03/cubic_tut3_positive_power.png

There are slight differences in their shapes — the higher the power, the wider it is at the bottom and the narrower it is at the top. Keep in mind that they all pass the points $(1, 1)$ and $(-1, 1)$.

We can also include transformations:

$$y = a(x - h)^n + k$$

where n is a positive even number.

These will have the following characteristics:

- Turning point at (h, k).
- If a is positive, the graph is upright.
- If a is negative, the graph is upside down.
- Zero, one or two x-intercepts.

Let's consider this example: $y = -3(x+1)^6 + 2$.

First, we can see that this is a negative graph, so the shape should be upside-down. Also, the location of the turning point can be easily observed as well — it's at $(-1, 2)$.

Next, we have to find the x-intercept(s) by making $y = 0$ and solve for x.

$$-3(x+1)^6 + 2 = 0 - 3(x+1)^6 \qquad = -2(x+1)^6 = \frac{2}{3}x + 1 \qquad = \pm\sqrt[6]{\frac{2}{3}}x = \pm\sqrt[6]{\frac{2}{3}} - 1$$

So the x-intercepts are $(\sqrt[6]{\frac{2}{3}} - 1, 0)$ and $(-\sqrt[6]{\frac{2}{3}} - 1, 0)$. Don't forget that to get rid of the power of 6, we have to raise both sides to the power of $\frac{1}{6}$. Since this is an even power, we have to include a **plus-minus** in our solution. All even powers produce two solutions: a plus and a minus solution.

To find the y-intercept, we just need to make $x = 0$ and solve for y.

$$y = -3(0+1)^6 + 2y \qquad = -3(1) + 2y = -1$$

Therefore, the y-intercept is at $(0, -1)$.

Positive, odd power

Graphs that have an equation in the form of:

$$y = x^n$$

where n is a positive odd number.

For example, $y = x^3$, $y = x^5$ and $y = x^7$ etc. These graphs have very similar shapes as they all look like a cubic.

Similar to even powers, the slight difference between their shapes is that the higher the power, the wider they are in the middle and the narrower they are at the top and bottom. Keep in mind that they all pass the points $(1, 1)$ and $(-1, -1)$.

Again, we can also include transformations:

$$y = a(x-h)^n + k$$

where n is a positive odd number.

They'll have the following characteristics:

- Stationary point of inflection at (h, k).
- If a is positive, the graph is upright.
- If a is negative, the graph is upside-down.

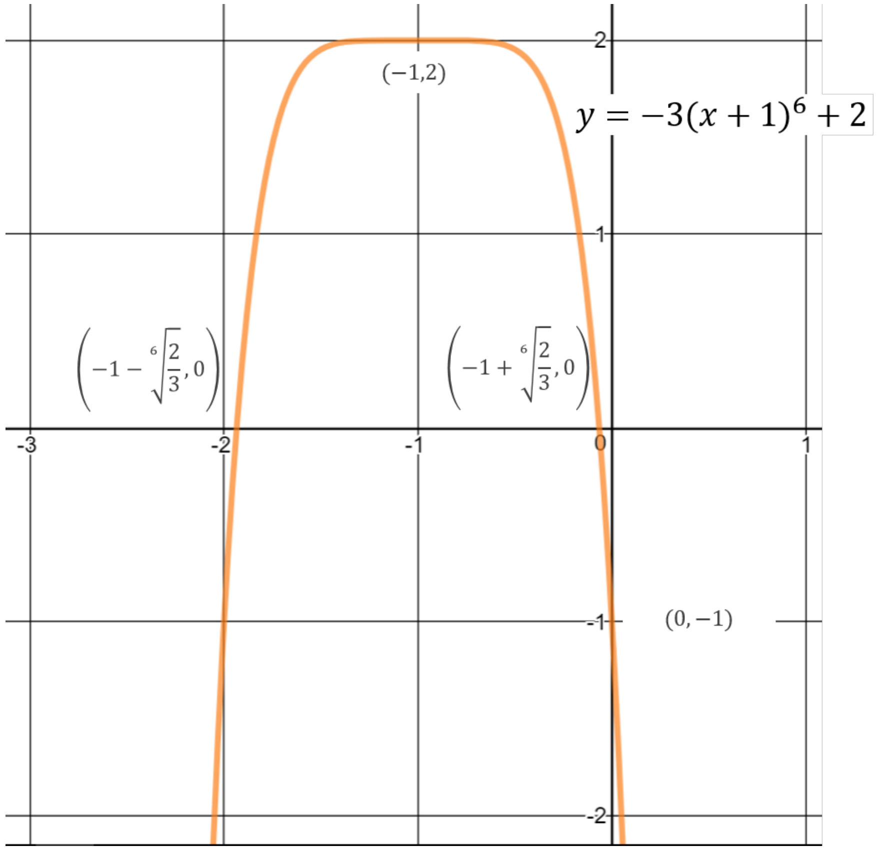

Figure 21: {{s3_url}}/tutorial-03/cubic_tut3_positive_power1.png

- Always one x-intercept.

Let's consider this example: $y = -2(x+2)^5 - 6$. From the equation, we can tell two things straight away: it's a negative graph, so it should end at the bottom right; and it has a stationary point of inflection at $(-2, -6)$.

Let try to get the x-intercept by making $y = 0$ and solve for x:

$$-2(x+2)^5 - 6 = 0(x+2)^5 \qquad = -3x + 2 = \sqrt[5]{-3}x \qquad = \sqrt[5]{-3} - 2$$

So the x-intercept is at $(\sqrt[5]{-3} - 2, 0)$.

To get the y-intercept:

$$y = -2(0+2)^5 - 6y = -2(2)^5 - 6y = -2(2)^5 - 6y = -70$$

Hence, the y-intercept is located at $(0, -70)$.

Now we can now sketch the graph.

Factorised form

Sometimes, we can be asked to sketch:

$$y = x(x+1)(x-2)^2(x-4)^3$$

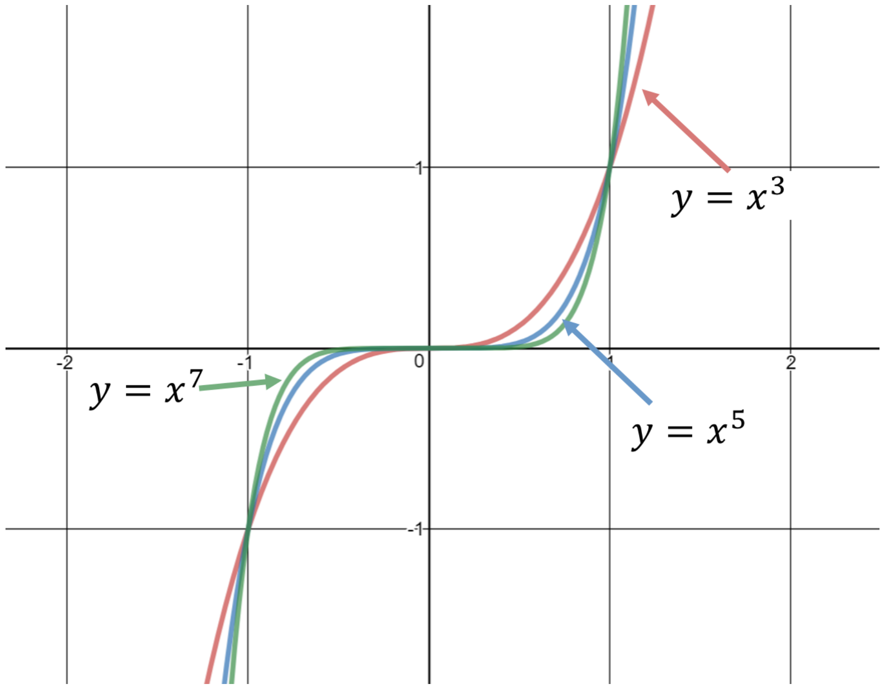

Figure 22: {{s3_url}}/tutorial-03/cubic_tut3_negative_power.png

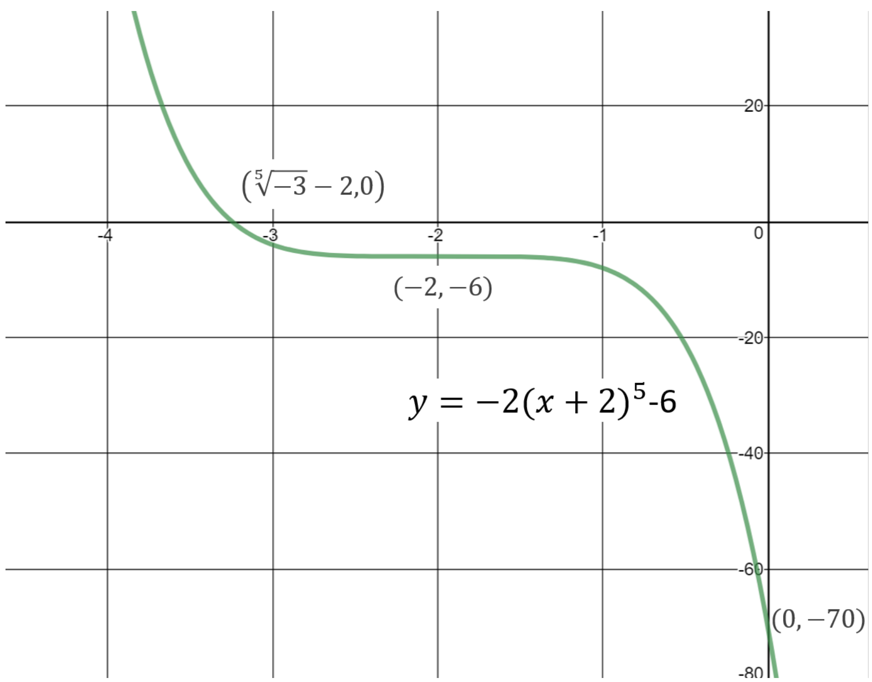

Figure 23: {{s3_url}}/tutorial-03/cubic_tut3_negative_power1.png

It can be a bit challenging to see what is going on. But we already know what to look for!

Let's look at each factor one by one:

- This is a positive graph as it doesn't have a negative sign. So we have a graph that ends on the top right.
- x and $(x+1)$ are linear factors. So the graph **crosses** the x-axis at $x=0$ and $x=-1$.
- $(x-2)^2$ is a quadratic factor. So there is a turning point and x-intercept at $x=2$. The graph **touches** the x-axis and turns back at $x=2$.
- $(x-4)^3$ is a cubic factor. So there is a stationary point of inflection and x-intercept at $x=4$. The graph looks like a **cubic** at the point $(4,0)$.

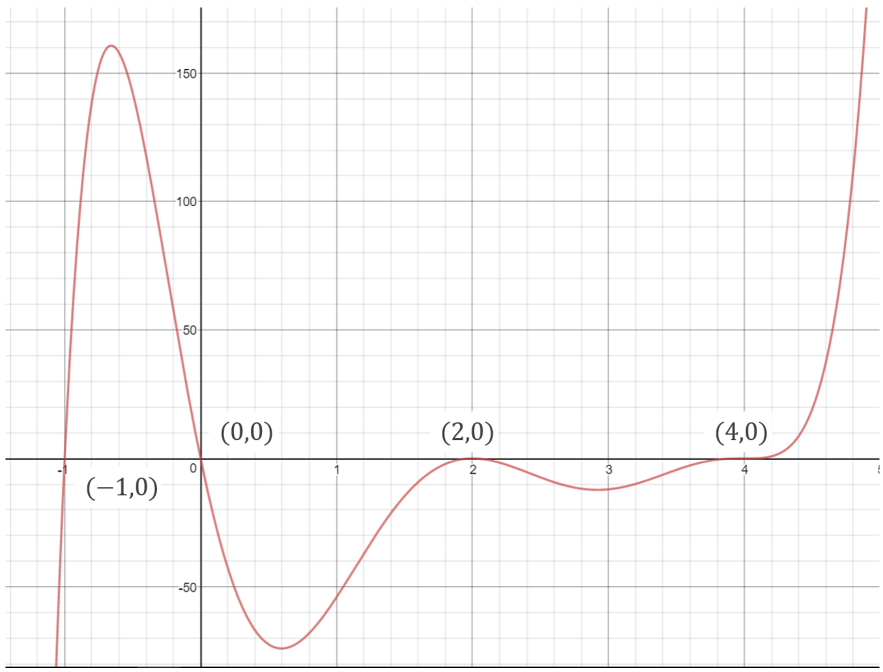

Figure 24: {{s3_url}}/tutorial-03/cubic_tut3_factorised.png

Keep in mind that a polynomial with a degree of n can have up to n number of solutions (x-intercepts). However, some polynomials can have less than n x-intercepts if it contains quadratic factors and cubic factors. Just like that example we looked at above: it has a degree of 7 but only contains 4 x-intercepts!

Question 1:

Title: Select graph when given equation in factorised form.

Tags: *Cubic graphs, factorised form, solving*

Prompt: Choose the graph that represents $y = -0.1\,(x-5)\,(x+9)\,(x-6)$.

Multiple choice options: (write 'correct' in brackets after the correct option)

A:

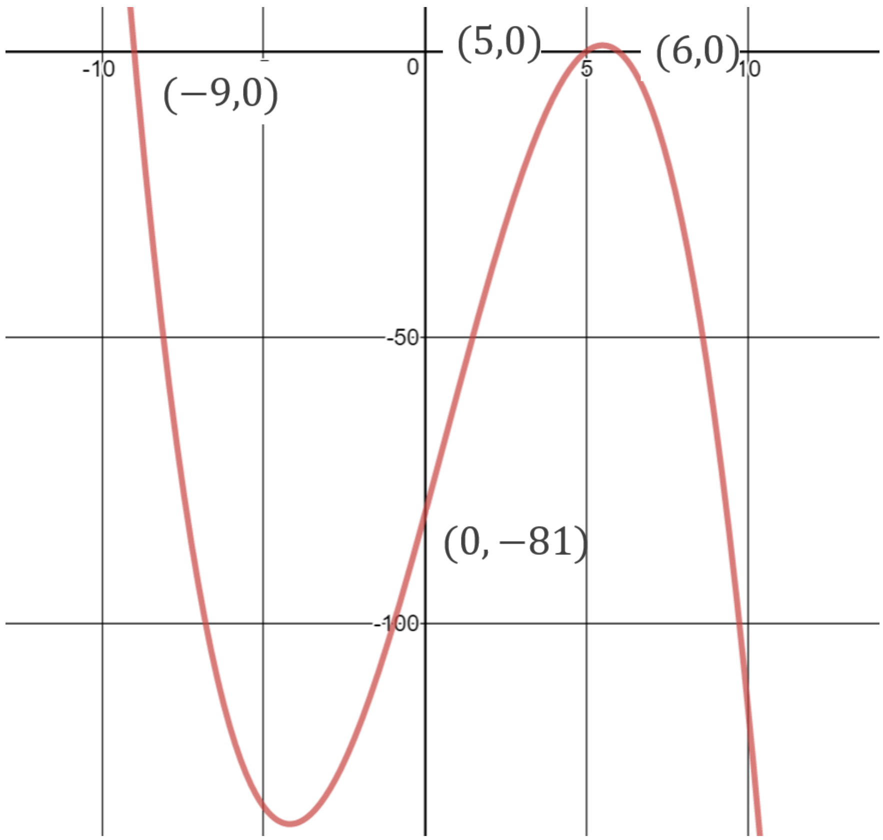

Figure 25: {{s3_url}}/mc-01/ cubic_mc1_a.png

Explanation:

No, this is not correct. But you are very close!

All the x-intercepts are at the right locations and the sign of the graph is correct. Great job!

However, you should check the y-intercept of this graph. To find the y-intercept, we can make $x = 0$ and then solve for y. Let me help you get started:

$$y = -0.1(0-5)(0+9)(0-6)y \qquad = -0.1(-5)(9)(-6)$$

And keep going from here!

B:

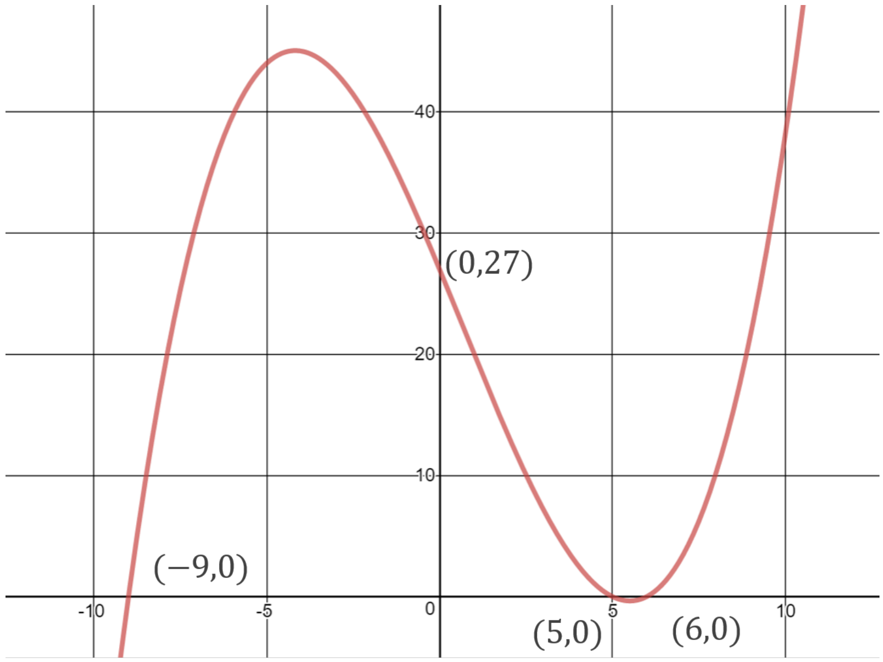

Figure 26: {{s3_url}}/mc-01/ cubic_mc1_b.png

Explanation:

No, this is not correct. But you are very close!

Although all the x-intercepts are at the right locations, the sign of the graph is incorrect.

Remember, if there is **no negative** in front of the equation, then the graph should end in the **top right**. If there is a negative sign in front of the equation, then the graph should end in the **bottom right**.

In this case, we have a -0.1 in front so we should have a graph that ends in the bottom right, not top right.

C:

Explanation:

No, this is not correct. But you are very close!

Although all the x-intercepts are at the right locations, the sign of the graph is incorrect.

Remember, if there is no negative in front of the equation, then the graph should end in the **top right**. If there is a negative sign in front of the equation, then the graph should end in the **bottom right**.

In this case, we have a -0.1 in front, we should have a graph that ends in the bottom right, not top right.

Also, check the location of the y-intercept. To find the y-intercept, we can make $x = 0$ and then solve for y. Let me help you get started:

$$y = -0.1(0-5)(0+9)(0-6)y \qquad = -0.1(-5)(9)(-6)$$

And keep going from here!

D:

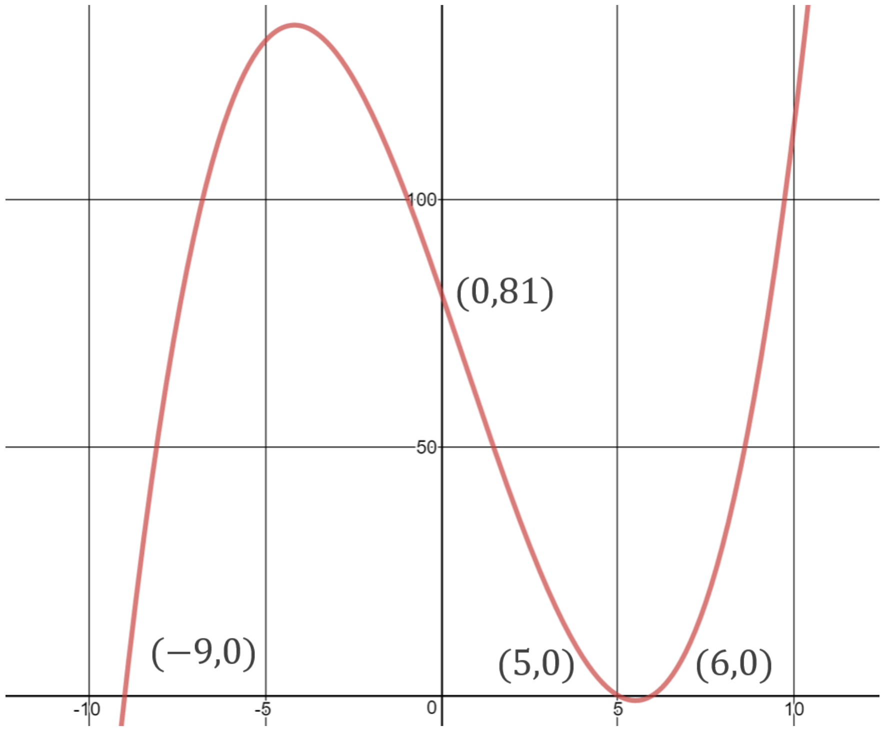

Figure 27: {{s3_url}}/mc-01/ cubic_mc1_c.png

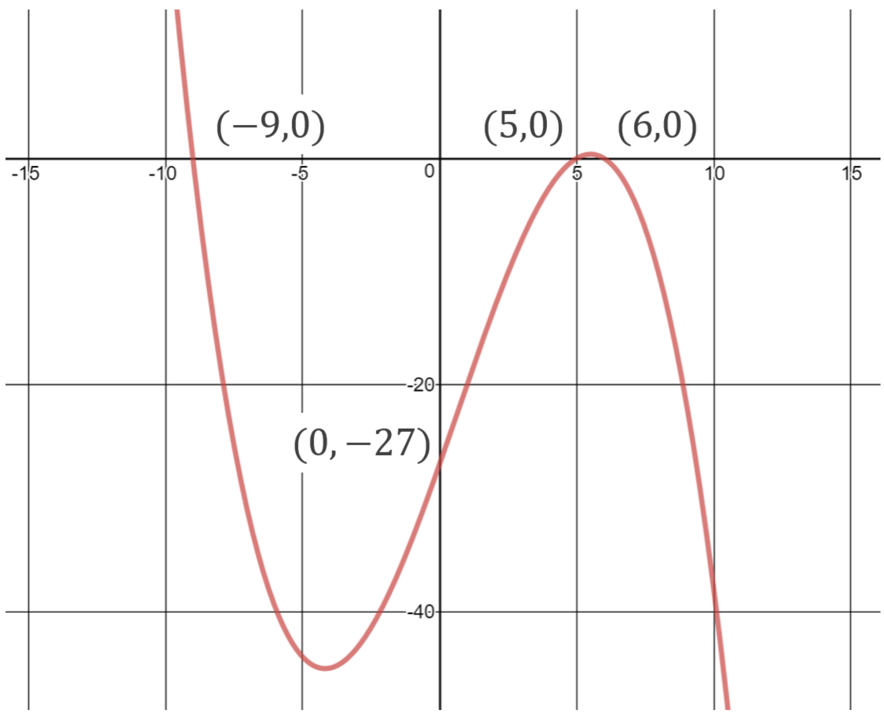

Figure 28: {{s3_url}}/mc-01/ cubic_mc1_d.png

Explanation:

Yes, this is correct!

From the equation, we can see that it is a negative graph, so the graph should go to the bottom right.

To find the x-intercepts, we need to use make $y = 0$ and then use the null factor law.

$-0.1(x-5)(x+9)(x-6) = 0 \Rightarrow x-5=0 \quad o \quad r \quad x+9=0 \quad or \quad x-6=0 \therefore x=5 \quad or \quad x=-9 \quad or \quad x=6$

Hence, the x-intercepts are located at $(-9, 0)$, $(5, 0)$ and $(6, 0)$.

To find the y-intercept, we need to make $x = 0$ and solve for y.

$$y = -0.1(0-5)(0+9)(0-6)y \qquad = -0.1(-5)(9)(-6)y = -27$$

Hence, the y-intercept is at $(0, -27)$.

With all the hints we can find from this cubic equation in factorised form, the graph in this option is the only one that satisfies all the requirements.

Question 1 Hint Menu

Copy/paste the following tutorials:

- *How do we sketch cubic functions?*

Question analysis

To see which one is correct. We need to know three things:

- shape: positive graph ends in the top right and negative graph ends in the bottom right.
- x-intercepts: by making y=0 and solving for x
- y-intercept: by making x=0 and solving for y

Without considering all this information, we cannot correctly sketch the graph. So let's find them all, then see which graph satisfy all the requirements.

Question 2:

Title: Identify the number of stationary points on a quartic

Tags: Turning points, stationary points, quartic

Prompt:

The number of stationary points that the graph of $y = -(x+2)(x-1)^2(x-3)$ contains on its maximal domain is/are:

Multiple choice options: (write 'correct' in brackets after the correct option)

A:

1

Explanation:

That's not quite right.

Just because the factorised equation of a quartic has one quadratic factor doesn't mean that it's the only turning point on the graph. We have to look at the entire graph before we can make that conclusion.

Keep in mind that the only way for a quartic to have one stationary point is if it can be written in power form. Since this factorised form implies that we have three x-intercepts, we won't be able to rewrite this in power form.

Another way of approaching this question is to visualise the graph. As with many aspects of Methods, if you don't know where to start it's a good idea to visualise what you're dealing with.

Figure 29: {s3_url}}/mc-02/quartics_mc02_graph1.png

As we're sketching this, especially by hand, it's important to remember the following:

- The sign of a, the constant in front of the factors, tells us the shape of the graph. Since the sign is negative, the graph is reflected upside-down.
- Linear factors tell us that the graph **cuts through** the x-axis at these points, just like a linear graph.
- Quadratic factors tell us that the graph **smoothly touches** the x-axis at these points, just like a parabola.

We can read the number of stationary points straight off this graph by looking for the points on the graph where it becomes flat. Have a look and try the question again!

B:

2

Explanation:

This isn't quite right.

Quartics with two stationary points are not as common as other types of quartics, though we can still come across them. Here's an example of a quartic graph with two stationary points.

Figure 30: {{s3_url}}/mc-02/quartics_mc02_graph2.png

We can see there are two stationary points here, one is a minimum turning point, and the other is a stationary point of inflection.

The same process of looking at the graph and identifying the stationary points can be done with our function, too.

Figure 31: {{s3_url}}/mc-02/quartics_mc02_graph1.png

As we're sketching this, especially by hand, it's important to remember the following:

- The sign of a, the constant in front of the factors, tells us the shape of the graph. Since the sign is negative, the graph is reflected upside-down.
- Linear factors tell us that the graph **cuts through** the x-axis at these points, just like a linear graph.
- Quadratic factors tell us that the graph **smoothly touches** the x-axis at these points, just like a parabola.

We can read the number of stationary points straight off this graph by looking for the points on the graph where it becomes flat. Have a look and try the question again!

C: (correct)

3

Explanation:

Yes, that's it! That's the correct answer.

We can easily see the number of stationary points a quartic has by examining its graph.

Figure 32: {{s_url}}/mc-02/quartics_mc02_graph1.png

As we're sketching this, especially by hand, it's important to remember the following:

- The sign of a, the constant in front of the factors, tells us the shape of the graph. Since the sign is negative, the graph is reflected upside-down.
- Linear factors tell us that the graph **cuts through** the x-axis at these points, just like a linear graph.
- Quadratic factors tell us that the graph **smoothly touches** the x-axis at these points, just like a parabola.

We can read the number of stationary points straight off this graph by looking for the points on the graph where it becomes flat. You've done this correctly!

D:

4

Explanation:

This option actually isn't possible for a quartic!

Note that a quartic can only have up to four x-intercepts, but only **up to three** stationary points. It's best to sketch the graph and examine the number of points where the graph becomes flat.

Figure 33: {{s3_url}}/mc-02/quartics_mc02_graph1.png

As we're sketching this, especially by hand, it's important to remember the following:

- The sign of a, the constant in front of the factors, tells us the shape of the graph. Since the sign is negative, the graph is reflected upside-down.
- Linear factors tell us that the graph **cuts through** the x-axis at these points, just like a linear graph.
- Quadratic factors tell us that the graph **smoothly touches** the x-axis at these points, just like a parabola.

We can read the number of stationary points straight off this graph by looking for the points on the graph where it becomes flat. Have a look and try the question again!

Question 2 Hint Menu

Copy/paste the following tutorials:

- *How do we know what quartic functions look like.*

Question analysis

This question is asking us to identify how many **stationary points** the quartic function has. In other words, at how many points does the graph become momentarily flat?

To answer this question we need a general idea of what the shape of the graph is. The fact that the function is in factorised form helps us with this. We can use the factors to identify where the graph hits the x-axis, meaning we can find the x-intercepts straight away.

Using these x-intercepts we can roughly sketch the graph. This sketch of the graph will help us in finding the point(s) where the graph is stationary, i.e. the stationary points!

Question 3:

Title: Identify the characteristics of higher power polynomials.

Tags: *Higher power polynomials, factorised form*

Prompt:

Which of the following statements about $y = (x+7)^3 (1-x)(x-3)^2$ is correct?

Multiple choice options: (write 'correct' in brackets after the correct option)

A:

Since the degree of this polynomial is 6, it has 6 x-intercepts.

Explanation:

This is not correct.

Polynomials with a degree of n can have **up to** n solutions. However, they can also have less than that. In this case, although the degree is 6, there is one cubic factor and one quadratic factor. This reduces the number of x-intercepts. Keeping this in mind, have another go at answering the question!

B:

There are two x-intercepts at the point $(3, 0)$.

Explanation:

No, this isn't the right logic to use.

The factor $(x-3)^2$ is a quadratic factor. Although this is the product of two linear factors, it still corresponds to one x-intercept. This x-intercept, however, smoothly touches the graph and turns back again, just like a parabola. Hence, there is only one x-intercept at $(3, 0)$.

C:

This is a positive graph, so the graph continues into the top-right quadrant.

Explanation:

Although there is no negative sign at the front, when we rearrange the equation we end up with:

$$y = (x+7)^3(1-x)(x-3)^2 y \qquad = (x+7)^3(-x+1)(x-3)^2 y = -(x+7)^3(x-1)(x-3)^2$$

Hence, it's actually a negative graph that should end in the bottom right.

D: (correct)

There is a stationary point of inflection at the point $(-7, 0)$.

Explanation:

Yes, this is correct!

A cube on the factor $(x+7)$ makes it a cubic factor. Thus, the graph will look like a standard cubic at the x-intercept $(-7, 0)$ and will have a stationary point of inflection at that point.

Question 3 Hint Menu

Copy/paste the following tutorials:

- *What about polynomials with a degree greater than four?*

Question analysis

To do this question, we have to understand the definition of each power. What does a factor with a power of two mean? What does a factor with a power of three mean?

Also knowing how to define a positive and negative graph can help us eliminate some options! If you are not sure what to do, revisit the tutorial, *What about polynomials with a degree greater than four?*.

Question 4:

Title: Identify the number of intercepts and stationary points on a cubic

Tags: x-intercepts, stationary points, cubics

Prompt:

Consider the following function.

$$f : R \to R, f(x) = x^3 - 3x^2 + 4x$$

The number of x-intercepts and stationary points that the graph of $y = f(x)$ has respectively are:

Multiple choice options: (write 'correct' in brackets after the correct option)

A: (correct)

1 and 0

Explanation:

Great work, that's correct! There are two parts to this question, so let's summarise them below. Don't forget that cubic graphs can have **up to three** x-intercepts and **up to two** stationary points.

x-intercepts

Finding x-intercepts involves subbing in $f(x) = 0$. Typically, we solve this equation by factorising the polynomial.

$$0 = x^3 - 3x^2 + 4x0 \qquad = x(x^2 - 3x + 4) \Rightarrow x = 0 \ or \ x^2 - 3x + 4 = 0$$

Note that we can't solve $x^2 - 3x + 4 = 0$. Calculating its discriminant gives us:

$$\Delta = (-3)^2 - 4(1)(4) = -7 \Rightarrow \Delta \qquad < 0$$

A negative discriminant means that the quadratic has no solutions. Thus, the only x-intercept is at $x = 0$.

Stationary points

Looking for stationary points is best done on a graph. As we haven't specifically covered differentiation, we can't use it to find stationary points just yet. Sketching the graph on a CAS gives us something that looks like this.

Figure 34: {{s3_url}}/mc-04/mc04_graph1.png

We can see that there is in fact no stationary point on this graph. In other words, the graph doesn't have a point where it momentarily has a gradient of zero. Hence, we can say that there are no stationary points on this graph.

B:

1 and 1

Explanation:

You almost had it, though it's not quite correct!

You've got the number of x-intercepts right, though the number of stationary points isn't right.

Stationary points

The best method to identify the number of stationary points is to look at the graph of the function. Although we learn how to calculate stationary points exactly using differentiation, we haven't covered the topic yet so we'll leave it aside.

Sketch the graph on a CAS and see what its shape looks like. What you're looking for are the points on the graph where it is **momentarily flat**. Count the number of points and select your new answer!

C:

2 and 2

Explanation:

Both the number of x-intercepts and stationary points aren't right, so let's take a look at the two parts of this question individually to see where we went wrong. Keep in mind that cubic graphs can have **up to three** x-intercepts and **up to two** stationary points.

x-intercepts

Finding x-intercepts involves subbing in $f(x) = 0$. Typically, we solve this equation by factorising the polynomial.

$$0 = x^3 - 3x^2 + 4x0 \qquad = x(x^2 - 3x + 4)$$

Continue this working out by factorising further and use the null factor law to solve the equation. The number of solutions to this equation tells us how many x-intercepts we have.

Stationary points

The best method to identify the number of stationary points is to look at the graph of the function. Although we learn how to calculate stationary points exactly using differentiation, we haven't covered the topic yet so we'll leave it aside.

Sketch the graph on a CAS and see what its shape looks like. What you're looking for are the points on the graph where it is **momentarily flat**. Count the number of points and select your new answer!

D:

3 and 2

Explanation:

Both the number of x-intercepts and stationary points aren't quite right, so let's take a look at the two parts of this question individually to see where we went wrong. Keep in mind that cubic graphs can have **up to three** x-intercepts and **up to two** stationary points. This answer represents the maximum number of x-intercepts and stationary points a cubic can have, though not all cubics (including this one) have this many intercepts and stationary points.

x-intercepts

Finding x-intercepts involves subbing in $f(x) = 0$. Typically, we solve this equation by factorising the polynomial.

$$0 = x^3 - 3x^2 + 4x0 \qquad\qquad = x(x^2 - 3x + 4)$$

Continue this working out by factorising further and use the null factor law to solve the equation. The number of solutions to this equation tells us how many x-intercepts we have.

Stationary points

The best method to identify the number of stationary points is to look at the graph of the function. Although we learn how to calculate stationary points exactly using differentiation, we haven't covered the topic yet so we'll leave it aside.

Sketch the graph on a CAS and see what its shape looks like. What you're looking for are the points on the graph where it is **momentarily flat**. Count the number of points and select your new answer!

Question 4 Hint Menu

Copy/paste the following tutorials:

- *How do we sketch cubic functions?*

Question analysis

This question is really asking us for two answers: the first one being the number of x-intercepts; and the second being the number of stationary points.

When it comes to **x-intercepts**, we can find them in two ways:

- Solve the equation when $f(x) = 0$, then count the number of solutions.

- Sketch the graph and count the number of x-intercepts you see.

Either of these methods will help us to find the total number of x-intercepts.

On the other hand, **stationary points** are best found by examining the graph itself. When looking at the graph, identify and count the number of points where the graph is momentarily flat; i.e. has a gradient of zero.

There is another way of finding the number of stationary points - differentiation. We haven't covered this topic just yet, but as an extension you can try to use your Unit 2 knowledge of differentiation to try this method!

Question 5:

Title: Identify a region of a graph

Tags: *higher power polynomial, factorised form*

Prompt:

The rule of a quintic function is given by $p(x) = x^2(x^2 - 1)(x - 2)$. For what values of x is $p(x) < 0$?

Multiple choice options: (write 'correct' in brackets after the correct option)

A: (correct)

$x \in (-\infty, -1) \cup (1, 2)$

Explanation:

Yes, that's correct! The best way to deal with this question is to visualise it — that means we're going to graph it!

If we try sketching $y = p(x)$ by hand, we'll notice that there's a strange $(x^2 - 1)$ factor in the rule. Notice that we can further factorise this using the difference of two squares.

$$p(x) = x^2(x^2 - 1)(x - 2) \Rightarrow p(x) \qquad = x^2(x + 1)(x - 1)(x - 2)$$

Now that it's in factorised form, we can sketch it by hand.

Figure 35: {{s3_url}}/mc-05/mc05_graph1.png

$p(x) < 0$ means that we're looking for the parts of the graph where it is less than zero. In other words, we're looking for the sections of the graph where it is negative, or **below the x-axis**.

We can see the graph comes up from below all the way until $x = -1$, and once again dips below the x-axis between $x = 1$ and $x = 2$. Don't forget that when writing the set of values, the brackets we use make a difference; in this case, we use exclusive brackets since we want the graph to be strictly less than zero, not less than or equal to ($p(x) < 0$, not $p(x) \leq 0$).

B:

$x \in (1, 2)$

Explanation:

Almost there!

You've identified one portion of the graph that satisfies $p(x) < 0$, but there's another region! The best way to go about this question is to **visualise the graph**.

If we try sketching $y = p(x)$ by hand, we'll notice that there's a strange $(x^2 - 1)$ factor in the rule. Notice that we can further factorise this using the difference of two squares.

$$p(x) = x^2(x^2 - 1)(x - 2) \Rightarrow p(x) \qquad = x^2(x + 1)(x - 1)(x - 2)$$

Now that it's in factorised form, we can sketch it by hand.

Figure 36: {{s3_url}}/mc-05/mc05_graph1.png

$p(x) < 0$ means that we're looking for the parts of the graph where it is less than zero. In other words, we're looking for the sections of the graph where it is negative, or **below the x-axis**.

The space in $x \in (1, 2)$ is definitely below the x-axis, but what other region is negative too?

C:

$x \in (-1, 0) \cup (0, 1) \cup (2, \infty)$

Explanation:

Looks like you've made a silly error and identified the opposite regions of what we're actually try to find!

What we've found here are actually the values of x for which $p(x) > 0$, rather than $p(x) < 0$. A good way to avoid making such mistakes is to sketch the graph to help **visualise the scenario**.

If we try sketching $y = p(x)$ by hand, we'll notice that there's a strange $(x^2 - 1)$ factor in the rule. Notice that we can further factorise this using the difference of two squares.

$$p(x) = x^2(x^2 - 1)(x - 2) \Rightarrow p(x) \qquad = x^2(x + 1)(x - 1)(x - 2)$$

Now that it's in factorised form, we can sketch it by hand.

Figure 37: {{s3_url}}/mc-05/mc05_graph1.png

$p(x) < 0$ means that we're looking for the parts of the graph where it is less than zero. In other words, we're looking for the sections of the graph where it is negative, or **below the x-axis**.

This chosen answer is what we'd get if we look at the graph above the x-axis. Highlight the values of x for which the graph is below the x-axis and we'll have our answer!

D:

$x \in R$

Explanation:

This isn't right, we need to work a bit harder to get the answer!

$x \in R$ represents the domain of p, meaning that $d_p = R$. However, this doesn't necessarily represent the values of x for which $p(x) < 0$. The best approach is to **visualise** the graph of $y = p(x)$ and use it to interpret the question.

If we try sketching $y = p(x)$ by hand, we'll notice that there's a strange $(x^2 - 1)$ factor in the rule. Notice that we can further factorise this using the difference of two squares.

$$p(x) = x^2(x^2-1)(x-2) \Rightarrow p(x) \qquad = x^2(x+1)(x-1)(x-2)$$

Now that it's in factorised form, we can sketch it by hand.

Figure 38: {{s3_url}}/mc-05/mc05_graph1.png

$p(x) < 0$ means that we're looking for the parts of the graph where it is less than zero. In other words, we're looking for the sections of the graph where it is negative, or **below the x-axis**.

Highlight the values of x for which the graph is below the x-axis and we'll have our answer!

Question 5 Hint Menu

Copy/paste the following tutorials:

- *What about polynomials with a degree greater than four?*

Question analysis

Answering this question involves understanding what the statement $p(x) < 0$ means. $p(x)$ is the function, and if we sketch out $y = p(x)$ we'd get a quintic polynomial. $p(x) < 0$ tells us that we're looking for all the y values of the graph that are less than zero. Don't forget that if something is less than zero, it means it's negative.

To summarise, we can do the following to help answer this question:

- Sketch the graph of $y = p(x)$.
- Look for the parts of the graph that are negative, or below the x-axis.

The values of x for the regions below the x-axis are the answer to this question!

Question 6:

Title: Calculate/identify/deduce

Tags:

Multiple choice options: (write ‘correct’ in brackets after the correct option)

A:

Explanation:

B:

Explanation:

C:

Explanation:

D:

Explanation:

Question 7:

Title: Calculate/identify/deduce

Tags:

Multiple choice options: (write 'correct' in brackets after the correct option)

A:

Explanation:

B:

Explanation:

C:

Explanation:

D:

Explanation:

Question 8:

Title: Calculate/identify/deduce

Tags:

Multiple choice options: (write ‘correct’ in brackets after the correct option)

A:

Explanation:

B:

Explanation:

C:

Explanation:

D:

Explanation:

Question 9:

Title: Calculate/identify/deduce

Tags:

Multiple choice options: (write 'correct' in brackets after the correct option)

A:

Explanation:

B:

Explanation:

C:

Explanation:

D:

Explanation:

Question 10:

Title: Calculate/identify/deduce

Tags:

Multiple choice options: (write ‘correct’ in brackets after the correct option)

A:

Explanation:

B:

Explanation:

C:

Explanation:

D:

Explanation:

Question 1

Summary Title: Calculate and work with the components of a higher power polynomial.

Tag: *Higher power polynomials, solving, sketching*

Source: Connect

Context: Consider the following equation:

$$f(x) = (ax + b)^4 - 1$$

Question Part (a):

Prompt:

In expanded form, the coefficient of x is two times the negative of the coefficient of x^2. Also, the coefficient of x^3 is -12.

Hence, find a and b.

Display type: Short answer-box

No. marks: 3

Question Part (b):

Prompt:

Sketch the graph of $y = f(x)$, labelling any stationary points and axial intercepts in coordinate form.

Display type: Short answer-box

No. marks: 4

Question Part (a)

Marking step 1

Description of marks: Find the values of a and b.

[1 mark for correctly expanding the expression]

[1 mark for writing out the correct simultaneous equations]

[1 mark for solving for a and b]

Option type: Multiple (checkboxes)

Marking prompt : Which of the following calculations did you perform?

Options:

- I correctly expanded the quartic equation
- I wrote two simultaneous equations: $4a^3b = -12$ and $4ab^3 = -12a^2b^2$.
- I solved for a and b and get $a = 1$ and $b = -3$ as my answers.

Worked solution:

We're told various bits of information about the coefficients in expanded form, so a logical first step would be to expand the equation:

$$f(x) = (ax+b)^4 - 1 \qquad = a^4x^4 + 4a^3bx^3 + 6a^2b^2x^2 + 4ab^3x + b^4 - 1$$

From the question, we know that the coefficient of x is two times the negative of the coefficient of x^2. So we can write:

$$4ab^3 = -2(6a^2b^2)ab^3 \qquad = -3a^2b^2b = -3a \tag{1}$$

The coefficient of x^3 is -12 can be written as:

$$4a^3b = -12 \tag{2}$$

What you should recognise now is that we have two simultaneous equations! Let's use the substitution method to solve them: Substitute $b = -3a$ into equation (2)

$$4a^3(-3a) = -12 - 12a^4 \qquad = -12a = 1$$

Substitute $a = 1$ into $b = -3a$

$$b = -3(1)b \qquad = -3$$

Question Part (b)

Marking step 1

Description of marks: Sketch the graph

[1 mark for sketching the graph with the correct shape: positive, U-shaped parabola and intersects the x-axis twice]

[1 mark for labeling the stationary point in coordinate form]

[1 mark for labeling both x-intercepts in coordinate form]

[1 mark for labeling the y-intercept in coordinate form]

Option type: Multiple (checkboxes)

Marking prompt: Which of the following calculations did you perform?

Options:

- I sketched the quartic as a positive, U-shaped parabola that intersects the x-axis twice.
- I labeled the turning point of the parabola with the coordinates $(3, -1)$.
- I labeled the two x-intercepts in coordinate form: one with $(2, 0)$ and the other with $(4, 0)$.
- I labeled the y-intercept with the coordinates $(0, 80)$, which was above the x-axis.

Worked solution:

To get the x-axis, we can make $y = 0$ and solve for x:

$$(x-3)^4 - 1 = 0(x-3)^4 \qquad = 1x - 3 = \pm 1x = 2 \quad o \qquad r \quad x = 4$$

So the x-intercepts are $(2, 0)$ and $(4, 0)$.

Figure 39: {{s3_url}}/sa-01/ cubic_sa1_graph1.png

To get the y-intercept:

$$y = (0-3)^4 - 1y \qquad = 81 - 1y = 80$$

Turning point is given by $(3, -1)$.

Question 2

Summary Title:

Tag:

Source: Connect

Context:

Question Part (a):

Prompt:

Display type: Short answer-box

No. marks:

Question Part (b):

Prompt:

Display type: Short answer-box

No. marks:

Question Part (c):

Prompt:

Display type: Short answer-box

No. marks:

Question Part (d):

Prompt:

Display type: Short answer-box

No. marks: