Finite-Time Performance Enhanced Bounded Control for Linear Systems With Input Saturation

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Abstract

Real-world dynamical control systems are often subject to input saturation due to the physical constraints of actuators. This causes the response performance of the system to be reduced due to the effect of input saturation. To this end, we propose a finite-time performance enhanced bounded control (FTPE-BC) method based on sliding mode control for finite time tracking of linear systems with input saturation. In this method, an approach rate based on inverse tangent function is proposed to make the system converge in finite time. A bounded controller based on sliding mode control is proposed, in which two control parameters are introduced to reduce steady-state error, settling time and overshoot. Meanwhile, the finite-time stability of the system is proved in a domain of attraction. The parameters selection principle of the controller is given to improve the transient and steady-state performance of the closed-loop system. The simulation results verify the superiority of the proposed method in the transient and steady-state performance enhancement.

KEYWORDS
input saturation, finite-time stability, performance enhancement

1 | INTRODUCTION

Input saturation is an inevitable problem in controller design due to the physical characteristics of actuators. At the same time, in order to obtain better performance, the problem of finite-time stability has been widely concerned.

Input saturation is a common phenomenon in mechanical, robotics, and other fields [1,2], which usually causes system performance deterioration or even instability. The methods to deal with saturation control are mainly divided into two categories: One-step methods and anti-windup methods. The one-step methods are to meet the performance specifications using saturation constraints directly [3,4]. The anti-windup methods introduce some compensators or governors to meet performance index in closed-loop systems [5,6]. Anti-windup methods include model recovery anti-windup designs (MRAW) [7] and direct linear anti-windup designs (DLAW) [8], reference menagement method [9], etc. The MRAW and DLAW methods introduce a compensator to adjust the controller to optimize the $\mathcal{L}_2$ performance [10,11] or enlarg the estimated domain of attraction of the system with input saturation [12,13,14]. However, the MRAW and DLAW methods are suitable for linear time-invariant systems with input saturation, and it is difficult to apply to nonlinear systems. The reference management method introduces an auxiliary system to adjust the reference input to maintain the transient performance of the system [15]. This method can be applied to nonlinear systems, but it can not improve the transient performance and domain of attraction of the system with input saturation. Summing up the above, there is no discussion of how to improve transient performance while guaranteeing the domain of attraction.

The settling time is an important performance specification of the control system, which characterizes the convergent speed of systems. Finite-time stability means that the settling time of the system is a finite value [16]. In the past decades, the problem of convergence in a shorter time became a hot topic in the control designs in order to improve transient performance in terms of a finite convergence time [17]. The definition and theorem of finite-time stability are given [18]. A finite-time stability theorem for faster convergence is given [19]. The definition and theorem of practical finite-time stability are given. The problem of attitude control for a spacecraft nonlinear system with inertia uncertainty and external disturbance is investigated. The sliding mode control method

Abbreviations: FTPE-BC, finite-time performance enhanced bounded control; MRAW, model recovery anti-windup; DLAW, direct linear anti-windup; LTI, linear time-invariant.

is proposed to make the closed-loop system finite-time stable[20]. A practical finite-time stability theorem for faster convergence is given. The finite time control problem of a class of uncertain nonlinear systems with unknown actuator fault is investigated. The controller is designed by fusing the techniques of command filter and backstepping control[21]. A fast finite-time stability theorem is given. The global finite-time adaptive stabilization problem for a class of high order uncertain nonlinear systems is investigated. A state feedback stabilizer with an adaptive mechanism is constructed by applying continuous domination to adaptive fashion of the systems[22]. However, these finite-time stabilization methods do not consider input saturation, and the controllers all contain sign functions, which will lead to chattering problems in the steady-state response of the system.

Finite-time stabilization under input saturation is a hot issue. The solution includes: bounded finite time control and reference management based sliding mode control method. Recently, a bounded linear time-varying feedback control is proposed for a linear time-invariant systems with actuator saturation, but it only achieves the stability goal in finite time[23]. The global finite-time stabilization problem for systems with bounded controls is investigated. A bounded global finite-time controller is proposed to stabilize the single integrator system with input saturation and to specify the saturation level of the control input[24]. However, these bounded control methods are only suitable for finite-time stable control of linear systems with input saturation. They can only satisfy determined finite time performance. At the same time, a finite-time observer-based adaptive sliding mode output feedback controller is developed for dynamic positioning ship with input saturation and unknown disturbances by constructing an auxiliary system[25]. To stabilize the UAV system with input saturation, an auxiliary system is designed to compensate for the saturation effect. A fast terminal sliding mode controller is developed to achieve trajectory tracking control of the UAV[26]. However, these reference management based methods can only maintain finite time performance, not improve it. A super-twisting algorithm is proposed for a first-order linear system with input saturation. This approach achieves finite-time convergence considering input saturation, but only for first-order linear systems[27]. A generic higher-order sliding mode with bounded integral control is proposed for a nonlinear affine systems with actuator saturation. This approach achieves finite-time stability for the nonlinear system, but only for single-input systems[28]. At present, the problem of finite time control to achieve better performance and faster convergence for systems with input saturation is still worth exploring.

In this paper, we propose a finite-time performance enhanced bounded control (FTPE-BC) based on sliding mode variable structure method for finite time tracking of linear systems with input saturation, which does not have these drawbacks. The main innovations are as follows: Firstly, in order to improve the steady-state performance of the system applied by the sliding mode variable structure method, an approach rate based on the inverse tangent function is proposed, so that the tracking error of the system reaches a small neighborhood of the sliding surface within a finite time. It converges exponentially to the sliding surface in this neighborhood to avoid chattering near the sliding surface. Secondly, in order to reduce the influence of input saturation on the transient performance of the system, two control parameters are introduced into the bounded controller. The transient performance of the closed-loop system can be improved by adjusting these parameter to reduce steady-state error, settling time and overshoot. Thirdly, by analyzing the transient and steady-state response of the closed-loop system, the selection principle of control parameters is given to improve the transient and steady-state performance of the closed-loop system. The advantage of this method is that it is simple and easy, and the transient and steady-state performance of the system can be highly controllable.

This paper is organized as follows. Section II gives definitions of symbols and fundamental theorems. Section III describes the design problems. A FTPE-BC control framework for a class of linear systems with input saturation is presented in Section IV. Section V gives analysis of the closed-loop system. Section VI verifies the superiority of the controller through a case study. Section VII concludes the paper.

2 | PRELIMINARIES

2.1 | Notation

A few notational conventions and definitions are first discussed. Throughout the paper, for a vector $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ common use of the abbreviation $\lfloor x \rceil^q = (|x_1|^q \mathrm{sign}(x_1), |x_2|^q \mathrm{sign}(x_2), \cdots, |x_n|^q \mathrm{sign}(x_n))^{\mathrm{T}}$ and the function $\arctan(x) = (\arctan(x_1), \arctan(x_2), \cdots, \arctan(x_n))$. x_i is defined as the i-th component of vector x. The norm is defined by $\|x\| = \sqrt{x^{\mathrm{T}} x}$. A^{-1} and A^{+} are the inverse and the pseudo-inverse of matrix $A \in \mathbb{R}^{m \times n}$, respectively.

The vector-valued decentralized saturation function $\sigma(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as

$$\sigma(u(t)) := (\mathrm{sat}(u_1(t)), \mathrm{sat}(u_2(t)), \cdots, \mathrm{sat}(u_m(t)))^{\mathrm{T}}, \tag{1}$$

where the saturation function $\mathrm{sat}(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined as

$$\mathrm{sat}(u_i(t)) := \begin{cases} u_i(t), & \text{if } |u_i(t)| \leq \bar{u}_i, \\ \mathrm{sign}(u_i(t))\bar{u}_i, & \text{if } |u_i(t)| > \bar{u}_i, \end{cases} \tag{2}$$

with the sign function $\mathrm{sign}(\cdot)$. $\bar{u}_i$ represents the maximum absolute value of the input signal of symmetric saturation $\mathrm{sat}(u_i(t))$. We denote $\bar{u} := (\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_m)^{\mathrm{T}}$.

2.2 | Stability theorem

Some useful definitions and Lemmas on finite time control are introduced as follows.

Definition 1. [18] The equilibrium $x(t) = 0$ of a system $\dot{x}(t) = f(t, x(t))$ is said to be finite-time stable if there are an open neighborhood Ω of the origin and a function $\mathrm{T}(x_0) : \Omega \backslash 0 \to (0, \infty)$, such that the following statements hold.
1) Finite-time convergence:

$$\lim_{t \to \mathrm{T}(x_0)} x(t, x_0) = 0, \text{ and } x(t, x_0) = 0 \text{ for } \forall\, t > \mathrm{T}(x_0).$$

2) Lyapunov stability: For each open neighborhood Γ_ε of the origin, there is an open subset Γ_δ of Ω including the origin such that for each $x_0 \in \Gamma_\delta \backslash 0$, $x(t, x_0) \in \Gamma_\varepsilon$ for all $t \in [0, \mathrm{T}(x_0)]$, where $\mathrm{T}(x_0)$ is called settling time or convergence time. Further, the equilibrium is globally finite-time stable when $\Omega = \mathbb{R}^n$.

Definition 2. [20] The equilibrium $x(t) = 0$ of a system $\dot{x}(t) = f(t, x(t))$ is practical finite-time stable, if for any initial condition x_0, there exist a constant $\varepsilon > 0$ and a settling time $\mathrm{T}(\varepsilon, x_0) < \infty$ such that

$$\|x(t)\| < \varepsilon, \text{ for } \forall\, t \leq \mathrm{T}(\varepsilon, x_0).$$

Lemma 1. [19] Suppose that there is a Lyapunov function $V(x(t)) : D \to \mathbb{R}^+$, some design constants $\alpha > 0$, $\beta > 0$, and $0 < q < 1$ such that

$$\dot{V}(x(t)) \leq -\alpha V(x(t)) - \beta V^q(x(t)), \ \forall\, x(t) \in D \backslash 0.$$

Then, the origin $x(t) = 0$ of system $\dot{x}(t) = f(t, x(t))$ is finite-time stable and the convergence time is given by $\mathrm{T}(x_0) \leq \frac{1}{\alpha(1-q)} \ln \frac{\alpha V^{1-q}(x_0)+\beta}{\beta}$. Moreover, the origin is globally finite-time stable if $D = \mathbb{R}^n$ and $V(x(t))$ is radially unbounded.

Lemma 2. [21] If there exist a Lyapunov function $V(x(t)) : D \to \mathbb{R}^+$, some design constants $\alpha > 0$, $\beta > 0$, $0 < q < 1$, and $0 < b < \infty$ such that

$$\dot{V}(x(t)) \leq -\alpha V(x(t)) - \beta V^q(x(t)) + b, \ \forall\, x(t) \in D \backslash 0,$$

then the trajectory of system $\dot{x}(t) = f(x(t), u(t))$ is practical finite-time stable and the convergence time is given by $\mathrm{T}(\varepsilon, x_0) = \frac{1}{\alpha(1-q)} \ln \frac{\alpha V^{1-q}(x_0)+\lambda\beta}{\lambda\beta}$, with constants $\varepsilon > 0$ and $0 < \lambda < 1$.

3 | PROBLEM FORMULATION

3.1 | A control system

We consider that a linear time-invariant system (LTI) with input saturation as follows

$$\begin{cases} \dot{x}(t) = Ax(t) + B\sigma(u(t)), \\ y(t) = Cx(t), \end{cases} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state. $u(t) \in \mathbb{R}^m$ is the control input. $y(t) \in \mathbb{R}^p$ is the output. The magnitude of the control input $u(t)$ is constrained by the saturation function $\sigma(\cdot)$ in Eq. (1) since any practical actuators can only implement bounded control signals[3]. $(A, B) \in (\mathbb{R}^{n\times n}, \mathbb{R}^{n\times m})$ is controllable, and $(A, C) \in (\mathbb{R}^{n\times n}, \mathbb{R}^{p\times n})$ is observable. The state $x(t)$, the control input $u(t)$, and the output $y(t)$ are measurable.

Our objective is to design a controller under input saturation constraints so that the output $y(t)$ of the system in Eq. (3) converges in a finite time to a small neighborhood of a given continuously differentiable reference signal $r(t) \in \mathbb{R}^p$, and converge exponentially in the neighborhood to achieve accurate tracking.

By introducing the error vector $e(t) := r(t) - y(t)$, the dynamics of the error $e(t)$ is given as follows

$$\dot{e}(t) = A_e x(t) + B_u \sigma(u(t)) + \dot{r}(t), \tag{4}$$

where matrices $A_e = -CA$ and $B_u = -CB$ are determined by Eq. (3).

Assumption 1. Assume that $p \leq m$ and $rank(B_u) = p$.

3.2 | Finite time control problem

Considering the input saturation in Eq. (1) of the system in Eq. (3), our objective is to design a controller to stabilized the error system in Eq. (4) in finite time. This control method can improve the settling time of transient response, i.e., $t_s \leq \mathrm{T}$, reduce the steady-state error i.e., $\|e_{ss}\| \leq \varepsilon$. The following problem will be addressed.

Problem 1. Consider the error system in Eq. (4) with the input saturation in Eq. (1) satisfying Assumption 1. Design a controller $u(t)$ to make the error $e(t)$ converge to a $\varepsilon > 0$ neighborhood containing the equilibrium point $e(t) = 0$ for a finite time, i.e. for $\forall\ |u(t)| \preceq \bar{u}$, there exist a convergence time T and a domain Ω such that

$$\|e(t)\| < \varepsilon, \text{ for } \forall\ t > \mathrm{T},\ e(0) \in \Omega. \tag{5}$$

Furthermore, there exist positive constants κ and λ such that

$$\|e(t)\| \leq \kappa \|e(t_0)\| e^{-\lambda(t-t_0)}, \text{ for } \forall\ \|e(t_0)\| < \varepsilon. \tag{6}$$

Meanwhile, the overshoot and settling time of the system are reduced simultaneously.

4 | THE CONTROL DESIGN METHOD OF FTPE-BC

To solve the Problem 1 formulated in Section 3.2, we propose a general approach to design a sliding mode control based finite time performance enhanced bounded control (FTPE-BC) strategy that takes input saturation into account so that the error system in Eq. (4) is practical finite-time stable.

4.1 | An overview of FTPE-BC

We propose a FTPE-BC framework for a class of LTI systems with input saturation in Figure 1. In this framework, the plant P consists of an actuator $\mathscr{P}_1$ subject to saturation constraint, and a LTI system $\mathscr{P}_2$ without input saturation. The controller C consists of two parts, which are an unconstrained subcontroller $\mathscr{C}_1$ and a bounded subcontroller $\mathscr{C}_2$. Firstly, we propose a finite time approach rate so that the tracking error of the system reaches near the sliding surface in finite time, at the same time eliminate the chattering near the sliding surface, and reduce the steady-state error. Secondly, we design an unconstrained controller so that the closed-loop system without input saturation is finite-time stable. Finally, a bounded controller is proposed to make the closed-loop system practical finite-time stable under input saturation, and improve the transient and steady-state performance of the closed loop system.

4.2 | A finite time approach rate

In order to guarantee the performance of the proposed approach rate, we present the following facts.

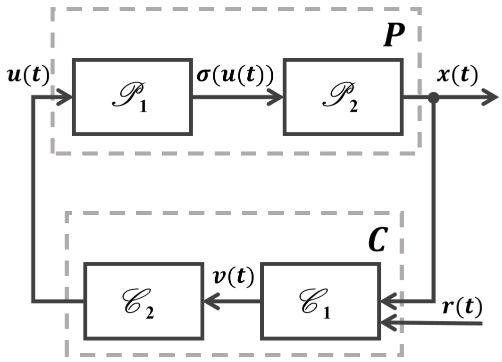


FIGURE 1 The FTPE-BC framework diagram.

Fact 1. According to the monotonicity and concavity of the function, if the inequality $\left(\frac{1}{2}(\frac{q_s-p_s}{q_s+p_s})^{1/2}\right)^{\frac{p_s}{q_s}} - \arctan\left((\frac{q_s-p_s}{q_s+p_s})^{1/2}\right) < 0$ holds, the following inequality is true,

$$|\arctan(a\zeta)| \geq \begin{cases} |\frac{a}{2}\zeta|, & \text{if } 0 \leq |\zeta| < \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}, \\ |(\frac{a}{2}\zeta)^{\frac{p_s}{q_s}}|, & \text{if } \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a} \leq |\zeta| \leq \frac{2}{a}, \\ 1, & \text{if } |\zeta| > \frac{2}{a}, \end{cases} \tag{7}$$

where $a > 0$ is a real number, $0 < p_s < q_s < N$ are positive integers, and $\zeta \in \mathbb{R}$ is a variable.

Proof. See Appendix A. □

Remark 1. There is some large positive integer N, when $0 < p_s < q_s < N$, to make the inequality $\left(\frac{1}{2}(\frac{q_s-p_s}{q_s+p_s})^{1/2}\right)^{\frac{p_s}{q_s}} - \arctan\left((\frac{q_s-p_s}{q_s+p_s})^{1/2}\right) < 0$ hold. So $\frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$ can be a enough small real number.

A finite time approach rate that guarantees a closed-loop system converges in finite time is presented as follows,

$$\dot{s}(t) = -\alpha_1 s(t) - \beta_1 \arctan(as(t)), \tag{8}$$

where $\alpha_1 > 0$, $\beta_1 > 0$, and $0 < a < 1$.

Lemma 3. Suppose that there is a approach rate $s(t) : \mathbb{R}^* \to \mathbb{R}$, some design constants $\alpha_1 > 0$, $\beta_1 > 0$, and $0 < a < 1$ such that Eq. (8). If there is a large integer $N > 0$ that makes Fact 1 true, then the origin $s(t) = 0$ is finite-time stable for $|s(t)| \geq \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$ and the origin $s(t) = 0$ is exponentially stable for $0 \leq |s(t)| \leq \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$.

Proof. See Appendix B. □

Remark 2. The approach rate in Eq. (8) makes the tracking error of the system reach a small neighborhood of the sliding surface in a finite time. In this neighborhood, it converges exponentially to the sliding surface to avoid chattering near the sliding surface.

4.3 | The FTPE-BC design

Let $s(t) \in \mathbb{R}^p$ be the following sliding surface expressed

$$s(t) = C_s e(t), \tag{9}$$

where $C_s \in \mathbb{R}$ is a positive constant.

When the input saturation of system is not considered, an unconstrained controller is given as follows

$$u(t) = B_u^+ C_s^{-1}(-\alpha_1 s(t) - \beta_1 \arctan(s(t))) + B_u^+(-A_e x(t) - \dot{r}(t)), \tag{10}$$

where $\alpha_1 > 0$ and $\beta_1 > 0$.

Now, considering the input saturation constraint and solving Problem 1, a bounded FTPE-BC control law is proposed by combining the finite time approach rate in Eq. (8) as follows

$$\begin{aligned} v(t) =&kB_u^{+}C_s^{-1}(-\alpha_1 s(t) - \beta_1 \arctan(as(t))) + B_u^{+}(-A_e x(t) - \dot{r}(t)),\\ u(t) =&\sigma(v(t)), \end{aligned} \tag{11}$$

where $1 < k < \infty$, $\alpha_1 > 0$, $\beta_1 > 0$, and $0 < a < 1$.

Remark 3. A new finite-time approach rate in Eq. (8) is proposed, in which parameter $0 < a < 1$ can reduce the overshoot and the steady-state oscillation of the closed-loop system.

Remark 4. The performance of the system decreases due to input saturation constraints. To this end, we add a parameter $k > 1$ to shorten the settling time and steady-state error to improve the response performance.

5 | ANALYSIS OF THE CLOSED-LOOP SYSTEM

Analysis of the closed-loop system includes stability analysis, performance analysis and parameter selection of the designed controller.

5.1 | Stability analysis

In order to investigate the stability properties of the closed-loop system in Eq. (4), the control input of the system is considered under the condition of unsaturated and saturated respectively.

5.1.1 | Unsaturated control input

First, we discuss the finite-time stability of the closed-loop system in Eq. (4) without input saturation, i.e., for $\sigma(u(t)) = u(t)$, is shown as follows.

Theorem 1. Consider the error system in Eq. (4) without the input saturation in Eq. (1), and the controller in Eq. (10). There exist positive constants ε and T, then after T, the error $e(t)$ converge to the neighborhood of the equilibrium point $e(t) = 0$, i.e., $\|e(t)\| < \varepsilon$ for $\forall\, t \geq \mathrm{T}$. Moreover, there exist positive constants κ and λ such that $\|e(t)\| \leq k\|e(t_0)\|e^{-\lambda(t-t_0)}$, for $\forall\ \|e(t_0)\| < \varepsilon$, then $e(t) = 0$ is exponentially stable in the ε neighborhood.

Proof. Consider Lyapunov function candidate $V(e(t)) = \frac{1}{2}s^{\mathrm{T}}(t)s(t)$ whose time derivative is

$$\begin{aligned} \dot{V}(e(t)) =&s(t)^{\mathrm{T}}\dot{s}(t)\\ =&s(t)^{\mathrm{T}}C_s(A_e x(t) + B_u\sigma(u(t)) + \dot{r}(t)). \end{aligned} \tag{12}$$

Since $\sigma(u(t)) = u(t)$ and there is a large integer $N > 0$ that makes Fact 1 true, by substituting the controller in Eq. (10) into Eq. (12) and using Fact 1, we obtain

$$\begin{aligned} \dot{V}(e(t)) =&s(t)^{\mathrm{T}}C_s(A_e x(t) + B_u B_u^{+}(-A_e x(t) - \dot{r}(t)) + B_u B_u^{+}C_s^{-1}(-\alpha_1 s(t) - \beta_1 \arctan(s(t))) + \dot{r}(t))\\ =&s(t)^{\mathrm{T}}(-\alpha_1 s(t) - \beta_1 \arctan(s(t)))\\ \leq& -\alpha V(e(t)) - \beta V^{q}(e(t)), \end{aligned}$$

where $\alpha_1 > 0$, $\beta_1 > 0$, and $\alpha = 2\alpha_1$. When $|s(t)| \succeq (\frac{q_s-p_s}{q_s+p_s})^{1/2}$, $\beta = 2^{q}\beta_1(\frac{1}{2})^{p_1}$, $q = \frac{p_1+1}{2}$, $p_1 = \frac{p_s}{q_s}$, and $0 < p_s < q_s < N$. According to Lemma 1, the system in Eq. (4) is finite-time stable with convergence time

$$\mathrm{T} \leq \frac{1}{\alpha(1-q)} \ln \frac{\alpha V^{1-q}((e(0))) + \beta}{\beta}. \tag{13}$$

When $0 \preceq |s(t)| \prec (\frac{q_s-p_s}{q_s+p_s})^{1/2}$, $\beta = \beta_1$, $q = 1$, and $0 < p_s < q_s < N$. We obtain

$$\dot{V}(e(t)) \leq -\lambda V(e(t)), \text{ for } \forall \ \|e(t)\| < \varepsilon$$

where $\lambda = \alpha + \beta$. The equilibrium point $e(t) = 0$ of the system in Eq. (4) is exponentially stable in the ε neighborhood. The proof is completed. □

Remark 5. We can take the positive integers p_s, q_s, N as large as possible in order to make $(\frac{q_s-p_s}{q_s+p_s})^{1/2}$ as small as possible. To ensure that the error of the closed-loop system converges to a smaller ε neighborhood in a finite time.

5.1.2 | Saturated control input

Next, we discuss the domain of attraction and finite-time stability of the closed-loop system with the input saturation.

Theorem 2. Consider the error system in Eq. (4) with the input saturation in Eq. (1) and the controller in Eq. (11). If the domain of attraction is

$$\Omega = \{e(t) = r(t) - Cx(t) : |B_u^{-1}(A_e x(t) + \dot{r}(t))| \prec \bar{u}\}, \tag{14}$$

then, the equilibrium point $e(t) = 0$ is practical finite-time stable.

Proof. See Appendix C. □

Theorem 3. Consider the error system in Eq. (4) with the input saturation in Eq. (1), and the controller in Eq. (11). There exists positive constants ε and T. If the domain Ω satisfy Eq. (14), then the equilibrium point $e(t) = 0$ is practical finite-time stable, i.e., $\|e(t)\| < \varepsilon$ for $\forall \ t \geq \mathrm{T}$ and $e(0) \in \Omega$. Moreover, there exist positive constants κ and λ such that $\|e(t)\| \leq k\|e(t_0)\|e^{-\lambda(t-t_0)}$, for $\forall \ \|e(t_0)\| < \varepsilon$, then $e(t) = 0$ is exponentially stable in the ε neighborhood.

Proof. Consider Lyapunov function candidate $V(e(t)) = \frac{1}{2}s^{\mathrm{T}}(t)s(t)$ whose time derivative is

$$\begin{aligned}\dot{V}(e(t)) =& s^{\mathrm{T}}(t)\dot{s}(t)\\ =& s(t)^{\mathrm{T}}C_s(A_e x(t) + B_u\sigma(u(t)) + \dot{r}(t))\\ =& s(t)^{\mathrm{T}}C_s(A_e x(t) + B_u v(t) - B_u q(t) + \dot{r}(t)),\end{aligned}$$

where $q(t) = v(t) - \sigma(v(t))$. Since $e(t) \in \Omega$ and there is a large integer $N > 0$ that makes Fact 1 true, by substituting the controller (11) into Eq. (15), we obtain

$$\begin{aligned}\dot{V}(e(t)) =& s^{\mathrm{T}}(t)\left(k(-\alpha_1 s(t) - \beta_1 \arctan(as(t))) - C_s B_u q(t)\right)\\ \leq& -\alpha V(t) - \beta V^q(t) + b,\end{aligned}$$

where $\alpha = 2k\alpha_1$ and $b = -s^{\mathrm{T}}(t)C_s B_u q(t)$. $|b| < \infty$ is ture because $\|C_s B_u q(t)\| < \alpha_1\|s(t)\| + \beta_1\|\arctan(as(t))\|$ when $e(t) \in \Omega$. When $|s(t)| \succeq \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$, $\beta = 2^q k\beta_1(\frac{a}{2})^{p_1}$, $q = \frac{p_1+1}{2}$, $p_1 = \frac{p_s}{q_s}$, and $0 < p_s < q_s < N$. According to Lemma 2 the error $e(t)$ converge to the neighborhood $\Omega_b = \{e(t) : V^q(e(t)) \leq \frac{b}{(1-\lambda)\beta}\}$ in finite time

$$\mathrm{T} \leq \frac{1}{\alpha(1-q)} \ln \frac{\alpha V^{1-q}(e(0)) + \lambda\beta}{\lambda\beta}, \tag{15}$$

where $\lambda \in (0, 1)$. There is a constant $\varepsilon > 0$ to make $|s(t)| \succeq \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$ hold, so $\|e(t)\| < \varepsilon$ for $\forall \ t \geq \mathrm{T}$, the equilibrium point $e(t) = 0$ is practical finite-time stable. When $0 \preceq |s(t)| \prec \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$, $\beta = k\beta_1 a$, $q = 1$, and $q(t) = 0$. We obtain

$$\dot{V}(e(t)) \leq -\lambda V(e(t)), \text{ for } \forall \ \|e(t)\| < \varepsilon$$

where $\lambda = \alpha + \beta$. The equilibrium point $e(t) = 0$ of the system in Eq. (4) is exponentially stable in the ε neighborhood. The proof is completed. □

5.2 | Performance analysis

Consider the closed-loop system consisting of the error dynimac in Eq. (4) of the plant in Eq. (3) with the input saturation in Eq. (1), and the controller in Eq. (11). The transient and steady-state performance analyses are given below.

5.2.1 | Transient performance analysis

The following theorem proves that the designed bounded controller can ensure finite-time stability of the closed-loop system.

Theorem 4. Consider the closed-loop system consisting of the error dynimac in Eq. (4) and the controller in Eq. (11). When $|s(t)| \succeq \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$, if $e(0) \in \Omega$ in Eq. (14), then the origin $e(t) = 0$ is practical finite-time stable, and the overshoot is smaller than a constant.

Proof. According to Theorem 2 and $e(0) \in \Omega$ in Eq. (14), $\|q(t)\| < k\|(C_sB_u)^{-1}(-\alpha_1 s(t) - \beta_1 \arctan(as(t)))\|$. Since there is a integer $N > 0$ that holds Fact 1, we obtain the closed-loop system as follows,

$$\begin{aligned}\dot{e}(t) =& kC_s^{-1}(-\alpha_1 s(t) - \beta_1 \arctan(as(t))) - B_u q(t)\\ =& \bar{k}C_s^{-1}(-\alpha_1 s(t) - \beta_1 \arctan(as(t)))\\ \preceq& -\alpha e(t) - \beta e^q(t),\end{aligned}$$

where constants $0 < \bar{k} < k$, $\alpha = \bar{k}\alpha_1$, $\beta = \bar{k}\beta_1(\frac{a}{2})^q$, $q = \frac{p_s}{q_s}$, and $0 < p_s < q_s < N$. The origin $e(t) = 0$ is finite-time stable when $|s(t)| \succeq \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$ according to Lemma 3. By solving Eq. (16) we obtain

$$\lfloor e_i(t) \rceil^{1-q} \leq -\frac{\beta}{\alpha} + \frac{\beta}{\alpha} e^{-\alpha(1-q)t} + \lfloor e_i(0) \rceil^{1-q} e^{-\alpha(1-q)t}, \tag{16}$$

where $i \in \{1, 2, \ldots, p\}$. Thus the overshoot of the system is $\lfloor e_i(t) \rceil^{1-q} \leq -\frac{\beta}{\alpha}$ when $|s(t)| \succeq \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$. The proof is completed. □

5.2.2 | Steady-state performance analysis

The following theorem proves that the designed bounded controller can enhance the steady-state performance of the closed-loop system.

Theorem 5. Consider the closed-loop system consisting of the error dynimac in Eq. (4) and the controller in Eq. (11). When $0 \preceq |s(t)| \prec \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$, if $e(0) \in \Omega$ in Eq. (14) and $\|q(t)\| = 0$, then the origin $e(t) = 0$ is exponentially stable, and the overshoot is zero.

Proof. Since $e(0) \in \Omega$ in Eq. (14), $\|q(t)\| = 0$, and there is a integer $N > 0$ that holds Fact 1. Therefore, we obtain the closed-loop system as follows,

$$\begin{aligned}\dot{e}(t) =& kC_s^{-1}(-\alpha_1 s(t) - \beta_1 \arctan(as(t)))\\ \preceq& -k\alpha_1 e(t) - k\beta_1 \frac{a}{2} e(t).\end{aligned} \tag{17}$$

The origin $e(t) = 0$ is finite-time stable when $0 \preceq |s(t)| \prec \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$ with $0 < p_s < q_s < N$ according to Lemma 3. By Solving Eq. (17) we obtain $e(t) \preceq e(0)e^{-(k\alpha_1 + k\beta_1 \frac{a}{2})t}$. Therefore the overshoot of the system is zero when $0 \preceq |s(t)| \prec \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$. The proof is completed. □

5.3 | Parameters selection for FTPE-BC

In order to make the overshoot, rise time and steady-state time of the closed-loop system response as small as possible, we give an analytical design method for the selection of controller parameters.

Furthermore, the constraints $u(t) \preceq \bar{u}$ for controller (11) are considered. In order to make $V(e(0)) = \frac{1}{2}s^2(0)$ with Eq. (9) and $b = -s^{\mathrm{T}}(t)C_sB_u q(t)$ as small as possible, C_s is taken as small as possible. To satisfy the T in Eq. (15) and the error

$e(t) \preceq e(0)e^{-(k\alpha_1 + k\beta_1 \frac{a}{2})t}$ in Eq. (17) while minimizing, select α_1 and k as large as possible. In order to make the overshoot $\lfloor e_i(t) \rceil^{1-q} \leq \frac{\beta_1 (\frac{a}{2})^q}{\alpha_1}$ in Eq. (16) as small as possible, select β_1 and a as small as possible.

6 | A CASE STUDY

The proposed FTPE-BC in Eq. (11) method is compared to windup sliding mode control (WSMC) in Eq. (10) method and Unconstrained sliding mode control (USMC) in Eq. (10) method in the plant in Eq. (3) without input saturation in Eq. (1) as well as to the super-twisting algorithm (STA) based[27] method and to the finite-time stabilization bounded control (FTSBC) based[24] method in the course of a simulation.

6.1 | System description

For this purpose, the plant in Eq. (3) with $\bar{u} = (0.3, 0.3)^{\mathrm{T}}$, $C = (1, 0, 0, 0)$ and

$$A = \begin{pmatrix} -3.2320 & 20 & 0 & 0 \\ -2.3040 & -20 & 20 & 0 \\ -0.3420 & 0 & -20 & 20 \\ 0.0960 & 0 & 0 & -20 \end{pmatrix}, \; B = \begin{pmatrix} 89.7480 & 41.3200 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is considered.

6.2 | Setting up

The functionality of the proposed controller has been tested through simulation on a personal computer with a CPU of 2.00 GHz. The simulation language is C language.

The reference $r(t) = 2\sin(t)$ is used. The FTPE-BC in Eq. (11), WSMC in Eq. (10), and USMC in Eq. (10) controller's parameters are chosen as $C_s = 0.5$, $\alpha_1 = 12$, $\beta_1 = 0.2$, $a = 0.1$, and $k = 80$. STA[27] control law is $u(t) = \sigma(v(t))$, where

$$\begin{cases} v(t) = k_1 B_u^+ \lfloor r - y(t) \rceil^{\frac{1}{2}} + \tilde{v}(t) \\ \dot{\tilde{v}}(t) = k_2 \operatorname{sign}(u(t) - \tilde{v}(t)), \end{cases} \tag{18}$$

where $k_1 = 12$ and $k_2 = 0.2$. FTSBC[24] control law is

$$u(t) = B_u^+(-\alpha(tanh(k_3 e(t)))^{\frac{4}{3}} - \beta(tanh(k_3 e(t)))^{\frac{2}{3}}) + B_u^+(-A_e x(t) - \dot{r}(t)), \tag{19}$$

where $\alpha = 12$, $\beta = 0.2$, and $k_3 = 1$.

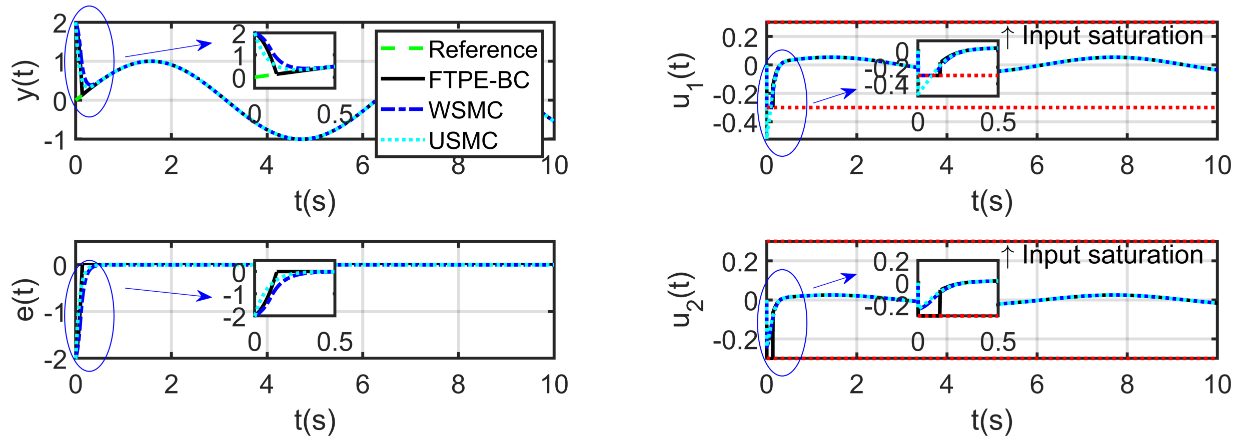
For the comparison of the transient performance in different closed-loop systems, we define some measures that include the rise time t_r when $\|e(t)\| \leq 0.1$ for all $t > t_r$, the overshoot μ as $\mu := e(t_p)$ where t_p is the peak time, the settling time t_s when $\|e(t)\| \leq 1\%$ for all $t > t_s > t_r$, and steady-state error $e_{ss} := \|e(t)\|$ for $t \to \infty$.

6.3 | Result

The response performance of the system in Eq. (3) under the action of different controllers is shown in Figure 2 and Figure 3. The response performance analysis of system in Eq. (3) under the action of different controllers is shown in Table 1 and Table 2.

6.4 | Analysis and discussion

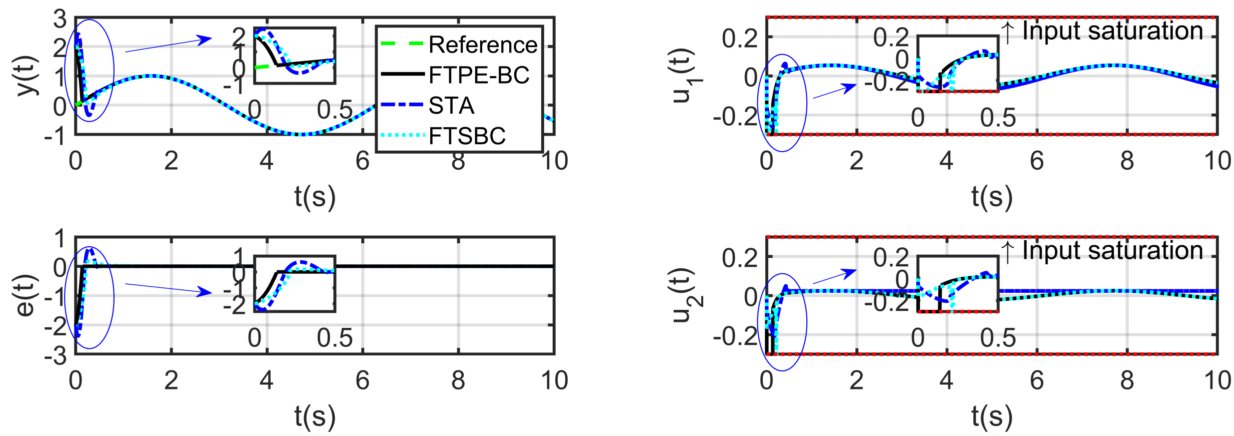
The effects of different controllers FTPE-BC in Eq. (11), WSMC in Eq. (10), and USMC in Eq. (10) in the plant in Eq. (3) without input saturation in Eq. (1) are compared. According to Table 1, $t_r = 0.2977$(s) of WSMC is twice larger than $t_r = 0.2415$(s) of



(a) The system output and tracking error

(b) The system control input

FIGURE 2 The response of the plant with input saturation under different controllers.



(a) The system output and tracking error

(b) The system control input

FIGURE 3 The response of the plant with input saturation under different controllers.

USMC and $t_r = 0.1336$(s) of FTPE-BC, $t_s = 0.4824$(s) of WSMC is larger than $t_s = 0.4231$(s) of USMC and $t_s = 0.1378$(s) of FTPE-BC, $e_{ss} = 2.3866 \times 10^{-4}$ of WSMC and $e_{ss} = 2.3866 \times 10^{-5}$ of USMC are much larger than $e_{ss} = 3.0620 \times 10^{-6}$ of FTPE-BC, $\mu = 1.0975 \times 10^{-4}$ of USMC is larger than $\mu = 9.6794 \times 10^{-5}$ of WSMC and $\mu = 8.4633 \times 10^{-5}$ of FTPE-BC. Therefore, the proposed FTPE-BC design method improves the transient and steady-state performance compared with the original SMC method. The effects of different controllers FTPE-BC in Eq. (11), STA[27], and FTSBC[24] are compared. According to Table 2, $t_r = 0.4104$(s) of STA is twice larger than $t_r = 0.2145$(s) of FTSBC and $t_r = 0.1336$(s) of FTPE-BC, $\mu = 0.6219$ of STA is much larger than $\mu = 0.1730$ of FTSBC and $\mu = 8.4633 \times 10^{-5}$ of FTPE-BC, $t_s = 0.4573$(s) of STA is twice larger than $t_s = 0.2171$(s) of FTSBC and $t_s = 0.1378$(s) of FTPE-BC, $e_{ss} = 1.6818 \times 10^{-3}$ of FTSBC is much larger than $e_{ss} = 4.8536 \times 10^{-5}$ of STA and $e_{ss} = 3.0620 \times 10^{-6}$ of FTPE-BC. Therefore, the proposed $u(t)$design has better transient and steady-state performance than the tool designs in[27] and[24]. In addition, the FTPE-BC design method takes full advantage of limited control capabilities. The finite-time stability of the system is guaranteed in the domain of attraction. The advantages of FTPE-BC design method in transient and steady-state performance are demonstrated.

TABLE 1 The performance comparison

Indexes	WSMC	USMC	FTPE-BC
Rise time t_r (s)	0.2977	0.2415	**0.1336**
Peak time t_p (s)	1.0030	0.9440	**0.1420**
Overshoot μ	9.7345×10^{-5}	1.1024×10^{-4}	$\mathbf{8.4633 \times 10^{-5}}$
Settling time t_s (s)	0.4824	0.4231	**0.1378**
Steady-state error e_{ss}	2.3866×10^{-4}	2.3866×10^{-4}	$\mathbf{3.0620 \times 10^{-6}}$

TABLE 2 The performance comparison

Indexes	STA [27]	FTSBC [24]	FTPE-BC
Rise time t_r (s)	0.4104	0.2145	**0.1336**
Peak time t_p (s)	0.2850	0.2770	**0.1420**
Overshoot μ	0.6219	0.1730	$\mathbf{8.4633 \times 10^{-5}}$
Settling time t_s (s)	0.4573	0.2171	**0.1378**
Steady-state error e_{ss}	4.8536×10^{-5}	1.6818×10^{-3}	$\mathbf{3.0620 \times 10^{-6}}$

7 | CONCLUSIONS

In this paper, a FTPE-BC framework is proposed for a class of linear time-invariant systems with input saturation to achieve finite-time convergence. In this method, an approach rate based on the inverse tangent function is proposed to make the tracking error of the system reach near the sliding surface in a finite time and avoid chattering near the sliding surface. A bounded controller based on sliding mode variable structure method is presented. By introducing a control parameter into the bounded controller, the transient performance of the closed-loop system is improved by reducing both the overshoot and the stabilization time. The finite-time stability of the system is proved in a certain domain of attraction. The method is simple and easy to implement, and the response performance of the closed-loop system is highly controllable. The simulation results verify the superiority of the design tool in the transient and stability indexes. The method realizes the highly controllable transient and steady-state performance. It is shown that the proposed control framework can effectively improve the response performance of the system.

AUTHOR CONTRIBUTIONS

Wenyu Ma: methodology; Data curation; software; writing-original draft; writing-review and editing. Guangyu Liu: Conceptualization; funding acquisition.

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FINANCIAL DISCLOSURE

None reported.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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□

APPENDIX

A PROOF OF FACT 1

Proof. Firstly, when $0 \leq |\zeta| < \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$, if $\zeta \geq 0$, then $\frac{a}{2}\zeta - \arctan(a\zeta) \leq 0$ according to the monotonicity. If $\zeta < 0$ then $\frac{a}{2}\zeta - \arctan(a\zeta) > 0$ is true by symmetry. Secondly, when $\frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a} \leq |\zeta| \leq \frac{2}{a}$, if $\zeta \geq 0$, take

$$f = \left(\frac{a}{2}\zeta\right)^{\frac{p_s}{q_s}} - \arctan(a\zeta). \tag{A1}$$

The derivative of f with respect to ζ is

$$\frac{df}{d\zeta} = \frac{p_s}{q_s}\left(\frac{a}{2}\right)^{\frac{p_s}{q_s}} \zeta^{\frac{p_s-q_s}{q_s}} - \frac{a}{1+a^2\zeta^2}. \tag{A2}$$

Let $\zeta = \frac{\xi}{a}$, therefore
$$\frac{df}{d\zeta} = a\left(\frac{p_s}{q_s}\left(\frac{1}{2}\right)^{\frac{p_s}{q_s}}\xi^{\frac{p_s-q_s}{q_s}} - \frac{1}{1+\xi^2}\right). \tag{A3}$$
Take $F = (1+\xi^2)\xi^{\frac{p_s-q_s}{q_s}} - \frac{q_s}{p_s}2^{\frac{p_s}{q_s}}$. The derivative of F with respect to ξ is
$$\frac{dF}{d\xi} = \left(2 - \frac{q_s-p_s}{q_s}(1+\xi^2)\xi^{-2}\right)\xi^{\frac{p_s}{q_s}}. \tag{A4}$$
When $(\frac{q_s-p_s}{q_s+p_s})^{1/2} \leq \xi \leq 2$, $\frac{dF}{d\xi} \geq 0$ is true. Therefore f is concave. Since $\left(\frac{1}{2}(\frac{q_s-p_s}{q_s+p_s})^{1/2}\right)^{\frac{p_s}{q_s}} - \arctan\left((\frac{q_s-p_s}{q_s+p_s})^{1/2}\right) < 0$ and $1-\arctan(2) < 0$ is true, $(\frac{a}{2}\zeta)^{\frac{p_s}{q_s}} - \arctan(a\zeta) < 0$. If $\zeta < 0$ then $(\frac{a}{2}\zeta)^{\frac{p_s}{q_s}} - \arctan(a\zeta) > 0$ is true by symmetry. Finally, when $|\zeta| > \frac{2}{a}$, $|\arctan(a\zeta)| > 1$ according to the monotonicity. The proof is completed. □

B PROOF OF LEMMA 3

Proof. When $s(t) > 0$, if $\frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a} \leq s(t) \leq \frac{2}{a}$, by using Fact 1 obtain
$$\dot{s}(t) \leq -\alpha_1 s(t) - \beta_1\lfloor\frac{a}{2}s(t)\rceil^{\frac{p_s}{q_s}}, \tag{B5}$$
then, the origin $s(t) = 0$ is finite-time stable and the convergence time is given by $\mathrm{T}(s(0)) \leq \frac{1}{\alpha_1(1-\frac{p_s}{q_s})}\ln\frac{\alpha_1 s^{1-\frac{p_s}{q_s}}(0)+\beta_1}{\beta_1}$ according to Lemma 1. If $s(t) > \frac{2}{a}$, by using Fact 1 obtain
$$\dot{s}(t) \leq -\alpha_1 s(t) - \beta_1. \tag{B6}$$
By multiplying both sides of equation (B6) by $e^{\alpha_1 t}$ to get
$$\frac{de^{\alpha_1 t}s(t)}{dt} \leq -\beta_1 e^{\alpha_1 t}. \tag{B7}$$
Solve equation (B7) to obtain
$$s(t) \leq -\frac{\beta_1}{\alpha_1} + \frac{\beta_1}{\alpha_1}e^{-\alpha_1 t} + s(0)e^{-\alpha_1 t}, \tag{B8}$$
therefore, the origin $s(t) = 0$ is finite-time stable and the convergence time is given by $\mathrm{T}(s(0)) < \frac{1}{\alpha_1}\ln\frac{\alpha_1 s(0)+\beta_1}{\beta_1}$ according to Eq. (B8) . If $0 \leq s(t) \leq \frac{(\frac{q_s-p_s}{q_s+p_s})^{1/2}}{a}$, by using Fact 1 obtain
$$\dot{s}(t) \leq -\alpha_1 s(t) - \beta_1\frac{a}{2}s(t). \tag{B9}$$
Solve equation (B9) to obtain
$$s(t) \leq s(0)e^{-(\alpha_1+\beta_1\frac{a}{2})t}. \tag{B10}$$
Therefore, the origin $s(t) = 0$ is exponentially stable. When $s(t) \leq 0$, the conclusion holds because of the symmetry of the function in Eq. (8). The proof is completed. □

C PROOF OF THEOREM 2

Proof. When $|B_u^{-1}(A_e x(t) + \dot{r}(t))| \prec \bar{u}$ holds, by the idea of cancellation we obtain $\alpha_1 s(t) \neq 0$ or $\beta_1\arctan(as(t)) \neq 0$ of the controller in Eq. (11). Consider Lyapunov function candidate $V(e(t)) = \frac{1}{2}s^{\mathrm{T}}(t)s(t) \in \mathbb{R}$ whose time derivative is as follows,
$$\begin{aligned}\dot{V}(e(t)) =& s^{\mathrm{T}}(t)\dot{s}(t)\\ =& s^{\mathrm{T}}(t)C_s(A_e x(t) + B_u\sigma(u(t)) + \dot{r}(t))\\ =& s^{\mathrm{T}}(t)\bar{k}\left(-\alpha_1 s(t) - \beta_1\arctan(as(t))\right)\\ <& 0,\end{aligned} \tag{C11}$$
where constant $0 < \bar{k} < k$ due to $|B_u^{-1}(A_e x(t) + \dot{r}(t))| \prec \bar{u}$. Therefore the equilibrium $e(t) = 0$ of the closed-loop system is practical finite-time stable. The proof is completed. □

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