Infi - Finals

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Week 3 - Lecture 7

Minima and maxima of a set

Let A be a non-empty subset of the real numbers. Let $M,\ m\ \in A$. Let's suppose that:

- $M\ =\ \max(A)\ \Leftrightarrow M\ \in A\ \wedge\ M\ =\ \sup(A)$
- $m\ =\ \min\ (A)\ \Leftrightarrow m\ \in A\ \wedge$ m $=\ \inf(A)$

Corrolary on the how two numbers are equal

Let $x\ \in \mathbb{R}$ and $m,\ n\ \in \mathbb{Z}$. The numbers m and n are said to be equal if:
$x-1\ <\ n\ \leq x$ and $x-1\ <\ m\ \leq x$ In other words, two integers cannot be in between a real number and its successor, unless they are the same number.

On a upper bounded set

Let $b\ \in \mathbb{R}$. Then, the set $A_b\ = \{n\ \in \mathbb{Z}\ ,\ n \leq b\}$. In other words, b is a real number and A_b is the set of all integers smaller than or equal to b. Then, A_b is bounded from above.

Floor and ceiling functions

These two functions make it possible to "round" the value of a number.

Integral part or integer part

Let b be a real number. The value $\sup(A_b)$ will be called the integer part of b and will be represented by: $\lfloor b \rfloor$.

Theorem - b in relation to its integer part

The number between $\lfloor b \rfloor\ \leq b\ <\ \lfloor b \rfloor + 1$ is b itself and $\lfloor b \rfloor$ is an integer number.

Conclusion derived from the previous theorem

It follows from the previous that in every interval greater than one contains an integer. Let $x,\ y\ \in \mathbb{R}$. If $y\ >\ x+1$, then:
$\exists\ n\ \in \mathbb{Z}$, such as:
$x\ < n < y < x+1$

Density

Definition of density in sets

Let A be a subset of the real numbers. This set will be called dense in $\mathbb{R}$ if between all two real numbers there is an element of A. In mathematical notation,
$\forall x,\ y\ \in \mathbb{R}\ (x <\ y\ \Rightarrow \exists\ a\ \in A,\ x\ <\ a\ < y)$

Theorem - $\mathbb{Q}$ is dense in $\mathbb{R}$

How to prove

Week 4 - Lectures 8 and 9

Roots and powers

Roots

Square roots

Let Fbe an ordered field. Let's define the set A and B as:
$A\ =\ \left\{X\ \in\ \mathbb{F}\ :\ x\ >\ 0\ \wedge\ x^2\ \leq 2\right\}$
$B\ =\ \left\{x\ \ in\, \mathbb{F}:\ x\ >\ 0\ \wedge\ x^2\ \ \geq\ 2\right\}$
If there was $c\ \in \mathbb{F}$, such that:
$\forall\ a\ \in A,\ \forall\ b\ \in B,\ a \leq c \leq b$, then $c^2\ =\ 2$

1 Irrational numbers

A real number is called irrational if and only if: $x\ \in \mathbb{R},\ x\ \notin \mathbb{Q}$

n-ordered roots

For any $n\ \in \mathbb{N}$, let's define a^n. And for all real number a:

- $a^1\ =\ a$
- $a^{n+1}\ =\ a^n \cdot a$

Theorem - the existence of the nth root of a real number

For all positive real number and for all natural number, there is another real number, such that the first real number is the root of the latter. in mathematical notation:
$\forall a \in \mathbb{R},\ \forall n \in \mathbb{N},\ \exists x \in \mathbb{R},\ x^n = a$

Power of a number

Definition

Let a be a positive number. And let $r$be any rational number, which can be written as: $r = \frac{m}{n}$, such that $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. We define a^r as:
$a^r = a^{\frac{m}{m}} = (\sqrt[n]{a})^m$

Real functions of a real variable

Definition of a function

Let there be two sets A and B, which are subsets of $\mathbb{R}$. A function is defined by any law that connects every $a \in A$ to an element $b \in B$. The mathematical notation used is: $f: A \to B$.

Injective function

Definition

A function $f: A \to B$ will be called injective if and only if:
$\forall a_1,\ a_2 \in A,\ f(a_1) = f(a_2) \implies a_1 = a_2$
This is equivalent to:
$\forall a_1,\ a_2 \in A,\ a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

Surjective function

Definition

A function $f: A \to B$ is surjective if and only if:
$f(A) = Im(f) = B$
or:
$\forall b \in B\ \exists a \in A,\ f(a) = b$

Bijective function

Definition

A function $f: A \to B$ is said to be bijective if it is injective and surjective at the same time. In other words,
$\forall b \in B\ \exists! a \in A,\ f(a) = b$

Inverse function

Let f be a function such that: $f: \ A \rightarrow \ B$, its inverse will be $g: \ B \ \rightarrow \ A$.

Important characteristics of a surjective function

If f is surjective, then:

- f^{-1}, i.e. its inverse, is also surjective
- $\forall a \ \in A, \ f^{-1}\left(f\left(a\right)\right) \ = \ a$
- $\forall b \ \in B, \ f^{-1}\left(f\left(b\right)\right) \ = \ b$
- $\left(f^{-1}\right)^{-1} \ = \ f$

Limit of a real function

Point limit

Definition

Let $x_0$be any real number. Let's define two different typse of neighborhood.

1. Deleted neighborhood

2. Complete neighborhood

Complete neighborhood

$(x_0 \ - \ \delta, \ x_0 \ + \ \delta)$, such that $0 \ < \ \delta \ \in \mathbb{R}$.
In other words, a complete niehgborhood is a subset that can be defined as:
$(x_0 \ + \ \delta, \ x_0 \ - \ \delta) \ = \ \{x \ \in \mathbb{R}, \ |\, x - x_0 \,|< \delta\} \ = \ \{x \ \in \mathbb{R}, \ dist\left(x, \ x_0\right) < \delta\}$
It means that for every given real number in this neighborhood, the distance between this number and x_0 is very small, i. e. small than δ.

Deleted neighborhood

For every given real number in this neighborhood, the distance between this number and x_0 is somewhere between 0 and δ. Or:
$(x_0 - \delta, \ x_0 + \delta) \ = \ \{x \in \mathbb{R}, \ 0 < |x - x_0| < \delta\} \ = \ \{x \in \mathbb{R}, \ 0 < dist\left(x, \ X_0\right) < \delta\}$

Theorem on limits - Uniqueness of the limit on a point

Let f be a function, such that $f: \ D \ \rightarrow \ \mathbb{R}$. This function is well defined in the deleted neighborhood of $x_0 \ \in \mathbb{R}$. Let there be $L_1, \ L_2 \ \in \mathbb{R}$. If L_1 and L_2 are limits of the function on a point x_0, then $L_1 \ = \ L_2$.

Definition of Limit

$\lim_{x\to x_0} f(x)$ Let $f : D \to \mathbb{R}$ be a function defined on the deleted neughborhood of x_0, such that: $\exists\delta < 0,\ (x_0-\delta,\ x_0+\delta) - \{x_0\} \subseteq D$. The number $L \in \mathbb{R}$ is said to be the limit of the function f on the point x_0 if and only inf:
$\forall\epsilon > 0,\ \exists\delta > 0,\ \forall x \in D,\ 0 < |x - x_0| < \delta \ \Rightarrow 0, |f(x) - L| < \epsilon$

Week 5

Lecture 10

Direct conclusions from the definition of limits

Upper bounded set

Let $f : D \to \mathbb{R}$ be a function well defined in the deleted neighborhood of x_0. suppose that the limit $\lim_{x\to\mathbb{R}} f(x) = L \in \mathbb{R}$ exists. There is a neighborhood U of x_0 in which the function is limited from above. It is easier to think about it as U being a upper-bounded set. In mathematical notation,
$\{f(U) = \{f(x)|\, x \in U\}$

Corollary on the sign of L

Let $f : D \to \mathbb{R}$ be a function defined in a deleted neighborhood of x_0 and the limit $\lim_{x\to x_0} f$ exists. Then, if:

1. $L > 0 :\ \frac{L}{2} < f(x) < \frac{3L}{2}$
2. $L < 0 :\ \frac{3L}{2} < f(x) < \frac{L}{2}$

In plain English, if the limit L is positive, the function is bounded between half of L and three-halves of L. And, if the limit L is negative, the function is always bounded between three-halves of L and half the value of L.

Conclusion on the sign of the function

Let $f : D \to \mathbb{R}$ be a function defined in a deleted neighborhood of x_0 and the limit $\lim_{x\to x_0} f(x)$ exists. If L is finite, i.e. different from zero, there is a deleted neighborhood U of x_0, such that the function and the limit will have the same sign to all $x \in$ U. especially for $f(x) \neq 0,\ \forall x \in U$.
In other words, if the limit exists and it is finite, the function and the limit have the same sign for every value in the subset $U \subseteq D$, which is a subset of the domain.

Another conclusion on the sign of L

Given a function $f : D \to \mathbb{R}$ defined on a deleted neighborhood of x_0 and given that the limit $\lim_{x\to\mathbb{R}} f(x) = L \neq 0$. then there is a deleted neighborhood of x_0, where the sign of the function $f(x)$ is constant.

Corrolary on the behavior of two functions given their limits

Let's call every deleted neighborhood of x_0: $B_s^*(x_0)$. And let there be two real functions $f,\ g$. Let them be functions defined on a deleted neighborhood of x_0. Given that the limits exist both for f and g. Let there be $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$, as $L_1,\ L_2 \in \mathbb{R}$. If $L_1 < L_2$, then $f(x) < g(x)$, $\forall x \in B_s^*(x_0)$. Explaining this in plain English: given two bounded functions, if the upper bound of one function is greater than the other function upper bound, then all the values of the latter will be under the values of the former for every given number in the deleted neighborhood where the limits are defined.

Corollary similar to the one above

This corollary is similar to the previous one, but deals with partial order.
Given two functions $f,\ g$ well defined on the deleted neighborhood of x_0. And the limits of both functions exist and are finite. If $L_1 \le L_2$, it implies that $f(x) \le g(x)$ for every value of x in the deleted neighborhood of x_0.

Sandwich theorem

Let there $f,\ g,\ h$ be three real functions defined on a deleted neighborhood of $x_0 \in \mathbb{R}$. If all the following conditions apply:

1. There is a subset of $\mathbb{R}$, namely a deleted neighborhood of x_0, such that for every value of x in this subset, the following is true: $f(x) \le g(x) \le h(x)$
2. $f,\ g$ are bounded from above, i.e. the limits $\lim_{x\to \mathrm{x}_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist and are finite
3. $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x)$

Then, if the limit $\lim_{x\to x_0} g(x)$ exist and is finite, then:
$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x)$

Theorem on the multiplication of limits

Let there be $f,\ g$ be two real functions, which are defined on a deleted neighborhood of $x_0 \in \mathbb{R}$. And the limits of these functions exist and that $\lim_{x\to x_0} f(x) = 0$. Then. $\lim_{x\to x_0} (f\cdot g) = 0$.

Lecture 10

Arithmetics of limits

Lemma on the finite limits

Let f be a real function. Let x_0 be any real number and the limit for the function, exists is finite and equal to L, such that $L \in \mathbb{R}$. Then, the following conditions are equivalent:

1. $\lim_{x\to x_0} f = L$, the limit exists and is finite.
2. $\lim_{x\to x_0} (f - L) = 0$
3. $\lim_{x\to x_0} |f - L| = 0$, from the definition of limits.

Theorem on the Arithmetics of limits

Let f, be two real functions defined on any deleted vicinity of $x_0 \ \in \mathbb{R}$. Given that their limits exist and are final, like $\lim_{x\to x_0} f \ = \ L$ and $\lim_{x\to x_0} g \ = \ M$, such that $L, \ M \ \in \mathbb{R}$. Then, the following hold true:

1. Multiplication of a function by a scalar: $\forall c \ \in \mathbb{R}, \ \lim_{x\to x_0} c \cdot f \ \ = \ c \cdot L$
2. Adding f and g: $\lim_{x\to x_0} f + g \ = \ L + M$
3. Similarly, $\lim_{x\to x_0} f - g \ = \ L - M$
4. Multiplication of functions: $\lim_{x\to x_0} f \cdot g \ = \ L \cdot M$
5. Division of functions: given that $M \neq 0$, $\lim_{x\to x_0} \left(\frac{f}{g}\right) \ = \ \frac{L}{M}$
6. The inverse of a function: $\lim_{x\to x_0} \left(\frac{1}{g}\right) \ = \ \frac{1}{M}$, given that $M \neq 0$, as stated above.

Limits of composite function

Let f be a function defined on a deleted neighborhood of $x_0 \in \mathbb{R}$ and the limit of the function exits at the point x_0, is finite and equals to $y_0 \in \mathbb{R}$. And let there be another real function g, which is defined in the deleted neighborhood of y_0. The limit of g exits, like: $\lim_{y\to y_0} g\left(y\right) \ = \ L \ \in \mathbb{R}$. In other words, the function is: $h \ = \ f\left(g\left(y\right)\right) \ = \ g \circ f$. Then, $\lim_{x\to x_0} f \circ g \ = \lim_{x\to x_0} g\left(f\left(x\right)\right) \ = \ L$.

Theorem on the limit of composite functions

Let f be a real function defined on a deleted neighborhood of x_0, such that $\lim_{x\to x_0} f\left(x\right) \ = \ y_0$. And let there g be a real function defined on a deleted neighborhood of y_0 and the limit exists, such that: $\lim_{y\to y_0} g\left(y\right) \ = \ L \in \mathbb{R}$. Let us suppose that there is a deleted neighborhood U of x_0, such that $\forall x \in U, \ f\left(x\right) \ \neq y_0$. Then, $\lim_{y\to y_0} g \circ f \ = \ L$. In other words, in a neighborhood U of x_0, in which the function f is never equal to y_0. There, the limit of the compisite function is L.

Lecture 11

One-sided limits

Definition of one-sided neighborhood

Let x_0 be any real number. There is a righ-sided (complete) neighborhood of $x_0 \ \in \mathbb{R}$, which can be defined as: $(x_0, \ x_0 + \delta_0]$, as $0 < \delta_0 \in \mathbb{R}$. A deleted right-sided of x_0 is defined by: $(x_0, \ x_0 + \delta_0)$. The definitions of left-side neighbothood, complete or deleted, can be defined in a similar way than he right-sided ones.

Definition of one-sided limit

Let f be a real function, such that: $f : D \to \mathbb{R}$. This function is defined on a right-sided neighborhood of x_0, the righ-sided limit exists if the following hokds true:
$\forall \epsilon > 0, \ \exists \delta > 0, \ \forall x \in \mathbb{R}, \ 0 < x - x_0 < \delta \ \Rightarrow \left|f\left(x\right) - L\right| < \epsilon$

Corollary on the existence and the uniqueness of the one-sided limit

$\exists! \ \lim_{x\to x_0^+} f$

Theorem on the existence of limits

Let f be the real function $f : D \to \mathbb{R}$, defined on a deleted function of $x_0 \in \mathbb{R}$. The limit exist if and only if the three following condition apply concomitantly:

1. $\exists \lim_{x \to x_0^+} f$
2. $\exists \lim_{x \to x_0^-} f$
3. $\lim_{x \to x_0^+} f \ = \ \lim_{x \to x_0^-} f$

Lecture 12

Continuity

Definition

Let $f : D \to \mathbb{R}$ be a real function, defined on a complete neighborhood of x_0. This function will be called continuous if and only if, as x tends to x_0, then the function takes values of $f(x_0)$. In mathematical notation: $\lim_{x \to x_0} f(x) \ = \ f(x_0)$

Conclusion derived from the definition

If two functions $f,\ g$ are continuous, adding them and subtracting them will generate a continuous function. The same thing is true for the multiplication of two continuous function. It is important to consider that in the case of the division $\frac{f}{g}$, $g(x_0) \ \neq 0$ is mandatory. All this is valid for a point continuity in x_0, average continuity and uniform continuity will be discussed later.

Important note

Every polynomial function is continuous at every point in its domain.

Week 6

Lecture 13

Theorem on continuity and the existence of the limit of the function

The real function f can be called continuous if and only if it is defined on the complete vicinity of x_0 and the limit of the function when x tends to x_0 exists.

Theorem on the continuity of a composite function

Let $f,\ g$ be two real functions, which are continuous in x_0, the composite function $g \circ f \ = \ g(f(x))$ is also continuous in x_0.

Characteristics of a continuous function

Let f be a real function, which is continuous on x_0.

1. The function is bounded around x_0
2. if $f(x_0) < 0,\ \exists B_s^*(x_0),\ f < 0,\ f < \frac{f(x_0)}{2}$
3.

Explanations on the previous list

1. Being continuous implies that the function is bounded at a point.
2. If the function is positive at the point where it is continuous, there is a deleted neighborhood of x_0, in which f will always be positive and there it will assume values that are greater than the average of $f(x_0)$.
3. If the function is negative at eh point where it is continuous, there is a deleted neighborhood of x_0, in which f will be always negative there and will never take values higher than the average of $f(x_0)$.

Corollary on the continuity of two functions in the vicinity of the same point

Let $f,\ g$ be real functions, continuous at x_0. If $f(x_0) < g(x_0),\ f(x) < g(x),\ \forall x \in B_s^*(x_0)$.
In plain English, it means that if g is bounded by f at point x_0, then in the whole deleted neighborhood of x_0, g will be bounded by f.

One-sided continuity

In the cases when the limit $\lim_{x \to x_0} f(x)$ does not exist, but either one-sided limit does exist, the function $f : D \to \mathbb{R}$ is said to be one-sided continuous if and only if the one-sided limits exist and are equal to the value of the function at that point. Basically, the same definition of continuity applies here, except that specifics apply. Then, the function f will be called right-sided continuous if and only if: $\lim_{x \to x_0^+} f(x) = f(x_0)$. The function will be called left-sided continuous if and only if: $\lim_{x \to x_0^-} f(x) = f(x_0)$. The same way that it goes for limits, it goes for continuity, i.e., a function can be called continuous if and only if it is continuous on both sides.

Sandwich theorem for continuous functions

Let $f,\ g,\ h$ be real functions, defined on a complete neighborhood of x_0. Let them be continuous and let $f \leq g \leq h$. If f and h are continuous at x_0, then g is also continuous at x_0.

Continuity on an interval

Definition of interval

Let I be a subset of $\mathbb{R}$ and $x \in I$. Then, x is considered to be an internal point of I if it is not on its edges.
Example: Let $I = [a,\ b]$, x will be a internal point of I if: $x \neq a,\ x \neq b$.

Definition of continuity in a interval

Let there be $f : D \to \mathbb{R}$. The function f will be continuous in I if all the conditions apply:

1. I is a subset of the domain of f
2. f is continuous in every internal point of I

3. In the case of the interval $I = [a,\ b)$, f will be right-sided continuous at point a
4. In the case of the interval $I = (a,\ b]$, f will be left-sided continuous at point b

Mean value theorem

Let f be a real function, which is continuous in the interval $[a,\ b]$. It means that no information can be derived for the continuity of $f(a)$ or $f(b)$. Let x be an internal point in this interval. Supposing that for every x the function at that point will always be smaller than the value it takes at point b. Then, there is a λ on the y axis, which is everywhere between $f(a)$ and $f(b)$. It is important to notice that λ is not on the edges of this interval, namely it cannot be neither $f(a)$ nor $f(b)$. Then, there is a x, which is an internal point at the interval (a, b), such that the function at that point will be equal to λ. In mathematical notation, it goes like that:
$f(x) < f(b),\ \exists\lambda \in (f(a),\ f(b)),\ \exists x \in (a,\ b),\ f(x) = \lambda.$

Corollary on the mean value theorem

Let I be a subset of $\mathbb{R}$ and let f be a real, continuous function, such that $f : I \to \mathbb{R}$. If there are any two real numbers $a,\ b$, which are on the image of the function, then any real number between them is also on the image of f.

Corollary on the roots of odd degree polynomial functions

Let f be a polynomial function with an odd-numbered degree. Then, there is at least one real solution for it.

First Weierstrass Theorem

Let $f : [a,\ b] \to \mathbb{R}$ be a function, continuous at the interval $[a,\ b]$. Then, the function is bounded, both from above and from beneath, in $[a,\ b]$. In other words, if the function is continuous on an interval, it is bounded there.

Definition of maximum

Let $f : D \to \mathbb{R}$ be a real function and x_0 be a point in its domain. This point will be called maximum if:
$\forall x \in D,\ f(x) \leq f(x_0)$

Second Weierstrass theorem

Given that $f : D \to \mathbb{R}$ is continuous in its domain, then it will have a minimum and a maximum there.

Theorem on the existence of maximum for polynomial functions

Let f be a polynomial function of an even degree. Then, there is a real minimum.

Lecture 15

Monotonuous function

Definition

A monotonically increasing function is any real function for which any two numbers x and y in the domain, the values of the function in these two points are: $f(x) \leq f(y)$. A strictly monotonically increasing function is defined by: $\forall x < y \in D,\ f(x) < f(y)$. The same definition can be used for monotonically decreasing and strictly monotonically decreasing functions.

Corollary on monotonicity and continuity

Let f be a real function defined on a domain, which is the closed interval $[a,\ b]$, such as: $f : [a,\ b] \to \mathbb{R}$. If f is continuous and increases, then its image is contained in the closed interval: $[f(a),\ f(b)]$. The same applies for a continuous and decreasing function.

Theorem on Injective Functions

If a real function is continuous and injective, then it is strictly monotonic.

Inverse function

If a real function is bijective, then there can be found an inverse function to it.

Lemma on the monotonically increasing functions

Let a real function be from a domain, which is a closed interval, to an image, which is also a closed interval. Let the function be $f : [a,\ b] \to [c,\ d]$. if it is continuous, strictly monotonically and surjective, then its inverse is strictly monotonically increasing.

Theorem on bijective continuous function

Let a real function be from a domain, which is a closed interval, to an image, which is also a closed interval. If this function is continuous and bijective, then its inverse is also continuous.

Weel 7

Lecture 16

Limits which tend to infinity and limits at the infinity

Definition of limitis of functions that tend to infinity

Let us define $\lim_{x \to x_0} f\ =\ \infty$:// $\forall M \in \mathbb{R}\ \exists \delta < 0\ \ \forall x \in D,$ // $0 < |x - x_0| < \delta\ \Rightarrow f(x) > M$// The same definition can be expanded for minus infinity.

Important theorems on limits at the infinity

Let f and g be two real functions, defined on the same domain and also defined at x_0. Suppose that $\lim_{x \to x_0} f\ =\ \infty$. Then,//

1. The function f is bounded from below, but not from above.

2. Let $f \leq g$. Then, $\lim_{x \to x_0} g \ = \ \infty$. In other words, if g is always above f, then g is not bounded from above.
3. If you multiply the function by -1, its limit will tend to minus infinity.
4. If g is bounded from below, and its limit exists and is either finite or not. Then, the limit of the sum of both functions also tends to infinity. Mathematically, $\lim_{x \to x_0} (f + g) \ = \ \infty$
5. Let $c > 0$ and $g > c$, then the product of both functions tend to infinity, such that: $\lim_{x \to x_0} f \cdot g \ = \ \infty$
6. Let $c < 0$ and $g < c$, then the product of both functions tend to infinity, such that: $\lim_{x \to x_0} f \cdot g \ = \ -\infty$
7. Let g be bounded from above and positive at a vicinity of x_0. The limit of g exists, us finite and positive. Then, $\lim_{x \to x_0} \frac{f}{g} \ = \ \infty$. And it happens because it was already defined that f is not bounded. If g is negative in the vicinity of x_0, then the limit of the division tends to minus infinity.
8. $\lim_{x \to x_0} \frac{1}{f(x_0)} \ = \ 0$
9. According to intuition, if the limit of function g as x goes to zero is zero, then the limit of $\frac{1}{|g(x)|}$ tends to infinity. But if g is positive at a vicinity of x_0, then $\lim_{x \to x_0} \frac{1}{g(x)} \ = \ \infty$.

Lecture 17

Limits that tend to infinity

Definition

$\lim_{x \to \infty} f(x)$:// $\forall \epsilon > 0 \ \ \exists N \in \mathbb{R} \ \forall x \in \mathbb{R}$// $x > N \ \Rightarrow |f(x) \ - L| < \epsilon$// If the limit exists, it is unique.

Theorem on the limits of two real functions

Let there be two real functions f and g, whose limits tend to the infinity exist and are finite. Then,//

- There is an open interval, not bounded from above, i.e. $(a, \ \infty)$, in which f is bounded.
- If $f \leq g$, then $L \leq M$. Being that $\lim_{x \to \infty} f \ = \ L$ and $\lim_{x \to \infty} g \ = \ M$
- Similarly, if $M < L$, then there is an open interval $(a, \ \infty)$, in which $f < g$.
- If the limit of the multiplication of both functions exists and is finite, then it is simply $L \cdot M$
- If the limit of the addition of both functions exists and is finite, then the result is simply $L + M$

Limit of a composite function

Let there be two real functions f and g. Suppose that f is not bounded at a vicinity of x_0, while g is bounded, i.e its limit is finite. Then, the limit of the composite function $g(f(x))$ will be equals to the finite limit. In other words, $\lim_{x \to x_0} f \ = \ \infty$, $\lim_{x \to \infty} g \ = \ L$, $\lim_{x \to \infty} g(f(x)) \ = \ L$.

Lecture 18

Derivatives

Relation between continuity and differentiation

Differentiation implies continuity. But continuity does not imply differentiation. It is important to note that the derivative only exists if the derivatives on both sides exist and are equal.

Week 8

Lecture 20

Derivatives

Theorem on the derivatives of inverse functions

Let there be f and g two real functions, defined on a domain which is a closed interval. Let g be the inverse of f. If f is differentiable at x_0 and the derivative at that point is not zero, then g is also differentiable at point x_0.

Corollary on the differentiation of polynomial functions

All polynoms are differentiable in their domain.

Week 9

Lecture 21

Critical points

Definition

Let f be a real function. It will be increasing at x_0 if $f(x) < f(x_0)$ in a left-sided complete vicinity of x_0 and also $f(x) > f(x_0)$ in a right-sided complete vicinity of x_0.

Corollary on the relation between the derivative and the direction of the function

If the derivative is positive at x_0, then the function is increasing at that point. If the derivative is negative, then the function is decreasing there.

Local maximum

Definition

There is a local maximum to the function f if $f(x) \leq f(x_0)$.

Local minimum

Definition

There is a local minimum to the function f if $f(x) \geq f(x_0)$.

Fermat's theorem: relation between derivative and critical points

Let f be a real function defined on a deleted vicinity of x_0. If there is a critical point for f at x_0. Then, the derivative of the function at that point is zero.

Rolle's theorem: existence of critical point when the edges of the function are equal

Let f be a real function defined on a domain which is a subset of $\mathbb{R}$. If f is differentiable in its domain, it is continuous in its domain and it is differentiable at any internal point of its domain. Suppose that the function f is defined as: $f : [a,\ b] \to \mathbb{R}$. Suppose that the image of the edges of the domain are equal, i.e. $f(a) = f(b)$. Then, there is a point inside the domain, which is not its edges, for which the function has a critical point. In other words, $\exists x \ \in \ (a,\ b)\,,\ f'(x_0) \ = \ 0$.

Lagrange's theorem: derivative as the slope of the function

Let there f be a real function defined as $f : [a, b] \to \mathbb{R}$. There is an internal point at the domain, for which the derivative of the function is its slope. The conclusion of this theorem is that if the derivative is always zero for all internal points in the domain, then the function is constant.

Conclusions based on the previous theorem

Let there be two real functions f and g, defined on the same domain. If both functions have the same derivative for all points of their domain, there is a real number c, such that $f \ = \ g + c$. If the derivative is positive, then the function is increasing in the closed interval of the domain. The same reasoning can be used if the derivative is negative.

Higher order derivatives

Theorem on the second derivative

Let f be a real function that can be derived twice. Suppose that x_0 is a critical point of f. Then,//

1. If the second derivative at that point is negative, then the function is decreasing there.
2. Something similar can be said if the second derivative is positive, then the function is increasing there.
3. If the second derivative is equal to zero and x_0 is a local maximum, then the function is decreasing there.
4. If the second derivative is equal to zero and x_0 is a local minimum, then the function is increasing at a vicinity of that point.

Week 10

Lecture 23

Darboux sums

Definition of partition

Let there a and b be two real numbers, such that $a < b$. The set P is defined as a partition of the interval $[a,\ b]$ when P can be written as: $P \ = \ \{x_0,\ x_1,\ ...,\ x_n\}$, such that: $a \ = \ x_0 \ < \ x_1 \ < \ ... \ < \ x_n \ = \ b$. In other words, P is a set of points from the interval $[a, b]$, which include a and b.

Definition of the parameter of a partition

It is the maximum distance between two consecutive points in the partition. It is defined as: $\Delta(P) \ = \ \max_{1 \leq i \leq n}(x_i \ - \ x_{i-1})$

Definition of Refinement

Let there be two sets P and Q, which are both partitions of the interval $[a,\ b]$. If Q is a subset of P, then it is called a refinement of P. The union of both sets is defined as the common refinement of the interval $[a,\ b]$.

Definition of Lower Darboux Sum

Let there f be a real function, whose domain is the interval $[a,\ b]$. Let the function be bounded on the domain. And let there P be a partition of the domain. The lower Darboux sum is defined as: $L(f,\ P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$. Being m_i the infimum of the function on a certain point. In plain English, the lower Darboux sum L of a function f according to a partition P is the multiplication of the infimum of the function on a point by the distance between a point and the point just before that, according to the partition.

Definition of Upper Darboux Sum

Considering true all the information from above, the definition of upper Darboux sum is: $U(f,\ P) = \sum_{i=1}^{n} M_i (x_i\ -\ x_{i-1})$. In other words, the upper Darboux sum L of a function f according to a partition P is the multiplication of the supremum of the function on a point by the distance between a point and the point just before that, according to the partition.

Corollary on the relation between lower and upper Darboux sums

To any partition P of the domain, the lower Darboux sum will always be smaller or equal than the upper one. or: $L(f,\ P) \leq U(f,\ P)$.

Corollary on the relation between the refinement of a partition and Darboux sums

Any refinements of the partition P will cause the lower sum of the refinement to be greater than the lower sum of P itself. And any refinement of the partition will cause the upper sum of the refinement to be lower or greater than the sum of the original partition. Being $\overline{P}$ be the refinement of the partition P, then: $L(f,\ P) \leq L\left(f,\ \overline{P}\right) \leq U\left(f,\ \overline{P}\right) \leq U(f,\ P)$.

Conclusions on the relation between infimum, supremum and Darboux sums

Let the infimum of the function f be $m = \inf_{x\in[a,\ b]} f(x)$ and the supremum of the function be $M = \sup_{x\in[a,\ b]} f(x)$. Any lower and upper Darboux sums of the function according to a partition P will always be in between the infimum of the function on the interval $[a,\ b]$ and the supremum of the function in the same interval. In mathematical notation: $m(b-a) \leq L(f,\ P) \leq U(f,\ P) \leq M(b-a)$.

Integral

Definition of integral

Let f be a real function, whose domain is the interval $[a\ b]$. Let this function be bounded on the domain. Let L and U be the lower and upper Darboux sums, respectively. The supremum of the lower sum is said to be the lower integral of f in the interval $[a,\ b]$ and can be written as: $\int_{\underline{a}}^{b} f(x)\,dx$. The infimum of the upper Darboux sum is said to be the upper integral of f on the interval $[a,\ b]$ and can be written as: $\int_{a}^{\overline{b}} f(x)\,dx$. A function is said to be integrable if and only if the lower and the upper integral have the same value.In such case, the integral of the function f over the interval $[a,\ b]$ is: $\int_{a}^{b} f(x)\,dx$. If the function is positive or zero for any point of the domain, then the integral is said to be the area under the curve.

Lecture 24

Integral

Lemma on the slices of a Darboux sum

Let there be f a real function, whose domain is the interval $[a,\ b]$. Let there be two partitions of the domain P_1 and P_2. Suppose the function is bounded, then:

- f is integrable over the domain
- There is a real number which is: $L(f,\ P_1) \leq c\ \ \leq U(f,\ P_2)$
- $\forall \epsilon > 0,\ U(f,\ P_2) - L(f,\ P_1) < \epsilon$

Darboux condition on integrability

let the function $f:\ [a,\ b] \to \mathbb{R}$ be bounded on the domain. It will be said to be integrable if and only if $U(f,\ P) - L(f,\ P)\ <\ \epsilon$. In other words, the difference between the upper and the lower Darboux sums cannot be zero.

Arithmetics of Integrals

Theorem on addition of integrals and multiplication of an integral by a scalar

Let f and g be two real function, whose domain is the interval is $[a,\ b]$. Then,

- Addition: $\int_a^b f(x)\,dx\ +\ \int_a^b g(x)\,dx\ =\ \int_a^b f(x) + g(x)\,dx$
- Let there be any real number. $\int_a^b \lambda \cdot f(x)\,dx\ =\ \lambda \int_a^b f(x)\,dx$

Characteristics of order in integrals

Let f and g be two real function, whose domain is the interval is $[a,\ b]$. Then,

- If f is positive or zero, its integral over the interval $[a,\ b]$ is also positive or zero.
- If $f\ \leq g$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$. Comment: if f is smaller or equal to g, then their integrals will have the same relation.
- the functions $\max\{f,\ 0\}$ and $min\{f,\ 0\}$ are integrable
- If the absolute value of the function is integrable, then $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

Lecture 25

Frequency and Integrals

Definition

Let there be a real function f bounded on a subset I of its domain. The frequency of this function on this interval is:// $\omega(f,\ I)\ =\ \sup\{f(x) - f(y):\ x,\ y \in I\}\ =\ \sup_{x\in I} f(x) - \inf_{y\in I} f(y)$// The frequency can also be calculated in relation to a partition P. The maximum frequency $\omega_{\max}$ is the supremum of all the frequencies on that partition.

Theorem on the frequency of a function on a partition

The frequency will always be positive and non-zero.

Integrability of continuous functions

Let I and J be subsets of the domain, being J a subset of I. Then, the frequency of the function in J will always be smaller or equal than the frequency in I. Or: $\omega(f,\ J) \leq \omega(f,\ I)$

Theorem on the relation between continuity and integrability

Continity implies integrability. But the other way around is not true. A function can have certain discontinuities and still be integrable. For it to be happen. The function will still be integrable even if it has a finite number of discontinuities of the first or the second type, which means that either one of the sided-limits do not exist or they are not equal.

Week 11

Lecture 26

Domain of integration

Theorem on integrability of a function over an interval

Let there be three real numbers $a,\ b,\ c$, such that $a < b < c$. Let f be the real function $f : [a,\ c] \to \mathbb{R}$. If the function is integrable over the intervals $[a,\ b]$ and $[b,\ c]$, then f is integrable over the whole domain.

Theorem on the integrability over subsets of the domain

If a function is integrable all over its domain, then it is integrable over every subset of its domain.

Darboux theorem

If the function f is integrable over its domain then the lower and the upper Darboux sums are approximation to the integral of the function over its domain.

Rieman Sums

Definition

Let f be a real function, P be a partition over the domain of f and ξ a subset of points in P. The Rieman sum is defined as: $S(f,\ P,\ \xi) = \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$. In other words, it is the summation of the value of the function in certain points, multiplied by the distance between the points in the partition.

Corollary on the relation between Rieman and Darboux sums

The Rieman sum of certain points over a partition is always in between the lower and the upper Darboux sums over the same partition. Or: $L(f,\ P) \leq S(f,\ P,\ \xi) \leq U(f,\ P)$.

Lecture 27

Order of the integration domain

Corollary on the order of integration domain

Inverting the order of the integration domain is the same as multiplying the integral by minus 1.

Corollary on the subdivision of integration domain

Let there be three real numbers $r,\ s,\ t$, such that $r < s < t$. Then, $\int_r^t f(x)\,dx = \int_r^s f dx + \int_s^t f dx$

Important note on addition and multiplication by a scalar

The rules for addition and multiplication by a scalar of definite integral are the same as for the indefinite integral, which were already discussed.

Fundamental Theorem of Calculus

$\int_s^t f(x)\,dx = F(t) - F(s)$, where $F(x)$ is called the primitive function of $f(x)$.

Lifschitz

If a function has a Lifschitz, then it is continuous. the Lifschitz of a function is: $|F(x) - F(y)| < M|x - y|$, for all x and y in the domain.

Theorem on the continuity of the primitive function

The primitive function is always continuous and integrable.

Theorem on the differentiability of the primitive function

If f is continuos, then its primitive fucntion is differentiable. And the derivative of the primitive function is the function itself. Or: $F'(x) = f(x)$. It means that for all function continuous in its domain there is a primitive function. In other words, any continuous function is integrable, as has been discussed before.

Integral Mean Value Theorem

Let f be a real function, which is also integrable. Then, there is a real number c that belongs to the domain of the function such that: $F'(c) = \frac{F(b)-F(a)}{b-a}$. In plain English, the value of the function f at point c is the slope between the extremes of the domain.

Integration techniques

1. Integration by parts
2. Integration by substitutions
3. Integration by inverse substitutions

Integration by substitution

Let there be two real functions f and ϕ, being $f : I \to \mathbb{R}$ and $\phi : [a,\ b] \to I$. Then, $\int_a^b f(\phi(x))\, \phi'(x)\, dx \ = \int_{\phi(a)}^{\phi(b)} f(x)\, dx$. Given the integral of a composite function multiplied by the derivative of its arguments, this method can be used to solve it.

Integration by inverse substitution

It means changing the dimension of the integration: $\int_a^b f(x)\, dx \ = \ \int_{\phi^{-1}(a)}^{\phi^{-1}} f(\phi(t))\, \phi'(t)\, dt$

Week 12

Lecture 29

Improper integrals

Definition

Let there be f be a real function $f :\ [a,\ \infty] \to \mathbb{R}$. The integral $\int_a^\infty f(x)\, dx$ will be said to converge to a value L if the function is integrable over the closed interval $[a,\ T]$, being a and T any real numbers. It is mandatory that the following limit exists: $\lim_{T\to\infty} \int_a^\infty f(x)\, dx$. In other words, the function has to tend to a certain, finite value. Then, the improper integral will be said to converge to the value of the limit. Another type of improper integral is when the function is not bounded on any side, i.e. $f :\ (a,\ \infty) \to \mathbb{R}$ or $f : (a,\ b] \to \mathbb{R}$. Let there λ be a real, positive number which is smaller than any value of the upper bound of the domain. The integral $\int_a^b f(x)\, dx$ is said to converge to a value L if the limit exists: $\lim_{\lambda\to 0^+} \int_{a+\lambda}^b f(x)\, dx \ = \ L$. Similarly, the improper integral of the function $f : [a,\ b) \to \mathbb{R}$ s said to converge to a certain value if the limit $\lim_{\lambda\to 0^-} \int_a^{b-\lambda} f(x)\, dx \ = \ L$ exists. Then, the improper integral $\int_a^{b-\lambda} f(x)\, dx$ is said to converge to L.

Improper integral from $-\infty$ to ∞

Definition

Let there be the real function f that can be defined as $f : \mathbb{R} \to \mathbb{R}$. Suppose that f is integrable over any closed interval. Let there be a real number x_0, such that: $\int_{-\infty}^\infty f(x)\, dx \ = \ \int_{-\infty}^{x_0} f(x)\, dx \ + \ \int_{x_0}^\infty f(x)\, dx$

Lecture 30

Taylor's Polynom

Polynomial approximation to a function

On the uniqueness and the existence of linear polynomial approximation

Let f be a real function defined on the vicinity of 0. If there is a linear approximation of order n than this approximation is unique. A MacLauren linear polynomial approximation is given around zero.

Theorem the uniqueness and existence of MacLauren polynom

Let the function f be differentiable n times around zero. Then, the residue of the MacLauren polynom exists and it is unique and can be defined as: $R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ A Taylor polynom is an approximation around any point and can be defined as: $f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} (x - x_0)^k$

Week 13

Lecture 31

Taylor series

Theorem on the residue of a Taylor polynom- Lagrange residue

Let there be any real number x_0 and let f be a function differentiable $n+1$ times at any point between 0 to x_0.

Critical points

Theorem on the relation between critical points and first and second derivatives of a function

Let the real function f be differentiable at least twice at a point x_0. Then, if the first derivative is zero, it is possible to use the following list in order to evaluate the critical points of the function.

1. If f' is descending around x_0 and the second derivative is negative, then x_0 is a local maximum. If f' is descending around x_0, but the second derivative is positive, then the point is a local minimum.
2. If the second derivative is zero and the point x_0 is local maximum, then the function is descending around x_0. Otherwise, the function is ascending there.

Theorem on the relation between different order derivatives and minima and maxima of a function

Let there f be a function and $f^{(k)}$ be its k^{th} derivative. The following list can be used to define critical points of f. Let there be $f^{(k)}(x_0) = a$.

1. If k is even and a is positive, then x_0 is a local minimum
2. If k is odd and a is negative, then x_0 is a local maximum.

Numerical series

Definition

A numerical series is in essence a function that can be defined as $f : \mathbb{N} \rightarrow \mathbb{R}$ and its symbol is $f_n = (f_n)_{n=1}^{\infty} = (f_1, f_2, ...)$

Limit of a series

Definition

The real series $(a_n)_{n=1}^{\infty}$ is said to converge to a real value L as n goes to infinity if and only if: // $\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N}, \; n > N \; \Rightarrow |a_n - L| < \epsilon$ if the limit exists, it is unique.

Lecture 32

Convergence

If a series converges it is somehow bounded, it can be either bounded from above, from below or both.

Let there be two series a_n and b_n. If they converge to A and B, respectively. Then, it is possible to draw the following conclusions:

1. If A¡B, then a_n¡b_n
2. For a general number N, such that $n < N$. If $a_n \leq b_n$, then $A \leq B$
3. If A is positive, the series a_n is also positive.
4. The sandwich theorem is valid also for series.

The arithmetics limits of series follow the same rules of arithmetics of limits of functions.

Theorem on the relation between limits and series

Let there be a_n a series that converges to a. And let there f be a real function for which the limit $\lim_{x\to a} f(x) = L$. Then, the following conclusions can be drawn:

1. f is continuous at a
2. $a_n \neq a, \ \forall n$

Heine's characteristics of the continuity of a series

Let there f be a function well defined at the point x_0. It will be said continuous at x_0 if and only if for all series x_n that converges to x_0 and the limit $\lim_{n\to\infty} f(x_n) = f(x_0)$

Increasing and decreasing series

Let a_n be a series. It will be said to be increasing if $a_n \leq a_{n+1}$. And it will be said strictly increasing if $a_n < a_{n+1}$

Subseries

Let there be two series a_n and b_n. If the limit $\lim_{n\to\infty} a_n = a$ exists and b_n is a subseries of a_n. Then, $\lim_{n\to\infty} b_n = a$ and $b_n$also converges.

Lecture 33

Bolzano-Weierstrass Theorem

Definition

Let a_n be a series that converges and let L be the sided limit of a_n. Any subseries of a_n also converge to L.

The theorem - on the subseries

Let a_n be a bounded series. Then, there are subseries which are also bounded.

Theorem on the relation between sided limit and the convergence of a series

A series is said to converge if and only if the the side limit exits and is unique.

Week 14

Lecture 34

Series

Sums

Let there be the numeric series a_n. The partial sum of the series is $\sum_{n=1}^{N} a_n$. The sum of the series is $\sum_{n=1}^{\infty} a_n$.

Taylor Series

$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Convergence of a sum

If the sum $\sum_{n=1}^{\infty} a_n$ converges, then a_n tends to zero.

Infinite sums

$\sum_{n=1}^{\infty} a_n \ = \ \sum_{n=1}^{m-1} a_n \ + \ \sum_{n=m}^{\infty} a_n$

Convergence on the addition of series

Let there be two series a_n and b_n. If both series converge, then the series $c_n \ = \ a_n + b_n$ also converge.

Comparison test

Definition of positive series

A series $\sum_{n=1}^{\infty} a_n$ will be said to be positive if $a_n > 0$, for every n.

Corollary on the positiveness and convergence of a series

Let the sum $\sum_{n=1}^{\infty} a_n$ is positive and has partial sums S_N. It will converge if and only if the series is bounded and $\sum_{n=1}^{\infty} a_n$ is the supremum of its partial sums.

Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ be positive and convergent. And let there be $\sum_{n=1}^{\infty} b_n$ be positive. If $0 \leq b_n \leq a_n$, then $\sum_{n=1}^{\infty} b_n$ also converges.

Absolute convergence

Definition

The sequence $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if and only if $\sum_{n=1}^{\infty} a_n$ also converges. If the sequence converges absolutely, then it converges.

Lecture 35

Sequences

Special cases

The sequence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ does not converge absolutely, but it does converge.

If a sequence is reached by another sequence, then if one converges, the other one also converges.

New ordering

If the series b_k is defined as a new ordering of the series a_n. Then, b_k converges, because it is bounded by a_n and $\sum_{k=1}^{\infty} b_k \ = \ \sum_{n=1}^{\infty} a_n$.