Option pricing for uncertain stock model based on optimistic value

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Abstract

Option pricing plays an important role in modern finance. This paper investigates the uncertain option pricing problems based on uncertainty theory for Liu's uncertain stock model and Peng's mean-reverting stock model which are two basic and representative uncertain stock models in uncertain finance. The pricing formulas of the European and American options are derived by applying the method to calculate the optimistic value of uncertain returns of options instead of the usual method of expected value in the sense of the weighted average. In the end, some numerical experiments are given to illustrate the effectiveness of the obtained results.

Keywords: option pricing, optimistic value, uncertainty

1 Introduction

Option pricing is one of the most fundamental yet most important problems in modern finance. In the past decades, it has received a lot of interest from many researchers. Options are a type of financial derivatives whose values depend on the value of underlying assets. It is a contract that gives its holder the right, but not the obligation, to buy or sell

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a certain amount of underlying asset, at a certain price, at a certain date. Finding the fair value of options is a core problem in modern finance. In 1973, Black and Scholes [1] proposed the famous Black-Scholes model based on the assumption that stock price follows geometric Brownian motion, and gave an option pricing formula. From then on, the prices of many kinds of options based on the Black-Scholes model have been investigated by many researchers and many useful results were obtained. Up to now, the study on stochastic option pricing has made considerable advances both in theory and application.

The Black-Scholes model was established based on probability theory in which the underlying asset price process follows the stochastic differential equation. It is well known that the basic premise of applying probability theory to describe indeterministic phenomena is that there must be sufficient available sample data to estimate probability distribution which is close enough to the frequency of indeterminate events occurring. However, in many cases of the practical financial market, we cannot always obtain adequate statistical data and even sometime there are no samples available owing to various reasons. For example, bridge strength, oil filed reserves and newly listed securities have few historical data. In this case, we cannot estimate a probability distribution by means of statistics and have to invite some domain experts to evaluate their belief degree that each indeterministic event will occur. Perhaps some people think that personal belief degree is a subjective probability or fuzzy concept. However, Liu [2] showed that it is inappropriate because both probability theory and fuzzy set theory may lead to counterintuitive results. In order to rationally deal with the indeterminacy, Liu [3] founded an uncertainty theory in 2007 which is a branch of axiomatic mathematics for modeling the uncertain behaviors of human beings. In 2008, for describing state evolution of dynamic systems with indeterministic disturbances over time, an uncertain process and canonical Liu process were introduced by Liu [4] as counterparts of stochastic process and Wiener process, respectively. Furthermore, uncertain calculus on the basis of Liu process and a type of uncertain differential equation driven by canonical Liu process were introduced by Liu [5]. Nowadays, the uncertain theory has been well developed and successfully applied to many areas such as uncertain finance [6, 7], uncertain risk analysis [8, 9], uncertain statistics [10, 11], uncertain programming [12, 13], and uncertain optimal control [14, 15], etc. For exploring the recent developments of uncertainty theory, the readers may consult Liu [16].

Uncertain differential equation was first applied into finance by Liu [5]. In 2009, Liu proposed an uncertain stock model in which the stock price is assumed to follow a geometric Liu process and derived European option pricing formula. Afterwards, many researchers have investigated the option pricing problems based on uncertainty theory. For example, Chen [7] studied the pricing method of American options for uncertain financial market and derived some pricing formulae of such options based on Liu's stock model. In 2015, Sun and Chen [17] discussed the pricing problems of uncertain Asian options and derived their pricing formulae based on Liu's stock model. Zhang and Liu [18] investigated the pricing problem of geometric average Asian option and derived its pricing formula. Gao et al. [19] studied Lookback option pricing problem of uncertain exponential Ornstein-Uhlenbeck model. Zhang et al. [20] presented the pricing formulas of Lookback options of fixed strike. Gao et al. [21] studied uncertain Barrier option pricing problem. Zhang et al. [22] proposed the power option formulas in uncertain financial market. Zhang et al. [23] studied the option pricing formulae of the interest rate ceiling. Liu et al. [24] introduced the uncertain currency model and presented the currency option pricing method. Besides, Peng and Yao [25] proposed a mean-reverting stock model for uncertain market and derived European and American option price formulae. Shen and Yao [26] presented mean-reverting currency model and gave its option pricing formulae. Wang and Ning [27] further discussed the currency model with floating interest rate and derived its option pricing formulae. Considering the influence of sudden jump on the stock price, Ji and Zhou [28] proposed an uncertain stock model with jumps which followed geometric Liu process and uncertain renewal process and then discussed the European option pricing formulas for presented model.

The above option pricing methods are based on expected value which is a traditional way of calculating uncertain returns of options and has been widely used in the study of option pricing problems. However, this method is obviously only suitable for the cases of underlying asset volatility being relatively stable. When the price of underlying asset of options fluctuates greatly, the expected value in the sense of the weighted average may not be considered only. In this case, critical value (optimistic value or pessimistic value) of calculating uncertain returns of options may be a good alternative. In 2019, Lu et al. [29] discussed critical value-based Asian European option pricing model based on a mean-reverting stock price model with uncertain fractional differential equation for uncertain financial market. In this paper, we will investigate the problem of European and American option pricing by applying the method of optimistic value for Liu's uncertain stock model and Peng's mean-reverting stock model which are two basic and representative uncertain stock models in uncertain finance. To our knowledge, no research has been done on the option pricing problems based on optimistic value for the two stock models.

The rest of the paper is organized as follows. Next section is preliminary in which we introduce some useful concepts and theorems of uncertainty theory as needed. In Section 3, the uncertain option pricing models under optimistic value criterion based on Liu's uncertain stock model and Peng's mean-reverting stock model are investigated and the option pricing formulaes are derived. In Section 4, some numerical examples are given to illustrate the results obtained. Finally, a brief conclusion is given in Section 5.

2 Preliminary

In this section, we will review some basic concepts and theorems in uncertainty theory [3]. Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathcal{M}$ defined on the σ-algebra $\mathcal{L}$ over Γ is called an uncertain measure if it satisfies $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$, and $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\{\Lambda_i\}$. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be an uncertainty space. An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that for any Borel set of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \ \xi(\gamma) \in B\}$ is an event. An uncertain variable ξ may be described by its uncertainty distribution Φ: $\Re \to [0, 1]$ which is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. The expected value of ξ is defined by $E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\}dr$ provided that at least one of the two integrals is finite. The variance of ξ is $V[\xi] = E[(\xi - E[\xi])^2]$. The uncertain variables $\xi_1, \xi_2, ..., \xi_m$ are said to be independent if $\mathcal{M}\{\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\} = \min_{1\leq i\leq m} \mathcal{M}\{\xi_i \in B_i\}$, for any Borel set $B_1, B_2, \cdots, B_m$ of real numbers.

Theorem 2.1 . *(Liu [16], Extreme Value Theorem) Let X_t be a sample-continuous independent increment process with with regular uncertainty distribution $\Phi_t(x)$ at each time t. Then the supremum*

$$\sup_{0\leq t\leq T} X_t$$

has an uncertainty distribution

$$\Psi_t(x) = \inf_{0 \le t \le T} \Phi_t(x).$$

Moreover, if f is a continuous and strictly increasing function, then the supremum

$$\sup_{0 \le t \le T} f(X_t)$$

has an uncertainty distribution

$$P_t(x) = \inf_{0 \le t \le T} \Phi_t(f^{-1}(x)),$$

and if f is a continuous and strictly decreasing function, then the uncertainty distribution is

$$Q_t(x) = 1 - \sup_{0 \le t \le T} \Phi_t(f^{-1}(x)).$$

Definition 2.1 . *(Liu [4]) A canonical Liu process C_t is an uncertain process if and only if*
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normally distributed uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

Theorem 2.2 . *(Liu [16]) The normal uncertain process $X_t \sim N(et, \sigma t)$ has an uncertainty distribution,*

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(et - x)}{\sqrt{3}\sigma t}\right)\right)^{-1}$$

and its inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.$$

Theorem 2.3 . *(Liu [16]) Let $f(t)$ be an integrable function with respect to t. Then the Liu integral*

$$\int_0^s f(t)dC_t$$

is a normal uncertain variable at each time s, and

$$\int_0^s f(t)dC_t \sim N\left(0, \int_0^s |f(t)|dt\right).$$

Theorem 2.4 . *(Liu [16]) Let X_t be an uncertain process with regular uncertainty distribution $\Phi_t(x)$, and let a and b be real numbers. Show that (i) if $a > 0$, then $aX_t + b$ has an inverse uncertainty distribution,*

$$\Psi_t^{-1}(\alpha) = a\Phi_t^{-1}(\alpha) + b;$$

and (ii) if $a < 0$, then $aX_t + b$ has an inverse uncertainty distribution,

$$\Psi_t^{-1}(\alpha) = a\Phi_t^{-1}(1-\alpha) + b.$$

Definition 2.2 . *(Liu [3]) Let ξ be an uncertain variable, and $\alpha \in (0, 1]$. Then $\xi_{\sup}(\alpha) = \sup\{r|\mathcal{M}\{\xi \geq r\} \geq \alpha\}$ is called the α-optimistic value to ξ; and $\xi_{\inf}(\alpha) = \inf\{r|\mathcal{M}\{\xi \leq r\} \geq \alpha\}$ is called the α-pessimistic value to ξ.*

Let C_t be a canonical Liu process, and $\xi = \Delta C_t = C_{t+\Delta t} - C_t$. Then for any $0 < \alpha < 1$, α-optimistic and α-pessimistic values of ξ are

$$\xi_{\sup}(\alpha) = -\frac{\sqrt{3}\Delta t}{\pi}\ln\frac{\alpha}{1-\alpha}, \tag{2.1}$$

$$\xi_{\inf}(\alpha) = \frac{\sqrt{3}\Delta t}{\pi}\ln\frac{\alpha}{1-\alpha}, \tag{2.2}$$

respectively.

Theorem 2.5 . *(Liu [3, 16]) Let ξ and η be are independent uncertain variables and $\alpha \in (0, 1]$. Then we have*

(i) if $c \geq 0$, then $(c\xi)_{\sup}(\alpha) = c\xi_{\sup}(\alpha)$ and $(c\xi)_{\inf}(\alpha) = c\xi_{\inf}(\alpha)$;

(ii) if $c < 0$, then $(c\xi)_{\sup}(\alpha) = c\xi_{\inf}(\alpha)$ and $(c\xi)_{\inf}(\alpha) = c\xi_{\sup}(\alpha)$;

(iii) $(\xi + \eta)_{\sup}(\alpha) = \xi_{\sup}(\alpha) + \eta_{\sup}(\alpha)$, $(\xi + \eta)_{\inf}(\alpha) = \xi_{\inf}(\alpha) + \eta_{\inf}(\alpha)$.

Theorem 2.6 . *(Liu [16]) Let ξ be are uncertain variable, whose uncertainty distribution is $\Phi(x)$. Then for any $0 < \alpha < 1$, α-optimistic and α-pessimistic values of ξ are*

$$\xi_{\sup}(\alpha) = \Phi^{-1}(1-\alpha),$$
$$\xi_{\inf}(\alpha) = \Phi^{-1}(\alpha),$$

respectively.

3 Uncertain option pricing model under optimistic value criterion

In this section, we will investigate the option pricing problems for Liu's uncertain stock model and Peng's mean-reverting stock model under optimistic value criterion.

In 2009, Liu [5] first presented an uncertain stock model in which the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} dX_t = rX_t dt \\ dY_t = eY_t dt + \sigma Y_t dC_t. \end{cases} \tag{3.1}$$

where r is the riskless interest rate, e is the log-drift, σ is the log-diffusion, and C_t is a canonical Liu process.

Liu's stock model (3.1) describes stock prices in short-run properly. It can not describe stock prices in long-run. When the stock prices fluctuate around some average price in long-run, as a counterpart of Black-Karasinski's model [30], another uncertain stock model for financial markets was proposed by Peng and Yao [25] in which the stock price is assumed following mean-reversion uncertain differential equations. In Peng-Yao's stock model, the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} dX_t = rX_t dt \\ dY_t = (a - bY_t)dt + \sigma dC_t. \end{cases} \tag{3.2}$$

where a, b are given positive constants and meaning of r, σ, C_t are the same as in (3.1).

Peng and Yao [25] also gave European formula and American option formula for this stock model based on expected value criterion.

3.1 European option pricing

This section will price European call and put options for the financial market determined by the uncertain stock models (3.1) and (3.2).

3.1.1 European call option pricing

An European call option is a contract between a buyer and a seller, which ensures that the buyer has the right but not the obligation to buy a quantity of a financial instrument from the seller at a certain time (expiration date) in a certain price (strike price). In this section, we will provide the formulas to calculate the prices of the European call option based on the uncertain stock models (3.1) and (3.2), respectively.

Assume an European call option ensures the buyer to buy a stock whose price follows the uncertain differential equation in stock model (3.1) with a strike price K and an expiration date T. Then the profit of such an European call option is

$$(Y_T - K)^+ = \max(Y_T - K, 0).$$

Considering the time value of money resulted from the bond, the present value of this payoff is

$$\exp(-rT)(Y_T - K)^+.$$

Definition 3.1 . *Assume an European call option has a strike price K and an expiration time T. Then for given confidence level $\alpha \in (0, 1]$, the price of the call option under optimistic value criterion is*

$$f_c = \exp(-rT)\left[(Y_T - K)^+\right]_{\sup}(\alpha). \tag{3.3}$$

Theorem 3.1 *Assume an European call option for the uncertain stock model (3.1) has a strike price K and an expiration time T. Then for given $\alpha \in (0, 1]$, the European call option price is*

$$f_c = \exp(-rT)\left[Y_0 \exp\left(eT - \frac{\sigma T\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right]^+. \tag{3.4}$$

Proof. It follows from the uncertain stock model (3.1) that the bond price is

$$X_t = X_0 \exp(rt), \tag{3.5}$$

and the stock price is

$$Y_t = Y_0 \exp(et + \sigma C_t), \tag{3.6}$$

whose inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = Y_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right). \tag{3.7}$$

By Theorem 2.4, we know that $(Y_T - K)^+$ has an inverse uncertainty distribution

$$\Psi_T^{-1}(\alpha) = \left[Y_0 \exp\left(eT + \frac{\sigma T\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right) - K\right]^+. \tag{3.8}$$

According to Theorem 2.6, we have

$$\left[(Y_T - K)^+\right]_{\sup}(\alpha) = \Psi_T^{-1}(1-\alpha) = \left[Y_0 \exp\left(eT + \frac{\sigma T\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha}\right) - K\right]^+. \tag{3.9}$$

Substituting (3.9) into (3.3), the pricing formula (3.4) is derived. The theorem is proved.

Following, we will provide a formula to calculate the price of the European call option based on the uncertain stock model (3.2).

Theorem 3.2 *Assume an European call option for the uncertain stock model (3.2) has a strike price K and an expiration time T. Then for given $\alpha \in (0, 1]$, the European call option price is*

$$f_c = \exp(-rT)\left[\frac{a}{b} + \exp(-bT)\left(Y_0 - \frac{a}{b} + \frac{(1-\exp(bT))\,\sigma\sqrt{3}}{b\pi} \ln \frac{\alpha}{1-\alpha}\right) - K\right]^+. \tag{3.10}$$

Proof. It follows from the chain rule and the uncertain stock model (3.2) that

$$d\left[\exp(bt)Y_t\right] = b\exp(bt)Y_t dt + \exp(bt)dY_t = a\exp(bt)dt + \sigma\exp(bt)dC_t.$$

Integration on both sides of above equation yields

$$Y_t = \frac{a}{b} + \exp(-bt)\left(Y_0 - \frac{a}{b}\right) + \sigma\exp(-bt)\int_0^t \exp(bs)dC_s. \tag{3.11}$$

Theorem 2.3 implies that $\int_0^t \exp(bs)dC_s \sim N\left(0, \int_0^t \exp(bs)ds\right)$.

Noting that

$$\int_0^t \exp(bs)ds = \frac{\exp(bt)-1}{b}.$$

Thus Theorem 2.2 indicates that $\int_0^t \exp(bs)dC_s$ has an inverse uncertainty distribution

$$\Phi_t^{-1}(\alpha) = \frac{(\exp(bt)-1)\sqrt{3}}{b\pi} \ln \frac{\alpha}{1-\alpha}. \tag{3.12}$$

According to Theorem 2.4, we know that $(Y_T - K)^+$ has an inverse uncertainty distribution

$$\Psi_T^{-1}(\alpha) = \left[\frac{a}{b} + \exp(-bT)\left(Y_0 - \frac{a}{b} + \frac{(\exp(bT)-1)\,\sigma\sqrt{3}}{b\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right]^+. \tag{3.13}$$

By applying Theorem 2.6, we get

$$\left[(Y_T - K)^+\right]_{\sup}(\alpha) = \left[\frac{a}{b} + \exp(-bT)\left(Y_0 - \frac{a}{b} + \frac{(1-\exp(bT))\,\sigma\sqrt{3}}{b\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right]^+. \tag{3.14}$$

By (3.14) and (3.3), we can obtain (3.10). The proof of theorem is finished.

3.1.2 European put option pricing

An European put option is a contract between a seller and a buyer, which ensures that the seller has the right but not the obligation to sell a quantity of a financial instrument to the buyer at a certain time (expiration date) in a certain price (strike price). In this section, we will provide the formulas to calculate the prices of the European put option with the uncertain stock models (3.1) and (3.2), respectively.

Assume an European put option ensures the seller to sell a stock whose price follows the uncertain differential equation in the uncertain stock model (3.1) with a strike price K and an expiration date T. Then the payoff of such an European put option is

$$(K - Y_T)^+ = \max(K - Y_T, 0).$$

Taking into account the time value of money resulted from the bond, the present value of this payoff is

$$\exp(-rT)(K - Y_T)^+.$$

Definition 3.2 . *Assume an European put option has a strike price K and an expiration time T. Then for given $\alpha \in (0,1]$, the price of the put option is*

$$f_p = \exp(-rT)\left[(K - Y_T)^+\right]_{\sup}(\alpha). \tag{3.15}$$

Theorem 3.3 *Assume an European put option for the uncertain stock model (3.1) has a strike price K and an expiration time T. Then for given $\alpha \in (0,1]$, the European put option price is*

$$f_p = \exp(-rT)\left[K - Y_0 \exp\left(eT + \frac{\sigma T\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+. \tag{3.16}$$

Proof. From the uncertain stock model (3.1), we know that the stock price Y_T has an inverse uncertainty distribution

$$\Phi_T^{-1}(\alpha) = Y_0 \exp\left(eT + \frac{\sigma T\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Thus it follows from Theorem 2.4 that $(K - Y_T)^+$ has an inverse uncertainty distribution

$$\Psi_T^{-1}(\alpha) = \left[K - Y_0 \exp\left(eT + \frac{\sigma T\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]^+ \tag{3.17}$$

By applying Theorem 2.6, we have

$$\left[(K - Y_T)^+\right]_{\sup}(\alpha) = \Psi_T^{-1}(1-\alpha) = \left[K - Y_0 \exp\left(eT + \frac{\sigma T\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+. \tag{3.18}$$

Substituting (3.18) into (3.15) yields (3.16). The proof of the theorem is completed.

Next, we will present a formula to calculate the price of the European put option in the uncertain stock model (3.2).

Theorem 3.4 *Assume an European put option for the uncertain stock model (3.2) has a strike price K and an expiration time T. Then for given $\alpha \in (0,1]$, the European put option price is*

$$f_p = \exp(-rT)\left[K - \frac{a}{b} - \exp(-bT)\left(Y_0 - \frac{a}{b} + \frac{(\exp(bT)-1)\,\sigma\sqrt{3}}{b\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+. \tag{3.19}$$

Proof. It follows from Eq.(3.11), Eq.(3.12) and Theorem 2.4 that Y_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = \frac{a}{b} + \exp(-bt)\left(Y_0 - \frac{a}{b} + \frac{(\exp(bt)-1)\,\sigma\sqrt{3}}{b\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Thus by Theorem 2.4 that $(K - Y_T)^+$ has an inverse uncertainty distribution

$$\Upsilon_T^{-1}(\alpha) = \left[K - \frac{a}{b} - \exp(-bT)\left(Y_0 - \frac{a}{b} + \frac{(1-\exp(bT))\,\sigma\sqrt{3}}{b\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+. \quad (3.20)$$

It follows from Theorem 2.6 that

$$\left[(K - Y_T)^+\right]_{\sup}(\alpha) = \left[K - \frac{a}{b} - \exp(-bT)\left(Y_0 - \frac{a}{b} + \frac{(\exp(bT)-1)\,\sigma\sqrt{3}}{b\pi}\ln\frac{\alpha}{1-\alpha}\right)\right]^+. \quad (3.21)$$

Substitute (3.21) into (3.15) yields (3.19). The theorem is verified.

3.2 American option pricing

In this section, we will considering the pricing problems of American call and put options for the financial market determined by the uncertain stock model (3.1) and (3.2).

3.2.1 American call option pricing

An American option gives one the right, but not the obligation, to buy or sell a stock before a specified time for a specified price. Assume that an American call option has a strike price K and an expiration time T. If Y_t is the price of the underlying stock at some time t, then the payoff from buying an American call option is

$$\sup_{0\le t\le T}(Y_t - K)^+.$$

Thinking about the time value of money resulted from the bond, the present value of this payoff is

$$\sup_{0\le t\le T}\exp(-rt)(Y_t - K)^+.$$

Definition 3.3 . *Assume an American call option has a strike price K and an expiration time T. Then for given $\alpha \in (0,1]$, this option has price*

$$f_c = \left[\sup_{0\le t\le T} \exp(-rt)(Y_t - K)^+\right]_{\sup}(\alpha). \tag{3.22}$$

Theorem 3.5 *Assume an American call option for the uncertain stock model (3.1) has a strike price K and an expiration time T. Then for given $\alpha \in (0,1]$, the American call option price is*

$$f_c = \sup_{\gamma>0}\left\{\gamma \middle| \sup_{0\le t\le T}\left(1+\exp\left(\frac{-\pi\left(et+\ln Y_0 - \ln(K+\gamma\exp(rt))\right)}{\sqrt{3}\sigma t}\right)\right)^{-1} \ge \alpha\right\}. \tag{3.23}$$

Proof. By Definition 2.1, C_t has an uncertainty distribution

$$\Theta_t(x) = \left(1+\exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}. \tag{3.24}$$

Then it follows from (3.6) that the stock price Y_t has an uncertainty distribution

$$\begin{aligned}\Phi_t(x) &= \mathcal{M}\{Y_t \le x\} = \mathcal{M}\{Y_0\exp(et+\sigma C_t)\le x\} = \mathcal{M}\left\{C_t \le \frac{\ln x - \ln Y_0 - et}{\sigma}\right\}\\ &= \Theta_t\left(\frac{\ln x - \ln Y_0 - et}{\sigma}\right) = \left(1+\exp\left(\frac{\pi\left(et+\ln Y_0-\ln x\right)}{\sqrt{3}\sigma t}\right)\right)^{-1}.\end{aligned} \tag{3.25}$$

Since $\exp(-rt)(Y_t-K)^+$ is an increasing function of Y_t, by Theorem 2.1, we know that

$$\sup_{0\le t\le T}\exp(-rt)(Y_t-K)^+$$

has an uncertainty distribution

$$\Psi_t(x) = \inf_{0\le t\le T}\left(1+\exp\left(\frac{\pi\left(et+\ln Y_0-\ln(K+x\exp(rt))\right)}{\sqrt{3}\sigma t}\right)\right)^{-1}. \tag{3.26}$$

Noting that $\sup_{0\le t\le T}\exp(-rt)(Y_t-K)^+ \ge 0$, according to Definition 2.2 and (3.26), (3.22), we have

$$\begin{aligned}f_c &= \sup_{\gamma>0}\left\{\gamma \middle| \mathcal{M}\left\{\sup_{0\le t\le T}\exp(-rt)(Y_t-K)^+\ge\gamma\right\}\ge\alpha\right\}\\ &= \sup_{\gamma>0}\left\{\gamma \middle| 1-\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-rt)(Y_t-K)^+\le\gamma\right\}\ge\alpha\right\}\\ &= \sup_{\gamma>0}\{\gamma|1-\Psi_t(\gamma)\ge\alpha\}\\ &= \sup_{\gamma>0}\left\{\gamma \middle| \sup_{0\le t\le T}\left(1+\exp\left(\frac{-\pi\left(et+\ln Y_0-\ln(K+\gamma\exp(rt))\right)}{\sqrt{3}\sigma t}\right)\right)^{-1}\ge\alpha\right\}.\end{aligned}$$

Thus the American call option pricing formula for the uncertain stock model (3.1) is derived.

Theorem 3.6 *Assume an American call option for the uncertain stock model (3.2) has a strike price K and an expiration time T. Then for given $\alpha \in (0,1]$, the American call option price is*

$$f_c = \sup_{\gamma>0}\left\{\gamma \middle| \sup_{0\le t\le T}\left(1+\exp\left(\frac{b\pi\left(K-\frac{a}{b}-\exp(-bt)\left(Y_0-\frac{a}{b}\right)+\gamma\exp(rt)\right)}{\sqrt{3}\sigma(1-\exp(-bt))}\right)\right)^{-1} \ge \alpha\right\} \tag{3.27}$$

Proof. It follows from (3.12) that $\int_0^t \exp(bs)dC_s$ has a normal uncertainty distribution

$$\Phi_t(x) = \left(1+\exp\left(\frac{b\pi x}{\sqrt{3}\,(1-\exp(bt))}\right)\right)^{-1}. \tag{3.28}$$

In accordance with (3.11), (3.28) and Theorem 2.1, we know that

$$\sup_{0\le t\le T} \exp(-rt)(Y_t-K)^+$$

has an uncertainty distribution

$$\Psi_t(x) = \inf_{0\le t\le T}\left(1+\exp\left(\frac{b\pi\left(K-\frac{a}{b}-\exp(-bt)\left(Y_0-\frac{a}{b}\right)+x\exp(rt)\right)}{\sqrt{3}\sigma(\exp(-bt)-1)}\right)\right)^{-1}. \tag{3.29}$$

By using (3.22), (3.29) and Definition 2.2, we get

$$\begin{aligned}
f_c &= \sup_{\gamma>0}\left\{\gamma \middle| \mathcal{M}\left\{\sup_{0\le t\le T}\exp(-rt)(Y_t-K)^+\ge\gamma\right\}\ge\alpha\right\}\\
&= \sup_{\gamma>0}\left\{\gamma \middle| 1-\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-rt)(Y_t-K)^+\le\gamma\right\}\ge\alpha\right\}\\
&= \sup_{\gamma>0}\{\gamma|1-\Psi_t(\gamma)\ge\alpha\}\\
&= \sup_{\gamma>0}\left\{\gamma \middle| \sup_{0\le t\le T}\left(1+\exp\left(\frac{b\pi\left(K-\frac{a}{b}-\exp(-bt)\left(Y_0-\frac{a}{b}\right)+\gamma\exp(rt)\right)}{\sqrt{3}\sigma(1-\exp(-bt))}\right)\right)^{-1}\ge\alpha\right\}.
\end{aligned}$$

Thus theorem 3.6 is verified.

3.2.2 American put option pricing

Assume that an American put option has a strike price K and an expiration time T. If Y_t is the price of the underlying stock at some time t, then the payoff from selling an American put option is

$$\sup_{0\le t\le T}(K-Y_t)^+. \tag{3.30}$$

In consideration of the time value of money resulted from the bond, the present value of this payoff is

$$\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+. \tag{3.31}$$

Definition 3.4 . *Assume an American put option has a strike price K and an expiration time T. Then for given $\alpha\in(0,1]$, this option has price*

$$f_p=\left[\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+\right]_{\sup}(\alpha). \tag{3.32}$$

Theorem 3.7 *Assume an American put option for the uncertain stock model (3.1) has a strike price K and an expiration time T. Then for given $\alpha\in(0,1]$, the American put option price is*

$$f_p=\sup_{\gamma>0}\left\{\gamma|\sup_{0\le t\le T}\left(1+\exp\left(\frac{\pi\left(et+\ln Y_0-\ln(K-\gamma\exp(rt))\right)}{\sqrt{3}\sigma t}\right)\right)^{-1}\ge\alpha\right\}. \tag{3.33}$$

Proof. Since $\exp(-rt)(K-Y_t)^+$ is a decreasing function of Y_t, by (3.25) and Theorem 2.1, we know that

$$\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+$$

has an uncertainty distribution

$$\Psi_t(x)=1-\sup_{0\le t\le T}\left(1+\exp\left(\frac{\pi\left(et+\ln Y_0-\ln(K-x\exp(rt))\right)}{\sqrt{3}\sigma t}\right)\right)^{-1}. \tag{3.34}$$

Since $\sup\limits_{0\le t\le T}\exp(-rt)(K-Y_t)^+\ge 0$, It follows from (3.32), (3.34) and Definition 2.2 that

$$\begin{aligned}
f_p &= \sup_{\gamma>0}\left\{\gamma|\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+\ge\gamma\right\}\ge\alpha\right\}\\
&= \sup_{\gamma>0}\left\{\gamma|1-\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+\le\gamma\right\}\ge\alpha\right\}\\
&= \sup_{\gamma>0}\{\gamma|1-\Psi_t(\gamma)\ge\alpha\}\\
&= \sup_{\gamma>0}\left\{\gamma|\sup_{0\le t\le T}\left(1+\exp\left(\frac{\pi\left(et+\ln Y_0-\ln(K-\gamma\exp(rt))\right)}{\sqrt{3}\sigma t}\right)\right)^{-1}\ge\alpha\right\}.
\end{aligned}$$

Thus the proof is finished.

Theorem 3.8 *Assume an American put option for the uncertain stock model (3.2) has a strike price K and an expiration time T. Then for given $\alpha\in(0,1]$, the American put option price is*

$$f_p=\sup_{\gamma>0}\left\{\gamma|\sup_{0\le t\le T}\left(1+\exp\left(\frac{b\pi\left(\frac{a}{b}+\exp(-bt)\left(Y_0-\frac{a}{b}\right)-K+\gamma\exp(rt)\right)}{\sqrt{3}\sigma(1-\exp(-bt))}\right)\right)^{-1}\ge\alpha\right\} \tag{3.35}$$

Proof. Noting that $\exp(-rt)(K-Y_t)^+$ is a decreasing function of Y_t, it follows from (3.11), (3.28) and Theorem 2.1 that

$$\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+$$

has an uncertainty distribution

$$\Psi_t(x)=1-\sup_{0\le t\le T}\left(1+\exp\left(\frac{b\pi\left(\frac{a}{b}+\exp(-bt)\left(Y_0-\frac{a}{b}\right)-K+x\exp(rt)\right)}{\sqrt{3}\sigma(1-\exp(-bt))}\right)\right)^{-1} \tag{3.36}$$

In the light of (3.32), (3.36) and Definition 2.2, we have

$$\begin{aligned}
f_p &= \sup_{\gamma>0}\left\{\gamma|\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+\ge\gamma\right\}\ge\alpha\right\}\\
&= \sup_{\gamma>0}\left\{\gamma|1-\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+\le\gamma\right\}\ge\alpha\right\}\\
&= \sup_{\gamma>0}\{\gamma|1-\Psi_t(\gamma)\ge\alpha\}\\
&= \sup_{\gamma>0}\left\{\gamma|\sup_{0\le t\le T}\left(1+\exp\left(\frac{b\pi\left(\frac{a}{b}+\exp(-bt)\left(Y_0-\frac{a}{b}\right)-K+\gamma\exp(rt)\right)}{\sqrt{3}\sigma(1-\exp(-bt))}\right)\right)^{-1}\ge\alpha\right\}.
\end{aligned}$$

The theorem 3.8 is proved.

4 Numerical experiments

Finally, we give some numerical examples to illustrate the effectiveness of the obtained results. Suppose that the stock price follows uncertain stock model (3.1). The riskless interest rate $r = 0.02$. The initial stock price is $Y_0 = 6$, the strike price $K = 7$ and the expiration time $T = 0.25$. For European call option with the parameters $e = 0.9, \sigma = 0.2$ and put option with the parameters $e = 0.1, \sigma = 0.4$, American call option with the parameters $e = 0.9, \sigma = 0.1$ and put option with the parameters $e = 0.1, \sigma = 0.2$, we can calculate the price of European call and put option by employing the pricing formulas (3.4), (3.16) and the price of American call and put option by employing the pricing formulas(3.23) and (3.33), respectively. The results is shown in table 1.

Table 1 The price of the option with different confidence level α

parameters α	European option		American option	
	call	put	call	put
$\alpha = 0.1$	0.9782	1.5422	0.7413	1.2036
$\alpha = 0.2$	0.8026	1.2943	0.6556	1.0734
$\alpha = 0.3$	0.6881	1.1232	0.5992	0.9852
$\alpha = 0.4$	0.5954	0.9792	0.5533	0.9119
$\alpha = 0.5$	0.5114	0.8439	0.5114	0.8439
$\alpha = 0.6$	0.4283	0.7055	0.4697	0.7751
$\alpha = 0.7$	0.3388	0.5511	0.4246	0.6992
$\alpha = 0.8$	0.2310	0.3577	0.3699	0.6054
$\alpha = 0.9$	0.0720	0.0556	0.2883	0.4617

Suppose that the stock price follows uncertain stock model (3.2). The values of r, Y_0, K and T are the same as those in model (3.1) above. For European call option with the parameters $a = 5, b = 0.01, \sigma = 0.6$ and put option with the parameters $a = 1, b = 0.2, \sigma = 0.7$, American call option with the parameters $a = 6, b = 0.08, \sigma = 0.7$ and put option with the parameters $a = 0.8, b = 0.1, \sigma = 0.8$, we can calculate the price of European call and put option by employing the pricing formulas (3.10) and (3.19), and the price of American call and put option by employing the pricing formulas (3.27) and (3.35), respectively. The results is shown in table 2.

Table 2 The price of the option with different confidence level α

parameters α	European option		American option	
	call	put	call	put
$\alpha = 0.1$	0.4129	1.2493	0.5733	1.1840
$\alpha = 0.2$	0.3462	1.1734	0.4962	1.0961
$\alpha = 0.3$	0.3019	1.1229	0.4450	1.0377
$\alpha = 0.4$	0.2656	1.0815	0.4030	0.9898
$\alpha = 0.5$	0.2323	1.0435	0.3645	0.9459
$\alpha = 0.6$	0.1990	1.0056	0.3259	0.9019
$\alpha = 0.7$	0.1627	0.9642	0.2839	0.8541
$\alpha = 0.8$	0.1184	0.9137	0.2327	0.7957
$\alpha = 0.9$	0.0517	0.8378	0.1556	0.7078

From Table 1 and Table 2, we know that the prices of European and American options decrease as the confidence levels $\alpha(\alpha \in (0, 1])$ increase. That's because as the confidence level increases, investors believe that the return of the option will decrease, so they is willing to pay less to buy or sell the underlying asset for exercising the option.

5 Conclusion

This paper explored the option pricing problems under the framework of uncertainty theory. The pricing formulas of the European and American options are obtained by applying the method to calculate the optimistic value of uncertain returns of options. In order to illustrate the effectiveness of the obtained results, several numerical examples are discussed. In future work, we will further study the optimistic value pricing problems of some exotic options.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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