Exact solution of Einstein's field Equation by metric of Hyperbolic 3D Space-time

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Abstract

n-dimensional **anti-de Sitter space** (AdSn) is a maximally symmetric Lorentzian manifold with constant negative scalar curvature.

Manifolds of constant curvature are most familiar in the case of two dimensions, where the surface of a sphere is a surface of constant positive curvature, a flat (Euclidean) plane is a surface of constant zero curvature, and a hyperbolic plane is a surface of constant negative curvature.

Einstein's general theory of relativity places space and time on equal footing, so that one considers the geometry of a unified spacetime instead of considering space and time separately. The cases of spacetime of constant curvature are de Sitter space (positive), Minkowski space (zero), and anti-de Sitter space (negative). As such, they are exact solutions of Einstein's field equations for an empty universe with a positive, zero, or negative cosmological constant, respectively.

I solved the Einstein field equation for hyperboilic spacetime with metric signature (+,-,-). i found that hyperbolic geometry has comnstant negative curvature which resembles with definition of anti-de sitter spacetime means this is an considerable exact solution of einstein field equation

Mathematical work:

A metric for poincare half model hyperbolic 3D space time can be written as follows;

$ds^2 = -dt^2 + \frac{dx^2}{y^2} + \frac{dy^2}{y^2}$

Now comparing this with general spacetime interval,

$ds^2 = g_{\alpha\beta} dx^\alpha dy^\beta$

where $g_{\alpha\beta}$ is a metric tensor of index 3,[where $\alpha, \beta = 1, 2, 3$]

$g_{11} = -1$, $g_{22} = \frac{1}{y^2}$, $g_{33} = \frac{1}{y^2}$ and rest of all mertrics is zero's

Now we can find all components of christoffel's symbols because christoffel's symbols are partial derivative of metrics given we can use a formulea,

we got,

$\gamma^2_{23} = \frac{1}{y}$, $\gamma^2_{32} = -\frac{1}{y}$, $\gamma^3_{22} = -\frac{1}{y}$, $\gamma^3_{33} = -\frac{1}{y}$ Rest of all components in christoffels symbols are automatically zero because all other metric are also zero's

Now we have to find Ricci tensor's components by tedious formulea but i am skipping this and writing ricci components,R

$R_{\alpha\beta}$

$R_{11}=R_{12}=R_{13}=R_{21}=R_{23}=R_{31}=R_{32}=0$, But we have some non-zer components too which is

$R_{22}=-\frac{1}{y^2}$, $R_{33}=-\frac{1}{y^2}$

Now i am finding Reimann tensor component after huge calculation i finally reached there is only one non zero Reimann component which is

$R_{2323}=-\frac{1}{y^4}$

Curvature scalar R can be found by following relations :

$R= g^{11}R_{11}+g^{22}R_{22}+g^{33}R_{33}$

$=-1\,(0)+y^2\left(-\frac{1}{y^2}\right)+y^2\left(-\frac{1}{y^2}\right)$ $\quad$ [as $g^{\alpha\beta} = \frac{1}{g_{\alpha\beta}}$]

= -2

Finally we have reached last step by calculating Einstein's Tensor which is equal to

$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ $\quad$ [where $\alpha, \beta = 1, 2, 3$]

$G_{11} = R_{11} - \frac{1}{2}g_{11}R$

$= 0 - \frac{1}{2}(-1)(-2)$

=-1

$G_{22} = R_{22} - \frac{1}{2}g_{22}R$

$=-\frac{1}{y^2} - \frac{1}{2}\left(\frac{1}{y^2}\right)(-2)$

$= 0$

$G_{33} = R_{33} - \frac{1}{2}g_{33}R$

$= -\frac{1}{y^2} - \frac{1}{2}\left(\frac{1}{y^2}\right)(-2)$

$= 0$

rest of all components in Einstein's tensor is zero in matrix form we can wrirte this in

$[-1 \quad 0 \quad 0]$

We can calculate Einstein's equation from these data

$G = kT_{\alpha\beta}$

G Can be calculated by following relation: $\Sigma^3_{\alpha,\beta=1} G_{\alpha\beta}g^{\alpha\beta}$

$G = G_{11}g^{11} + G_{22}g^{22} + G_{33}g^{33}$

$= -1\,(-1) + 0 + 0 = 1$

final Einstein's field equation is

$kT_{\alpha\beta} = 1$(Sharan, 2009; Ahsan, 1995; Padmanabhan, 2014; 201, 2011a,b)

Conclusion:

I showed that hyperbolic geometry has constant negtaive curvature Thus, this metric **is** a solution for the vacuum Einstein equations. This seems strange because this is a space of constant negative curvature, which would seem to require some sort of matter/energy.

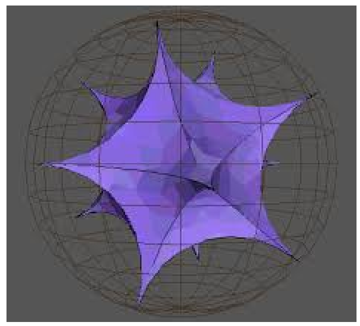

Figure 1: 1This is a caption

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