TRANSLATION-INVARIANT GENERALIZED P-ADIC GIBBS MEASURES FOR THE ISING MODEL ON CAYLEY TREES

FARRUKH MUKHAMEDOV AND OTABEK KHAKIMOV

Abstract. Main aim of the present paper is explore certain physical phenomena by means of p-adic probability theory. To overcome this study, we deal with a more general setting to define p-adic Gibbs measures. For the sake of simplicity of explanations, we restrict ourselves to the Ising model on the Cayley tree, since such a model has broad theoretical and practical applications. To study p-adic quasi Gibbs measures, we reduce the problem to the description of the fixed points of the Ising-Potts mapping. Finding fixed points is not an easy job as in the real setting. Furthermore, the phase transition for the model is established. In the real case, the phase transition yields the the singularity of the limiting Gibbs measures. However, we show that the p-adic quasi Gibbs measures do not exhibit the mentioned type of singularity, such kind of phenomena is called strong phase transition. Finally, we deal with the solvability and the number of solutions of ceratin p-adic equation depending on several parameters. Such a description allows us to find all possible translation-invariant pa-adic quasi Gibbs measures.

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1. Introduction

It is well-known that the modern axiomatics of probability theory was given by A.N. Kolmogorov [30]. Now, this theory was reduced to the theory of normalized σ-additive measures taking values in the segment $[0, 1]$ of the field of real numbers $\mathbb{R}$. There is another path to the probability which is von Mises' frequency approach to probability [31]. It is stressed that von Mises' approach (as many others) could not compete with the precisely and simply formulated Kolmogorov theory. Here, the von Mises' approach is mentioned not only because of its interest for applications, but also because his model with frequency probabilities acted an essential part in the process of formulation of conventional axiomatics of probability theory. It is natural to aks: is Kolomgorov's theory enough? From the pure mathematical point of view such theory is satisfactory. However, as a physicist, it is not be so optimistic. It seems that Kolmogorov's model, despite its generality, does not provide a reasonable mathematical description of all probabilistic structures that appear in physics (as well as other natural and social sciences) (see [25] for discussion). In particular, one can recall the old problem of negative probabilities. There are many objects that must be probabilities by their physical origin, but they can take negative values (as well as values larger than 1), (see, e.g., [17, 32]). As a consequence, physicists should work with such objects at the physical level of rigor. However, negative probabilities appear again and again in different domains of physics.

We also pay attention to another probability-like structure that recently appeared in theoretical physics. This is the so called *p-adic probability* [24]. Such probabilities have naturally appeared in p-adic physical models, namely, the p-adic string, was suggested by I. Volovich [59]. Furthermore, numerous applications of the p-adic analysis to mathematical physics have been proposed in [1]-[5], [22]. We refer the reader to [13, 14] for recent development of the subject.

In fact, there are two types of p-adic physical models:

(A) the variables are p-adic, but the functions are $\mathbb{C}$-valued;
(B) both the variables and functions take p-adic values.

The (A)-models of p-adic physics and their relation to conventional probability theory on locally compact groups (especially, totally disconnected) are briefly discussed in [8, 26, 58, 63], whereas the (B)-models are the most interesting for our present considerations. In this setting, the probabilities (by their physical origin) belong to fields of p-adic numbers $\mathbb{Q}_p$. However, in such a approach, Kolmogorov's axiomatics can not be employed, see [23]. Furthermore, in [22,25,28,51], p-adic probability theory was developed as the first example of mathematically rigorous formalism for probabilities taking values in a topological group which differs from $\mathbb{R}$. Its interpretation plays the fundamental role in applications of probability (e.g., to physics). We stress that the main attention to a statistical interpretation of these generalized probabilities was given [25]. Using that p-adic measure theory in [25,27], the theory of p-adic and non-Archimedean stochastic processes has been further advanced. In the present paper, we are going to employ the last model to the investigation of phase transitions in statistical mechanical models.

One of the main aims of the present paper is explore certain physical phenomena by means of p-adic probability theory. To overcome this study, we deal with a more general setting to define p-adic Gibbs measures. For the sake of simplicity of explanations, we restrict ourselves to the Ising model on the Cayley tree, since such a model has broad theoretical and practical applications [10]. We note that in [19–21], [33]- [41], [47–49] it has been started analysis of p-adic Gibbs measures within p-adic probability framework, for spacial kinds of measures. It is known [15], in the classical setting, the ferromagnetic Ising model with nearest-neighbor interaction on a Cayley tree exhibits a phase transition. The occurrence of the phase transition is detected in a variety of models on hierarchical lattices by means of renormalization group (RG) technique. Clearly, RG transformation basically depends on the construction of hierarchical lattice and model. The simplest one is governed by rational functions. In [12] an interesting relation between the phase transition and Julia sets of the RG has been discovered (see [43]).

In [6, 36, 39] it has been developed the RG method to study phase transitions for several p-adic models on Cayley trees. The RG method is closely related to the investigation of p-adic dynamical system associated with a given model (see [9,28]). In [42,44] a connection between the existence of the phase transition and chaoticity of the corresponding p-adic dynamical system has been established. In this way, we notice that chaotic p-adic chaotic dynamical systems have immense applications in coding theory [16,57,62].

In this paper, our main purpose to study the set of p-adic Gibbs measures of the Ising model on the Cayley tree. To study such a model, in Section 3, we reduce the problem to the description of the fixed points of the Ising-Potts mapping (3.8). Finding fixed points is not an easy job as in the real setting. We refer the reader to [9, 46, 50, 55] for the differences. Furthermore, in this section, the phase transition for the model is established. In the real case, the phase transition yields the the singularity of the limiting Gibbs measures. However, in this paper, we show that the p-adic quasi Gibbs measures do not exhibit the mentioned type of singularity, such kind of phenomena is called strong phase transition. We notice that such kind of transition occurs if one considers more complex interactions (see e.g. [33,36,41]). In Section 4, we deal with the solvability and the number of solutions of ceratin p-adic equation depending on several parameters. Such a description allows us to find all possible translation-invariant pa-adic quasi Gibbs measures.

2. Preliminaries

In this section, we recall some definitions related to the p-adic analysis and we introduce the necessary notations.

2.1. **p-adic numbers.** Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p: $(p, n) = 1$, $(p, m) = 1$. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

This norm is non-Archimedean and satisfies the so called *strong triangle inequality*

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$. We point out that $\mathbb{Q}_p$ is not an ordered field [56].

Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \dots), \tag{2.1}$$

where $\gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 > 0$, $0 \leq x_j \leq p-1$. In this case, $|x|_p = p^{-\gamma(x)}$.

In [45] we have introduced new symbols "O" and "o" which allow us to simplify our calculations. Roughly speaking, these symbols replace the notation $\equiv \ (mod\ p^k)$ without noticing about power of k. Let us recall such notions. A given p-adic number x by $O[x]$ we mean a p-adic number with the norm $p^{-\gamma(x)}$, i.e. $|x|_p = |O(x)|_p$. By $o[x]$, we mean a p-adic number with a norm strictly less than $p^{-\gamma(x)}$, i.e. $|o(x)|_p < |x|_p$. For instance, if $x = 1 - p + p^2$, we can write $O[1] = x$, $o[1] = x - 1$ or $o[p] = x - 1 + p$. Therefore, the symbols $O[\cdot]$ and $o[\cdot]$ make our work easier when we need to calculate the p-adic norm of p-adic numbers. It is easy to see that $y = O[x]$ if and only if $x = O[y]$.

We give some basic properties of $O[\cdot]$ and $o[\cdot]$, which will be used later on.

Lemma 2.2. *[45] Let $x, y \in \mathbb{Q}_p$. Then the following statements hold:*

1°. *$O[x]O[y] = O[xy]$;*

2°. *$xO[y] = O[y]x = O[xy]$;*

3°. *$O[x]o[y] = o[xy]$;*

4°. *$o[x]o[y] = o[xy]$;*

5°. *$xo[y] = o[y]x = o[xy]$;*

6°. *$\frac{O[x]}{O[y]} = O\left[\frac{x}{y}\right]$, if $y \neq 0$;*

7°. *$\frac{o[x]}{O[y]} = o\left[\frac{x}{y}\right]$, if $y \neq 0$.*

For each $a \in \mathbb{Q}_p$ and $r > 0$ we denote

$$B_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}.$$

We recall that $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ and $\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}$ are the set of all *p-adic integers* and *p-adic units*, respectively.

The following result is known as the Hensel's lemma

Lemma 2.3. *[11, 29] Let $F(x)$ be a polynomial whose coefficients are p-adic integers. Let x^* be a p-adic integer such that for some $i \geq 0$ one has*

$$F(x^*) \equiv 0 (\operatorname{mod} p^{2i+1}), \quad F'(x^*) \equiv 0 (\operatorname{mod} p^i), \quad F'(x^*) \not\equiv 0 (\operatorname{mod} p^{i+1}).$$

Then $F(x)$ has a p-adic integer root x_* such that $x_* \equiv x^* (\operatorname{mod} p^{i+1})$.

Remark 2.4. *We emphasize that in [60, 61] a generalization of Hensel Lemma is given for 1-Lipschitz functions using van der Put decomposition.*

Recall that the p-adic exponential is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for every $x \in B_1(0)$ if $p \neq 2$ and $x \in B_{\frac{1}{2}}(0)$ if $p = 2$. Denote

$$\mathcal{E}_p = \left\{ x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)} \right\}.$$

This set is the range of the p-adic exponential function [29,56]. In the sequel, the following well known fact will be frequently used without noticing.

Lemma 2.5. *[56] The set $\mathcal{E}_p$ has the following properties:*
(a) $\mathcal{E}_p$ is a group under multiplication;
(b) $|a-b|_p < 1$ for all $a, b \in \mathcal{E}_p$;
(c) if $a, b \in \mathcal{E}_p$ then $|a+b|_p < 1$ if $p = 2$ and $|a+b|_p = 1$ if $p > 2$.
(d) if $a \in \mathcal{E}_p$, then there is an element $h \in B_{p^{-1/(p-1)}}(0)$ such that $a = \exp_p(h)$.

2.6. **p-adic measures.** Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a *p-adic measure* if for any $A_1, \dots, A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ $(i \neq j)$ the equality holds

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

A p-adic measure is called a *probability measure* if $\mu(X) = 1$. One of the important condition (which was already invented in the first Monna–Springer theory of non-Archimedean integration [51]) is boundedness, namely a p-adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$. We pay attention to an important special case in which boundedness condition by itself provides a fruitful integration theory (see for example [51]). Note that, in general, a p-adic probability measure need not be bounded [23,27,29]. For more detail information about p-adic measures we refer to [9,28].

2.7. **Cayley tree.** Let $\Gamma^k_+ = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k+1$ edges, except for the root x^0, which has k edges). Here V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y\rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1\rangle, \dots, \langle x_{d-1}, y\rangle$ is called a *path* from the point x to the point y. The distance $d(x, y)$ on the Cayley tree, is the length of the shortest path from x to y.

Let us set

$$W_n = \{x \in V : \ d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m, \quad L_n = \{\langle x, y\rangle \in L : \ x, y \in V_n\}.$$

Recall a coordinate structure in Γ^k_+: for the vertex x^0 we put (0) and for the vertex $x \in W_n$, $n \geq 1$ we put $(i_1, \dots, i_n)$, here $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$. For $x = (i_1, \dots, i_n)$ we denote

$$S(x) = \{(x, i) : \ 1 \leq i \leq k\}, \tag{2.2}$$

here (x, i) means that $(i_1, \dots, i_n, i)$. This set is called a set of *direct successors* of x.

Using the coordinate system one can define translations of Γ^k_+ by

$$\tau_{(i_1,\dots,i_n)}(j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m). \tag{2.3}$$

Let $H \subset \Gamma^k_+$ be a sub-semigroup of Γ^k_+ and $h : \Gamma^k_+ \to Y$ be a Y-valued function defined on Γ^k_+. We say that h is *$H-$ periodic* if $h(\tau_g(x)) = h(x)$ for all $g \in H$ and $x \in \Gamma^k_+$. A function h is called *translation invariant* if it is a Γ^k_+- periodic.

3. p-Adic generalized quasi Gibbs measures for the p-adic Ising model

In this section we define a notion of p-adic generalized quasi Gibbs measure in a general setting, i.e. for the Ising model (see [18,53] for the real setting).

Let $\Phi = \{-1, 1\}$, (Φ is called a *state space*) and is assigned to the vertices of the tree $\Gamma^k_+ = (V, \Lambda)$. A configuration σ on V is then defined as a function $x \in V \mapsto \sigma(x) \in \Phi$; in a similar manner one defines configurations σ_n and $\omega_{[n]}$ on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega_{[n]} \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega_{[n]})(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } \ x \in V_{n-1}, \\ \omega_{[n]}(x), & \text{if } \ x \in W_n. \end{cases}$$

It is clear that $\sigma_{n-1} \vee \omega_{[n]} \in \Omega_{V_n}$.

The (formal) Hamiltonian of the p-adic Ising model on Ω_{V_n} is given by

$$H_n(\sigma) = N \sum_{\langle x,y\rangle \in L_n} \sigma(x)\sigma(y), \qquad \forall \sigma \in \Omega_{V_n}, \tag{3.1}$$

where $N \in \mathbb{Z}$ ($N \neq 0$) is a coupling constant.

Let us construct p-adic generalized quasi Gibbs measures of for the model (3.1) on Γ^k_+.

Assume that $\mathbf{h} : V \setminus \{x^{(0)}\} \to \mathbb{Q}_p^\Phi$ is a function, i.e. $\mathbf{h}_x = (h_{-1,x}, h_{1,x})$, where $h_{\pm 1,x} \in \mathbb{Q}_p$, $x \in V \setminus \{x^{(0)}\}$. Given $\rho \in \mathbb{Q}_p \setminus \{0\}$ let us consider a p-adic probability measure $\mu^{(n)}_{\mathbf{h},\rho}$ on Ω_{V_n} defined by

$$\mu^{(n)}_{\mathbf{h},\rho}(\sigma) = \frac{1}{Z^{(\mathbf{h})}_{n,\rho}} \rho^{H_n(\sigma)} \prod_{x \in W_n} h_{\sigma(x),x} \tag{3.2}$$

Here $Z^{(\mathbf{h})}_{n,\rho}$ is the corresponding normalizing factor or *partition function* given by

$$Z^{(\mathbf{h})}_{n,\rho} = \sum_{\sigma \in \Omega_{V_n}} \rho^{H_n(\sigma)} \prod_{x \in W_n} h_{\sigma(x),x}. \tag{3.3}$$

Remark 3.1. *Note that in general, $Z^{(\mathbf{h})}_{n,\rho}$ could be zero for some $\mathbf{h}$. In this case in formal we may assume that $\mu^{(n)}_{\mathbf{h},\rho}(\sigma) = \infty$ for all $\sigma \in \Omega_{V_n}$. But such kind of measures are not interested. Hence, when it occurs we say that for $\mathbf{h}$ there is no measure.*

In the present we are interested in a construction of an infinite volume distribution with given finite-dimensional distributions in a p-adic setting. More exactly, we want to define a p-adic probability measure $\mu_{\mathbf{h},\rho}$ on Ω which is compatible with defined ones $\mu^{(n)}_{\mathbf{h},\rho}$, i.e.

$$\mu_{\mathbf{h},\rho}(\{\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n\}) = \mu^{(n)}_{\mathbf{h},\rho}(\sigma_n), \quad \text{for all } \ \sigma_n \in \Omega_{V_n}, \ n \in \mathbb{N}. \tag{3.4}$$

In general, à priori the existence such a kind of measure μ is not known, since there is not much information on topological properties, such as compactness, of the set of all p-adic measures defined even on compact spaces. To find we employ the p-adic Kolmogorov's extension Theorem (see [27]) which is based on so called *compatibility condition* for the measures $\mu^{(n)}_{\mathbf{h},\rho}$, $n \geq 1$, i.e.

$$\sum_{\omega_{[n]} \in \Omega_{W_n}} \mu^{(n)}_{\mathbf{h},\rho}(\sigma_{n-1} \vee \omega_{[n]}) = \mu^{(n-1)}_{\mathbf{h},\rho}(\sigma_{n-1}), \qquad \forall \sigma_{n-1} \in \Omega_{V_{n-1}}. \tag{3.5}$$

This condition according to the theorem implies the existence of a unique p-adic measure $\mu_{\mathbf{h},\rho}$ defined on Ω with a required condition (3.4). Such a measure $\mu_{\mathbf{h},\rho}$ is said to be *a genaralized p-adic quasi Gibbs measure* corresponding to the model [35,36].

For given Hamiltonian H by $GQ\mathcal{G}_\rho(H)$ we denote the set of all p-adic generalized quasi Gibbs measures associated with functions $\mathbf{h} = \{\mathbf{h}_x, \ x \in V\}$. If there are at least two distinct p-adic generalized quasi Gibbs measures $\mu, \nu \in GQ\mathcal{G}_\rho(H)$ such that μ is bounded and ν is unbounded, then we say that *a phase transition* occurs. By another words, one can find two different functions $\mathbf{s}$ and $\mathbf{h}$ defined on $\mathbb{N}$ such that there exist the corresponding measures $\mu_{\mathbf{s}}$ and $\mu_{\mathbf{h}}$, for which one is bounded, another one is unbounded. Moreover, if there is a sequence of sets $\{A_n\}$ such that $A_n \in \Omega_{V_n}$ with $|\mu(A_n)|_p \to 0$ and $|\nu(A_n)|_p \to \infty$ as $n \to \infty$, then we say that there occurs a *strong phase transition.* It is said to occur *a quasi phase transition* if there are two different functions $\mathbf{s}$ and $\mathbf{h}$ defined on $\mathbb{N}$ such that there exist the corresponding measures $\mu_{\mathbf{s},\rho}$, $\mu_{\mathbf{h},\rho}$, and they are either bounded or unbounded.

Remark 3.2. *1. We point out, in the real case [18], that at low temperature for the classical Ising model the phase transition is reflected in a mathematical model by the non-uniqueness of the Gibbs measures, i.e. if one finds two different Gibbs measures $\mu_\pm$ such that*

$$\mu_+(\sigma(0) = 1) > 1/2, \quad \mu_-(\sigma(0) = 1) < 1/2.$$

This yields that the measures μ_+ and μ_- are mutually singular. The strong phase transition (see definition above), in the p-adic setting, has a similar meaning as singularity, i.e. the p-adic measures μ and ν are "singular" (in the above given sense). Here, we have to emphasizes that the absolutely *continuity and the singularity of p-adic measures cannot be directly defined in a similar manner with the real case.*

Remark 3.3. *Note that in [43] we have considered the following sequence of p-adic measures defined by*

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{\tilde{Z}_n^{(\mathbf{h})}} \exp_p\{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}, \tag{3.6}$$

here as usual $\tilde{Z}_n^{(\mathbf{h})}$ is the corresponding normalizing factor. A limiting p-adic measures generated by (3.6) *was called p-adic generalized Gibbs measure. By $G\mathcal{G}(H)$ we denote the set of all p-adic generalized Gibbs measures associated with a given Hamiltonian. In [20, 21, 41] it was found conditions for the uniqueness of p-adic generalized Gibbs measures, i.e. $|G\mathcal{G}(H)| = 1$. In [49] we have shown that $|G\mathcal{G}(H)| \geq 2$ for the p-adic Ising -Vannimenus model.*

The following statement describes conditions on $\mathbf{h}$ guaranteeing compatibility of the sequence of probability distributions $\{\mu_{\mathbf{h}}^{(n)}\}_{n\geq 1}$.

Theorem 3.4. *[20] The sequence of probability distributions $\{\mu_{\mathbf{h}}^{(n)}\}_{n\geq 1}$ given by* (3.2)*, are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$\tilde{h}_x = \prod_{y \in S(x)} \frac{\theta \tilde{h}_y + 1}{\tilde{h}_y + \theta} \tag{3.7}$$

where $\tilde{h}_x = \frac{h_{1,x}}{h_{-1,x}}$, $x \in V \setminus \{x^0\}$ and $\theta = \rho^{2N}$.

Remark 3.5. *There are several approaches to derive the solutions of the equation which describe the limit Gibbs measures for the lattice models on the Cayley tree. One approach is based on properties of Markov random fields on Cayley tree [54]. Another approach is based on recurrence equations for partition functions [10, 15].*

We recall that a function $\mathbf{h}$ is translation-invariant if $\mathbf{h}_x = \mathbf{h}_y$ for all $x, y \in V$. The corresponding p-adic measure is also called translation-invariant. If $\mathbf{h}$ is translation-invariant then (3.7) reduces to the equation $f_{\theta,k}(h) = h$, where $h = h_x$ for all x. Here, the function $f_{\theta,k}$ is called *Ising-Potts mapping* and is defined by

$$f_{\theta,k}(x) = \left(\frac{\theta x + 1}{x + \theta}\right)^k, \quad \theta = \rho^{2N} \in \mathbb{Q}_p \setminus \{-1, 0, 1\}. \tag{3.8}$$

In what follows, by $Fix(f_{\theta,k})$ we denote the set of all fixed points of (3.8). The following Proposition shows a relation between the set $Fix(f_{\theta,k})$ and the set of all translation-invariant p-adic generalized quasi Gibbs measures for Ising model.

Proposition 3.6. *Let $h \neq -1$ be a fixed point of* (3.8)*. Then $\mu_h := \mu_{\mathbf{h},\rho}$ is a translation-invariant p-adic generalized quasi Gibbs measure for Ising model, where $\mathbf{h}_x = (1, h)$ for every $x \in V \setminus \{x^0\}$. Moreover, it holds*

$$\mu_h(\{\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n\}) = \frac{\rho^{H_n(\sigma)} h^{\sum_{x \in W_n} \delta_{1\sigma(x)}}}{(\rho^{-N} h + \rho^N)^{\frac{k(k^n-1)}{k-1}} (h+1)}, \qquad \forall n \in \mathbb{N}, \tag{3.9}$$

where δ_{ij} is a Kronecker symbol.

Proof. We notice that $-1 \in Fix(f_{\theta,k})$ if and only if k is odd. Let $h \in Fix(f_{\theta,k}) \setminus \{-1\}$. Put $h_{-1,x} = 1$ and $h_{1,x} = h$ for every $x \in V \setminus \{x^0\}$. Clearly, $\mathbf{h}_x = (h_{-1,x}, h_{1,x})$ is a solution of (3.7). Then thanks to Theorem 3.4 we infer that $\mu_{\mathbf{h},\rho}$ is a p-adic generalized quasi Gibbs measure for Ising model. Due to $\mathbf{h}_x = \mathbf{h}_y$ for every $x, y \in V \setminus \{x^0\}$ we obtain that $\mu_{\mathbf{h},\rho}$ is a translation-invariant. Since $\mathbf{h}$ depends only h we denote $\mu_h := \mu_{\mathbf{h},\rho}$.

In order to show (3.9) it is enough to prove the following

$$\mu^{(n)}_{\mathbf{h},\rho}(\sigma)=\frac{\rho^{H_n(\sigma)}h^{\sum_{x\in W_n}\delta_{1\sigma(x)}}}{\left(\rho^{-N}h+\rho^{N}\right)^{\frac{k(k^n-1)}{k-1}}(h+1)},\qquad \forall\sigma\in\Omega_{V_n},\ \forall n\in\mathbb{N}. \tag{3.10}$$

Let $n\geq 1$ and $\sigma\in\Omega_{V_n}$. Then we have

$$\begin{aligned}\mu^{(n)}_{\mathbf{h},\rho}(\sigma) &= \frac{1}{Z^{(\mathbf{h})}_{n,\rho}}\rho^{H_n(\sigma)}\prod_{\substack{x\in W_n:\\ \sigma(x)=1}}h_{1,x}\prod_{\substack{x\in W_n:\\ \sigma(x)=-1}}h_{-1,x}\\ &= \frac{1}{Z^{(\mathbf{h})}_{n,\rho}}\rho^{H_n(\sigma)}h^{\sum_{x\in W_n}\delta_{1\sigma(x)}}. \end{aligned}\tag{3.11}$$

On the other hand keeping in mind $h(\rho^{-N}h+\rho^{N})^k=(\rho^{N}h+\rho^{-N})^k$, for any $\omega\in\Omega_{n-1}$, $n\geq 2$ from (3.7) one finds

$$\prod_{\substack{x\in W_{n-1}:\\ \omega(x)=1}}\prod_{y\in S(x)}\left(\rho^{N}h+\rho^{-N}\right)\prod_{\substack{x\in W_{n-1}:\\ \omega(x)=-1}}\prod_{y\in S(x)}\left(\rho^{-N}h+\rho^{N}\right)=\prod_{x\in W_{n-1}}\left(\rho^{-N}h+\rho^{N}\right)\prod_{\substack{x\in W_{n-1}:\\ \omega(x)=1}}h.$$

Multiplying by $\rho^{H_{n-1}(\omega)}$ both side of the last one and denoting by $card(W_{n-1})$ a cardinality of W_{n-1} we obtain

$$\sum_{\substack{\sigma\in\Omega_{V_n}:\\ \sigma|_{V_{n-1}}\equiv\omega}}\rho^{H_n(\sigma)}\prod_{\substack{x\in W_n:\\ \sigma(x)=1}}h=\left(\rho^{-N}h+\rho^{N}\right)^{card(W_{n-1})}\rho^{H_{n-1}(\omega)}\prod_{\substack{x\in W_{n-1}:\\ \omega(x)=1}}h.$$

Since arbitrariness of ω, from the last equality we immediately get the following recurrence formula:

$$Z^{(\mathbf{h})}_{n,\rho}=\left(\rho^{-N}h+\rho^{N}\right)^{card(W_{n-1})}Z^{(\mathbf{h})}_{n-1,\rho},\qquad \forall n\geq 2. \tag{3.12}$$

For $n=1$ we have

$$\begin{aligned}Z^{(\mathbf{h})}_{1,\rho} &= \left(\rho^{-N}h+\rho^{N}\right)^k+\left(\rho^{N}h+\rho^{-N}\right)^k\\ &= \left(\rho^{-N}h+\rho^{N}\right)^k(h+1).\end{aligned}\tag{3.13}$$

Keeping in mind $card(W_{n-1})=k^{n-1}$ from (3.12), (3.13) one finds

$$Z^{(\mathbf{h})}_{n,\rho}=\left(\rho^{-N}h+\rho^{N}\right)^{\frac{k(k^n-1)}{k-1}}(h+1),\qquad \forall n\in\mathbb{N}. \tag{3.14}$$

Putting (3.14) into (3.11) we obtain (3.10). □

For the sake of convenience, we call the model *ferromagnetic* if $|\theta|_p>1$, and *antiferromagnetic* if $|\theta|_p<1$. Such notions will be helpful when we consider several distinct cases w.r.t. $|\theta|_p$.

In order to study the existence phase transition (or strong, quasi types of phase transition) for Ising model we need know boundedness (or unboundedness) given measures. For this reasons the following Lemma plays a crucial role in our further investigations.

Lemma 3.7. *Let $f_{\theta,k}$ be a function given by* (3.8)*. Then the following statements are true:*

(A.) *if $|\theta|_p\leq 1$ then $Fix(f_{\theta,k})\subset\mathbb{Z}_p^*$;*

(B.) *if $|\theta|_p>1$ then $Fix(f_{\theta,k})\subset\bigcup_{t\in\{-1,0,1\}}\theta^{kt}\mathbb{Z}_p^*$.*

Proof. (A.) Let $|\theta|_p\leq 1$. Take arbitrary $x\in\mathbb{Q}_p$ such that $|x|_p\neq 1$. First we assume that $|x|_p<1$. Then due to strong triangle inequality we find $|\theta x+1|_p=1$ and $|x+\theta|_p\leq 1$. Hence, $|f_{\theta,k}(x)|_p\geq 1$, which implies that $f_{\theta,k}(x)\neq x$. Now we suppose that $|x|_p>1$. Again using strong triangle inequality we get $|x+\theta|_p=|x|_p$ and $|\theta x+1|_p\leq\max\{|\theta x|_p,1\}$. From these one finds $|f_{\theta,k}(x)|_p\leq\max\{|\theta^k|_p,|x^{-k}|_p\}\leq 1$. Consequently, we infer that $f_{\theta,k}(x)\neq x$. Thus, we have shown that $Fix(f_{\theta,k})\cap(\mathbb{Q}_p\setminus\mathbb{Z}_p^*)=\emptyset$. This means that

$$Fix(f_{\theta,k})\subset\mathbb{Z}_p^*.$$

(B.) Let $|\theta|_p>1$. First we show that $f(\theta^{-k}\mathbb{Z}_p^*)\subset\theta^{-k}\mathbb{Z}_p^*$ and if $y\in Fix(f_{\theta,k})\cap p\mathbb{Z}_p$ then $y\in\theta^{-k}\mathbb{Z}_p^*$.

Take an arbitrary $y \in p\mathbb{Z}_p$, i.e. $|y|_p < 1$. If $|y|_p > |\theta^{-k}|_p$ then we have

$$|f_{\theta,k}(y)|_p = \left|\left(y+\theta^{-1}\right)^k\right|_p \leq \max\left\{\left|y^k\right|_p, \left|\theta^{-k}\right|_p\right\} < |y|_p.$$

This means that $y \notin Fix(f_{\theta,k})$. If $|y|_p \leq |\theta^{-k}|_p$ then we immediately get $|\theta y|_p < 1$. Keeping in mind that fact we obtain $|f_{\theta,k}(y)|_p = |\theta^{-k}|_p$. It yields that $f(\theta^{-k}\mathbb{Z}_p^*) \subset \theta^{-k}\mathbb{Z}_p^*$ and there is no fixed point in $\theta^{-k}\mathbb{Z}_p^* \setminus p\theta^{-k}\mathbb{Z}_p$. So, we conclude that

$$Fix(f_{\theta,k}) \cap p\mathbb{Z}_p \subset \theta^{-k}\mathbb{Z}_p^*. \tag{3.15}$$

Now we show that

$$Fix(f_{\theta,k}) \cap (\mathbb{Q}_p \setminus \mathbb{Z}_p) \subset \theta^{k}\mathbb{Z}_p^*. \tag{3.16}$$

Pick an arbitrary $z \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, i.e. $|z|_p > 1$. If $|z|_p < |\theta^k|_p$ then we obtain

$$|f_{\theta,k}(z)|_p \geq \frac{|\theta z|_p^k}{\max\{|z^k|_p, |\theta^k|_p\}} > |z|_p,$$

which yields that $z \notin Fix(f_{\theta,k})$. If $|z|_p \geq |\theta^k|_p$ then using strong triangle inequality one gets $|f_{\theta,k}(z)|_p = |\theta^k|_p$. From the last one we immediately obtain (3.16).

We notice that $f_{\theta,k}(\mathbb{Z}_p^*) \subset \mathbb{Z}_p^*$. Keeping in mind that fact and from (3.15), (3.16) we infer that

$$Fix(f_{\theta,k}) \subset \bigcup_{t\in\{-1,0,1\}} \theta^{kt}\mathbb{Z}_p^*.$$

$\square$

Theorem 3.8. *Let $\rho^{2N} \in \mathbb{Q}_p \setminus \{-1,0,1\}$ and μ_h be a translation-invariant p-adic generalized quasi Gibbs measures for Ising model. Then the followings are true:*

(I) *Let $|\rho|_p \neq 1$. Then μ_h is bounded;*
(II) *Let $|\rho|_p = 1$. Then μ_h is unbounded iff $0 < |h+\rho^{2N}|_p < 1$.*

Proof. (I) We notice $|\rho|_p \neq 1$ if and only if $|\rho^{2N}|_p \neq 1$. In this case thanks to Lemma 3.7 we obtain $|h|_p \in \left\{1, |\rho|_p^{-2kN}, |\rho|_p^{2kN}\right\}$ for every $h \in Fix(f_{\theta,k})$.

Case $|h|_p = 1$. Due to Proposition 3.6 for corresponding translation-invariant measure μ_h we obtain

$$|\mu_h(\{\omega\in\Omega : \omega|_{V_n} \equiv \sigma_n\})|_p = \frac{\left|\rho^{H_n(\sigma)}\right|_p}{|(h+1)|_p} \min\left\{\left|\rho^{\frac{k(k^n-1)}{k-1}N}\right|_p, \left|\rho^{-\frac{k(k^n-1)}{k-1}N}\right|_p\right\}, \quad \forall\sigma_n \in \Omega_{V_n}.$$

Keeping in mind the following fact

$$-\frac{k(k^n-1)}{k-1}N \leq H_n(\sigma) \leq \frac{k(k^n-1)}{k-1}N \tag{3.17}$$

from the last one we get

$$|\mu_h(\omega)|_p \leq \frac{1}{|(h+1)|_p}, \qquad \forall\omega\in\Omega,$$

which implies boundedness of μ_h.

Case $|h|_p \in \left\{|\rho|_p^{-2kN}, |\rho|_p^{2kN}\right\}$. According to Lemma 3.7 we have $|\rho^{2N}|_p > 1$. For any $n \geq 1$ and $\sigma \in \Omega_{V_n}$ using strong triangle inequality one has

$$\begin{cases} |h|_p^{\sum_{x\in W_n}\delta_{1\sigma(x)}} \leq 1, & \\ |h+1|_p = 1, & \text{for } |h|_p = |\rho|_p^{-2kN}, \\ \left|\rho^{-N}h+\rho^N\right|_p = |\rho|_p^N, & \end{cases} \tag{3.18}$$

and

$$\begin{cases} |h|_p^{\sum_{x\in W_n}\delta_{1\sigma(x)}} \leq |\rho|_p^{2k^{n+1}N}, \\ |h+1|_p = |\rho|_p^{2kN}, & \text{for } |h|_p = |\rho|_p^{2kN}. \\ \left|\rho^{-N}h+\rho^N\right|_p = |\rho|_p^{(2k-1)N}, \end{cases} \tag{3.19}$$

Thanks to Proposition 3.6 for corresponding translation-invariant measure μ_h after using (3.18), (3.19) we obtain

$$|\mu_h(\{\omega\in\Omega : \omega|_{V_n}\equiv\sigma_n\})|_p \leq \begin{cases} |\rho|_p^{H_n(\omega)-\frac{k(k^n-1)}{k-1}N}, & \text{if } |h|_p = |\rho|_p^{-2kN}; \\ |\rho|_p^{H_n(\omega)+2k^{n+1}N-\frac{(2k-1)k(k^n-1)}{k-1}N-2kN}, & \text{if } |h|_p = |\rho|_p^{2kN}. \end{cases}$$

Again keeping in mind (3.17) from the last one for $|h|_p \in \left\{|\rho^{-2kN}|_p, |\rho^{2kN}|_p\right\}$ we get

$$|\mu_h(\omega)|_p \leq 1, \qquad \forall\omega\in\Omega.$$

This means that μ_h is bounded.

(II) Let us suppose that $|\rho|_p = 1$. Then due to Lemma 3.7 (A.), from $h \in Fix(f_{\theta,k})$ we have $|h|_p = 1$. Using $|h|_p = |\rho|_p = 1$, thanks to Proposition 3.6 one has

$$|\mu_h(\{\sigma\in\Omega : \sigma|_{V_n}\equiv\sigma_n\})|_p = \frac{\left|\rho^{-N}h+\rho^N\right|_p^{-\frac{k(k^n-1)}{k-1}}}{|h+1|_p}, \qquad \forall\sigma_n\in\Omega_{V_n},\ \forall n\geq 1. \tag{3.20}$$

Hence,

$$\mu_h \text{ is unbounded } \iff 0 < \left|h+\rho^{2N}\right|_p < 1. \tag{3.21}$$

The proof is completed. $\square$

Corollary 3.9. *Let $p = 2$ and μ_h be a translation-invariant p-adic generalized quasi Gibbs measures for Ising model. Then μ_h is unbounded iff $|\rho|_2 = 1$.*

Proof. If $|\rho|_2 \neq 1$ then due to Theorem 3.8 (I) we infer that μ_h is bounded. If $|\rho|_2 = 1$ then thanks to Lemma 3.7 one has $|h|_2 = 1$. Using 2-adic representation of ρ and h we can easily check that $0 < |h+\rho^{2N}|_2 < 1$. According to (3.21) we conclude that μ_h is unbounded. $\square$

From Theorem 3.8 and Corollary 3.9 we immediately get the following result.

Corollary 3.10. *On the set of all translation-invariant p-adic generalized quasi Gibbs measures for Ising model a phase transition does not occur if $p = 2$ or $|\rho|_p \neq 1$ for $p > 2$.*

Now, keeping in mind (3.20) and Theorem 3.8 we infer the following important fact.

Corollary 3.11. *On the set of all translation-invariant p-adic generalized quasi Gibbs measures for Ising model the strong phase transition does not occur.*

This result implies that the p-adic quasi Gibbs measures do not exhibit the "singularity" which mentioned in the definition of strong phase transition.

By Corollary 3.10 we infer under what conditions a phase transition does not occur. However, a natural question arises: Can we find some p and $\rho \in \mathbb{Q}_p$ such that for Ising model on Γ_+^k a phase transition occurs? We are going to answer this question below.

Example 1. *Let $k = 2$, $p = 5$ and $\rho \in \mathcal{E}_p$. Then (3.8) has exactly three fixed points: $h_0 = 1$ and*

$$h_{\pm 1} = \frac{-2+(\rho^{2N}-1)^2 \pm (\rho^{2N}-1)\sqrt{-4+(\rho^{2N}-1)^2}}{2}.$$

We notice that $\sqrt{-4+(\rho^{2N}-1)^2} \in \mathbb{Q}_5$. It is easy to check that

$$|h_0+\rho^{2N}|_5 = 1, \qquad 0 < |h_\pm+\rho^{2N}|_5 < 1.$$

Then due to (3.21), μ_{h_0} is bounded and $\mu_{h_{-1}}$, μ_{h_+} are unbounded.

Remark 3.12. *We notice that if $\theta \in \mathcal{E}_p$, then in [43] we have investigated the dynamics of the p-adic Potts-Bethe mapping and showed that the function has infinitely many periodic points which yields the existence infinitely many periodic p-adic quasi Gibbs measures for the model.*

4. Description of the set of all translation-invariant p-adic quasi Gibbs measures

This section is devoted to the existence of translation-invariant p-adic generalized quasi Gibbs measures. To describe them, we are going to find the set of all fixed points of the function given by (3.8). In what follows, we always assume that $|\rho|_p > 1$ and $k \geq 2$.

In what follows, we need some auxiliary facts.

Lemma 4.1. *Let $k \geq 2$. Then for any $\alpha, \beta \in \mathcal{E}_p$ it holds*

$$\sum_{j=0}^{k-1} \alpha^{k-j-1}\beta^j = O[k]. \tag{4.1}$$

Proof. It is enough to prove (4.1) for the case $p = 2$, since the assertion of the lemma was established in [44] for $p \geq 3$.

Let $p = 2$. Take any $\alpha, \beta \in \mathcal{E}_2$ which is equivalent to $\alpha, \beta \in 1 + 4\mathbb{Z}_2$. If $\alpha = \beta$ then

$$\sum_{j=0}^{k-1} \alpha^{k-1-j}\beta^j = k\alpha^{k-1}.$$

By Lemma 2.5 (a) we get $\alpha^{k-1} \in \mathcal{E}_2$ which implies (4.1).

Now, let us assume that $\alpha \neq \beta$. Then, Lemma 2.5 yields $\frac{\alpha}{\beta} \in \mathcal{E}_2$. For convenience, we denote $\delta := \frac{\alpha}{\beta}$. Then

$$\sum_{j=0}^{k-1} \alpha^{k-1-j}\beta^j = \alpha^{k-1}\sum_{j=0}^{k-1}\delta^j = \alpha^{k-1}\frac{\delta^k - 1}{\delta - 1} = \alpha^{k-1}\left[k + \sum_{j=2}^{k} C_k^j(\delta-1)^{j-1}\right]. \tag{4.2}$$

The last equality implies that to prove (4.1) it is suffices to establish

$$\sum_{j=2}^{k} C_k^j(\delta-1)^{j-1} = o[k]. \tag{4.3}$$

First, we show that $|n!|_2 > \frac{1}{4^{n-1}}$ for any $n \geq 2$. Indeed, it is clear that

$$|n!|_2 \geq 2^{-\frac{n}{2}(1+\frac{1}{2}+\frac{1}{2^2}\cdots)} = 2^{-n} > 4^{1-n}.$$

Consequently, keeping in mind $|\delta - 1|_2 \leq \frac{1}{4}$ one gets

$$|(\delta-1)^{n-1}|_2 < |n!|_2, \quad \text{for any } n \geq 2. \tag{4.4}$$

It is easy to check that

$$\frac{|k!|_2}{|(k-n)!|_2} \leq |k|_2, \quad \text{for any } 2 \leq n \leq k.$$

Therefore, the last one together with (4.4) yields

$$|C_k^n(\delta-1)^{n-1}|_2 < |k|_2, \quad \text{for any } 2 \leq n \leq k. \tag{4.5}$$

By the strong triangle inequality from (4.5) we infer that

$$\left|\sum_{j=2}^{k} C_k^j(\delta-1)^{j-1}\right|_2 < |k|_2,$$

which is equivalent to (4.3). This completes the proof. $\square$

It is well known [29] that one can decompose $\mathbb{Z}_p^* = \bigcup_{j=1}^{p-1} B_1(j)$. Hence, as a corollary of Lemma 4.1 we obtain the following fact.

Corollary 4.2. *Let $k \geq 2$. If $x, y \in B_1(j)$ for some $j \in \{1, 2, \ldots, p-1\}$ then one has*

$$x^k - y^k = o\left[pk(x-y)\right].$$

4.3. **Ferromagnetic case.** In this subsection, we consider the case when $N > 0$, which yields $|\theta|_p > 1$.

Proposition 4.4. *Let $|\theta|_p > 1$ then*

$$card\left(Fix(f_{\theta,k}) \cap (\mathbb{Q}_p \setminus \mathbb{Z}_p^*)\right) = 2, \tag{4.6}$$

here $card(H)$ stands for the cardinality of a set H.

Proof. From $|\theta|_p > 1$ and $\theta = \rho^{2N}$ we infer that $|\theta|_p \geq p^2$. Due to Lemma 3.7 we already know that

$$Fix(f_{\theta,k}) \cap \left(\mathbb{Q}_p \setminus \mathbb{Z}_p^*\right) \subset \theta^{-k}\mathbb{Z}_p^* \cup \theta^k\mathbb{Z}_p^*.$$

Therefor, one needs to consider only two cases.

Case $Fix(f_{\theta,k}) \cap \theta^{-k}\mathbb{Z}_p^*$**.** Let us denote $r = \left|\theta^{-k}\right|_p$. One can see that $f_{\theta,k}(S_r(0)) \subset S_r(0)$. We clearly have $|\theta x + 1|_p = |\theta^{-1}x + 1|_p = 1$ if $|x|_p = r$. This means that for any $x \in S_r(0)$ there is $\alpha_x \in S_1(0)$ such that $f_{\theta,k}(x) = \theta^{-k}\alpha_x^k$. It is obvious that α_x depends only to x and it has the following form

$$\alpha_x = \frac{1 + \theta x}{1 + x\theta^{-1}}.$$

From $|\theta|_p \geq p^2$, we have $|\theta^{-1}x|_p < |\theta x|_p \leq p^{-2}$ for any $x \in S_r(0)$. Hence, the strong triangle inequality implies $1 + \theta x \in \mathcal{E}_p$ and $1 + \theta^{-1}x \in \mathcal{E}_p$. Therefore, by Lemma 2.5 one get $\alpha_x \in \mathcal{E}_p$.

For any $y, z \in S_r(0)$, we obtain

$$\begin{aligned} f_{\theta,k}(y) - f_{\theta,k}(z) &= \theta^{-k}\left(\alpha_y^k - \alpha_z^k\right) \\ &= \theta^{-k}\frac{(\theta^2 - 1)(y - z)}{\theta(1 + y\theta^{-1})(1 + z\theta^{-1})}\sum_{j=0}^{k-1}\alpha_y^{k-j-1}\alpha_z^j \\ &= \frac{y - z}{O[\theta^{k-1}]}\sum_{j=0}^{k-1}\alpha_y^{k-j-1}\alpha_z^j. \end{aligned}$$

Hence, by Lemma 4.1 one finds

$$\frac{f_{\theta,k}(y) - f_{\theta,k}(z)}{y - z} = \frac{O[k]}{O\left[\theta^{k-1}\right]} = o[1],$$

this means that $f_{\theta,k}$ is a contraction of $S_r(0)$. Consequently, we get

$$card\left(Fix(f_{\theta,k}) \cap S_r(0)\right) = 1. \tag{4.7}$$

Case $Fix(f_{\theta,k}) \cap \theta^k\mathbb{Z}_p^*$**.** Let us denote $R = \left|\theta^k\right|_p$. It is easy to check that $S_R(0)$ is invariant with respect to $f_{\theta,k}$. Moreover, for any $x \in S_R(0)$, we have $f_{\theta,k}(x) = \theta^k\beta_x^k$, here

$$\beta_x = \frac{1 + \frac{1}{\theta x}}{1 + \frac{\theta}{x}}.$$

Clearly, $\beta_x \in \mathcal{E}_p$. Then, for every $y, z \in S_r(0)$ one finds

$$\begin{aligned} f_{\theta,k}(y) - f_{\theta,k}(z) &= \theta^k\left(\beta_y^k - \beta_z^k\right) \\ &= \theta^k\frac{(\theta^2 - 1)(y - z)}{\theta(1 + y^{-1}\theta)(1 + z^{-1}\theta)}\sum_{j=0}^{k-1}\beta_y^{k-j-1}\beta_z^j \\ &= \frac{y - z}{O[\theta^{k-1}]}\sum_{j=0}^{k-1}\beta_y^{k-j-1}\beta_z^j. \end{aligned}$$

Again by Lemma 4.1 we obtain

$$\frac{f_{\theta,k}(y)-f_{\theta,k}(z)}{y-z}=\frac{O[k]}{O\left[\theta^{k-1}\right]}=o[1]$$

which yields that $f_{\theta,k}$ is a contraction of $S_R(0)$. Therefore,

$$card\left(Fix(f_{\theta,k})\cap S_R(0)\right)=1. \tag{4.8}$$

Finally, from (4.7) and (4.8) we get (4.6). This completes the proof. □

Now we are going to find fixed points of (3.8) on $\mathbb{Z}_p^*$. In order to find such kind of points, we consider the following polynomial

$$F_k(x)=x(\theta^{-1}x+1)^k-(x+\theta^{-1})^k,$$

with p-adic integer coefficients. We notice that all roots of F_k are fixed points of $f_{\theta,k}$ and visa versa. It is evident that 1 is a root of F_k for any $k\geq 1$ and -1 is a root only for odd values of k.

By $\mathcal{N}_{p,k}$ we denote a number of k-th roots of unity, i.e.

$$\mathcal{N}_{p,k}=card\left(\{x\in\mathbb{Q}_p : x^k=1\}\right).$$

We point out that the description of this set has been carried out in [45, 50].

Proposition 4.5. *Let $|\theta|_p>1$. If $|\theta^{-1}|_p<|(k-1)^2|_p$ then*

$$card\left(Fix(f_{\theta,k})\cap\mathbb{Z}_p^*\right)=\mathcal{N}_{p,k-1}. \tag{4.9}$$

Proof. It is obvious that $f_{\theta,k}(\mathbb{Z}_p^*)\subset\mathbb{Z}_p^*$. Let $a\in\mathbb{Z}_p^*$. Then $\frac{f_{\theta,k}(a)}{a^k}\in 1+p\mathbb{Z}_p$. Hence,

$$f_{\theta,k}(a)-a^k=o[1].$$

This means that if $a\in\mathrm{Fix}(f_{\theta,k})$ then $a^k-a=o[1]$ or $a^{k-1}-1=o[1]$. In other words, there is no fixed point of $f_{\theta,k}$ in $a+p\mathbb{Z}_p$ if $a^{k-1}-1=O[1]$.

Now, we assume that $x_0^{k-1}=1$ and show that the function $f_{\theta,k}$ has exactly one fixed point belonging to $x_0+p^n\mathbb{Z}_p$, where $p^{-n+1}=|k-1|_p$.

One has

$$\begin{aligned} F_k(x_0) &= x_0\left(\left(1+x_0\theta^{-1}\right)^k-x_0^{k-1}\left(1+x_0^{-1}\theta^{-1}\right)^k\right)\\ &= x_0\left(\left(1+x_0\theta^{-1}\right)^k-\left(1+x_0^{-1}\theta^{-1}\right)^k\right). \end{aligned}$$

So, Lemma 4.1 together with the last one implies

$$|F_k(x_0)|_p\leq|k\theta^{-1}|_p. \tag{4.10}$$

For F_k' at point x_0 using $|\theta^{-1}|_p<|(k-1)^2|_p\leq|k-1|_p$ we find

$$\begin{aligned} F_k'(x_0) &= \left(1+x_0\theta^{-1}\right)^k-kx_0^{k-1}\left(1+x_0^{-1}\theta^{-1}\right)^{k-1}+kx_0\theta^{-1}\left(1+x_0\theta^{-1}\right)^{k-1}\\ &= (1-k)\left(1+x_0^{-1}\theta^{-1}\right)^{k-1}+\left(1+x_0\theta^{-1}\right)^k-\left(1+x_0^{-1}\theta^{-1}\right)^{k-1}+kx_0\theta^{-1}\left(1+x_0\theta^{-1}\right)^{k-1}\\ &= 1-k+\left(1+x_0\theta^{-1}\right)^{k-1}-\left(1+x_0^{-1}\theta^{-1}\right)^{k-1}+O[(k+1)\theta^{-1}]\\ &= 1-k+o[p\theta^{-1}]\\ &= O[k-1]. \end{aligned}$$

Hence, the last one together with (4.10) implies that F_k satisfies all conditions of the Hensel's Lemma. Then the polynomial F_k has only one root belonging to $x_0+p^n\mathbb{Z}_p$, here, as before, $p^{-n+1}=|k-1|_p$. Hence, $f_{\theta,k}$ has only one fixed point in $x_0+p^n\mathbb{Z}_p$. Consequently,

$$card\left(Fix(f_{\theta,k})\cap\mathbb{Z}_p^*\right)=\mathcal{N}_{p,k-1}.$$

which completes the proof. □

Corollary 4.6. *Let $|\theta|_p > 1$. If $p \nmid (k-1)$ then*

$$card\left(Fix(f_{\theta,k}) \cap \mathbb{Z}_p^*\right) = \mathcal{N}_{p,k-1}.$$

Proof. The condition $p \nmid (k-1)$ implies that $|k-1|_p = 1$. From $|\theta|_p > 1$ we get $|\theta^{-1}|_p < |(k-1)^2|_p$. Then due to Proposition 4.5 one has (4.9). □

By $TIpGQGM$ we denote the set of all translation-invariant p-adic generalized quasi Gibbs measures for the Ising model. As a corollary of Propositions 3.6, 4.4 and 4.5 we formulate the following result.

Theorem 4.7. *Let $k \geq 2$ and $|\rho^N|_p > 1$. Then for the Ising model on Γ_+^k the following statements hold:*

(i) *$card\,(TIpGQGM) \geq 3$;*

(ii) *If $|\rho^{-N}|_p < |k-1|_p$ then*

$$card\,(TIpGQGM) = \begin{cases} \mathcal{N}_{p,k}+2, & \text{if } k \text{ is even;} \\ \mathcal{N}_{p,k}+1, & \text{if } k \text{ is odd.} \end{cases}$$

4.8. **Antiferromagnetic case.** In this subsection, we assume that $|\theta|_p < 1$.

Proposition 4.9. *Let $|\theta|_p < 1$. If $|\theta|_p < |(k+1)^2|_p$ then*

$$card\,(Fix(f_\theta)) = \mathcal{N}_{p,k+1}.$$

Proof. Thanks to Lemma 3.7 we infer that there is no fixed points of (3.8) in $\mathbb{Q}_p \setminus \mathbb{Z}_p^*$. One can see that $\mathbb{Z}_p^*$ is an invariant w.r.t. $f_{\theta,k}$.

Now, let us consider the polynomial

$$P_k(x) = x(\theta + x)^k - (\theta x + 1)^k.$$

According to $\theta \in \mathbb{Z}_p$ all coefficients of P_k are p-adic integers. Let x_0 be a $k+1$-th root of unit. Then

$$\begin{aligned} P_k(x_0) &= x_0^{k+1}\left(1+\theta x_0{}^{-1}\right)^k - (1+\theta x_0)^k \\ &= \left(1+\theta x_0{}^{-1}\right)^k - (1+\theta x_0)^k. \end{aligned}$$

Hence, thanks to Lemma 4.1 one gets

$$|P_k(x_0)|_p \leq |k\theta|_p. \tag{4.11}$$

On the other hand, we obtain

$$P_k'(x_0) = x_0^k\left((1+\theta x_0^{-1})^{k-1}(k+1+\theta x_0^{-1}) - k\theta x_0^{-k}(1+\theta x_0)^{k-1}\right).$$

Using $|\theta|_p < |(k+1)^2|_p \leq |k+1|_p$ from the last one, one finds

$$|P_k'(x_0)|_p = |k+1|_p. \tag{4.12}$$

Consequently, (4.11) and (4.12) imply that P_k satisfies all condition of the Hensel's Lemma. Hence, P_k has exactly one root in $x_0 + O[p(k+1)]\mathbb{Z}_p$. So, we infer that

$$card\,(Fix(f_{\theta,k})) = card\left(\{x \in \mathbb{Q}_p : x^{k+1} = 1\}\right).$$

□

Corollary 4.10. *Let $|\theta|_p < 1$. If $p \nmid (k+1)$ then*

$$card\,(Fix(f_\theta)) = \mathcal{N}_{p,k+1}.$$

As a corollary of Propositions 3.6 and 4.9 we formulate the following result.

Theorem 4.11. *Let $k \geq 2$ and $|\rho^N|_p < 1$. Then for the Ising model on Γ_+^k the following statements hold:*

(i) *$card\,(TIpGQGM) \geq 1$;*

(ii) *If* $|\rho^N|_p < |k+1|_p$ *then*

$$card\,(TIpGQGM) = \begin{cases} \mathcal{N}_{p,k}+1, & \text{if } k \text{ is even;} \\ \mathcal{N}_{p,k}, & \text{if } k \text{ is odd.} \end{cases}$$

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Farrukh Mukhamedov, Department of Mathematical Sciences, College of Science, The United Arab Emirates University, P.O. Box, 15551, Al Ain, Abu Dhabi, UAE

Email address: far75m@gmail.com farrukh.m@uaeu.ac.ae

Otabek Khakimov, Laboratory of Algebra and its applications, Institute of Mathematics, P.O. Box, 100125, Tashkent, Uzbekistan

Email address: hakimovo@mail.ru