An efficient block Gauss-Seidel overrelaxation iteration method for the space fractional coupled nonlinear Schrödinger equations

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An efficient block Gauss-Seidel overrelaxation iteration method for the space fractional coupled nonlinear Schrödinger equations

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Abstract. A linearly implicit difference scheme for the space fractional coupled nonlinear Schrödinger (CNLS) equation is proposed. The resulting coefficient matrix of the discretized linear system consists of the sum of a complex scaled identity and a symmetric positive definite, diagonal-plus-Toeplitz, matrix. An efficient block Gauss-Seidel overrelaxation method (BGSOR) method has been established to solve the discretized linear system. It is worth noting that the proposed method solves the linear equations without the need to any system solution, which is beneficial for reducing computational cost and memory requirements. Theoretical analysis implies that the BGSOR method is convergent under a suitable condition. Moreover, an appropriate approach to compute the optimal parameter in the BGSOR method is exploited. Finally, the theoretical analysis is validated by some numerical experiments.

Keywords: The space fractional Schrödinger equations, Toeplitz matrix, Block Gauss-Seidel overrelaxation method, Convergence analysis.
AMS Subject Classification: 65F10, 81Q05, 81V99.

1 Introduction

The Schrödinger equation is one of the most important equation in the science of submicroscopic phenomena, known as quantum mechanics. It can be arisen from the path integral over Brownian paths. In [6], the path integral method to the Lévy-α process is generalized and the space fractional equations are derived.

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Consider the space fractional coupled nonlinear Schrödinger equations

$$\begin{cases} \imath u_t + \gamma(-\Delta)^{\frac{\alpha}{2}} u + \rho\left(|u|^2 + \beta|v|^2\right) u = 0, \\ \imath v_t + \gamma(-\Delta)^{\frac{\alpha}{2}} v + \rho\left(|v|^2 + \beta|u|^2\right) v = 0, \end{cases} \qquad a \le x \le b, 0 < t \le T, \tag{1}$$

with the initial boundary value conditions

$$\begin{aligned} u(x,0) &= u_0(x), & v(x,0) &= v_0(x), & a \le x \le b, \\ u(a,t) &= u(b,t) = 0, & v(a,t) &= v(b,t) = 0, & 0 \le t \le T, \end{aligned}$$

where $\imath$ is the imaginary unit, $1 < \alpha < 2$ and the parameters $\gamma > 0$, $\rho > 0$, $\beta \ge 0$ are some constants. In [5], the fractional Laplacian were characterized as

$$(-\Delta)^{\frac{\alpha}{2}} u(x,t) = \mathcal{F}^{-1}\left(|\xi|^{\alpha} \mathcal{F}(u(x,t))\right),$$

in which $\mathcal{F}$ is the Fourier transform acting on the spatial variable x. Assuming that

$$-\infty D_x^{\alpha} u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{x} (x-\tau)^{n-1-\alpha} u(\tau,t) d\tau,$$

$${}_x D_{+\infty}^{\alpha} u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{x}^{+\infty} (\tau-x)^{n-1-\alpha} u(\tau,t) d\tau,$$

are the left and right Riemann-Liouville fractional derivatives, respectively, the Riesz fractional derivative can be considered as

$$\frac{\partial^{\alpha}}{\partial |x|^{\alpha}} u(x,t) = -(-\Delta)^{\frac{\alpha}{2}} u(x,t) = -\frac{1}{2\cos\frac{\pi\alpha}{2}} \left[-\infty D_x^{\alpha} u(x,t) + {}_x D_{+\infty}^{\alpha} u(x,t)\right].$$

In general, analyzing and understanding the behavior of the exact solutions of the space fractional coupled nonlinear Schrödinger equations is so challenging. During recent years, some numerical methods have been proposed to solve the CNLS equations. The difference method [11–13], the Crank-Nickelson scheme [1], and the collocation method [2] have been presented to solve the CNLS equations.

The discretization of the CNLS equations leads to the solution of the complex symmetric linear systems. The coefficient matrix is equal to the sum of the complex identity scaled matrix and the symmetric positive definite, diagonal-plus-Toeplitz, matrix. Recently, Dai et al. [4] constructed a proper two-by-two linear system and employed the block Gauss-Seidel (BGS) iteration method to solve the obtained linear systems. Then they analyzed the convergence of the BGS method for the corresponding two-by-two linear system. In this work, we establish an efficient block Gauss-Seidel overrelaxation (BGSOR) method for solving the two-by-two linear system that arises from the discretization of CNLS equations. Notably, the new method allows the corresponding systems to be solved without the need to compute the inverse of the coefficient matrices. Moreover, it should be pointed out that the

block Gauss-Seidel method can be regarded as a special case of the new method when the relaxation parameter is set to be one.

The organization of this work is as follows. In Section 2, we will study the model problem and bring a linearly implicit difference scheme. Application, convergence theory, and finding the optimal parameter for the BGSOR method are proposed in Section 3. Section 4 is devoted to giving some numerical experiments. Finally, we made some conclusions in Section 5.

2 Model problem and a linearly implicit difference scheme

The domain $\Omega = (a, b) \times (0, T)$ is divided into a uniform grid of mesh points (x_j, t_k), where

$$x_j = jh, \quad h = \frac{b-a}{M+1}, \quad 0 \le j \le M+1,$$

and

$$t_k = k\tau, \quad \tau = \frac{T}{N}, \quad 0 \le k \le N.$$

At grid points, the values of functions $u(x,t), v(x,t)$ are respectively denoted by $u_j^k = u(x_j, t_k)$, $v_j^k = v(x_j, t_k)$, and U_j^k, V_j^k are the approximate solutions of (1).

The following equation gives a discrete approximation to the $\frac{\partial^\alpha}{\partial|x|^\alpha}u(x,t)$ [10]:

$$\frac{\partial^\alpha}{\partial|x|^\alpha}u(x_j,t_k) = -\frac{\Psi_\alpha}{h^\alpha}\left[\sum_{l=0}^{\infty}\tilde{w}_k^{(\alpha)}u(x_{j-l+1},t_k) + \sum_{l=0}^{\infty}\tilde{w}_k^{(\alpha)}u(x_{j+l-1},t_k)\right] + \mathcal{O}(h^2), \tag{2}$$

where $\Psi_\alpha = \frac{1}{2\cos(\frac{\pi\alpha}{2})}$ and $\{\tilde{w}_k^\alpha\}$ is defined as follows:

$$\tilde{w}_0^{(\alpha)} = \frac{\alpha}{2}g_0^{(\alpha)}, \quad \tilde{w}_l^{(\alpha)} = \frac{\alpha}{2}g_l^{(\alpha)} + \left(1-\frac{\alpha}{2}\right)g_{l-1}^{(\alpha)}, \quad l \ge 1,$$
$$g_0^{(\alpha)} = 1, \quad g_l^{(\alpha)} = \left(1-\frac{\alpha+1}{l}\right)g_{l-1}^{(\alpha)}, \quad l = 1, 2, \ldots.$$

Ortigueira [7] proposed the following fractional central difference operator:

$$\Delta_h^\alpha u(x) = \sum_{l=-\infty}^{\infty}\hat{g}_l^{(\alpha)}u(x-lh),$$

where

$$\hat{g}_l^{(\alpha)} = \frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-l+1)\Gamma(\frac{\alpha}{2}+l+1)}.$$

As stated in [7], the coefficient $\{\hat{g}_l^{(\alpha)}\}$ satisfies

$$\left|2\sin\left(\frac{x}{2}\right)\right|^2 = \sum_{l=-\infty}^{\infty}\hat{g}_l^{(\alpha)}e^{\imath lx}, \quad x \in \mathbb{R}.$$

When $\alpha > -1$, the recursive relations for $\{\hat{g}_l^{(\alpha)}\}$ are as follows:

$$\hat{g}_0^{(\alpha)} = \frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)},\ \hat{g}_l^{(\alpha)} = \Big(1 - \frac{\alpha+1}{\alpha/2+l}\Big)\hat{g}_{l-1}^{(\alpha)},\ \ l \geq 1;$$
$$\hat{g}_{-l}^{(\alpha)} = \hat{g}_l^{(\alpha)},\ \ l \geq 1.$$

Lemma 1. *[10] Assume that $u(x) \in \mathbb{C}^5(\mathbb{R})$ and its all derivatives of order up to 5 belong to $L^1(\mathbb{R})$. Then, it holds*

$$-\frac{\Delta_h^\alpha u(x)}{h^\alpha} = \frac{\partial^\alpha u(x)}{\partial|x|^\alpha} + \mathcal{O}(h^2). \tag{3}$$

From Lemma 1, it follows that

$$(-\Delta)^{\frac{\alpha}{2}} u(x_j, t_k) = \frac{\Delta_h^\alpha u(x)}{h^\alpha} + \mathcal{O}(h^2) = \frac{1}{h^\alpha}\sum_{l=1}^{M} \hat{g}_{j-l}^{(\alpha)} u(x_j, t_k) + \mathcal{O}(h^2).$$

Now, we consider the following numerical scheme for solving (1) [11]:

$$\begin{aligned}
&\imath\frac{U_j^{k+1} - U_j^{k-1}}{2\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{M}\hat{g}_{j-l}^{(\alpha)}\Big(\frac{U_l^{k+1} + U_l^{k-1}}{2}\Big) + \rho(|U_j^k|^2 + \beta|V_j^k|^2) + \frac{U_l^{k+1} + U_l^{k-1}}{2} = 0,\\
&\imath\frac{V_j^{k+1} - V_j^{k-1}}{2\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{M}\hat{g}_{j-l}^{(\alpha)}\Big(\frac{V_l^{k+1} + V_l^{k-1}}{2}\Big) + \rho(|V_j^k|^2 + \beta|U_j^k|^2) + \frac{V_l^{k+1} + V_l^{k-1}}{2} = 0,
\end{aligned} \tag{4}$$

where $1 \leq j \leq M$, $1 \leq k \leq N-1$. Another scheme should be provided for the numerical solution at $k = 1$. We consider the following scheme for this purpose (see [3])

$$\begin{aligned}
&\imath\frac{U_j^1 - U_j^0}{\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{M}\hat{g}_{j-l}^{(\alpha)}U_l^{(1)} + \rho\big(|U_j^0|^2 + \beta|V_j^0|^2\big)U_j^1 = 0,\\
&\imath\frac{V_j^1 - V_j^0}{\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{M}\hat{g}_{j-l}^{(\alpha)}V_l^1 + \rho\big(|V_j^0|^2 + \beta|U_j^0|^2\big)V_j^{(1)} = 0,\\
&\imath\frac{U_j^1 - U_j^0}{\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{M}\hat{g}_{j-l}^{(\alpha)}\Big(\frac{U_l^1 + U_l^0}{2}\Big) + \rho\Big(\frac{3}{2}|U_j^1|^2 - \frac{1}{2}|U_j^0|^2 + \beta\Big(\frac{3}{2}|V_j^{(1)}|^2 - \frac{1}{2}|V_j^0|^2\Big)\Big)\frac{U_j^1 + U_j^0}{2} = 0,\\
&\imath\frac{V_j^1 - V_j^0}{\tau} + \frac{\gamma}{h^\alpha}\sum_{l=1}^{M}\hat{g}_{j-l}^{(\alpha)}\Big(\frac{V_l^1 + V_l^0}{2}\Big) + \rho\Big(\frac{3}{2}|V_j^1|^2 - \frac{1}{2}|V_j^0|^2 + \beta\Big(\frac{3}{2}|U_j^1|^2 - \frac{1}{2}|U_j^0|^2\Big)\Big)\frac{V_j^1 + V_j^0}{2} = 0.
\end{aligned}$$

The first and the second difference equations in (4) have the same structure. Set

$$\begin{aligned}
U^{k+1} &= [U_1^{k+1}, \ldots, U_M^{k+1}]^T, \quad b^{k+1} = [b_1^{k+1}, \ldots, b_M^{k+1}]^T,\\
\mu &= \frac{\gamma\tau}{h^\alpha}, \quad d_j^{k+1} = \rho\tau(|U_j^k|^2 + \beta|V_j^k|^2), \quad D^{k+1} = diag(d_1^{k+1}, \ldots, d_M^{k+1}).
\end{aligned}$$

So, at each time step, we need to solve the following systems of linear equations:

$$\begin{aligned} A^{k+1}U^{k+1} &= b^{k+1}, \ \ 1 \le k \le N-1, \\ B^{k+1}V^{k+1} &= c^{k+1}, \ \ 1 \le k \le N-1, \end{aligned} \tag{5}$$

where $A^{k+1} = \imath I + D^{k+1} + T$ and b^{k+1} is as follows:

$$b^{k+1} = \begin{pmatrix} \imath U_1^{k-1} - \mu \sum_{l=1}^{M} \hat{g}_{1-l}^{(\alpha)} U_l^{k-1} - d_1^{k+1} U_1^{k-1} \\ \imath U_2^{k-1} - \mu \sum_{l=1}^{M} \hat{g}_{2-l}^{(\alpha)} U_l^{k-1} - d_2^{k+1} U_2^{k-1} \\ \vdots \\ \imath U_{M-1}^{k-1} - \mu \sum_{l=1}^{M} \hat{g}_{M-1-l}^{(\alpha)} U_l^{k-1} - d_{M-1}^{k+1} U_{M-1}^{k-1} \\ \imath U_M^{k-1} - \mu \sum_{l=1}^{M} \hat{g}_{M-l}^{(\alpha)} U_l^{k-1} - d_M^{k+1} U_M^{k-1} \end{pmatrix}.$$

T is the Toeplitz matrix, which has the following structure:

$$T = \mu \begin{pmatrix} \hat{g}_0^{(\alpha)} & \hat{g}_{-1}^{(\alpha)} & \cdots & \hat{g}_{2-M}^{(\alpha)} & \hat{g}_{1-M}^{(\alpha)} \\ \hat{g}_1^{(\alpha)} & \hat{g}_0^{(\alpha)} & \cdots & \hat{g}_{3-M}^{(\alpha)} & \hat{g}_{2-M}^{(\alpha)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{g}_{M-2}^{(\alpha)} & \hat{g}_{M-3}^{(\alpha)} & \cdots & \hat{g}_0^{(\alpha)} & \hat{g}_{-1}^{(\alpha)} \\ \hat{g}_{M-1}^{(\alpha)} & \hat{g}_{M-2}^{(\alpha)} & \cdots & \hat{g}_1^{(\alpha)} & \hat{g}_0^{(\alpha)} \end{pmatrix}. \tag{6}$$

Also, it should be noted that B^{k+1} and c^{k+1} can be obtained by changing the roles of U and V in A^{k+1} and b^{k+1}.

3 The BGSOR iteration method

To establish the BGSOR iteration method, we need to give some preliminaries. Let us first consider the iterative solution of the linear equation

$$AU = b, \quad A \in \mathbb{C}^{M\times M} \text{ nonsingular}, \quad \text{and } U, b \in \mathbb{C}^M, \tag{7}$$

in which A is a complex symmetric matrix as follows

$$A = \imath I + T + D,$$

where $D = \mathrm{diag}(d_1, d_2, \ldots, d_M)$ with $d_i \ge 0$, $i = 1, 2, \ldots, M$, is the diagonal matrix and T is the SPD and Toeplitz matrix designated in (6). Let $U = x + \imath y$ and $b = f + \imath g$ be comlex

vectors, where $y, z, p, q \in \mathbb{R}^M$. So, the system can be rewritten as a particular form, namely,

$$\mathcal{A}\mathbf{x} \equiv \begin{pmatrix} -I & W \\ W & I \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv \mathcal{P}, \tag{8}$$

where $W = D + T$. We are now in a position to design a new method for solving (8).

To introduce the BGSOR iteration method, we consider the following decomposition for the coefficient matrix (8)

$$\mathcal{A} = (\omega\mathcal{D} - \mathcal{E}) - (\mathcal{E}^T - (1-\omega)\mathcal{D}) =: \mathcal{M} - \mathcal{N}, \tag{9}$$

where

$$\mathcal{D} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \quad \text{and} \quad \mathcal{E} = \begin{pmatrix} 0 & 0 \\ -W & 0 \end{pmatrix},$$

and ω is a positive parameter, which is known as the relaxation parameter. Using the decomposition (9), the BGSOR iteration method is stated as

$$\mathcal{M}z^{(k+1)} = \mathcal{N}z^{(k)} + \mathcal{P}, \quad k = 0, 1, 2, \cdots,$$

where $\mathcal{M}$ and $\mathcal{N}$ are defined as (9), and $z^{(k)} = (y^{(k)}; x^{(k)})$. Notice that $y^{(k)}$ and $x^{(k)}$ are two M-vectors that stand for the iterations. Also, $z^{(0)}$ is an arbitrary initial guess. Thereupon, the iterations take the following procedure:

$$\begin{cases} y^{(k+1)} = \dfrac{1}{\omega}\left((\omega-1)y^{(k)} + Wx^{(k)} - f\right), \\ x^{(k+1)} = \dfrac{1}{\omega}\left((\omega-1)x^{(k)} + g - Wy^{(k+1)}\right). \end{cases} \tag{10}$$

As can be seen there is not any system solution in each iteration and only two matrix-vector multiplication are needed. This can be very important because the new scheme requires insignificant computational efforts, just contains the matrix-vector multiplications. Furthermore, if $\omega = 1$, the iteration scheme (10) reduces to

$$\begin{cases} y^{(k+1)} = Wx^{(k)} - f, \\ x^{(k+1)} = g - Wy^{(k+1)}. \end{cases} \tag{11}$$

which is presented in [4] and is known as the BGS iteration method. Therefore, BGS iteration method is a special case of the BGSOR iteration method.

Next, we investigate the convergence of the BGSOR method for solving (8), and then we obtain the optimal value of the relaxation parameter ω. In the following, we recall a result that will be useful later.

Lemma 2. *[14] Consider the quadratic equation $x^2 - bx + c = 0$, where b and c are real numbers. Both roots of the equation are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

Theorem 1. *Suppose that $A = \imath I + D + T \in \mathbb{R}^{M\times M}$ is a matrix where D and T are diagonal and Toeplitz SPD matrices, respectively. The necessary and sufficient condition for convergence of the BGSOR iteration method to the solution of (8) for any initial guess, is*

$$\omega > \frac{1+\mu_{\max}(W)}{2},$$

where $\mu_{\max}(W)$ is the largest eigenvalue of W.

Proof. Assume that λ is an eigenvalue of the iteration matrix $\mathcal{G} = \mathcal{M}^{-1}\mathcal{N}$, and $\mathbf{x} = [\mathbf{u};\mathbf{v}]$ is the corresponding eigenvector. Without loss of generality, let $\lambda \neq 0$. So,

$$(\mathcal{D} - \omega\mathcal{E})^{-1}(\mathcal{E}^T - (1-\alpha\mathcal{D}))\mathbf{x} = \lambda\mathbf{x},$$

equivalently,

$$(1-\omega)u - Wv = -\lambda\omega u, \tag{12}$$
$$(\omega-1)v = \lambda(Wu + \omega v). \tag{13}$$

From Eq. (12) and in view of positive definiteness of W, we can deduce that

$$v = ((\lambda-1)\omega + 1)W^{-1}u. \tag{14}$$

Substituting (14) into (13), gives

$$-\lambda W^2 u = ((\lambda-1)\omega+1)^2 u. \tag{15}$$

this shows that if μ is an eigenvalue of W, then

$$\lambda\mu^2 = -\left((\lambda-1)\,\omega+1\right)^2 \tag{16}$$
$$= -(\lambda^2\omega^2 + 2\omega(1-\omega)\lambda + (\omega-1)^2). \tag{17}$$

From Eq. (17) we get

$$\lambda^2 - \left(\frac{2\omega^2 - 2\omega - \mu^2}{\omega^2}\right)\lambda + (\frac{\omega-1}{\omega})^2 = 0. \tag{18}$$

Now it follows from Lemma 2 that $|\lambda| < 1$ if and only if

$$\begin{cases} |\omega - 1| < \omega, \\ |2\omega^2 - 2\omega - \mu^2| < 2\omega^2 - 2\omega + 1, \end{cases}$$

It is straightforward to see that $|\omega-1| < \omega$ is equivalent to $\omega > \frac{1}{2}$. By some easy manipulations we can observe, whenever

$$(2\omega-1)^2 > \mu^2, \tag{19}$$

the second inequality of (19) holds. The inequality (19) is ensured, if

$$|2\omega - 1| > \mu \quad \text{or} \quad |2\omega - 1| < -\mu$$

equivalently,

$$\omega < \frac{1-\mu}{2} \quad \text{or} \quad \omega > \frac{1+\mu}{2} \tag{20}$$

Evidently, the first inequality of (20) can not be true. On the other hand, holding the second inequality of (20) ensures $\omega > \frac{1}{2}$, and then it completes the proof. □

In the following, we would like to find the optimal value of the relaxation parameter ω, denoted by ω^*. To do so, ω^* should be computed to minimize the spectral radius of the iteration matrix of the BGSOR method, i.e.,

$$\rho\left(\mathcal{G}_{\omega^*}\right) = \arg \min_{\omega > \frac{1+\mu_{\max}(W)}{2}} \rho\left(\mathcal{G}_{\omega}\right).$$

To compute the optimal value of w we state and prove the next theorem.

Theorem 2. *Suppose that the assumptions of Theorem 1 are satisfied. Then the optimal value of the relaxation parameter and the corresponding optimal convergence factor in the BGSOR iteration method are as follows*

$$\omega^* = \frac{1}{2}\left(1 + \sqrt{1+\rho^2(W)}\right), \tag{21}$$

and

$$\rho\left(\mathcal{G}_{\omega^*}\right) = 1 - \frac{1}{\omega^*} = \left(\frac{\rho(W)}{1+\sqrt{1+\rho^2(W)}}\right)^2.$$

Proof. From Eq. (16), it can be observed that if λ is an eigenvalue of the iteration matrix $\mathcal{G}_\omega$, then $\lambda < 0$ or $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Besides, there exists an eigenvalue μ of W such that Eq. (18) holds true. The discriminant of this quadratic equation is

$$\Delta = \left(\frac{2\omega^2 - 2\omega - \mu^2}{\omega^2}\right)^2 - 4\left(\frac{\omega-1}{\omega}\right)^2,$$

and the roots of (17) are as following

$$\lambda_{1,2}(\omega) = \frac{-2\omega^2 + 2\omega + \mu^2}{2\omega^2} \pm \frac{\sqrt{\Delta}}{2}.$$

From Eq. (15) we get

$$(\lambda - 1)\omega + 1 = \pm\mu\sqrt{-\lambda}. \tag{22}$$

Set

$$\begin{aligned} f_\omega(\lambda) &= (\lambda-1)\omega+1 = \omega\lambda+1-\omega, \\ g(\lambda) &= \pm\mu\sqrt{-\lambda}. \end{aligned}$$

Clearly, the function f_ω passes through the point $(1,1)$, i.e., $f_\omega(1)=1$ and the slope of $f_\omega(\lambda)$ is ω. Fig. 1 displays the points of intersections of the functions $f_\omega(\lambda)$ and $g(\lambda)$ for an arbitrary value of ω. This figure shows that by increasing ω, the absolute values of abscissae of the points of intersection of the functions $f_\omega(\lambda)$ and $g(\lambda)$, i.e., λ_1 and λ_2, increase, while $f_\omega(\lambda)$ gets tangent to $g(\lambda)$. In the tangent case, we have $\lambda_1=\lambda_2$ and it indicates that $\Delta=0$ From $\Delta=0$, it is straightforward to verify that $\mu=0$ or $4\omega^2-4\omega-\mu^2=0$. The case $\mu=0$ is impossible, because of the positive definiteness of W. Thus, $4\omega^2-4\omega-\mu^2=0$. This quadratic equation has two roots, as follows:

$$\omega_\pm = \frac{1}{2}\left(1\pm\sqrt{1+\mu^2}\right).$$

Due to the condition $\omega > \frac{1+\mu_{\max}(W)}{2}$, ω_- is not acceptable. So, we consider

$$\omega_+ = \frac{1}{2}\left(1+\sqrt{1+\mu^2}\right),$$

and in this case we have

$$\lambda_1=\lambda_2=\lambda_+ = 1-\frac{1}{\omega_+}.$$

Now suppose that $\omega>\omega_+$. In this case, the roots of the quadratic equation (17) are complex and conjugate, which are as follows

$$\lambda_{1,2}(\omega) = \frac{(2\omega^2-2\omega-\mu^2)}{2\omega^2} \pm \imath\frac{\sqrt{\Delta'}}{2},$$

where

$$\Delta' = 4\left(\frac{\omega-1}{\omega}\right)^2 - \left(\frac{2\omega^2-2\omega-\mu^2}{\omega^2}\right)^2.$$

Then

$$|\lambda_{1,2}| = 1-\frac{1}{\omega}.$$

By recalling that $\omega>\omega_+$ and having in mind that $w_+>1$, we have

$$1-\frac{1}{\omega_+} < 1-\frac{1}{\omega},$$

and this shows that ω_+ is the best choice for ω. On the other hand, the curve $g(\lambda) = \pm\rho(W)\sqrt{-\lambda}$ serves an upper bound for each curve as $\pm\mu\sqrt{-\lambda}$, where $0\le\lambda\le\rho(W)$. Summarizing the above results, we see that

$$\rho\left(\mathcal{G}_{\omega^*}\right) = \min_{\omega}\max_{\omega>\frac{1+\mu_{\max}}{2}} |1-\frac{1}{\omega}| = 1-\frac{1}{\omega^*} = \left(\frac{\rho(W)}{1+\sqrt{1+\rho^2(W)}}\right)^2,$$

which ω^* was considered in (21). □

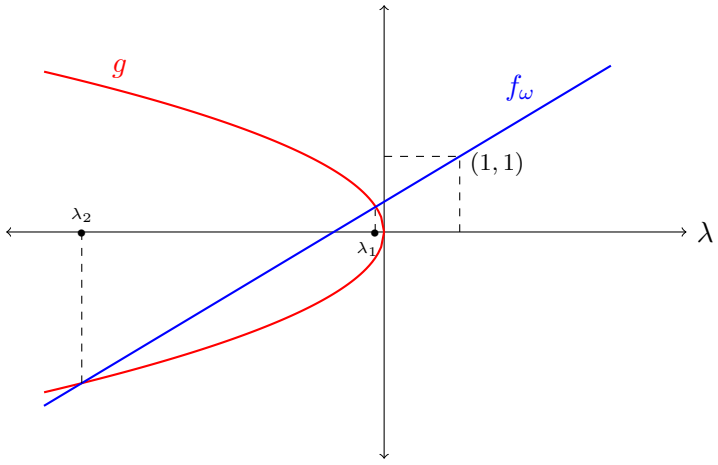

Figure 1: The graph of the functions $f_\omega(\lambda)$ and $g(\lambda)$.

Remark 1. *In Theorem 2, for computing ω^* we need to compute $\rho(W)$. One may use a few iterations of the power method to compute $\lambda_{\max}(W)$. On the other hand, because of positive definiteness of W, we have*

$$\rho(W) = \lambda_{\max}(W) = \|W\|_2.$$

So, we can compute $\|W\|_2$ instead of $\rho(W)$. In practice the `normest` *command of* MATLAB *can be used to compute an estimation of $\|W\|_2$.*

4 Numerical experiments

This section is devoted to give some numerical experiments to test the effectiveness of the BGSOR iteration method when it is employed to solve linear systems (8). Numerical results of the proposed method are compared with those of the GMRES [8,9] and the BGS methods. In all the test problems, we use the restart version of GMRES with a restarting number 10. The initial guess is taken to be a random vector and the iterations are stopped as soon as

$$Res = \frac{\|r_k\|_2}{\|r_0\|_2} \leq 10^{-9},$$

where $r_k = \mathcal{P} - \mathcal{A}z^{(k)}$ is the residual at the k-th iteration or if the maximum number of iterations $maxit = 1000$ is exceeded. In the tables, "IT" and "CPU" refer to the total number of iterations and elapsed CPU time in second for the convergence. We comment that five runs were performed for each test and then the average of CPU-times and iterations are reported (The average of the iteration numbers were rounded). For the BGSOR method the optimal parameter is computed according to the rule (21). The numerical results were

Table 1: The optimal parameters ω^* for BGSOR method with $\alpha = 1.3$ and $N = 4M$ at $t = 2$ for Example 1.

M	800	1600	3200	6400
ω^*	1.002	1.004	1.006	1.009

Table 2: Numerical results with $\alpha = 1.3$ and $N = 4M$ at $t = 2$ for Example 1.

Method	M	800	1600	3200	6400
BGSOR	IT	5	5	5	5
	CPU	0.016	0.051	0.171	0.955
BGS	IT	5	6	6	7
	CPU	0.018	0.072	0.228	1.705
GMRES(10)	IT	6	7	7	7
	CPU	0.080	0.112	0.352	3.610

carried out under MATLAB-R2017 on a Laptop with an intel (R) Core(TM) i5-8265U CPU @ 1.60 GHz 8 GB and the windows 10 operating system.

Example 1. Let $\beta = 0$. Then the system (1) is decoupled and becomes

$$\imath u_t + (-\Delta)^{\frac{\alpha}{2}} u + 2|u|^2 u = 0,$$

subject to the initial value

$$u(x, 0) = \mathrm{sech}(x) \cdot \exp(2\imath x).$$

In this example, we truncate the original problem in $[-20, 20]$ and set $u(-20, t) = u(20, t) = 0$. For this problem, we choose the parameters $\gamma = 1.3$ and $\rho = 1.2$.

We set $M = 800, 1600, 3200, 6400$ and examine two values of α, $\alpha = 1.3, 1.6$. When $\alpha = 1.3$, we set $N = 4M$, otherwise we choose $N = 6M$. The optimal values of the relaxation parameter in the BGSOR method for $\alpha = 1.3$ are given in Table 1 and the ones for $\alpha = 1.6$ are given in Table 3.

In Tables 2 and 4, we have listed the numerical results at $t = 2$. From these tables, we observe that the BGSOR method is superior to the examined methods in terms of both the iterations and the elapsed CPU times.

Example 2. For the coupled system with $\beta \neq 0$,

$$\begin{cases} \imath u_t + (-\Delta)^{\frac{\alpha}{2}} u + 2\left(|u|^2 + |v|^2\right) u = 0, \\ \imath v_t + (-\Delta)^{\frac{\alpha}{2}} v + 2\left(|v|^2 + |u|^2\right) v = 0, \end{cases} \quad -20 \leq x \leq 20, 0 < t \leq 2, \tag{23}$$

we take the initial conditions

$$\begin{cases} u(x, 0) = \mathrm{sech}(x + D_0) \cdot \exp(\imath V_0 x), \quad v(x, 0) = \mathrm{sech}(x - D_0) \cdot \exp(-\imath V_0 x), \\ u(-20, 0) = u(20, 0) = 0, \quad v(-20, 0) = v(20, 0) = 0. \end{cases} \tag{24}$$

Table 3: The optimal parameters ω^* for the BGSOR method with $\alpha = 1.6$ and $N = 6M$ at $t = 2$ for Example 1.

M	800	1600	3200	6400
ω^*	1.010	1.022	1.050	1.108

Table 4: Numerical results with $\alpha = 1.6$ and $N = 6M$ at $t = 2$ for Example 1.

Method	M	800	1600	3200	6400
BGSOR	IT	6	7	8	10
	CPU	0.018	0.068	0.311	2.015
BGS	IT	7	9	14	28
	CPU	0.022	0.093	0.571	4.462
GMRES(10)	IT	8	9	10	13
	CPU	0.112	0.185	0.235	6.941

Table 5: The optimal parameters ω^* of A and B for the BGSOR method with $\alpha = 1.3$ and $N = 4M$ at $t = 2$ for Example 2.

M	800	1600	3200	6400
$\omega^*(A)$	1.002	1.004	1.006	1.008
$\omega^*(B)$	1.002	1.004	1.006	1.008

Table 6: Numerical results with $\alpha = 1.3$ and $N = 4M$ at $t = 2$ for Example 2.

Method	M	800		1600		3200		6400	
		A	B	A	B	A	B	A	B
BGSOR	IT	5	5	5	5	5	5	5	5
	CPU	0.013	0.010	0.052	0.023	0.173	0.145	0.938	0.841
BGS	IT	5	5	6	6	6	6	7	7
	CPU	0.020	0.014	0.069	0.064	0.213	0.228	1.641	1.145
GMRES(10)	IT	6	6	7	7	7	7	8	8
	CPU	0.064	0.017	0.093	0.049	0.155	0.139	2.812	1.377

In this case, we choose the parameters $D_0 = 1$, $V_0 = 2$, $\gamma = 1.4$, and $\rho = 1.2$.

The discretization of the coupled system of (23), leads to the solution of the linear systems of equations of the form (5). We assume that these coefficient matrices are A and B. These matrices have the same structure. Tables 5 and 7 show the optimal values of the relaxation parameter of A and B in the BGSOR method for different values of α and M.

In Tables 6 and 8, we report the results for the BGSOR, BGS, and GMRES(10) iterative methods at $t = 2$. These results clearly show that the BGSOR method leads to a faster overall convergence time than the other examined methods. Besides, the BGSOR method gets less iteration numbers.

Table 7: The optimal parameters ω^* of A and B for the BGSOR method with $\alpha = 1.6$ and $N = 6M$ at $t = 2$ for Example 2.

M	800	1600	3200	6400
$\omega^*(A)$	1.010	1.022	1.050	1.122
$\omega^*(B)$	1.010	1.022	1.050	1.122

Table 8: Numerical results with $\alpha = 1.6$ and $N = 6M$ at $t = 2$ for Example 2.

Method	M	800		1600		3200		6400	
		A	B	A	B	A	B	A	B
BGSOR	IT	6	6	7	7	9	9	10	10
	CPU	0.021	0.017	0.071	0.069	0.346	0.248	1.941	2.003
BGS	IT	7	7	10	10	15	15	35	35
	CPU	0.025	0.020	0.106	0.112	0.607	0.592	6.832	5.483
GMRES(10)	IT	8	8	9	9	11	11	13	13
	CPU	0.088	0.061	0.093	0.082	0.448	0.412	3.376	3.251

5 Conclusion

In this paper, we have presented the block Gauss-Seidel overrelaxation scheme for solving the complex symmetric linear systems arising from the discretization of the space fractional coupled nonlinear Schrödinger equation. We have proposed the convergence theory of the method and we have shown that the method is convergent under a suitable condition. The optimal value of the relaxation parameter and the rate of convergence factor for the BGSOR method were also provided. Our results have verified that the BGSOR method performs better than some existing methods.

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