Modely s Weibullovým rozdělením

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Cíle

- Popsat vlastnosti Weibullova rozdělení
- Odvodit metody odhadu parametrů v modelech s Weibullovým rozdělením s důrazem na modely typu jednoduché ANOVY
- Studium testů hypotéz o parametrech v modelech s Weibullovým rozdělením
- Užití poznatků na simulovaných a reálných datech

The Weibull model

Three-parameter Weibull model

The tree-parameter Weibull distribution function has the form

$$F(t;\alpha,\beta,\tau) = 1 - exp\left[-\left(\frac{t-\tau}{\alpha}\right)^{\beta}\right], t \geq \tau, \alpha > 0, \beta > 0, \tau \geq 0. \quad (1)$$

The parameter α is named the *scale*, β is the *shape* and τ is called the *location parameter*.

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The Standard Weibull model

The two-parameter Weibull distribution is a special case of (1)

$$F(t;\alpha,\beta) = 1 - exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right], t \geq 0, \alpha > 0, \beta > 0. \tag{2}$$

Parameter estimation

The Method of Moments - first variant

- $I_j = [t_{j-1}, t_j)$ for $j = 1, ...p$ be nonoverlapping time intervals
- n_i the number of fails in the i-th time interval

$$P_j = P(T < t_j) = 1 - P(x_i \geq t_j, ..., X_n \geq t_j) = 1 - exp\left[-\left(\frac{t_j}{\alpha}\right)^{\beta}\right], \tag{3}$$

where $T = \min(X_1, ..., X_n)$.

The Method of Moments - first variant

From (3), for $j = 1, ..., p-1$

$$P_{j+1} = 1 - exp\left[-\left(\frac{t_{j+1}}{\alpha}\right)^{\beta}\right], \; P_j = 1 - exp\left[-\left(\frac{t_j}{\alpha}\right)^{\beta}\right]$$

$$\ln\left(\ln\left(1-P_{j+1}\right)^{-1}\right) - \ln\left(\ln\left(1-P_j\right)^{-1}\right) = \beta\left[\ln t_{j+1} - \ln t_j\right],$$

$$\beta = \frac{\ln\left(\ln\left(1-P_{j+1}\right)^{-1}\right) - \ln\left(\ln\left(1-P_j\right)^{-1}\right)}{\left[\ln t_{j+1} - \ln t_j\right]} \tag{4}$$

and for $j = 1, ..., p$

$$\alpha = \frac{t_j}{\sqrt[\beta]{\ln\left(1-P_j\right)^{-1}}}. \tag{5}$$

The Method of Moments - first variant

$$\hat{\beta} = \frac{\ln\left(\ln\left(1-\hat{P}_p\right)^{-1}\right) - \ln\left(\ln\left(1-\hat{P}_1\right)^{-1}\right)}{\ln t_p - \ln t_1}, \tag{6}$$

$$\hat{\alpha} = \frac{1}{p}\sum_{j=1}^{p} \frac{t_j}{\sqrt[\hat{\beta}]{\ln\left(1-\hat{P}_j\right)^{-1}}}, \tag{7}$$

where

$$\hat{P}_j = \sum_{k=1}^{j} \frac{n_k}{n}, \tag{8}$$

is the estimation of the distribution function.

The Method of Moments - second variant

For the second variant, use the following:

$$\hat{F}\left(X_j\right) = \frac{j - 0,3}{n - 0,4} \tag{9}$$

$$\hat{\beta} = \frac{\ln\left(\ln\left(1 - \hat{F}\left(X_n\right)\right)^{-1}\right) - \ln\left(\ln\left(1 - \hat{F}\left(X_1\right)\right)^{-1}\right)}{\ln X_n - \ln X_1} \tag{10}$$

$$\hat{\alpha} = \frac{1}{n}\sum_{j=1}^{n} \frac{X_j}{\sqrt[\hat{\beta}]{\ln\left(1 - \hat{F}\left(X_j\right)\right)^{-1}}}, \tag{11}$$

The Method of Percentile

A *Percentile* for a two-parametric Weibull is given by

$$t_p = \alpha \left[-\ln\left(1 - p\right)\right]^{\frac{1}{\beta}}, \tag{12}$$

for $p \in \langle 0; 1 \rangle$.
$100(1 - e^{-1}) = 63.2$th percentile is respond to α as before.

$$\hat{\alpha} = t_{(1-e^{-1})}, \tag{13}$$

where $t_{(1-e^{-1})}$ percentile is calculated from the distribution function for $\mathbf{X}$.
From (12):

$$\hat{\beta} = \frac{\ln\left(-\ln\left(1-p\right)\right)}{\ln\left(\frac{t_p}{t_{(1-e^{-1})}}\right)} \tag{14}$$

for $0 < t_p < t_{(1-e^{-1})}$.
Seki and Yokoyama (1993): proposed $p = 0,31$ for the estimation.

Data and Simulation

The simulation

Parameters of the simulation

$$n = 200, \ \alpha = 1.0, \ \beta = 50$$

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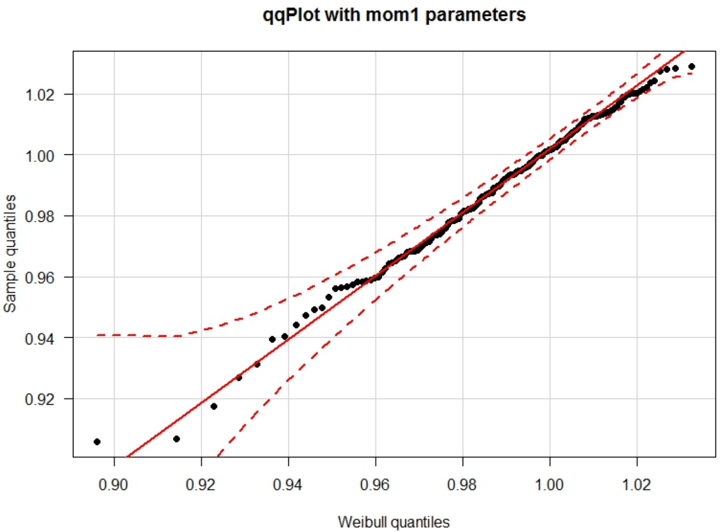
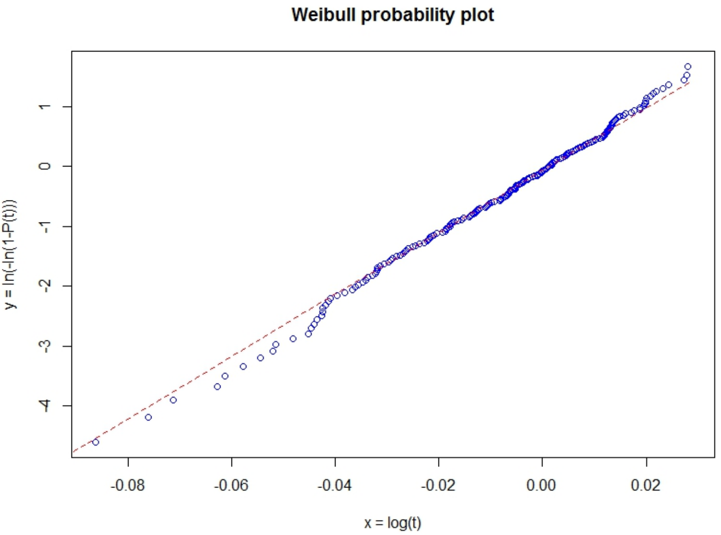
$$n = 200, \ \alpha = 1.0, \ \beta = 50$$

- MOM1: $\hat{\alpha} = 0.9994709, \ \hat{\beta} = 54.8687848$
- MOM2: $\hat{\alpha} = 1.0001792, \ \hat{\beta} = 57.9161131$
- MOP: $\hat{\alpha} = 1.0012489, \ \hat{\beta} = 51.7435920$


qqPlot with mom1 parameters
Sample quantiles
0.92
0.94
0.96
0.98
1.00
1.02
0.90
0.92
0.94
0.96
0.98
1.00
1.02
Weibull quantiles


Weibull probability plot
y = ln(-ln(1-P(t)))
1
0
-1
-2
-3
-4
-0.08
-0.06
-0.04
-0.02
0.00
0.02
x = log(t)

The real data

Information

Location: Yarra Valley, Position: Trunk, $n_t = 136$, Survey $= 1, \ldots, 9$

Survey 1

$n_1 = 24$

The real data

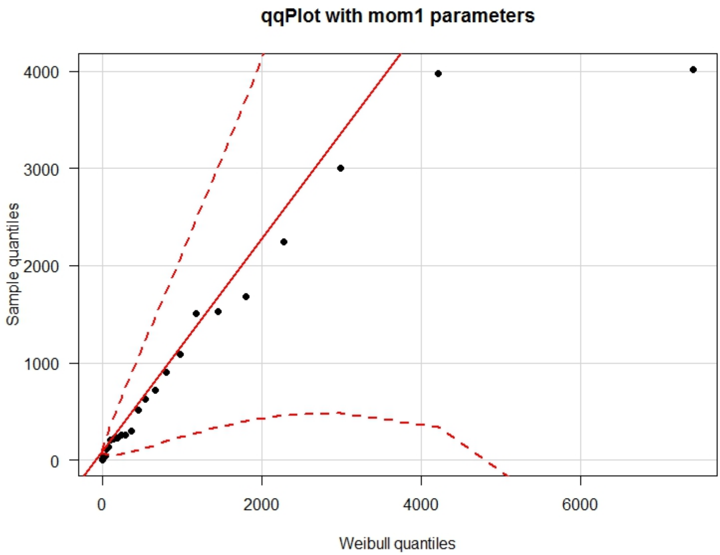
Information

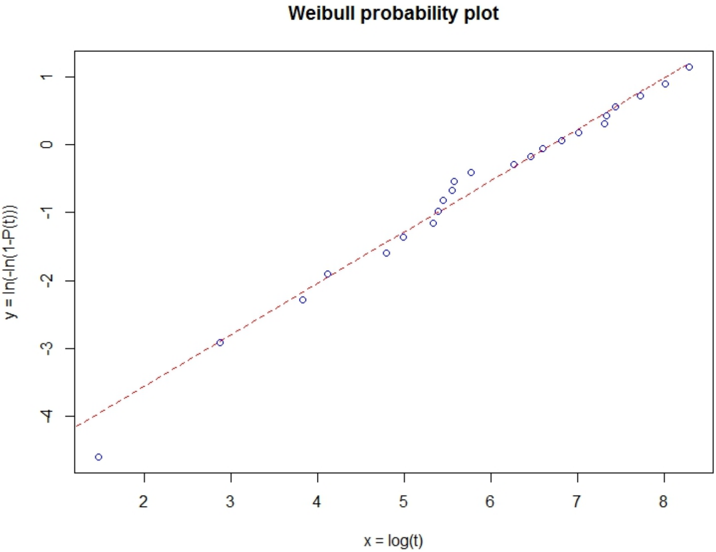
Location: Yarra Valley, Position: Trunk, $n_t = 136$, Survey $= 1, \ldots, 9$

Survey 1

$n_1 = 24$

- MOM1: $\hat{\alpha} = 752.2419883$, $\hat{\beta} = 0.5912263$
- MOM2: $\hat{\alpha} = 842.0691402$, $\hat{\beta} = 0.6576249$
- MOP: $\hat{\alpha} = 817.7864634$, $\hat{\beta} = 0.7572847$


qqPlot with mom1 parameters
Sample quantiles
4000
3000
2000
1000
0
0
2000
4000
6000
Weibull quantiles


Weibull probability plot
y = ln(-ln(1-P(t)))
1
0
-1
-2
-3
-4
2
3
4
5
6
7
8
x = log(t)

The Real data

Survey 5

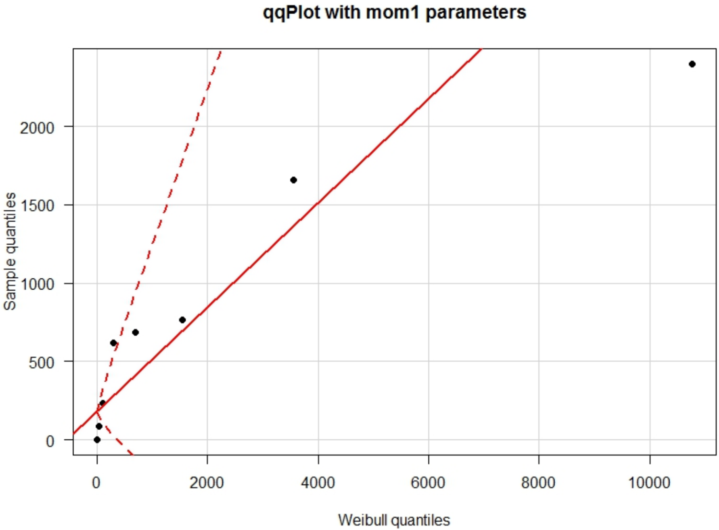
$n_8 = 8$

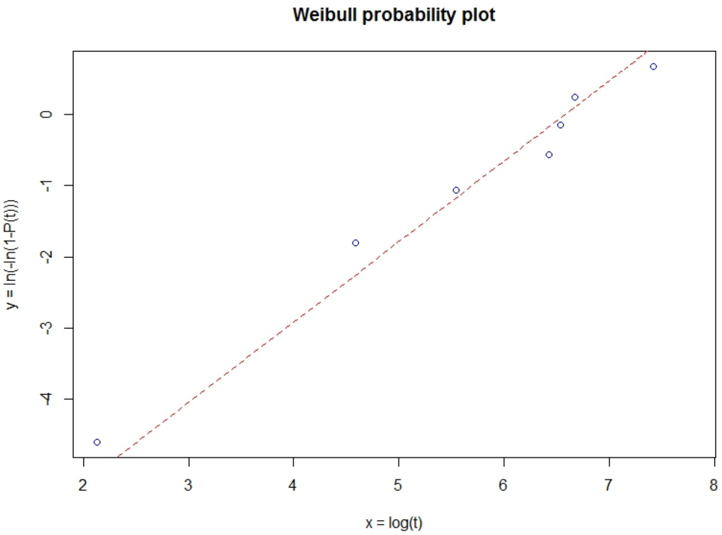
The Real data

Survey 5

$n_8 = 8$

- MOM1: $\hat{\alpha} = 1114.1340210, \ \hat{\beta} = 0.4177922$
- MOM2: $\hat{\alpha} = 1063.0428253, \ \hat{\beta} = 0.4829192$
- MOP: $\hat{\alpha} = 721.4105304, \ \hat{\beta} = 1.1306050$


qqPlot with mom1 parameters
Sample quantiles
0
500
1000
1500
2000
0
2000
4000
6000
8000
10000
Weibull quantiles


Weibull probability plot
y = ln(-ln(1-P(t)))
0
-1
-2
-3
-4
2
3
4
5
6
7
8
x = log(t)

Thank you for your attention.