A diffusive predator-prey system with hunting cooperation in predators and prey-taxis: II stationary pattern formation

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This paper is a continuation of the study conducted by Ko and Ryu (2024) [5], which introduces and analyzes a generalized predator-prey reaction-diffusion system incorporating (repulsive) prey-taxis and a hunting cooperation effect in predators, under homogeneous Neumann boundary conditions. In the study, the existence and uniqueness of global and classical solutions for the time- and space-dependent system are analytically examined. Furthermore, the study examines the local and global stability and convergence rate at the constant predator-extinction and coexistence states. In our paper, we analyze the stationary system corresponding to the system in [5], with a specific focus on examining the existence and nonexistence of positive and nonconstant solutions. The nonexistence occurs when the diffusion rate of prey is sufficiently high. On the other hand, the existence occurs when the prey-tactic rate is sufficiently high, indicating a strong repulsive prey-taxis, and the diffusion rate of prey is sufficiently low. For this investigation, we separately employ the energy method and the Leray-Schauder degree theory.

Keywords: Stationary pattern; Predator-prey; Hunting cooperation; Prey-taxis

MSC 2020: 35B36, 35J57, 92D25, 92D40

1. Introduction

In [5], we have provided a qualitative study on the following reaction-diffusion system modeling predator-prey interactions incorporating a hunting-cooperation functional response and prey-taxis:

$$\begin{cases} u_t - \tau\Delta u = \mathcal{F}(u,v) & \text{in } (0,\infty)\times\Omega, \\ v_t - \Delta v - \chi\nabla\cdot(q(v)\nabla u) = \mathcal{G}(u,v) & \text{in } (0,\infty)\times\Omega, \\ \partial_n u = \partial_n v = 0 & \text{on } (0,\infty)\times\partial\Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{F}(u,v) = g(u) - vf(u,v) \quad\text{and}\quad \mathcal{G}(u,v) = \beta v f(u,v) - \gamma(v);$$

$\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary denoted by $\partial\Omega$; $n(x)$ is the outward unit normal vector on $\partial\Omega$ with $\partial_n = \partial/\partial n$; and the parameters β, τ, and χ are positive constants. Moreover, on the given functions g, f, γ and q, we have assumed that there exist positive constants K, M_g, M_f, M_γ, and M_q such that the following hypotheses hold:

- (H1) $g \in C^1([0,\infty),\mathbb{R})$, $g(0) \geq 0$, $g(K) = 0$, $0 < g(u) \leq M_g$ for all $0 < u < K$, while $g(u) < 0$ for any $u > K$.
- (H2) $f \in C^1([0,\infty)^2,[0,\infty))$, $f(u,0) \geq 0$, $f(0,v) = 0$, $f_u(u,v) > 0$ and $f_v(u,v) > 0$ for any u, $v > 0$, and $f(u,v) \leq M_f$ for all u, $v \geq 0$.
- (H3) $\gamma \in C^1([0,\infty),[0,\infty))$, $\gamma(0) = 0$, $\gamma'(0) = M_\gamma$, $\gamma(v) \geq M_\gamma v$ and $\gamma'(v) > 0$ for any $v \geq 0$.
- (H4) $q \in C^1([0,\infty),[0,\infty))$, $q(0) = 0$, and $q(v) \leq M_q v$ for all $v \geq 0$.

Here, u and v represent the densities of the prey and its predator, respectively. Moreover, τ represents the diffusion rate, indicating that each prey species exhibits random movement within Ω, and β signifies the conversion rate of prey into predators. The function g, satisfying (H1), reflects the prey species' intrinsic growth rate, demonstrating the logistic property. The functional response f, satisfying (H2), illustrates the cooperative effect among predators when hunting their own prey. The function γ represents the predator's net growth rate, influenced by the death rate of predators or by competition among predators. The term $-\chi\nabla\cdot(q(v)\nabla u)$ denotes a repulsive prey-taxis (e.g., [3]), indicating an ecological situation where predators in Ω tend to move in the opposite direction of the increasing prey species gradient, suggesting that the prey possesses a defense mechanism against its predators. Here, q and χ are respectively referred to as the prey-tactic sensitivity function and the intrinsic prey-tactic rate. Synthetically, the system (1.1) is a model of predator-prey interaction characterized by three features: both prey and predators having a generalized growth rate, the prey possessing a group defense mechanism implemented in the form of prey-taxis diffusion against their predators, and the predators exhibiting a cooperative effect (e.g., [1, 13, 14]) in the form of a response function when hunting such prey. For a detailed derivation process and biological background regarding (1.1), refer to [5].

Following Turing's groundbreaking work, one of the most intriguing questions in the field of PDEs is whether spatially inhomogeneous steady states can be generated. As a result, a remarkably large number of interesting studies have been conducted on the occurrence and non-occurrence of stationary patterns in reaction-diffusion systems, observed in various fields such as ecology, biology, and chemistry. Building on this trend, the predator-prey systems with (attractive) prey-taxis, which have been studied for pattern formation, exhibit the following form:

$$\begin{cases} u_t - d\Delta u = u\phi(u) - v\xi(u,v) & \text{in } (0,\infty)\times\Omega, \\ v_t - \Delta v + \chi\nabla\cdot(q(v)\nabla u) = cv\xi(u,v) - v\delta(v) & \text{in } (0,\infty)\times\Omega, \\ \partial_n u = \partial_n v = 0 & \text{on } (0,\infty)\times\partial\Omega, \end{cases} \tag{1.2}$$

where the coefficients are positive constants; $u\phi(u)$ represents the prey's growth rate; $\xi(u,v)$ is the functional response; $v\delta(v)$ represents the predator's death rate; and the term $+\chi\nabla\cdot(q(v)\nabla u)$ signifies that the predators have a tendency to move in the direction of the increasing prey species gradient. When

$$\phi(u) = r\left(1 - \frac{u}{K}\right),\ \xi(u,v) = \frac{mu}{1+au},\ \delta(v) = b, \text{ and } q(v) = vq_1(v), \tag{1.3}$$

where $q_1 \in C^2([0,1])$, $q_1(v) = 0$ for $v \geq v_m$, $q_1(v) > 0$ for $0 \leq v < v_m$, and all coefficients used here are positive constants, the authors in [7] investigated stationary pattern formation in the system (1.2) using index theory. In [11], global bifurcation theory was employed to establish that nonconstant positive solutions of (1.2) are bifurcated when ϕ, ξ and δ satisfy (1.3); and $q_1(v)$ is a constant, rather than a truncated function as in (1.3). Moreover, the authors in [6] studied instability driven by diffusion and small taxis in (1.2) when Ω is an interval; $\phi(u) = K(1-u)(u-a_1)$ or $1-u$ with $0 < a_1 < 1$ and $K = 4/(1-a_1)^2$; $\xi(u,v) = u$ or $(a_2+1)u/(a_2+u)$ with $a_2 > 0$; $\delta(v) = b_1 + b_2 v$ with $b_i > 0$ $(i = 1,2)$; and $q(v) = v$. For further analytical or numerical research results on stationary pattern formation in prey-taxis or predator-taxis predator-prey models similar to (1.2), we refer to [3, 4, 12]. Furthermore, in [10], the study examined local bifurcation results of nonconstant positive steady states for a prey-taxis model with more generalized reaction functions and prey-tactic sensitivity functions (satisfying specific assumptions) than those in (1.2), over a one-dimensional domain.

Exploring generalized models that illustrate various dynamics based on different ecological scenarios is always intriguing and provides valuable insights. The results obtained from the models are likely to capture common characteristics shared by each model in different scenarios. These findings can serve as a preliminary study for specific models that may be considered in future scenarios. Despite this significance, research on pattern formation in generalized predator-prey systems with prey-taxis and nonconstant reaction rates is rare. Furthermore, considering that (1.1) is a model with repulsive prey-taxis and cooperative reaction functions that has recently gained considerable attention due to its significant ecological implications (see [5] and references therein) mentioned earlier, study on the stationary pattern formation in (1.1) is necessary. Thus, in this paper, our interest extends beyond the solutions of the time- and space-dependent system (1.1) to include the solutions of the steady-state corresponding to (1.1), namely the coupled elliptic system:

$$\begin{cases} -\tau\Delta u = \mathcal{F}(u,v) & \text{in } \Omega, \\ -\Delta v - \chi\nabla\cdot(q(v)\nabla u) = \mathcal{G}(u,v) & \text{in } \Omega, \\ \partial_n u = \partial_n v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Most of the study on (1.4) is dedicated to investigating pattern formation, specifically the existence of nonconstant positive solutions and the nonexistence of such solutions. The primary purposes of the study outlined in this paper are as follows:

(i) We present sufficient conditions that yield the nonexistence of positive and nonconstant solutions of (1.4) where q is in its simplest form, i.e., $q(v) = v$. Specifically, when τ is sufficiently large, we establish nonexistence by imposing additional assumptions on f, g, and γ. To achieve this, we prove that positive solutions to (1.4) have *a-priori* bounds by using standard elliptic regularity theory. Moreover, we study the convergence of positive solutions of the system to the constant positive solutions in the $L^\infty(\Omega)$-norm as $\tau \to \infty$.

(ii) We are investigating conditions that guarantee the existence of positive and nonconstant solutions of (1.4) where $q(v) = v$. Additional assumptions are required for f, g, and γ to achieve this purpose, particularly when τ is small and χ is large. To achieve this existence, we need to prove that the positive solutions have a positive and uniform lower bound. We further use the previously obtained result on the nonexistence in (i) along with the Leray-Schauder degree theory.

The brief overview of the paper's structure is as follows. In Section 2, we introduce additional assumptions required for the proposed study and present results on the constant coexistence of (1.4) from [5]. In Section 3, we present the main theorems of the results corresponding to the objectives of this study without providing detailed proofs. In Section 4, we perform detailed proofs of these theorems.

2. Preliminary

In this section, we sequentially list the additional assumptions necessary to obtain the results of this paper and also review the sufficient conditions from [5, Theorem 2.3] for (1.1) to have a positive constant solution of (1.4).

First, in addition to the assumptions (H1)-(H4) given in Section 1, we will invoke and utilize the following assumptions as necessary:

(H1b) $g'(K) < 0$ and $g'(0) > 0$ hold when $g(0) = 0$; $g'(K) < 0$ holds when $g(0) > 0$.

(H2e) $f(u, v) \in C^2([0, \infty)^2)$, and $f_{vv}(u, v) \leq 0$ for any u, $v \geq 0$.

(H3c) $\frac{\gamma(v)}{v} \in C^1([0, \infty))$, $\left.\frac{\gamma(v)}{v}\right|_{v=0} = M_\gamma$, and $\frac{d}{dv}\frac{\gamma(v)}{v} \geq \gamma_*$ for any $v > 0$, where $\gamma_* > 0$ is a constant independent of M_γ.

We next use the following notation to express the sufficient conditions for the constant coexistence of (1.4) and the properties satisfied by this coexistence:

$$\xi(u) := \gamma^{-1}(\beta g(u)) \quad \text{and} \quad \mathcal{H}(u) := g(u) - f(u, \xi(u))\xi(u).$$

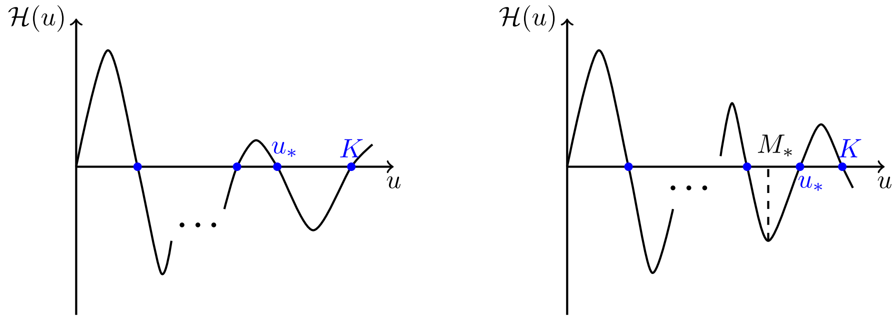
Theorem 2.1. *Assume that assumptions (H1)-(H3) and (H1b) hold. Then, there exists at leat one positive constant solution, denoted by* $\mathbf{u}_* := (u_*, v_*)$*, of (1.1) if either one of the followings holds:*

$$\begin{aligned} &(i)\ \mathcal{H}'(K) > 0 \ (i.e.,\ \beta f(K, 0) - \gamma'(0) > 0); \\ &(ii)\ \mathcal{H}'(K) \leq 0 \text{ and } \mathcal{H}(M_*) < 0 \text{ for some constant } M_* \in (0, K). \end{aligned} \tag{2.1}$$

Moreover, $\mathbf{u}_*$ *satisfies*

$$\mathcal{H}(u_*) = 0,\ 0 < u_* < K,\ \textit{and}\ 0 < v_* = \xi(u_*) \le \beta \frac{M_g}{M_\gamma}.$$

For the readers' convenience, we have extracted the proof of the theorem from [5] and included it in the appendix of this paper. For reference, Figure 1 provides an example of how the positive constant solution of (1.4) is determined in the function H when the former in (H1b) is satisfied. Furthermore, concrete examples satisfying Theorem 2.1 are provided in [5, Remark 2.4].



(a) (i) in (2.1)

(b) (ii) in (2.1)

Figure 1. The graphs of $\mathcal{H}(u)$ when $g(0) = 0$

3. Main results

We investigate not only the existence of positive and nonconstant solutions of (1.4) with $q(v) = v$ but also their nonexistence. We denote $\Theta := (\beta, \chi, f, g, \gamma, N, \Omega)$ in the sequel for notational convenience. Moreover, in proving the theorem below, we prepare the followings: let

$$0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots \le \mu_i \le \cdots \quad \text{and} \quad \lim_{i\to\infty} \mu_i = \infty$$

be all eigenvalues of the problem $-\Delta\phi = \mu\phi$ in Ω and $\partial_n \phi = 0$ on $\partial\Omega$. Additionally, let m_i denote the multiplicity corresponding to μ_i. These notations will continue to be used in the study of (1.4).

Theorem 3.1. *Suppose that (H1)-(H3) and (H3c) hold, and let* $q(v) = v$*. If*

$$\beta M_f > M_\gamma, \quad \mu_1 > (\beta M_f - M_\gamma)\left(1 + \beta \frac{e^{\chi K}}{\gamma_*} \max_{\substack{u\in[0,K],\\ v\in[0, e^{\chi K}(\beta M_f - M_\gamma)/\gamma_*]}} f_v(u,v)\right) \tag{3.1}$$

hold, there exists a constant $\widetilde{\tau}_1 = \widetilde{\tau}_1(\Theta) > 0$ *such that for* $\tau \ge \widetilde{\tau}_1$*, (1.4) has no nonconstant positive solution.*

We present another condition giving the nonexistence of nonconstant positive solutions of (1.4), dropping the second inequality in (3.1).

Theorem 3.2. *Assume that (H1)-(H3), (H1b), (H2e), and (H3c) are satisfied, and let* $q(v) = v$. *Then, there exists a constant* $\widetilde{\tau}_2 = \widetilde{\tau}_2(\Theta) > 0$ *such that if*

$$\beta \max_{u\in[0,K]} f_v(u,0) < \gamma_*, \tag{3.2}$$

then (1.4) has no nonconstant positive solution for $\tau \geq \widetilde{\tau}_2$.

As specific examples for the functions f, g, and γ satisfying all the assumptions in Theorem 3.2 when γ_* is large enough to satisfy (3.2), we can provide

$$f(u,v) = \frac{M_f u(1+av)}{u(1+av)+1},\ g(u) = ru(1-u/K),\ \gamma(v) = v(M_\gamma + \gamma_* v) \tag{3.3}$$

with positive constants a and r. Here, we note that the references for the functional response f can be found in [5]; and the large γ_* implies $\mathcal{H}'(K) > 0$, indicating the existence of $\mathbf{u}_*$ (see the proof of Lemma 4.3).

Before ending this section, we provide the sufficient conditions for (1.4) with $q(v) = v$ to possess a nonconstant positive solution. In stating and proving the main result for the nonconstant coexistence state, we need the following notations and a simple result. To the end, we first introduce the notations for $\mathbf{u}_* = (u_*, v_*)$:

$$\begin{aligned} &L_{11}(\mathbf{u}_*) := g'(u_*) - v_* f_u(\mathbf{u}_*), && L_{12}(\mathbf{u}_*) := -v_* f_v(\mathbf{u}_*) - f(\mathbf{u}_*), \\ &L_{21}(\mathbf{u}_*) := \beta v_* f_u(\mathbf{u}_*), && L_{22}(\mathbf{u}_*) := \beta v_* f_v(\mathbf{u}_*) + \beta f(\mathbf{u}_*) - \gamma'(v_*). \end{aligned}$$

Lemma 3.3. *Assume that assumptions (H1)-(H3), (H1b) and (H3c), and (i) in* (2.1) *hold. Assume, additionally, that*

(H5) $\mathcal{H}'(u_*) \neq 0$ *for all positive constant solutions* (u_*, v_*) *of (1.4).*

Then, (1.4) has an odd number of positive constant solutions, denoted as

$$(u_k^*, v_k^*) := \mathbf{u}_k^* \quad \textit{for } k = 1, 2, \ldots, 2n+1,$$

with $u_1^* < u_2^* < \cdots < u_k^* < \cdots < u_{2n+1}^*$. *In addition, assume that* $\beta f_v(\mathbf{u}_k^*) - \gamma_* < 0$ *for all k. Then, for odd k, there exists a constant* $\overline{\tau}_k = \overline{\tau}_k(\Theta) > 0$ *such that*

$$\begin{aligned} Q(\mu, \mathbf{u}_k^*) := \mu^2 &- \left(\frac{L_{11}(\mathbf{u}_k^*) - \chi v_k^* L_{12}(\mathbf{u}_k^*) + L_{22}(\mathbf{u}_k^*)\tau}{\tau}\right)\mu \\ &+ \frac{L_{11}(\mathbf{u}_k^*)L_{22}(\mathbf{u}_k^*) - L_{12}(\mathbf{u}_k^*)L_{21}(\mathbf{u}_k^*)}{\tau} = 0 \end{aligned} \tag{3.4}$$

possesses two distinct positive roots, denoted as $\mu^+(\mathbf{u}_k^*)$ *and* $\mu^-(\mathbf{u}_k^*)$, *with* $\mu^+(\mathbf{u}_k^*) > \mu^-(\mathbf{u}_k^*)$, *provided that*

$$\tau < \overline{\tau}_k \quad \textit{and} \quad \chi > \frac{L_{11}(\mathbf{u}_k^*)}{v_k^* L_{12}(\mathbf{u}_k^*)} := \overline{\chi}_k. \tag{3.5}$$

On the other hand, for even k, $\mu^+(\mathbf{u}_k^*)$ *is a unique positive root of* $Q(\mu, \mathbf{u}_k^*) = 0$.

Theorem 3.4. *Suppose that (H1)-(H3), (H1b), (H2e), (H3c), and (H5) hold, and let $q(v) = v$. Assume, additionally, that (3.2) holds, and*

$$\tau < \min\{\overline{\tau}_k : k = 1, 3, \ldots, 2n+1\}, \quad \chi > \max\{\overline{\chi}_k : k = 1, 3, \ldots, 2n+1\}. \tag{3.6}$$

Let $\mu^+(\mathbf{u}_k^*) \in (\mu_{b_k}, \mu_{b_k+1})$, and $\mu^-(\mathbf{u}_k^*) \in (\mu_{a_k}, \mu_{a_k+1})$ (only when k is odd) for some integers $b_k > a_k \geq 0$. Then, (1.4) possesses at least one nonconstant positive solution, if

$$|\{\sigma_k : 1 \leq k \leq 2n+1,\ \sigma_k = \mathit{even}\}| \neq |\{\sigma_k : 1 \leq k \leq 2n+1,\ \sigma_k = \mathit{odd}\}| + 1, \tag{3.7}$$

where

$$\sigma_k := \begin{cases} \displaystyle\sum_{i=a_k+1}^{b_k} m_i & \text{if } k = 1, 3, \ldots, 2n+1, \\ \displaystyle\sum_{i=0}^{b_k} m_i & \text{if } k = 2, 4, \ldots, 2n, \end{cases}$$

and $|\cdot|$ represents the number of elements in the given set.

Remark 3.5. (i) As an example satisfying the assumptions in Theorem 3.4, we focus on the system (1.4) with f, g, and γ given in (3.3). We choose a γ_* in (H3c) to be large enough to satisfy (3.2), which in turn implies that $\mathcal{H}'(K) > 0$. According to the discussion in [5, Remark 2.8], (1.4) has only one positive constant solution $\mathbf{u}_*$ if γ_* in the function γ is sufficiently large. Therefore, in Theorem 3.4, we have $n = 0$, which implies that k can only be 1. Hence, if χ, γ_* and $1/\tau$ are large enough, we can conclude that if $\sum_{i=a_1+1}^{b_1} m_i$ is odd, then (1.4) admits a nonconstant positive solution.

(ii) Investigating sufficient conditions for pattern formation when (ii) of (2.1) is satisfied is so complex (e.g., see [13]). Furthermore, our main model is composed of generalized terms. Therefore, there are limitations in finding direct (or verifiable) conditions like (2.1) when (ii) in (2.1) is satisfied. The investigation of pattern formation when (ii) of (2.1) is satisfied is planned for future research.

4. Proofs of main results

To begin, we derive *a-priori* estimates for the positive solutions to (1.4).

Lemma 4.1. *Suppose that $\beta M_f > M_\gamma$ and assumptions (H1)-(H3) and (H3c) are satisfied, and let $q(v) = v$. Then, all positive solutions $(u(x), v(x))$ to (1.4) satisfy*

$$\max_{\overline{\Omega}} u(x) \leq K, \quad \max_{\overline{\Omega}} v(x) \leq \widetilde{V} := \frac{\beta M_f - M_\gamma}{\gamma_*} e^{\chi K}. \tag{4.1}$$

Proof. By directly applying the maximum principle (e.g., see [9]) to the first equation in (1.4), we can immediately have that $\max_{\overline{\Omega}} u \leq K$.

We let $v = we^{-\chi u}$. Then, we can obtain from the second equation in (1.4) that

$$-\nabla \cdot \left(e^{-\chi u}\nabla w\right) = \mathcal{G}(u, we^{-\chi u}) \ \text{ in } \Omega, \ \ \partial_n w = 0 \ \text{ on } \partial\Omega.$$

Letting $\overline{w} = \max_{\overline{\Omega}} w(x)$, using assumptions (H2), (H3), and (H3c), and applying the maximum principle once again to the above boundary value problem, one can derive that

$$\begin{aligned}0 &\leq \mathcal{G}(u(x), e^{-\chi u(x)}\overline{w}) \\ &\leq \beta M_f e^{-\chi u(x)}\overline{w} - \gamma(e^{-\chi u(x)}\overline{w}) \\ &\leq \beta M_f e^{-\chi u(x)}\overline{w} - e^{-\chi u(x)}\overline{w}\left(\gamma'(0) + \gamma_* e^{-\chi u(x)}\overline{w}\right).\end{aligned}$$

Thus, by virtue of the first desired result in (4.1), one see that

$$\overline{w} \leq \frac{\beta M_f - M_\gamma}{\gamma_*} e^{\chi K},$$

which completes the proof. $\square$

Recall that we denote $\Theta = (\beta, \chi, f, g, \gamma, N, \Omega)$ before Theorem 3.1.

Lemma 4.2. *Let $\tau_* > 0$ be a fixed constant, and suppose that all assumptions given in Lemma 4.1 hold. Then, every positive solution (u, v) to (1.4) satisfies that for any $\tau \geq \tau_*$,*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})}, \|v\|_{C^{1,\alpha}(\overline{\Omega})} \leq C, \tag{4.2}$$

where $C = C(\Theta, \tau_*) > 0$ is a constant.

Proof. We denote and use C_i as generic positive constants depending only on Θ and τ_*. By multiplying u in the first equation in (1.4), integrating the resulting equation on Ω, and utilizing the uniform L^∞-estimate (4.1), one can obtain

$$\int_\Omega |\nabla u|^2\, dx \leq \frac{C_1}{\tau}. \tag{4.3}$$

Furthermore, by employing the elliptic regularity theory (e.g., see [2]) to the first equation in (1.4):

$$-\Delta u = \frac{\mathcal{F}(u, v)}{\tau} \ \text{ in } \Omega, \quad \partial_n u = 0 \ \text{ on } \partial\Omega,$$

and by using (4.1) and (4.3), one have that

$$\|u\|_{W^{2,2}(\Omega)} \leq C_2 \left(\|u\|_{W^{1,2}(\Omega)} + \frac{1}{\tau}\|\mathcal{F}(u, v)\|_{L^2(\Omega)}\right) \leq C_3.$$

We apply the Sobolev embedding theorems (e.g., see [2]) and bootstrapping (i.e., repeating this argument finitely many times) to show that u belongs to $W^{2,p}(\Omega)$ (with any $p > 1$), satisfying $\|u\|_{W^{2,p}(\Omega)} \leq C_4$. Furthermore, we can use the Sobolev embedding theorems once again to conclude that u belongs to $C^{1,\alpha}(\overline{\Omega})$, and moreover, the norm $\|u\|_{C^{1,\alpha}(\overline{\Omega})}$ is independent of τ. Thus, the first estimate in (4.2) has been established.

We next prove the second estimate in (4.2). By multiplying v to the second equation of (1.4) and integrating the resulting equation on Ω, and then using (4.1) and Young's inequality, we derive

$$
\begin{aligned}
\int_\Omega |\nabla v|^2\,dx &= \int_\Omega \mathcal{G}(u,v)v dx - \chi\int_\Omega v\nabla u\cdot\nabla v dx\\
&\le C_6 + C_5\int_\Omega |\nabla u\cdot\nabla v|\,dx\\
&\le C_6 + \frac{C_5^2}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega |\nabla v|^2\,dx.
\end{aligned}
$$

Thus, in this derivation, using (4.3), we obtain

$$
\int_\Omega |\nabla v|^2\,dx \le C_7. \tag{4.4}
$$

Similar to before, with the aid of (4.1), (4.4), and the fact that $\|u\|_{W^{2,p}(\Omega)} \le C_4$ for any $p>1$, applying the elliptic regularity theory to

$$
-\Delta v - \chi\nabla u\cdot\nabla v = \mathcal{G}(u,v) + \chi v\Delta u \ \text{ in } \Omega, \quad \partial_n v = 0 \ \text{ on } \partial\Omega
$$

which is a reformulation of the second equation in (1.4), and using the Sobolev embedding theorems and bootstrapping, one conclude that for any $p>1$, v belongs to $W^{2,p}(\Omega)$, so that the desired second estimate holds. □

Proof of Theorem 3.1. We first denote (u,v) as a positive solution to (1.4), and for convenience, we let and use

$$
\overline{\psi} = \frac{1}{|\Omega|}\int_\Omega \psi dx \ \text{ for } \psi\in L^1(\Omega), \quad \mathcal{U} = u-\overline{u}, \quad \mathcal{V} = v-\overline{v}.
$$

Moreover, below, we will use an arbitrary constant $\epsilon>0$ and generic constants $M_i>0$ that are independent of τ.

Multiplying $\mathcal{U}$ and $\mathcal{V}$ to the first and second equations of (1.4), respectively, and then integrating over Ω by parts, the followings can be derived:

$$
\begin{aligned}
\tau\int_\Omega |\nabla\mathcal{U}|^2 dx &= \int_\Omega \mathcal{U}\mathcal{F}(u,v)dx\\
&= \int_\Omega (\mathcal{F}(u,v)-\mathcal{F}(\overline{u},v))\mathcal{U}dx - \int_\Omega (vf(\overline{u},v)-\overline{v}f(\overline{u},\overline{v}))\mathcal{U}dx\\
&= \int_\Omega (g'(\phi_1)-vf_u(\phi_1,v))\mathcal{U}^2 dx - \int_\Omega (\psi_1 f_v(\overline{u},\psi_1)+f(\overline{u},\psi_1))\mathcal{U}\mathcal{V}dx
\end{aligned} \tag{4.5}
$$

and

$$
\begin{aligned}
\int_\Omega |\nabla\mathcal{V}|^2 dx + \chi\int_\Omega v\nabla\mathcal{U}\cdot\nabla\mathcal{V}dx &= \int_\Omega \mathcal{V}\mathcal{G}(u,v)dx\\
&= \int_\Omega \frac{\mathcal{G}(u,v)}{v}\mathcal{V}^2 + \overline{v}\left(\frac{\mathcal{G}(u,v)}{v} - \frac{\mathcal{G}(u,\overline{v})}{\overline{v}}\right)\mathcal{V}dx\\
&\quad + \int_\Omega \overline{v}\left(\frac{\mathcal{G}(u,\overline{v})}{\overline{v}} - \frac{\mathcal{G}(\overline{u},\overline{v})}{\overline{v}}\right)\mathcal{V}dx\\
&= \int_\Omega B(u,v)\mathcal{V}^2 dx + \int_\Omega \overline{v}\beta f_u(\phi_2,\overline{v})\mathcal{U}\mathcal{V}dx,
\end{aligned} \tag{4.6}
$$

where

$$B(u,v) = \frac{\mathcal{G}(u,v)}{v} + \overline{v}\beta f_v(u,\psi_2) - \overline{v}\left(\frac{\gamma(z)}{z}\right)'\Bigg|_{z=\psi_2},$$

and ϕ_i (where $i=1,2$) and ψ_i (where $i=1,2$) are within the range of u to $\overline{u}$, and v to $\overline{v}$, respectively. We note that ϕ_i and ψ_i (for $i=1,2$) are obtained using the mean-value theorem.

Using assumptions (H1) and (H2) and (4.1) in (4.5), we can derive that

$$\tau\int_\Omega |\nabla\mathcal{U}|^2 dx \le \int_\Omega M_1\mathcal{U}^2 + M_2\,|\mathcal{U}|\,|\mathcal{V}|\,dx. \tag{4.7}$$

Thus, applying the Poincaré inequality and Young's inequality to (4.7), one can conclude that

$$\left(\tau - \frac{M_1}{\mu_1} - \frac{M_2}{2\mu_1\epsilon}\right)\int_\Omega |\nabla\mathcal{U}|^2 dx \le \frac{M_2}{2\mu_1}\epsilon\int_\Omega |\nabla\mathcal{V}|^2 dx. \tag{4.8}$$

Similarly, using assumptions (H2), (H3), and (H3c), along with (4.1), and applying the Poincaré inequality and Young's inequality to (4.6), one can deduce

$$\begin{aligned}
&\int_\Omega |\nabla\mathcal{V}|^2 dx - \chi\epsilon\frac{\widetilde{V}}{2}\int_\Omega |\nabla\mathcal{V}|^2 dx - \chi\frac{\widetilde{V}}{2\epsilon}\int_\Omega |\nabla\mathcal{U}|^2 dx\\
&\le \int_\Omega \widetilde{B}\mathcal{V}^2 dx + \frac{M_3}{2}\epsilon\int_\Omega \mathcal{V}^2 dx + \frac{M_3}{2\epsilon}\int_\Omega \mathcal{U}^2 dx\\
&\le \int_\Omega\left(\frac{\widetilde{B}}{\mu_1} + \frac{M_3}{2\mu_1}\epsilon\right)|\nabla\mathcal{V}|^2 dx + \frac{M_3}{2\mu_1\epsilon}\int_\Omega |\nabla\mathcal{U}|^2 dx,
\end{aligned}\tag{4.9}$$

where

$$\widetilde{B} = \beta M_f - M_\gamma + \beta\widetilde{V}\max_{u\in[0,K],\,v\in[0,\widetilde{V}]} f_v(u,v).$$

Hence, we have from (4.8) and (4.9) that

$$\left(\tau - \frac{M_1}{\mu_1} - \frac{M_2}{2\mu_1\epsilon} - \chi\frac{\widetilde{V}}{2\epsilon} - \frac{M_3}{2\mu_1\epsilon}\right)\int_\Omega |\nabla\mathcal{U}|^2 dx + \left(1 - \frac{\widetilde{B}}{\mu_1} - \frac{M_3}{2\mu_1}\epsilon - \chi\frac{\widetilde{V}}{2}\epsilon - \frac{M_2}{2\mu_1}\epsilon\right)\int_\Omega |\nabla\mathcal{V}|^2 dx \le 0.$$

Due to (3.1), if we choose a small enough value for ϵ, then the term in the second round bracket is positive. Correspondingly, for large τ, the term in the first round bracket is also positive. This means that $\nabla\mathcal{U}\equiv 0$ and $\nabla\mathcal{V}\equiv 0$, indicating the desired result. □

We obtain the result concerning the asymptotic behavior of positive solutions of (1.4) as $\tau\to\infty$, which will be used in proving Theorem 3.2.

Lemma 4.3. *Assume that (3.2) and all assumptions in Theorem 3.2 are satisfied, and let (u_n, v_n) be a positive solution to (1.4) with $\tau=\tau_n$ and $q(v)=v$. Then, as $\tau_n\to\infty$, (u_n,v_n) converges to a positive constant solution $\mathbf{u}_*$ of (1.4) in $[L^\infty(\Omega)]^2$.*

Proof. By integrating the equations in (1.4) with $(u,v)=(u_n,v_n)$ and $\tau=\tau_n$ over Ω, we obtain that

$$0 = \int_\Omega \mathcal{F}(u_n,v_n)dx,\quad 0 = \int_\Omega \mathcal{G}(u_n,v_n)dx\ \text{ for all } n. \tag{4.10}$$

Furthermore, based on assumptions (H2), (H2e), and (H3c) and inequality (3.2), we can see that

$$\frac{\partial}{\partial v}\frac{\mathcal{G}(u,v)}{v} = \beta f_v(u,v) - \left(\frac{\gamma(v)}{v}\right)' \le \beta f_v(u,0) - \gamma_* < 0 \tag{4.11}$$

for any $v > 0$ and $0 < u \le K$. Using this in the second equation of (4.10) gives that

$$0 = \int_\Omega v_n \frac{\mathcal{G}(u_n, v_n)}{v_n} dx < (\beta f(K,0) - \gamma'(0)) \int_\Omega v_n dx$$

because of (H2), (H3), (H3c), and the fact that u_n satisfies the first estimate in (4.1). Thus, $\beta f(K,0) - \gamma'(0) > 0$, that is, $\mathcal{H}'(K) > 0$. In turn, since $0 < \beta f(K,0) - \gamma'(0) \le \beta M_f - M_\gamma$, we can conclude that $\beta M_f > M_\gamma$, which allows us to use the second estimate in (4.1) as well. Furthermore, due to the given assumptions and the derived inequality $\mathcal{H}'(K) > 0$, we know from Theorem 2.1 that (1.4) has an $\mathbf{u}_* = (u_*, v_*)$.

Now, contrarily, suppose that there exists a constant $\epsilon_0 > 0$ and a subsequence $\{(u_n, v_n)\}$ (which we will still denote by itself) satisfying that for any $\mathbf{u}_*$,

$$\|u_n - u_*\|_{L^\infty(\Omega)} + \|v_n - v_*\|_{L^\infty(\Omega)} \ge \epsilon_0. \tag{4.12}$$

We see from (4.2) that there is a function $\hat{u} \ge 0$ so that $u_n \to \hat{u}$ in $C^1(\overline{\Omega})$ as $n \to \infty$, passing to a subsequence. Moreover, due to the regularity of elliptic equations and the fact of $\tau_n \to \infty$, it follows that $\hat{u}$ is a constant. Furthermore, according to (4.1), $\hat{u} \le K$ is satisfied. Similarly, due to (4.2), there exists a function $\hat{v} \ge 0$ so that passing to a subsequence, $v_n \to \hat{v}$ in $C^1(\overline{\Omega})$ as $n \to \infty$. Thus, considering the weak form of the second equation in (1.4) with $(u, v) = (u_n, v_n)$, and taking the limit as $n \to \infty$, we see that $\hat{v}$ satisfy the following equation weakly:

$$-\Delta \hat{v} = \mathcal{G}(\hat{u}, \hat{v}) \ \text{ in } \Omega, \quad \partial_n \hat{v} = 0 \ \text{ on } \partial\Omega. \tag{4.13}$$

Furthermore, the theory of elliptic regularity guarantees that $\hat{v}$ belongs to $C^2(\overline{\Omega})$ and is a classical solution to (4.13). We notice that the term $\mathcal{G}(\hat{u}, \hat{v})$ in (4.13) satisfies the logistic property because of (4.11). Thus, $\hat{v}$ must also be constant to satisfy $\mathcal{G}(\hat{u}, \hat{v}) = 0$. By letting $n \to \infty$ in the first integral equation of (4.10), one can see that $\mathcal{F}(\hat{u}, \hat{v}) = 0$ as $\hat{u}$ and $\hat{v}$ are constants. Thus, if $\hat{u} \neq 0$ and $\hat{v} \neq 0$, then $(u_n, v_n) \to (\hat{u}, \hat{v})$ uniformly on $\overline{\Omega}$ as $n \to \infty$, where $(\hat{u}, \hat{v})$ satisfies

$$\hat{u} > 0, \ \hat{v} > 0, \ \mathcal{F}(\hat{u}, \hat{v}) = 0 = \mathcal{G}(\hat{u}, \hat{v}).$$

As a consequence, $(\hat{u}, \hat{v})$ is a desired constant positive solution of (1.4). This contradicts (4.12), thereby completing the proof.

To finish this proof, we consider two cases presented below:

Case 1. $\hat{u} = 0$. It follows from (H3) that $\hat{v} = 0$, because (H2) gives $\mathcal{G}(\hat{u}, \hat{v}) = 0 = -\gamma(\hat{v})$. Thus, one can deduce from (H2), (H3), and (H3c) that

$$\frac{\mathcal{G}(u_n, v_n)}{v_n} = \beta f(u_n, v_n) - \frac{\gamma(v_n)}{v_n} \to -\gamma'(0) = -M_\gamma < 0 \ \text{ as } n \to \infty,$$

so that we encounter a contradiction with the second equation in (4.10) for large n.

Case 2. $\hat{v}=0$. We see that since $\mathcal{F}(\hat{u},\hat{v})=0$, $\hat{u}=0$ or K. Especially, $\hat{u}=0$ occurs only when $g(0)=0$ in (H1) and (H1b). If $\hat{u}=0$, then a contradiction occurs as in the previous case. If $\hat{u}=K$, then one have from (H2), (H3), and the previously derived inequality $\mathcal{H}'(K)>0$ that

$$\beta f(u_n,v_n)-\frac{\gamma(v_n)}{v_n}\to\beta f(K,0)-\gamma'(0)>0 \ \text{ as } n\to\infty.$$

This, in turn, implies a contradiction with the second equation in (4.10) for large values of n, thereby completing the proof. □

Proof of Theorem 3.2. We begin with (4.8) in the proof of Theorem 3.1. From (4.8) with $\epsilon=1$, one see that there is a τ-independent constant $M_3>0$ so that

$$\int_\Omega|\nabla\mathcal{U}|^2dx\le\frac{M_3}{\tau}\int_\Omega|\nabla\mathcal{V}|^2dx \tag{4.14}$$

for τ sufficiently large. Similarly, using (4.1) and (H2) in (4.6), we obtain that

$$\int_\Omega|\nabla\mathcal{V}|^2dx\le M_4\int_\Omega|\nabla\mathcal{U}|\,|\nabla\mathcal{V}|\,dx+\int_\Omega B(u,v)\mathcal{V}^2dx+M_5\int_\Omega|\mathcal{U}||\mathcal{V}|dx,$$

where $M_4>0$ and $M_5>0$ are τ-independent constants. Furthermore, the Poincaré inequality and Young's inequality imply

$$\left(1-\frac{M_4}{2}\epsilon-\frac{M_5}{2\mu_1}\epsilon\right)\int_\Omega|\nabla\mathcal{V}|^2dx\le\left(\frac{M_4}{2\epsilon}+\frac{M_5}{2\mu_1\epsilon}\right)\int_\Omega|\nabla\mathcal{U}|^2dx+\int_\Omega B(u,v)\mathcal{V}^2dx, \tag{4.15}$$

where $\epsilon>0$ is a small constant such that $1>(\frac{M_4}{2}+\frac{M_5}{2\mu_1})\epsilon$. In particular, we note that $B(u,v)\le0$ when τ is sufficiently large, because assumptions (H2), (H3), (H2e), and (H3c), inequality (3.2) and Lemma 4.3 give that

$$\begin{aligned}B(u,v)&\to\frac{\mathcal{G}(\mathbf{u}_*)}{v_*}+v_*\left(\beta f_v(\mathbf{u}_*)-\left(\frac{\gamma(z)}{z}\right)'\Bigg|_{z=v_*}\right)\\&\le v_*\left(\beta f_v(u_*,0)-\gamma_*\right)<0\end{aligned}$$

as $\tau\to\infty$. Thus, we see from (4.15) that there is a τ-independent constant $M_6>0$ such that

$$\int_\Omega|\nabla\mathcal{V}|^2dx\le M_6\int_\Omega|\nabla\mathcal{U}|^2dx \ \text{ for large } \tau.$$

Hence, this, together with (4.14), establish that $\nabla\mathcal{U}\equiv0$ and $\nabla\mathcal{V}\equiv0$, indicating the desired result for τ sufficiently large. □

Lemma 4.4. *Assume that all assumptions in Lemma 4.1 and $\mathcal{H}'(K)\neq0$ are satisfied, and let $\tau_*>0$ be a fixed constant. Then, all positive solutions (u,v) to (1.4) satisfy that*

$$\min_{\overline{\Omega}}u(x),\ \min_{\overline{\Omega}}v(x)\ge C_*$$

for $\tau\ge\tau_*$, where $C_*=C_*(\Theta,\tau_*)>0$ is a constant.

Proof. Suppose, contrarily, that our conclusion is not valid. Then, there is a sequence $\{\tau_n\}$ satisfying $\tau_n \geq \tau_*$, and so correspondingly, the positive solutions (u_n, v_n) to (1.4) with $\tau = \tau_n$ and $q(v_n) = v_n$ satisfy that

$$\min_{\overline{\Omega}} u_n \to 0 \quad \text{or} \quad \min_{\overline{\Omega}} v_n \to 0 \quad \text{as } n \to \infty. \tag{4.16}$$

As $\tau_n \geq \tau_*$, we can assume, after extracting a subsequence, that $\tau_n \to \hat{\tau} \in [\tau_*, \infty]$. By Lemma 4.2, we may also assume that there is a subsequence $\{(u_n, v_n)\}$ (continuing to denote as itself) and two nonnegative functions $\hat{u}, \hat{v} \in C^1(\overline{\Omega})$ so that as $n \to \infty$, $(u_n, v_n) \to (\hat{u}, \hat{v})$ in $[C^1(\overline{\Omega})]^2$. We then see from (4.16) that

$$\min_{\overline{\Omega}} \hat{u} = 0 \quad \text{or} \quad \min_{\overline{\Omega}} \hat{v} = 0,$$

and from Lemma 4.1 that

$$\max_{\overline{\Omega}} \hat{u} \leq K \quad \text{and} \quad \max_{\overline{\Omega}} \hat{v} \leq \widetilde{V}. \tag{4.17}$$

Moreover, we notice that (u_n, v_n) solves (1.4) with $\tau = \tau_n$ and $q(v_n) = v_n$, and therefore, as in the proof of Lemma 4.3, it also satisfies (4.10).

Next, for every possible case, we induce a contradiction.

Case 1. $\min_{\overline{\Omega}} \hat{u} = 0$. We first consider the subcase where $\hat{\tau} = \infty$. Then, we see that $\hat{u} \geq 0$ is a constant, and thus $\hat{u} = 0$ as $\min_{\overline{\Omega}} \hat{u} = 0$. From assumptions (H2) and (H3), and the second equation of (4.10), one have in turn that $\hat{v} \equiv 0$, since

$$0 = \int_\Omega \beta \hat{v} f(\hat{u}, \hat{v}) dx = \int_\Omega \gamma(\hat{v}) dx \geq \int_\Omega M_\gamma \hat{v} dx.$$

However, assumptions (H2), (H3), and (H3c) give that

$$\beta f(u_n, v_n) - \frac{\gamma(v_n)}{v_n} \to \gamma'(0) = M_\gamma > 0 \quad \text{as } n \to \infty.$$

This gives a contradiction to the second equation of (4.10) with sufficiently large n. We next consider the subcase where $\hat{\tau} < \infty$. Then, $(\hat{u}, \hat{v})$ satisfies the first equation in (1.4) with $\tau = \tilde{\tau}$, which can be written as

$$-\hat{\tau} \Delta \hat{u} = \mathcal{F}(\hat{u}, \hat{v}) = g(0) + g'(\phi_1)\hat{u} - \hat{v}\hat{u} f_u(\phi_1, \hat{v}) \quad \text{in } \Omega, \quad \partial_n \hat{u} = 0 \quad \text{on } \partial\Omega,$$

due to (H1) and (H2). Here, ϕ_1 lying between 0 and $\hat{u}$ arises from the mean-value theorem. Thus, because of (4.17), (H1), and (H2), we see that there is a constant $M_1 > 0$ so that $-\hat{\tau}\Delta \hat{u} + M_1 \hat{u} \geq 0$ in Ω. Using $\min_{\overline{\Omega}} \hat{u} = 0$ and employing the strong maximum principle along with the Hopf boundary lemma, one can further induce that $\hat{u} \equiv 0$ in Ω. Then, as before, we attain $\hat{v} \equiv 0$, which once again leads to a contradiction.

Case 2. $\min_{\overline{\Omega}} \hat{v} = 0$. We first consider the subcase where $\hat{\tau} = \infty$. We then notice that $\hat{u}$ is a constant and $\hat{v}$ satisfies

$$-\Delta \hat{v} = \mathcal{G}(\hat{u}, \hat{v}) \quad \text{in } \Omega, \quad \partial_n \hat{v} = 0 \quad \text{on } \partial\Omega.$$

Thus, using (4.17), (H2), and (H3c), we see that there is a constant $M_2 > 0$ satisfying $-\Delta \hat{v} + M_2 \hat{v} \geq 0$ in Ω, so that, as before, we have that $\hat{v} \equiv 0$ in Ω. In turn, from the first equation in (4.10), one obtain that $0 = \int_\Omega g(\hat{u}) dx$, which, together with (H1) and the first estimate in (4.17), gives $\hat{u} = 0$ or $\hat{u} = K$. Here,

$\hat{u} = 0$ is observed only when $g(0) = 0$. If $\hat{u} = 0$, then, as in Case 1, we arrive at a contradiction. If $\hat{u} = K$, then we can derive from (H3) and the given assumption $\mathcal{H}'(K) \neq 0$ that

$$\beta f(u_n, v_n) - \frac{\gamma(v_n)}{v_n} \to \beta f(K, 0) - \gamma'(0) \neq 0 \quad \text{as } n \to \infty,$$

which contradicts with the second equation in (4.10) for large n. We next consider the subcase where $\hat{\tau} < \infty$. We see that $\hat{v}$ satisfies

$$-\Delta \hat{v} - \chi \nabla \hat{u} \cdot \nabla \hat{v} = -\frac{\chi}{\hat{\tau}} \hat{v} \mathcal{F}(\hat{u}, \hat{v}) + \mathcal{G}(\hat{u}, \hat{v}) \quad \text{in } \Omega, \quad \partial_n \hat{v} = 0 \quad \text{on } \partial\Omega.$$

Then, we see from (4.17) and assumptions (H1), (H2), and (H3c) that there exists a constant $M_3 > 0$ satisfying $-\Delta \hat{v} - \chi \nabla \hat{u} \cdot \nabla \hat{v} + M_3 \hat{v} \geq 0$ in Ω. Analogously, as before, one have that $\hat{v} \equiv 0$ in Ω, which in turn gives that $\hat{u} \equiv 0$ or $\hat{u} \equiv K$. Thus, similarly to the previous subcases, we once again reach at a contradiction. □

By virtue of Theorem 2.1, we provide straightforward information on the positive constant solutions to (1.4). We recall the definitions of $L_{ij}(\mathbf{u}_*)$ (where $i, j = 1, 2$) as previously stated before Lemma 3.3. Moreover, we denote $L_{ij} = L_{ij}(\mathbf{u}_*)$ (where $i, j = 1, 2$) for simplicity.

Lemma 4.5. *Assume that all assumptions in Lemma 3.3 hold.*

(i) The number of positive constant solutions to (1.4) is odd: these solutions are denoted by $\mathbf{u}_k^* = (u_k^*, v_k^*)$ *for* $k = 1, 2, \dots, 2n+1$ *in Lemma 3.3.*

(ii) If k *is odd, then* $\mathcal{H}'(u_k^*) < 0$*, whereas if* k *is even, then* $\mathcal{H}'(u_k^*) > 0$.

Proof. Since $\mathcal{H}'(K) > 0$ and (H5) are given, $\mathcal{H}(u) = 0$ has an odd number of positive roots u_* in $(0, K)$ (e.g., see Figure 1(a)). Moreover, due to (H5), we have $\mathcal{H}'(u_k^*) \neq 0$ for all k. Thus, our second claim also holds. □

Lemma 4.6. *Assume that all assumptions in Lemma 3.3 hold.*

(i) If k *is odd, there exists a constant* $\widehat{\tau}_k = \widehat{\tau}_k(\Theta) > 0$ *such that if* $\tau \geq \widehat{\tau}_k$*, then* $Q(\mu, \mathbf{u}_k^*) > 0$ *for all* $\mu \geq 0$.

(ii) If k *is odd, there exists a constant* $\overline{\tau}_k = \overline{\tau}_k(\Theta) > 0$ *such that the quadratic equation* $Q(\mu, \mathbf{u}_k^*) = 0$ *attains two distinct positive roots* $\mu^-(\mathbf{u}_k^*)$ *and* $\mu^+(\mathbf{u}_k^*)$*, provided that (3.5) is satisfied.*

(iii) If k *is even, then* $Q(\mu, \mathbf{u}_k^*) = 0$ *possesses only one positive root* $\mu^+(\mathbf{u}_k^*)$ *for any* $\tau > 0$.

Proof. Consider the case where $\mathbf{u}_* = \mathbf{u}_k^*$. From straightforward calculations, we derive that

$$\mathcal{H}'(u_k^*) = g'(u_k^*) - v_k^* f_u(\mathbf{u}_k^*) - \xi'(u_k^*) \left(v_k^* f_v(\mathbf{u}_k^*) + f(\mathbf{u}_k^*) \right) \quad \text{and} \quad \xi'(u_k^*) = \frac{\beta g'(u_k^*)}{\gamma'(v_k^*)}.$$

Using this, we can obtain from (H3) and Lemma 4.5(ii) that

$$-\gamma'(v_k^*) \mathcal{H}'(u_k^*) = L_{11} L_{22} - L_{12} L_{21} \begin{cases} > 0 & \text{if } k \text{ is odd,} \\ < 0 & \text{if } k \text{ is even.} \end{cases} \tag{4.18}$$

Moreover, we easily see that (H3c) and the inequality $\beta f_v(\mathbf{u}_k^*) - \gamma_* < 0$ give

$$L_{22} = \beta f_v(\mathbf{u}_k^*)v_k^* + \frac{\gamma(v_k^*)}{v_k^*} - \gamma'(v_k^*) = \left(\beta f_v(\mathbf{u}_k^*) - \left(\frac{\gamma(v)}{v}\right)'\Bigg|_{v=v_k^*}\right) v_k^*$$
$$\leq (\beta f_v(\mathbf{u}_k^*) - \gamma_*)v_k^* < 0.$$

(i) Let k be odd. Then it follows from (4.18) that $L_{11}L_{22} - L_{12}L_{21} > 0$. Moreover, if we choose a $\widehat{\tau}_k > 0$ such that

$$\frac{L_{11} - \chi v_k^* L_{12}}{-L_{22}} \leq \widehat{\tau}_k,$$

then we see that when $\tau \geq \widehat{\tau}_k$, $Q(\mu, \mathbf{u}_k^*) > 0$ for all $\mu \geq 0$.

(ii) Let k be odd. Because of $\mathcal{H}'(u_k^*) < 0$, it is straightforward to know that if

$$\frac{L_{11} - \chi v_k^* L_{12}}{-L_{22}} > \tau \quad \text{and} \quad D(\tau) > 0, \tag{4.19}$$

where

$$D(\tau) = (L_{11} - \chi v_k^* L_{12} + L_{22}\tau)^2 + 4\gamma'(v_k^*)\mathcal{H}'(u_k^*)\tau,$$

then the desired result follows. Furthermore, assuming that $\chi > \overline{\chi}_k$ and given that $L_{22} < 0$, it is evident that $\frac{L_{11}-\chi v_k^* L_{12}}{-L_{22}} > 0$. Here, $\overline{\chi}_k$ is defined in Lemma 3.3. Thus, $D(\tau) = 0$ attains two positive roots since $D(0) > 0$ and $D(\frac{L_{11}-\chi v_k^* L_{12}}{-L_{22}}) < 0$. In particular, let $\overline{\tau}_k$ denote the smaller of the two roots. Then, obviously, (4.19) holds for any $\tau < \overline{\tau}_k$.

(iii) Due to (4.18), the fact of $\mathcal{H}'(u_k^*) > 0$ directly leads to the desired assertion. □

Proof of Lemma 3.3. According to Lemma 4.5(i), the system (1.4) has $2n+1$ distinct positive constant solutions $\mathbf{u}_k^*$ with $k = 1, 2, \ldots 2n+1$. Moreover, from (ii) and (iii) in Lemma 4.6, it follows that $\mu^-(\mathbf{u}_k^*)$ (only when $k =$ odd) and $\mu^+(\mathbf{u}_k^*)$ exist. This completes the proof. □

We finally study the global existence of nonconstant positive solutions to (1.4). To achieve this, we employ the degree argument. We note that when (3.2) and all assumptions in Theorem 3.4 hold, it follows that

$$\beta M_f > M_\gamma \quad \text{and} \quad \mathcal{H}'(K) > 0,$$

as in the proof of Lemma 4.3. This implies that (1.4) admits a constant and positive solution $\mathbf{u}_*$ (see Theorem 2.1), and the L^∞-estimate in Lemma 4.1 holds. Moreover, we note that (H2e) and (3.2) imply the inequality $\beta f_v(\mathbf{u}_k^*) - \gamma_* < 0$ in Lemma 3.3, so that we can apply Lemma 3.3 below.

We recall the definitions of the eigenvalue μ_i and the multiplicity m_i, as stated before Theorem 3.1. For simplicity, we denote $\mathbf{u} = (u(x), v(x))$, $C_n^1(\overline{\Omega}) = \{\phi \in C^1(\overline{\Omega}) : \partial_n \phi = 0 \text{ on } \partial\Omega\}$, $E = C_n^1(\overline{\Omega}) \oplus C_n^1(\overline{\Omega})$, and

$$\Lambda = \left\{ \mathbf{u} \in E : \frac{C_*}{2} < u, \; v < 2\max\left\{K, \widetilde{V}\right\} \right\},$$

where $\widetilde{V}$ and C_* are respectively defined in Lemmas 4.1 and 4.4. We define

$$\mathcal{A}(\tau, \mathbf{u}) = (I - \Delta)^{-1} \begin{pmatrix} \dfrac{\mathcal{F}(u,v)}{\tau} + u \\ \chi \nabla u \cdot \nabla v - \chi v \dfrac{\mathcal{F}(u,v)}{\tau} + \mathcal{G}(u,v) + v \end{pmatrix}$$

for $\tau > 0$ and $\mathbf{u} \in \Lambda$. Here, I is the identity map on $C^1(\overline{\Omega})$ and the operator $(I - \Delta)^{-1}$ represents the inverse of $I - \Delta$ subject to homogeneous Neumann boundary condition. One can observe, using the standard method, that for each $\tau > 0$, $\mathcal{A}(\tau, \mathbf{u})$ is completely continuous on Λ. Furthermore, $\mathbf{u}$ is a positive fixed point of problem $\mathbf{u} = \mathcal{A}(\tau, \mathbf{u})$ if and only if it is a positive solution to (1.4). According to Lemmas 4.1 and 4.4, the positive fixed point $\mathbf{u}$ is always included in Λ, and due to the choice of Λ, $\mathbf{u} \neq \mathcal{A}(\tau, \mathbf{u})$ for all $\mathbf{u} \in \partial\Lambda$ and $\tau > 0$, and thus, the Leray-Schauder degree $\deg(I - \mathcal{A}(\tau, \cdot), \Lambda, 0)$ is well-defined. When $\mathbf{u} = \mathcal{A}(\tau, \mathbf{u})$ (i.e., (1.4)) has only positive constant solutions $\mathbf{u}_*$ in Λ, to show that by calculating the degree value, (1.4) has a positive and nonconstant solution, we shall calculate the fixed-point index of $\mathcal{A}(\tau, \mathbf{u})$ at $\mathbf{u}_*$, denoted by $\mathrm{index}(\mathcal{A}(\tau, \cdot), \mathbf{u}_*)$.

To find the index value, we start with the eigenvalue problem

$$\lambda(\phi, \psi)^T + (I - \mathcal{A}_{\mathbf{u}}(\tau, \mathbf{u}_*))(\phi, \psi)^T = 0, \tag{4.20}$$

where $(\phi, \psi) \not\equiv (0, 0)$, and we recall Theorem 2.8.1 in [8], known as the Leray-Schauder Theorem: if problem (4.20) does not have 0 as an eigenvalue, then

$$\mathrm{index}(\mathcal{A}(\tau, \cdot), \mathbf{u}_*) = (-1)^\sigma.$$

Here, $\sigma = \sum_{\lambda > 0} n_\lambda$, where n_λ represents the algebraic multiplicity of each eigenvalue $\lambda > 0$ of (4.20). Upon doing some computations, we can express (4.20) as follows:

$$\begin{cases} -(1+\lambda)\Delta\phi + \left(\lambda - \dfrac{L_{11}}{\tau}\right)\phi - \dfrac{L_{12}}{\tau}\psi = 0 & \text{in } \Omega, \\ -(1+\lambda)\Delta\psi + \left(\chi v_* \dfrac{L_{11}}{\tau} - L_{21}\right)\phi + \left(\lambda + \chi v_* \dfrac{L_{12}}{\tau} - L_{22}\right)\psi = 0 & \text{in } \Omega, \\ \partial_n \phi = \partial_n \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.21}$$

Using the Fourier expansions of ϕ and ψ in (4.21) and setting

$$\begin{aligned} P_i(\lambda, \mathbf{u}_*) &:= \det \begin{pmatrix} (1+\lambda)\mu_i + \lambda - \dfrac{L_{11}}{\tau} & -\dfrac{L_{12}}{\tau} \\ \chi v_* \dfrac{L_{11}}{\tau} - L_{21} & (1+\lambda)\mu_i + \lambda + \chi v_* \dfrac{L_{12}}{\tau} - L_{22} \end{pmatrix} \\ &= (\mu_i + 1)^2 \lambda^2 + \left(2\mu_i - \frac{L_{11}}{\tau} + \chi v_* \frac{L_{12}}{\tau} - L_{22}\right)(\mu_i + 1)\lambda + Q(\mu_i, \mathbf{u}_*), \end{aligned}$$

where $Q(\mu, \cdot)$ was defined in Lemma 3.3, we see that (4.21) has a nontrivial solution if and only if $P_i(\lambda, \mathbf{u}_*) = 0$ for some $i \geq 0$ and $\lambda \geq 0$. Thus, according to the Leray-Schauder Theorem,

$$\mathrm{index}(\mathcal{A}(\tau, \cdot), \mathbf{u}_*) = (-1)^\sigma, \quad \sigma = \sum_{i \geq 0} \sum_{\lambda_i > 0} m_{\lambda_i} m_i \tag{4.22}$$

if $P_i(0,\mathbf{u}_*) \neq 0$ for all $i \geq 0$. Here, m_{λ_i} is the multiplicity of λ_i and m_i is the multiplicity of μ_i when λ_i is a positive root of $P_i(\lambda,\mathbf{u}_*) = 0$. The sum of $m_{\lambda_i} m_i$ for all i satisfying $Q(\mu_i,\mathbf{u}_*) > 0$ is even. Therefore, to determine the index value in (4.22), it is sufficient to verify that for all $i \geq 0$, $Q(\mu_i,\mathbf{u}_*) \neq 0$, and find all i for which $Q(\mu_i,\mathbf{u}_*) < 0$.

Our next task is to compute $\mathrm{index}(\mathcal{A}(\tau,\cdot),\mathbf{u}_k^*)$ with respect to τ.

Lemma 4.7. *Assume that (3.2) and all assumptions in Theorem 3.4 hold.*

(i) If k is odd and $\tau \geq \widehat{\tau}_k$, where $\widehat{\tau}_k$ was defined in Lemma 4.6(i), then

$$\mathrm{index}(\mathcal{A}(\tau,\cdot),\mathbf{u}_k^*) = 1.$$

(ii) If k is odd, and χ and τ satisfy (3.5), then

$$\mathrm{index}(\mathcal{A}(\tau,\cdot),\mathbf{u}_k^*) = (-1)^{\sigma_k},$$

provided that

$$\mu^-(\mathbf{u}_k^*) \in (\mu_{a_k},\mu_{a_k+1}) \quad \text{and} \quad \mu^+(\mathbf{u}_k^*) \in (\mu_{b_k},\mu_{b_k+1}) \quad \text{for some integers } b_k > a_k \geq 0,$$

where σ_k was defined in Theorem 3.4.

(iii) If k is even, then for any $\tau > 0$,

$$\mathrm{index}(\mathcal{A}(\tau,\cdot),\mathbf{u}_k^*) = (-1)^{\sigma_k},$$

provided that

$$\mu^+(\mathbf{u}_k^*) \in (\mu_{b_k},\mu_{b_k+1}) \quad \text{for some integer } b_k \geq 0.$$

Proof. (i) According to Lemma 4.5(ii), $\mathcal{H}'(u_k^*) < 0$ since k is odd. Moreover, Lemma 4.6(i) shows that $Q(\mu_i,\mathbf{u}_k^*) > 0$ for all $i \geq 0$. Thus, σ in (4.22) is even, which yields the desired index value.

(ii) By Lemmas 4.5 and 4.6(ii), we see that $Q(\mu_i,\mathbf{u}_k^*) < 0$ is true only when $a_k + 1 \leq i \leq b_k$. This implies that $\sigma = \sum_{i=a_k+1}^{b_k} m_i + \text{even} = \sigma_k + \text{even}$, which proves our assertion.

(iii) Similarly to the proof of part (ii), it can be proven. □

We have now reached the point where we can prove the final theorem on pattern formation.

Proof of Theorem 3.4. As mentioned before, we see that $\mathbf{u} \neq \mathcal{A}(\tau,\mathbf{u})$ for all $\mathbf{u} \in \partial\Lambda$ and $\tau > 0$. Thus, by using the homotopy invariance property of the degree, one obtain that

$$\deg(I - \mathcal{A}(\tau,\cdot),\Lambda,0) = \text{constant} \quad \text{for any } \tau > 0. \tag{4.23}$$

We notice from (3.4) and Lemma 4.6(iii) that there exists a constant $t_k = t_k(\Theta) > 0$ so that if $\tau \geq t_k$,

$$0 < \mu^+(\mathbf{u}_k^*) < \mu_1 \quad \text{for } k = 2,4,\ldots,2n. \tag{4.24}$$

We first take $a_* = \max\{\widetilde{\tau}_2, \widehat{\tau}_{k_1}, t_{k_2} : k_1 = 1,3,\ldots,2n+1,\ k_2 = 2,4,\ldots,2n\}$, where $\widetilde{\tau}_2$ and $\widehat{\tau}_k$ were respectively defined in Theorem 3.2 and Lemma 4.6(i). Then, from Lemma 4.7(i), we see

$$\mathrm{index}(\mathcal{A}(a_*,\cdot),\mathbf{u}_k^*) = 1 \quad \text{for } k = 1,3,\ldots,2n+1. \tag{4.25}$$

Moreover, using (4.24) in Lemma 4.7(iii), we find that

$$\mathrm{index}(\mathcal{A}(a_*, \cdot), \mathbf{u}_k^*) = (-1)^{m_0} = -1 \quad \text{for } k = 2, 4, \ldots, 2n. \tag{4.26}$$

We apply the additivity property of the degree, if necessary (when $n \geq 1$), to obtain that

$$\deg(I - \mathcal{A}(a_*, \cdot), \Lambda, 0) = \sum_{k=1}^{2n+1} \mathrm{index}(\mathcal{A}(a_*, \cdot), \mathbf{u}_k^*), \tag{4.27}$$

because of $a_* \geq \widetilde{\tau}_2$. As a result, by inserting the index values from (4.25) and (4.26) into (4.27), we can determine $\deg(I - \mathcal{A}(a_*, \cdot), \Lambda, 0) = 1$. Therefore, using (4.23), we can have that

$$\deg(I - \mathcal{A}(\tau, \cdot), \Lambda, 0) = 1 \quad \text{for any } \tau > 0. \tag{4.28}$$

Suppose, contrariwise, that (1.4) admits no positive and nonconstant solution when χ and τ satisfy (3.6). Then, similar to the above, by virtue of (ii), (iii) in Lemma 4.7 and the additivity property of the degree, one can derive that

$$\deg(I - \mathcal{A}(\tau, \cdot), \Lambda, 0) = \sum_{k=1}^{2n+1} (-1)^{\sigma_k}.$$

However, from the assumption given in (3.7), it can be concluded that $\deg(I - \mathcal{A}(\tau, \cdot), \Lambda, 0) \neq 1$, which contradicts (4.28), thereby completing the proof. □

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Appendix: Proof of Theorem 2.1

Proof. The existence of positive constants u and v satisfying

$$0 = \mathcal{F}(u,v) = g(u) - vf(u,v) \quad \text{and} \quad 0 = \mathcal{G}(u,v) = \beta vf(u,v) - \gamma(v)$$

is ensured.

Obviously, $\beta g(u) = \gamma(v)$, and (H1) gives $u \in (0,K)$. Moreover, since (H3) implies the existence of the inverse of γ when $v \geq 0$, we have

$$v = \gamma^{-1}(\beta g(u)) = \xi(u).$$

We can easily observe that

$$\xi(0) = \gamma^{-1}(\beta g(0)) \geq 0, \;\; \xi(K) = \gamma^{-1}(\beta g(K)) = 0, \tag{4.29}$$

and further,

$$\xi'(0) = \frac{\beta g'(0)}{\gamma'(\xi(0))} \quad \text{and} \quad \xi'(K) = \frac{\beta g'(K)}{\gamma'(0)}. \tag{4.30}$$

Plugging $v = \xi(u)$ in $\mathcal{F}(u,v)$, we obtain $\mathcal{H}(u) = \mathcal{F}(u,\xi(u))$. Now, we confirm that (2.1) guarantees the existence of $u \in (0,K)$ such that $\mathcal{H}(u) = 0$. Consequently, such an u is denoted as u_*, and in turn, $v_* = \xi(u_*)$ is determined. To the end, we divide and consider two cases: $g(0) = 0$ and $g(0) > 0$.

Case 1. $g(0) = 0$. In this case, $\xi(0) = 0$ follows, so that, from (4.29), (4.30), and (H2), we can derive that $\mathcal{H}(0) = 0 = \mathcal{H}(K)$, and

$$\begin{aligned}
\mathcal{H}'(0) &= g'(0) - \xi(0)f_u(0,\xi(0)) - \xi'(0)\xi(0)f_v(0,\xi(0)) - \xi'(0)f(0,\xi(0)) \\
&= g'(0), \\
\mathcal{H}'(K) &= g'(K) - \xi(K)f_u(K,\xi(K)) - \xi'(K)\xi(K)f_v(K,\xi(K)) - \xi'(K)f(K,\xi(K)) \\
&= -\frac{g'(K)}{\gamma'(0)}\left(\beta f(K,0) - \gamma'(0)\right).
\end{aligned}$$

Thus, from (i) in (2.1), (H1b) and (H3), we see that $\mathcal{H}'(0) > 0$ and $\mathcal{H}'(K) > 0$ (i.e, $\beta f(K,0) - \gamma'(0) > 0$), which ensures that $\mathcal{H}$ is positive when $u > 0$ is close to 0, but negative when $u < K$ is close to K, so that there exists an $u_* \in (0,K)$ satisfying $\mathcal{H}(u_*) = 0$. In turn, $\xi(u_*) = v_* \leq \beta \frac{M_g}{M_\gamma}$ since $M_\gamma v_* \leq \gamma(v_*) = \beta g(u_*) \leq \beta M_g$. Even if the second option in (2.1) is met, we can still achieve the desired result.

Case 2. $g(0) > 0$. In this case, $0 < g(0) = \mathcal{H}(0)$ follows from (H2), so that $\mathcal{H}(u) > 0$ for small u. Thus, the desired assertion holds true because $\mathcal{H}'(K) > 0$ or the existence of M_* in (2.1) implies that there is an $u \in (0,K)$ such that $\mathcal{H}(u) < 0$. □