Leibniz rule for the high q -derivatives of a quotient of two functions

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Abstract

In this paper, we introduce some explicit formulas for the positive integer order q-derivative of a quotient of two functions. We establish the connections between linear algebra, the homomorphisms between the commutative algebras and the properties of q-operators. Especially, the third formula is a beautiful one involving the determinant of the functions and their q-derivatives. Then, we show that the formulas in the classical mathematical analysis are their special cases. Finally, the last formula is used for proving of some interesting identities. **MSC CLASSIFICATION:** 05A30, 26A24

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Leibniz rule for the high q-derivatives of a quotient of two functions

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In this paper, we introduce some explicit formulas for the positive integer order q-derivative of a quotient of two functions. We establish the connections between linear algebra, the homomorphisms between the commutative algebras and the properties of q-operators. Especially, the third formula is a beautiful one involving the determinant of the functions and their q-derivatives. Then, we show that the formulas in the classical mathematical analysis are their special cases. Finally, the last formula is used for proving of some interesting identities.

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1 | INTRODUCTION

The *Leibniz product rule for derivatives*

$$D^n(fg)(x) = \sum_{k=0}^{n} \binom{n}{k} (D^{n-k} f)(x)\,(D^k g)(x) \tag{1}$$

is a well known and useful formula. Its generalizations were going to various directions. So, its analog was considered in the fractional calculus in [1] and [2]. Also, the positive integer and the fractional version were formulated in the q-calculus [3, 4, 5].

The explicit formulas for the nth derivative of a quotient of two functions are less in usage and less known in comparison to the product formula. But, they are very interesting in formulations and applications, see, for example [6, 7, 8, 9]. In this article, we introduce the explicit formulas for the nth q-derivative of a quotient of two functions. Of course, they reduce to the classical ones after passage to the limit as q tends to 1. All presented formulas are stated for an arbitrary $q \in \mathbb{R} \setminus \{1\}$ and $x \in \mathbb{R}$ (including the case $x = 0$). But, it is worth noticing that all formulas are true in the case of complex functions of a complex variable as well.

The paper is organized as follows. We remind on the basics of q-calculus and q-differential calculus, and give some needed well-known results in Section 2. We introduce two q-Leibniz quotient rules in Section 3. The first one is in a recurrent form and it is the easiest one for derivation. The second one uses a recursively defined sequence of functions and can be proved by induction. In Section 4, we set a suitable system of linear equations and derive a q-Leibniz quotient formula as a solution of the system obtained by Cramer's rule. In Section 5, we present the main result of the paper. We prove the fourth formula where the nth q-derivative of u/v is given as a linear combination of nth q-derivatives of uv^k, $k = 0, 1, \dots, n$. Compared to the other rules, this one has the most difficult proof, but it seems to be the most useful one. We also establish an algebra homomorphism from the set of all q-differentiable functions to the set of lower triangular matrices. Finally, in Subsection 5.1, we apply the fourth rule to the power function and obtain some interesting q-binomial identities.

2 | BASICS OF Q–CALCULUS

In the theory of q-calculus (see for example [10, 11, 12, 13]), for a parameter $q \in \mathbb{R}$ and an integer $n \in \mathbb{Z}$, we introduce a q-integer $[n]_q$ by

$$[n]_q := \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1\,. \end{cases} \tag{2}$$

Note that

$$[-n]_q = -q^{-n}[n]_q, \qquad \text{for all} \quad n \in \mathbb{Z}, \tag{3}$$

and if $n > 0$ then $[n]_q = 1+q+q^2+\cdots+q^{n-1}$. It follows that for every $n \in \mathbb{Z}$, $[n]_q \to n$ when $q \to 1$. If $q > 0$ then we can use the definition (2) to define q-real for any real number. The q-analog of the Pochhammer symbol (q–shifted factorial) is defined by:

$$(a;q)_0 = 1\,, \qquad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i), \qquad a \in \mathbb{R},\ q \neq 1,\ k \in \mathbb{N}.$$

For $n, k \in \mathbb{N} \cup \{0\}$, $0 \le k \le n$ and $q \neq 1$, the q-factorial and q-binomial coefficient are given by

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q, \qquad [0]_q! = 1,$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

respectively. It is easy to see that $\lim_{q\to 1}[n]_q! = n!$ and

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}. \tag{4}$$

It holds

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n} \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

The q-binomial coefficient satisfies the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \,.$$

The q-derivative was introduced by Jackson [14] in 1908.

Definition 1. Let $q \in \mathbb{R} \setminus \{1\}$. The q-derivative of a function $f : G \to \mathbb{R}$ at a point $x \in G$, $x \neq 0$ is

$$D_q(f)(x) \equiv (D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

if the expression on the right-hand side is defined. If $0 \in G$ then the q-derivative of f at zero is defined by

$$(D_q f)(0) = f'(0)$$

provided that $f'(0)$ exists.

We say that a function $f : G \to \mathbb{R}$ is q-differentiable at $x \in G$ if $(D_q f)(x)$ exists.
It is easy to see that if f is a differentiable function, then

$$\lim_{q\to 1}(D_q f)(x) = (Df)(x) = f'(x)\,.$$

Note that if f has q-derivative at zero then f is defined in some open interval containing zero.

Lemma 1. For any q-differentiable functions $f, g : G \to \mathbb{R}$ and $x \in G$, it is true

$$\begin{aligned} D_q\left(\alpha f + \beta g\right)(x) &= \alpha(D_q f)(x) + \beta(D_q g)(x), \qquad \text{for all} \quad \alpha, \beta \in \mathbb{R}, \\ D_q\left(f g\right)(x) &= f(qx)(D_q g)(x) + (D_q f)(x) g(x) \\ &= f(x)(D_q g)(x) + (D_q f)(x) g(qx), \\ D_q\left(\frac{f}{g}\right)(x) &= \frac{(D_q f)(x) g(x) - f(x)(D_q g)(x)}{g(x) g(qx)}, \qquad \text{provided} \quad g(x)g(qx) \neq 0. \end{aligned}$$

The higher order q-derivatives of f at $x \neq 0$ are defined recursively by

$$(D_q^n f)(x) = D_q \left(D_q^{n-1} f \right)(x), \qquad n \in \mathbb{N}, \tag{5}$$

where $(D_q^0 f)(x) = f(x)$. If $(D_q^n f)(x)$ exists then we say that f is n times q-differentiable at x.

The formula for nth q-derivative at $x = 0$ was proved by Koekoek [15] in 1993. It is valid both for the real and complex functions.

Lemma 2 ([15])**.** Let n be a positive integer, $q \neq 1$ and let f be a function for which $f^{(n)}(0)$ exists. Then we have

$$(D_q^n f)(0) = \lim_{x\to 0}(D_q^n f)(x) = \frac{[n]_q!}{n!} f^{(n)}(0).$$

Remark 1. It follows that f is n times q-differentiable at $x = 0$ if and only if $f^{(n)}(0)$ exists. On the other hand, f is n times q-differentiable at $x \neq 0$ if and only if f is defined on the set $\{q^i x : i = 0, 1, \ldots, n\}$. Because of that, we will assume throughout the paper that the considered functions are defined on a q-geometric set G_q. Recall, that for a given $q \neq 1$, a subset $G_q \subseteq \mathbb{R}$ is called q-geometric set if $qx \in G_q$ whenever $x \in G_q$. For example, for $q \in (0, 1)$ and fixed $b > 0$, any function $f : (0, b] \to \mathbb{R}$ is infinitely q-differentiable at every $x \in (0, b]$.

Remark 2. By $(D_q^n f)(q^k x)$ we mean $(D_q^n f)(t) \mid_{t=q^k x}$.

The following expected result with nice proof was presented in 2002 by Ash et al. [16].

Lemma 3 ([16])**.** If the ordinary nth derivative $f^{(n)}(x)$ exists then for every $k \in \mathbb{N}$, the nth q-derivative $(D_q^n f)(q^k x)$ also exists and

$$\lim_{q\to 1}(D_q^n f)(q^k x) = f^{(n)}(x).$$

Proof. For $x \neq 0$, the proof follows along the same lines as the proof of Proposition 1 in [16]. Since $\lim_{q\to 1} \frac{(q;q)_n}{n!(1-q)^n} = 1$, from Lemma 2, it follows that the result is valid for $x = 0$ as well. □

The analog of the well-known Leibniz product rule in the q-calculus is given in the next theorem.

Theorem 1. Let $n \in \mathbb{N}$, $q \neq 1$ and let $f, g : G_q \to \mathbb{R}$ be n times q-differentiable at $x \in G_q$. Then

$$D_q^n(fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (D_q^{n-k} f)(q^k x)\, (D_q^k g)(x)\,. \tag{6}$$

The formula (6) is known as the *q-Leibniz product rule* and it is valid both for real and complex functions. Its proof for $x \neq 0$ can be found in [4]. The proof for $x = 0$ follows when we apply Lemma 2 to the case $x \neq 0$. It follows from Lemma 3 and (4) that the formula (6) reduces to ordinary Leibniz product rule when $q \to 1$.

3 | THE FIRST AND SECOND Q-LEIBNIZ QUOTIENT RULE

Let us denote by

$$h(x) = \frac{u(x)}{v(x)}. \tag{7}$$

By the product q-Leibniz rule (6), we have

$$\left(D_q^n u \right)(x) = D_q^n(hv)(x) = (D_q^n h)(x)v(x) + \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (D_q^{n-k} h)(q^k x)(D_q^k v)(x).$$

Hence, we can directly prove the the first q-Leibniz quotient formula.

Theorem 2. Let $q \neq 1$ and let $u, v : G_q \to \mathbb{R}$ be n times q-differentiable functions at $x \in G_q$. If $v(q^k x) \neq 0$, $k \in \{0, 1, \ldots, n\}$ then

$$D_q^n \left(\frac{u}{v} \right)(x) = \frac{1}{v(x)} \left((D_q^n u)(x) - \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^{n-k} \left(\frac{u}{v} \right)(q^k x)\, (D_q^k v)(x) \right). \tag{8}$$

When q tends to 1, from Lemma 3 and limit (4), we obtain that

$$\left(\frac{u}{v}\right)^{(n)}(x) = \frac{1}{v(x)}\left(u^{(n)}(x) - \sum_{k=1}^{n}\binom{n}{k}\left(\frac{u}{v}\right)^{(n-k)}(x)\, v^{(k)}(x)\right).$$

For the second q-Leibniz quotient formula we need the following lemma.

Lemma 4. Let $q \neq 1$ and let $v : G_q \to \mathbb{R}$ be q-differentiable function at $x \in G_q$. If

$$\psi_{0,n}(x) = \prod_{j=0}^{n} v(q^j x) \qquad \text{and} \qquad \psi_{1,n}(x) = \prod_{j=1}^{n} v(q^j x)$$

then

$$\left(D_q\psi_{0,n}\right)(x) = [n+1]_q \psi_{1,n}(x)\left(D_{q^{n+1}} v\right)(x). \tag{9}$$

Proof. Let $x \neq 0$. Obviously,

$$\psi_{0,n}(qx) = \psi_{1,n}(x) v(q^{n+1}x).$$

According to the definition of the q-derivative, we have

$$\begin{aligned}\left(D_q\psi_{0,n}\right)(x) &= \frac{\psi_{0,n}(x) - \psi_{0,n}(qx)}{(1-q)x} = \psi_{1,n}(x)\frac{v(x) - v(q^{n+1}x)}{(1-q)x}\\ &= \psi_{1,n}(x)\frac{v(x) - v(q^{n+1}x)}{(1-q^{n+1})x}\frac{(1-q^{n+1})x}{(1-q)x}\\ &= \psi_{1,n}(x)\left(D_{q^{n+1}} v\right)(x)[n+1]_q .\end{aligned}$$

Suppose now that $x = 0$. Using the rule for the derivative of a product of $n+1$ functions, we have

$$\frac{d}{dx}\psi_{0,n}(x) = \sum_{j=0}^{n} \prod_{\substack{0\le i\le n\\ i\neq j}} v(q^i x)\frac{d}{dx}(v(q^j x)),$$

so

$$\frac{d}{dx}\psi_{0,n}(0) = \sum_{j=0}^{n} v^n(0) q^j v'(0) = v^n(0)v'(0)\frac{1-q^{n+1}}{1-q} = v^n(0)v'(0)[n+1]_q.$$

The formula (9) now follows by the definition of q-derivative at zero. □

Lemma 5. Let $q \neq 1$ and let $v : G_q \to \mathbb{R}$ be n times q-differentiable function at $x \in G_q$. If $v(q^j x) \neq 0$, $j \in \{0, 1, \dots, n\}$, then

$$D_q^n\left(\frac{1}{v}\right)(x) = (-1)^n\frac{A_n(x)}{\prod_{j=0}^{n} v(q^j x)}\ , \tag{10}$$

where

$$\begin{aligned}&A_0(x) = 1,\\ &A_n(x) = [n]_q (D_{q^n} v)(x) A_{n-1}(x) - \left(D_q A_{n-1}\right)(x) v(x), \qquad n \ge 1.\end{aligned} \tag{11}$$

Proof. We will prove the equation (10) by the mathematical induction on n. For $n = 0$ the equation is trivially satisfied. Suppose that (10) is true for $n = k$ and let us prove it for $n = k + 1$. Let $\psi_{0,n}$ be as in Lemma 4. From the inductive hypothesis and the rule for the q-derivative of the quotient given in Lemma 1 we have

$$D_q^{k+1}\left(\frac{1}{v}\right)(x) = D_q\left((-1)^k\frac{A_k(x)}{\psi_{0,k}(x)}\right) = (-1)^k\frac{(D_qA_k)(x)\psi_{0,k}(x) - A_k(x)(D_q\psi_{0,k})(x)}{\psi_{0,k}(x)\psi_{0,k}(qx)}\ .$$

By Lemma 4, we obtain

$$\begin{aligned}D_q^{k+1}\left(\frac{1}{v}\right)(x) &= (-1)^k\frac{(D_qA_k)(x)\psi_{0,k}(x) - A_k(x)[k+1]_q\frac{\psi_{0,k}(x)}{v(x)}\left(D_{q^{k+1}} v\right)(x)}{\psi_{0,k}(x)\psi_{0,k}(qx)}\\ &= (-1)^k\frac{(D_qA_k)(x)v(x) - A_k(x)[k+1]_q\left(D_{q^{k+1}} v\right)(x)}{\psi_{0,k+1}(x)}\\ &= (-1)^{k+1}\frac{A_{k+1}(x)}{\psi_{0,k+1}(x)}\ ,\end{aligned}$$

so the formula (10) is true for $n = k + 1$. The proof is now complete. □

Lemma 6. Let k be a positive integer, $q \neq 1$, and let $f : G_q \to \mathbb{R}$ be a q-differentiable function. Then the following holds:

$$\left(D_{q^k} f\right)(x) = \frac{1}{[k]_q} \sum_{j=0}^{k-1} \left(D_q f\right)(q^j x).$$

Proof. By definition of the q^k-derivative, we have

$$\begin{aligned}\left(D_{q^k} f\right)(x) &= \frac{f(x) - f(q^k x)}{(1-q^k)x} = \frac{1-q}{1-q^k} \sum_{j=0}^{k-1} \frac{f(q^j x) - f(q^{j+1} x)}{(1-q)x} \\ &= \frac{1}{[k]_q} \sum_{j=0}^{k-1} \left(D_q f\right)(q^j x).\end{aligned}$$

□

Remark 3. Due to the previous lemma, we are able to avoid different basis of derivative in (11). In that way, the numerator $A_n(x)$ in formula (10) can be presented as

$$A_n(x) = A_{n-1}(x) \sum_{j=0}^{n-1} (D_q v)(q^j x) - \left(D_q A_{n-1}\right)(x) v(x), \qquad n \geq 1. \tag{12}$$

Theorem 3. Let $q \neq 1$ and let $u, v : G_q \to \mathbb{R}$ be n times q-differentiable functions at $x \in G_q$. If $v(q^j x) \neq 0$, $j \in \{0, 1, \dots, n\}$, then the second q-Leibniz quotient rule holds:

$$D_q^n \left(\frac{u}{v}\right)(x) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q (D_q^{n-k} u)(q^k x) \frac{A_k(x)}{\prod_{j=0}^{k} v(q^j x)}\ , \tag{13}$$

where $A_k(x)$ is given by (11), i.e. (12).

Proof. According to (6), we can write

$$D_q^n \left(\frac{u}{v}\right)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (D_q^{n-k} u)(q^k x) \cdot D_q^k \left(\frac{1}{v}\right)(x)\ .$$

By the Lemma 5, we finish the proof. □

Returning to the classical mathematical analysis, according to Lemma 3 and identity (4), we confirm the following formula.

Theorem 4. The Leibniz quotient rule can be written in the form

$$\left(\frac{u}{v}\right)^{(n)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u^{(n-k)}(x)\ \frac{A_k(x)}{v^{k+1}(x)}\ ,$$

where

$$A_0(x) = 1, \qquad A_k(x) = k v'(x) A_{k-1}(x) - v(x) A'_{k-1}(x), \qquad 1 \leq k \leq n.$$

4 | THE THIRD Q-LEIBNIZ QUOTIENT RULE

The third q-Leibniz quotient formula is the most interesting one.

Theorem 5. Let $q \neq 1$ and let $u, v : G_q \to \mathbb{R}$ be n times q-differentiable functions at $x \in G_q$. If $v(q^k x) \neq 0$, $k \in \{0, 1, \dots, n\}$, then

$$D_q^n \left(\frac{u}{v}\right)(x) = \frac{(-1)^n}{\prod_{k=0}^{n} v(q^k x)} \det W_{n+1}, \tag{14}$$

where W_{n+1} is an $(n+1)\times(n+1)$ matrix defined by

$$\left[W_{n+1}\right]_{i,j} = \begin{cases} (D_q^{i-1}u)(x), & \text{if} \quad j=1, \\ {i-1 \brack j-2}_q (D_q^{i-j+1}v)(q^{j-2}x), & \text{if} \quad i+1 \geq j \geq 2, \\ 0, & \text{if} \quad i+1 < j, \end{cases}$$

i.e. the matrix W_{n+1} looks like

$$W_{n+1} = \begin{bmatrix} u(x) & v(x) & 0 & \cdots & 0 \\ (D_q u)(x) & (D_q v)(x) & v(qx) & & 0 \\ (D_q^2 u)(x) & (D_q^2 v)(x) & {2 \brack 1}_q (D_q v)(qx) & & 0 \\ \vdots & & & \ddots & \\ (D_q^n u)(x) & (D_q^n v)(x) & {n \brack 1}_q (D_q^{n-1}v)(qx) & \cdots & {n \brack n-1}_q (D_q v)(q^{n-1}x) \end{bmatrix}.$$

Proof. Let $h(x) = u(x)/v(x)$. By q-Leibniz rule (6) we have

$$D_q^r(u)(x) = D_q^r(hv)(x) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_q (D_q^{r-k}v)(q^k x)\,(D_q^k h)(x).$$

Considering this formula for $r = 0, 1, \ldots, n$, we get the left triangular system of $n+1$ equations with unknowns $h(x), (D_q h)(x), \ldots, (D_q^n h)(x)$:

$$\begin{aligned} v(x)h(x) &= u(x), \\ (D_q v)(x)h(x) + v(qx)(D_q h)(x) &= (D_q u)(x), \\ &\vdots \\ \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q (D_q^{n-k}v)(q^k x)\,(D_q^k h)(x) + v(q^n x)(D_q^n h)(x) &= (D_q^n u)(x). \end{aligned} \tag{15}$$

Let M_v be the $(n+1)\times(n+1)$ matrix of this system. Its determinant is

$$\det M_v = \prod_{k=0}^{n} v(q^k x).$$

By the Cramer's rule, we have

$$(D_q^n h)(x) = \frac{\det M_{n+1}}{\det M_v},$$

where M_{n+1} is the $(n+1)$th algebraic complement of M_v, i.e. M_{n+1} is formed from the M_v by replacing its last column with $\begin{bmatrix} u(x) & (D_q u)(x) & \cdots & (D_q^n u)(x) \end{bmatrix}^T$. Finally, if we move the columns in M_{n+1} such that its last column becomes the first one then we obtain the matrix W_{n+1}, so the formula (14) is proved. □

Taking $q \to 1$, and applying Lemma 3 and formula (4), we obtain as a consequence the well-known formula for ordinary derivative which is proved by Gerrish [7] in 1980. In fact, the proof of Theorem 5 is based on the idea in [7].

Theorem 6 ([7])**.** If functions u and v are n-times differentiable at x and $v(x) \neq 0$ then

$$\left(\frac{u}{v}\right)^{(n)}(x) = \frac{(-1)^n}{v^{n+1}(x)} \det W_{n+1},$$

where

$$W_{n+1} = \begin{bmatrix} u(x) & v(x) & 0 & \cdots & 0 & 0 \\ u'(x) & v'(x) & v(x) & & 0 & 0 \\ u''(x) & v''(x) & \binom{2}{1} v'(x) & & 0 & 0 \\ \vdots & & & \ddots & & \\ u^{(n-1)}(x) & v^{(n-1)}(x) & \binom{n-1}{1} v^{(n-2)}(x) & \cdots & \binom{n-1}{n-2} v'(x) & v(x) \\ u^{(n)}(x) & v^{(n)}(x) & \binom{n}{1} v^{(n-1)}(x) & \cdots & \binom{n}{n-2} v''(x) & \binom{n}{n-1} v'(x) \end{bmatrix}.$$

5 | THE FOURTH Q-LEIBNIZ QUOTIENT RULE

In the year 1991 Leslie [6] provided the following nice formula for the nth derivative of the quotient:

$$\left(\frac{u}{v}\right)^{(n)}(x) = \frac{1}{v(x)}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\frac{\left(uv^k\right)^{(n)}(x)}{v^k(x)}. \tag{16}$$

In this section, we will generalize this formula in q-calculus. Compared to the previous formulas, it seems that this one is the most efficient. To get a better insight, let us see what we get in the case of the first and the second q-derivative. According to the product and the quotient q-derivative rules given in Lemma 1, we can write

$$\begin{aligned} D_q\left(\frac{u}{v}\right)(x) &= \frac{(D_q u)(x)v(x) - u(x)(D_q v)(x)}{v(x)v(qx)} \\ &= \frac{(D_q u)(x)v(x) - \left[D_q(uv)(x) - (D_q u)(x)v(qx)\right]}{v(x)v(qx)}, \end{aligned}$$

wherefrom

$$D_q\left(\frac{u}{v}\right)(x) = \frac{1}{v(x)v(qx)}\left[\left(v(x)+v(qx)\right)(D_q u)(x) - D_q(uv)(x)\right].$$

We find similarly that

$$D_q^2\left(\frac{u}{v}\right)(x) = \frac{1}{v(x)v(qx)v(q^2x)}\left[B_{2,0}(D_q^2 u)(x) - B_{2,1}D_q^2(uv)(x) + B_{2,2}D_q^2(uv^2)(x)\right],$$

where

$$\begin{aligned} B_{2,0} &= v(x)v(qx) + v(x)v(q^2x) + v(qx)v(q^2x) = \sum_{0\le i<j\le 2} v(q^i x)v(q^j x), \\ B_{2,1} &= v(x) + v(qx) + v(q^2x) = \sum_{0\le i\le 2} v(q^i x), \\ B_{2,2} &= 1 = \sum_{i\in\emptyset} v(q^i x). \end{aligned}$$

We need the following lemma which may be of independent interest.

Lemma 7. Let $q \neq 1$ and $x \in G_q$. Let $\mathcal{A}$ be the set of all functions $f : G_q \to \mathbb{R}$ which are n times q-differentiable at $x \in G_q$. For $v \in \mathcal{A}$ let $M_v = M_v(x)$ be the $(n+1)\times(n+1)$ matrix defined by

$$[M_v]_{i,j} = \begin{cases} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_q (D_q^{i-j}v)(q^{j-1}x), & 1 \le j \le i \le n+1, \\ 0, & 1 \le i < j \le n+1. \end{cases} \tag{17}$$

Then the map $v \mapsto M_v$ is a homomorphism from the commutative algebra $\mathcal{A}$ to the commutative algebra of $(n+1)\times(n+1)$ lower-triangular matrices. That is, the following hold for arbitrary $u, v \in \mathcal{A}$ and $c \in \mathbb{R}$:

(i) $M_{u+v} = M_u + M_v$,

(ii) $M_{cu} = cM_u$,

(iii) $M_{uv} = M_u M_v = M_v M_u$,

and consequently

(iv) $M_{1/v} = M_v^{-1}$, provided $v(x) \neq 0$,

(v) $M_{u/v} = M_u M_v^{-1} = M_v^{-1} M_u$, provided $v(x) \neq 0$.

Proof. The properties (i) and (ii) directly follow from the linearity of the map $v \mapsto D_q^k v$, $k \in \mathbb{N} \cup \{0\}$. We have

$$
\begin{aligned}
[M_u M_v]_{i,j} &= \sum_{p=1}^{n+1} [M_u]_{i,p} [M_v]_{p,j} = \sum_{p=j}^{i} [M_u]_{i,p} [M_v]_{p,j} \\
&= \sum_{p=j}^{i} \begin{bmatrix} i-1 \\ p-1 \end{bmatrix}_q (D_q^{i-p} u)(q^{p-1} x) \begin{bmatrix} p-1 \\ j-1 \end{bmatrix}_q (D_q^{p-j} v)(q^{j-1} x) \\
&= \sum_{p=0}^{i-j} \begin{bmatrix} i-1 \\ p+j-1 \end{bmatrix}_q (D_q^{i-(p+j)} u)(q^{p+j-1} x) \begin{bmatrix} p+j-1 \\ j-1 \end{bmatrix}_q (D_q^{p} v)(q^{j-1} x).
\end{aligned}
$$

Like for the ordinary binomial coefficients, it is easy to show that

$$
\begin{bmatrix} i-1 \\ p+j-1 \end{bmatrix}_q \begin{bmatrix} p+j-1 \\ j-1 \end{bmatrix}_q = \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} i-j \\ p \end{bmatrix}_q .
$$

Now, applying the q-Leibniz product formula (6), we obtain

$$
\begin{aligned}
[M_u M_v]_{i,j} &= \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_q \sum_{p=0}^{i-j} \begin{bmatrix} i-j \\ p \end{bmatrix}_q (D_q^{(i-j)-p} u)(q^p \cdot q^{j-1} x)(D_q^p v)(q^{j-1} x) \\
&= \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_q D_q^{i-j}(uv)(q^{j-1} x) = [M_{uv}]_{i,j}
\end{aligned}
$$

so the property (iii) follows. To prove the item (iv) notice that for the constant function $\mathbf{1} : z \to 1$, $z \in G_q$, we have $(D_q^k \mathbf{1})(z) = 0$ for every $k \in \mathbb{N}$ and, of course, $(D_q^k \mathbf{1})(z) = \mathbf{1}(z) = 1$ for $k = 0$. Now, by the property (iii) and (17) we see at once that $M_v M_{1/v} = M_{\mathbf{1}} = I$ where I is the identity matrix of order $n+1$. It follows that $M_{1/v} = M_v^{-1}$. The property (v) follows by (iii) and (iv). □

We need the following notation. Let $F_n = \{0, 1, 2, \ldots, n\}$ and let

$$
\mathcal{F}_{n,j} = \left\{ S \subseteq F_n : |S| = j \right\}, \qquad 0 \le j \le n+1,
$$

that is, $\mathcal{F}_{n,j}$ is the collection of all subsets of F_n with j elements.

We are now in a position to prove the fourth q-Leibniz quotient rule.

Theorem 7. Let $q \neq 1$ and let $u, v : G_q \to \mathbb{R}$ be n times q-differentiable functions at $x \in G_q$. If $v(q^i x) \neq 0$, $i \in \{0, 1, \ldots, n\}$, then

$$
D_q^n \left(\frac{u}{v} \right)(x) = \frac{1}{\prod_{i=0}^{n} v(q^i x)} \sum_{k=0}^{n} (-1)^k B_{n,k} D_q^n \left(u v^k \right)(x), \tag{18}
$$

where

$$
\begin{aligned}
B_{n,k} &= \sum_{S \in \mathcal{F}_{n,n-k}} \prod_{i \in S} v\left(q^i x\right), \qquad 0 \le k \le n-1, \\
B_{n,n} &= 1 .
\end{aligned}
$$

Remark 4. We can rewrite $B_{n,k}$ as

$$
\begin{aligned}
B_{n,k} &= \sum_{0 \le i_1 < i_2 < \cdots < i_{n-k} \le n} \prod_{j=1}^{n-k} v(q^{i_j} x), \qquad 0 \le k \le n-1, \\
B_{n,n} &= 1 .
\end{aligned}
$$

Proof. Let's first notice that $\mathcal{F}_{n,0} = \{\emptyset\}$, so the given definition for $B_{n,k}$, $0 \le k \le n-1$, is also valid for $k = n$. Furthermore, note that the matrix M_v from Lemma 7 is the matrix of the system of linear equations

$$
\sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix}_q (D_q^{r-k} v)(q^k x)\,(D_q^k h)(x) = (D_q^r u)(x), \qquad r = 0, 1, \ldots, n, \tag{19}
$$

with unknowns $(D_q^k h)(x)$, $k = 0, 1, \ldots n$, where $h = u/v$. The matrix form of this system is

$$
M_v X = L, \tag{20}
$$

where X and L are $(n+1)\times 1$ column vectors defined by $[X]_{i,1} = (D_q^{i-1}h)(x)$ and $[L]_{i,1} = (D_q^{i-1}u)(x)$, $i \in \{1,2,\dots,n+1\}$. For a fixed $x \in G_q$, the characteristic polynomial of lower triangular matrix M_υ is

$$p(t) = \det(M_\upsilon - tI) = \prod_{i=1}^{n+1}\left([M_\upsilon]_{i,i} - t\right) = \prod_{i=1}^{n+1}\left(\upsilon(q^{i-1}x) - t\right).$$

Since $\upsilon(q^i x) \neq 0$, $i \in \{0,1,\dots,n\}$, we conclude that the matrix M_υ is invertible. By the Cayley-Hamilton theorem, we know that M_υ satisfies its characteristic polynomial equation, that is,

$$\prod_{i=0}^{n}\left(\upsilon(q^i x)I - M_\upsilon\right) = \mathbf{0} \tag{21}$$

where I is the identity matrix of order $n+1$ and $\mathbf{0}$ is the square null matrix of order $n+1$. Each term in the expansion of the product (21) is of the form

$$(-1)^k M_\upsilon^k \prod_{i\in S}\upsilon\left(q^i x\right)$$

where $k \in \{0,1,\dots,n+1\}$ is the number of times we choose M_υ as a factor in the term and $S \in \mathcal{F}_{n,n-k+1}$ contains the positions where we do not choose M_υ as a factor in the term. Of course, when $k = n+1$, the corresponding term reduces to $(-1)^{n+1}M_\upsilon^{n+1}$. It follows that

$$\begin{aligned}
\mathbf{0} &= \sum_{k=0}^{n+1}\sum_{S\in\mathcal{F}_{n,n-k+1}}(-1)^k\prod_{i\in S}\upsilon\left(q^i x\right)M_\upsilon^k \\
&= \prod_{i=0}^{n}\upsilon(q^i x)I + \sum_{k=1}^{n+1}(-1)^k\sum_{S\in\mathcal{F}_{n,n-k+1}}\prod_{i\in S}\upsilon\left(q^i x\right)M_\upsilon^k \\
&= \prod_{i=0}^{n}\upsilon(q^i x)I + \sum_{k=0}^{n}(-1)^{k+1}B_{n,k}M_\upsilon^{k+1}.
\end{aligned} \tag{22}$$

By Lemma 7, we conclude by induction that $M_\upsilon^k = M_{\upsilon^k}$, for every $k \in \mathbb{N}$. Therefore, multiplying (22) by M_υ^{-1} we obtain

$$M_\upsilon^{-1} = \frac{1}{\prod_{i=0}^{n}\upsilon(q^i x)}\sum_{k=0}^{n}(-1)^k B_{n,k}M_{\upsilon^k}. \tag{23}$$

Now, from (20) and q-Leibniz product rule (6), it follows that

$$\begin{aligned}
D_q^n\left(\frac{u}{\upsilon}\right)(x) &= [X]_{n+1,1} = [M_\upsilon^{-1}L]_{n+1,1} = \sum_{p=1}^{n+1}[M_\upsilon^{-1}]_{n+1,p}[L]_{p,1} \\
&= \sum_{p=0}^{n}\frac{1}{\prod_{i=0}^{n}\upsilon(q^i x)}\sum_{k=0}^{n}(-1)^k B_{n,k}\left[M_{\upsilon^k}\right]_{n+1,p+1}[L]_{p+1,1} \\
&= \frac{1}{\prod_{i=0}^{n}\upsilon(q^i x)}\sum_{p=0}^{n}\sum_{k=0}^{n}(-1)^k B_{n,k}\begin{bmatrix} n \\ p\end{bmatrix}_q\left(D_q^{n-p}\upsilon^k\right)(q^p x)\left(D_q^p u\right)(x) \\
&= \frac{1}{\prod_{i=0}^{n}\upsilon(q^i x)}\sum_{k=0}^{n}(-1)^k B_{n,k}\sum_{p=0}^{n}\begin{bmatrix} n \\ p\end{bmatrix}_q\left(D_q^{n-p}\upsilon^k\right)(q^p x)\left(D_q^p u\right)(x) \\
&= \frac{1}{\prod_{i=0}^{n}\upsilon(q^i x)}\sum_{k=0}^{n}(-1)^k B_{n,k}D_q^n(u\upsilon^k)(x).
\end{aligned}$$

□

It is easy to see that

$$\lim_{q\to 1}\frac{1}{\prod_{i=0}^{n}\upsilon(q^i x)}B_{n,k} = \binom{n+1}{n-k}\upsilon^{n-k-(n+1)}(x) = \frac{1}{\upsilon^{k+1}(x)}\binom{n+1}{k+1}.$$

Now, from Lemma 3, it follows that q-Leibniz quotient rule (18) reduces to the ordinary Leibniz quotient rule (16) when $q \to 1$.

5.1 | Some interesting identities

Applying formula (18) to some concrete functions, we can derive interesting q-binomial identities.

Suppose that $q \neq 1, 0$, and let $u(x) = 1$ and $v(x) = x^m$, $x \neq 0$, where m is an arbitrary integer. Since $D_q(x^k) = [k]_q x^{k-1}$ we have that

$$D_q^n(x^k) = [k]_q[k-1]_q \cdots [k-n+1]_q x^{k-n}, \qquad k \in \mathbb{Z}.$$

Applying the identity (3), the left hand side of the formula (18) becomes

$$L = D_q^n\left(\frac{u}{v}\right)(x) = D_q^n(x^{-m}) = \frac{(-1)^n}{q^{mn+\binom{n}{2}} x^{m+n}} [m]_q[m+1]_q \cdots [m+n-1]_q.$$

The right hand side of the formula (18) is

$$\begin{aligned} R &= \frac{1}{\prod_{i=0}^{n}(q^i x)^m} \sum_{k=0}^{n} (-1)^k \sum_{S \in \mathcal{F}_{n,n-k}} \prod_{i \in S} (q^i x)^m D_q^n(x^{km}) \\ &= \frac{1}{q^{m\binom{n+1}{2}} x^{m(n+1)}} \sum_{k=1}^{n} (-1)^k \sum_{S \in \mathcal{F}_{n,n-k}} q^{mw(S)} x^{m(n-k)} [km]_q[km-1]_q \cdots [km-n+1]_q x^{km-n}, \end{aligned}$$

where

$$w(S) = \sum_{i \in S} i.$$

The following identity in a slightly different form is proved in Theorem 6.1, page 19, in [11]:

$$\sum_{S \in \mathcal{F}_{n,j}} q^{w(S)} = q^{\binom{j}{2}} \begin{bmatrix} n+1 \\ j \end{bmatrix}_q, \qquad 0 \leq j \leq n+1\,. \tag{24}$$

Applying the identity (24) and equating the expressions for L and R, after arranging, we obtain the following identity which is valid for every integers m and $n \geq 1$ and every real $q \neq 0, 1$:

$$(-1)^n q^{(m-1)\binom{n}{2}} [m]_q[m+1]_q \cdots [m+n-1]_q = \sum_{k=1}^{n} (-1)^k q^{m\binom{n-k}{2}} [km]_q[km-1]_q \cdots [km-n+1]_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q^m}. \tag{25}$$

If $m \geq 1$ then dividing both sides of (25) by $[n]_q!$, we get

$$(-1)^n q^{(m-1)\binom{n}{2}} \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q = \sum_{k=1}^{n} (-1)^k q^{m\binom{n-k}{2}} \begin{bmatrix} km \\ n \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q^m}.$$

where we assume that $\begin{bmatrix} i \\ j \end{bmatrix}_q = 0$ when $i < j$.

Suppose now that $m < 0$ and $n \geq -m+1$. Then $L = 0$, and by applying identity (3), we obtain:

$$0 = \sum_{k=1}^{n} (-1)^k q^{m\binom{k+1}{2}} \begin{bmatrix} n-km-1 \\ n \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q^m}.$$

Similarly, if $m < 0$ and $1 \leq n \leq -m$ then

$$(-1)^n q^{mn+\binom{n}{2}} \begin{bmatrix} -m \\ n \end{bmatrix}_q = \sum_{k=1}^{n} (-1)^k q^{m\binom{k+1}{2}} \begin{bmatrix} n-km-1 \\ n \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q^m}.$$

When $q \to 1$, we obtain the binomial identities from [6].

6 | CONCLUSION

We have proved q-analogs of several results about high derivatives of a quotient of two functions which are mostly based on related Leibniz product rule. Derived formulas include auxiliary functions which are recursively determined what reduces the number of operations. In the last one this derivative is expressed as the linear combination of the same order q-derivatives of product some functions. It is interesting that these conclusions can be used in proving the identities in the number theory and combinatorics.

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$\square$