Econometrics Homework 1

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(Textbook Question 10 on page 55)

Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the OLS intercept and slope estimators, respectively, and let $\overline{u}$ be the sample average of the errors (not the residuals!).

(i) Show that $\hat{\beta}_1$ can be written as $\hat{\beta}_1 = \beta_1 + \sum_{i=1}^n w_i u_i$, where $w_i = \frac{d_i}{SST_x}$ and $d_i = x_i - \overline{x}$.

The population model regression is $y_i = \beta_0 + \beta_1 x_i + u_i$ and the sample regression model is $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{u}_i$.

To find the estimate of $\hat{\beta}_1$ we solve below:

$y_i = \beta_0 + \beta_1 x_i + u_i$ $\qquad$ $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{u}_i$

$\hat{\beta}_1 = \frac{Cov(y_i, x_i)}{Var(x_i)} = \frac{\Sigma_{i=1}^n (x_i - \overline{x}) y_i}{\Sigma_{i=1}^n (x_i - \overline{x})^2} = \frac{\Sigma_{i=1}^n (x_i - \overline{x})(\beta_0 + \beta_1 x_i + u_i)}{SST_x}$

$\hat{\beta}_1 = \frac{\Sigma_{i=1}^n (x_i - \overline{x}) \beta_0 + \beta_1 \Sigma_{i=1}^n (x_i - \overline{x}) x_i + \Sigma_{i=1}^n (x_i - \overline{x}) u_i}{SST_x}$

Also, $\Sigma_{i=1}^n (x_i - \overline{x}) = 0$ $\qquad$ $\Sigma_{i=1}^n (x_i - \overline{x}) x_i = \Sigma_{i=1}^n (x_i - \overline{x})(x_i - \overline{x}) = \Sigma_{i=1}^n (x_i - \overline{x})^2 = SST_x$

Which gives $\hat{\beta}_1 = \frac{\Sigma_{i=1}^n (x_i - \overline{x}) \beta_0 + \beta_1 \Sigma_{i=1}^n (x_i - \overline{x}) x_i + \Sigma_{i=1}^n (x_i - \overline{x}) u_i}{SST_x} = \frac{0 + \hat{\beta}_1 SST_x + \Sigma_{i=1}^n (x_i - \overline{x}) u_i}{SST_x}$

$\hat{\beta}_1 = \beta_1 + \frac{\Sigma_{i=1}^n (x_i - \overline{x}) u_i}{SST_x} = \beta_1 + \Sigma_{i=1}^n w_i u_i$

Since $w_i = \frac{d_i}{SST_x}$ and $d_i = x_i - \overline{x}$

(ii) Use part (i), along with $\Sigma_{i=1}^n w_i = 0$, to show that $\hat{\beta}_1$ and $\overline{u}$ are uncorrelated. [Hint: You are being asked to show that $E[(\hat{\beta}_1 - \beta_1) \cdot \overline{x}] = 0$.]

Since $\Sigma_{i=1}^n w_i = 0$ and $\hat{\beta}_1 = \beta_1 + \Sigma_{i=1}^n w_i u_i$, the correlation between $\hat{\beta}_1$ and $\overline{u}$ can be shown by: $Corr(\hat{\beta}_1, \overline{u}) = \frac{Cov(\hat{\beta}_1, \overline{u})}{\sigma_{\hat{\beta_1}} \sigma_{\overline{u}}} = \frac{E[[(\hat{\beta}_1 - \beta_1)(\overline{u} - \overline{\overline{u}})]}{\sigma_{\hat{\beta_1}} \sigma_{\overline{u}}}$

And since $\overline{\overline{u}} = \overline{u}$, this means that $E[\hat{\beta}_1 - \beta_1)(\overline{u} - \overline{\overline{u}})] = 0$ and the $Corr(\hat{\beta}_1, \overline{u}) = 0$

(iii) Show that $\hat{\beta}_0$ can be written as $\hat{\beta}_0 = \beta_0 + \overline{u} - (\hat{\beta}_1 - \beta_1)\overline{x}$.

$y_i = \beta_0 + \beta_1 x_i + u_i$ and $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{u}_i$

Taking the average gives $\overline{y} = \beta_0 + \beta_1 \overline{x} + \overline{u} = \hat{\beta}_0 + \hat{\beta}_1 \overline{x}$

$\hat{\beta}_0 = \beta_0 + \beta_1 \overline{x} + \overline{u} - \hat{\beta}_1 \overline{x} = \beta_0 + \overline{u} - (\hat{\beta}_1 - \beta_1)\overline{x}$

(iv) Use parts (ii) and (iii) to show that $Var(\hat{\beta}_0) = \frac{\sigma^2}{n} + \frac{(\sigma^2)(\overline{x})^2}{SST_x}$

Since $Corr(\hat{\beta}_1, \overline{u}) = E(\hat{\beta}_1 \overline{u}) = 0$ and $\hat{\beta}_0 = \beta_0 + \overline{u} - (\hat{\beta}_1 - \beta_1)\overline{x}$, the $Var(\hat{\beta}_1) = \frac{\sigma^2}{SST_x}$

This means that $\hat{\beta}_0 - \beta_0 = \overline{u} - (\hat{\beta}_1 - \beta_1)\overline{x}$

$E(\hat{\beta}_0 - \beta_0)^2 = E[\overline{u} - (\hat{\beta}_1 - \beta_1)\overline{x}]^2$

$= E[\overline{u}^2 - 2\overline{u}(\hat{\beta}_1 - \beta_1)\overline{x} + ((\hat{\beta}_1 - \beta_1)\overline{x})^2]$

$= E(\overline{u}^2) - 2E[\overline{u}(\hat{\beta}_1)\overline{x} + 2\overline{x}E(\overline{u}) + \overline{x}^2 E[(\hat{\beta}_1 - \beta_1)^2]$

$= \frac{\sigma^2}{n} - 0 + 0 + \overline{x}^2 Var(\hat{\beta}_1) = \frac{\sigma^2}{n} + \frac{\overline{x}^2\sigma^2}{SST_x}$

Thus $Var(\hat{\beta}_0) = E(\hat{\beta}_0 - \beta_0)^2 = \frac{\sigma^2}{n} + \frac{\overline{x}^2\sigma^2}{SST_x}$

(v) Do the algebra to simplify the expression in part (iv) to equation (2.58) [Hint: $\frac{SST_x}{n} = n^{-1}\Sigma_{i=1}^n x_i^2 - (\overline{x})^2$**.]**

Because $\frac{SST_x}{n} = n^{-1}\Sigma_{i=1}^n x_i^2 - (\overline{x})^2$, $\quad Var(\hat{\beta}_0) = \frac{\sigma^2}{n} + \frac{\sigma^2(\overline{x})^2}{SST_x} = \frac{\sigma^2}{n} + \frac{\sigma^2(\overline{x})^2}{\Sigma_{i=1}^n x_i^2 - n(\overline{x})^2}$

$SST_x = \Sigma_{i=1}^n (x_i - \overline{x})^2 = \Sigma_{i=1}^n x_i^2 - 2\overline{x}\Sigma_{i=1}^n x_i + n\overline{x}^2$

$= \Sigma_{i=1}^n x_i^2 - 2\frac{\Sigma_{i=1}^n x_i}{n}\Sigma_{i=1}^n x_i + n(\frac{\Sigma_{i=1}^n x_i}{n})^2 = \Sigma_{i=1}^n x_i^2 - 2\frac{(\Sigma_{i=1}^n x_i)^2}{n} + \frac{(\Sigma_{i=1}^n x_i)^2}{n}$

$= \Sigma_{i=1}^n x_i^2 - \frac{(\Sigma_{i=1}^n x_i)^2}{n} = \Sigma_{i=1}^n x_i^2 - n(\overline{x})^2$

The gives $\Sigma_{i=1}^n (x_i - \overline{x})^2 = \Sigma_{i=1}^n (x_i^2) - \frac{(\Sigma_{i=1}^n x_i)^2}{n} = \Sigma_{i=1}^n x_i^2 - n(\overline{x})^2$

$Var(\hat{\beta}_0) = \frac{\sigma^2}{n} + \frac{\sigma^2(\overline{x})^2}{SST_x} = \frac{\sigma^2}{n} + \frac{\sigma^2(\overline{x})^2}{\Sigma_{i=1}^n x_i^2 - n(\overline{x})^2}$

$= \frac{\sigma^2}{n}(1 + \frac{(\overline{x})^2}{\Sigma_{i=1}^n x_i^2 - n(\overline{x})^2})$

$= \frac{\sigma^2}{n}(\frac{\Sigma_{i=1}^n x_i^2 - n(\overline{x})^2 + n(\overline{x})^2}{\Sigma_{i=1}^n x_i^2 - n(\overline{x})^2)}) = \frac{\sigma^2 n^{-1}\Sigma_{i=1}^n x_i^2}{\Sigma_{i=1}^n s_i^2 - n(\overline{x})^2}$

$\Sigma_{i=1}^n (x_i - \overline{x})^2 = \Sigma_{i=1}^n x_i^2 - \frac{(\Sigma_{i=1}^n x_i)^2}{n} = \Sigma_{i=1}^n x_i^2 - n(\overline{x})^2$

$Var(\hat{\beta}_0) = \frac{\sigma^2 n^{-1}\Sigma_{i=1}^n x_i^2}{\Sigma_{i=1}^n (x_i - \overline{x})^2}$