Study on generalized variable coefficient fifth-order KdV equation based on higher order dispersion term

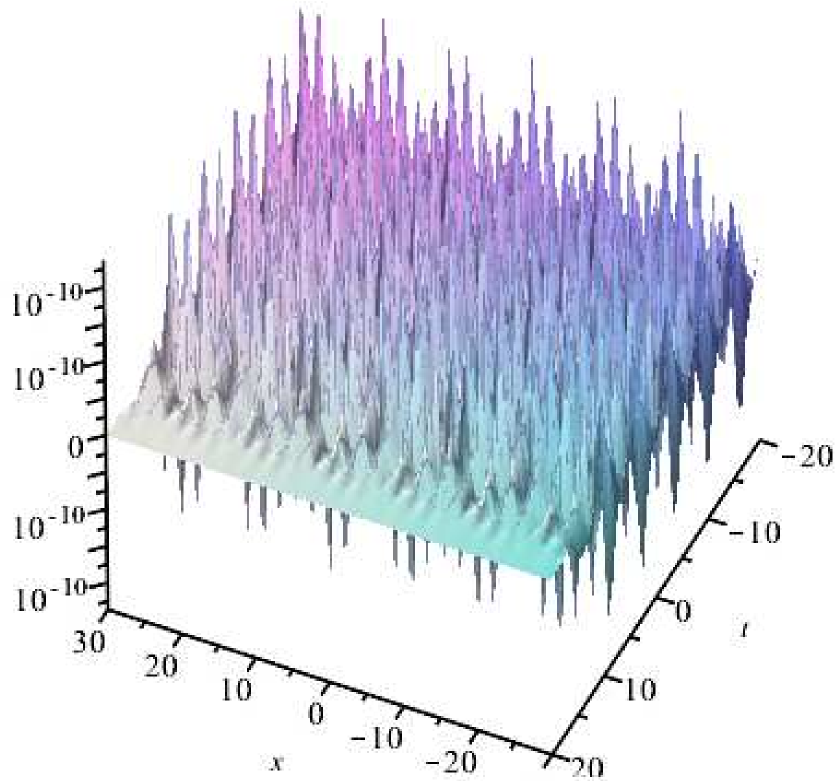
Zhen Zhao[1] and Jing Pang[1]

[1]Inner Mongolia University of Technology

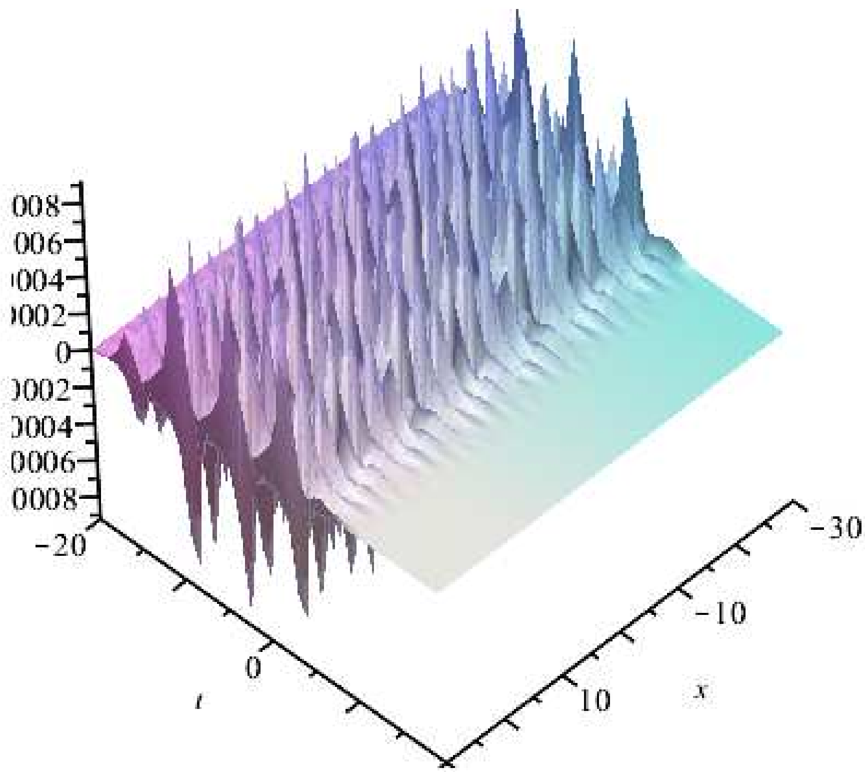
April 16, 2024

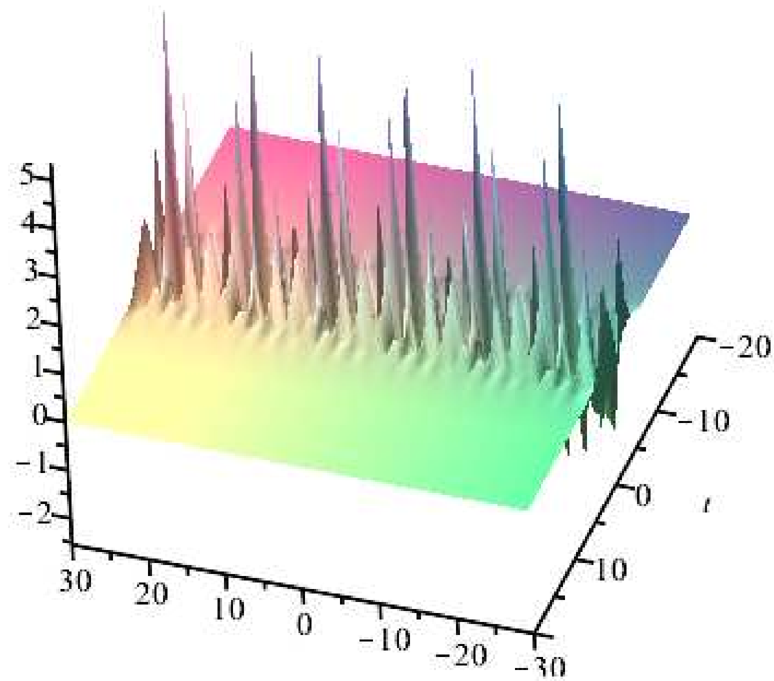
Abstract

Nonlinear partial differential equations with higher order dispersion terms play an important role in dynamics research. In this paper, the fifth order KdV equation with high order dispersion term is studied and discussed. Firstly, the bilinear form of the fifth order KdV equation with high order dispersion term is derived by Hirota bilinear form. Then, the combined test function of the positive quartic function, quadratic function, exponential function and the interaction solution of the hyperbolic function of the fifth order KdV equation with variable coefficients is constructed, and the resonance multi-soliton test function of the equation is constructed by using the linear superposition principle.By means of mathematical symbol calculation, the interaction solution between high-order Lump solution and periodic cross kink solution of the fifth order KdV equation with variable coefficients and its resonance multi-solitons are solved.And by observing its corresponding graph analysis of its physical phenomenon.

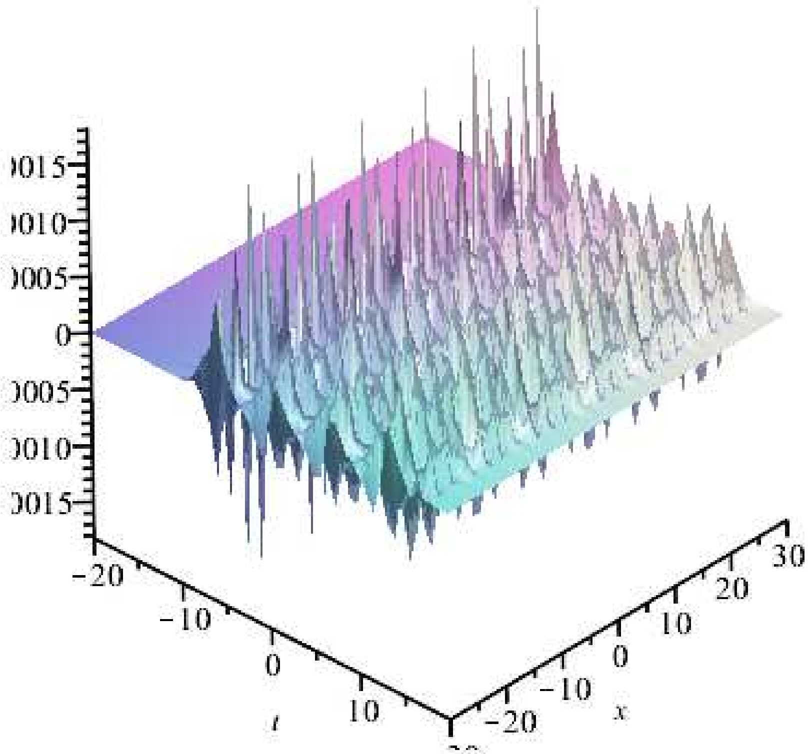


10^{-10}
10^{-10}
0
10^{-10}
10^{-10}
30
20
10
0
−10
−20
x
−20
−10
0
10
20
t

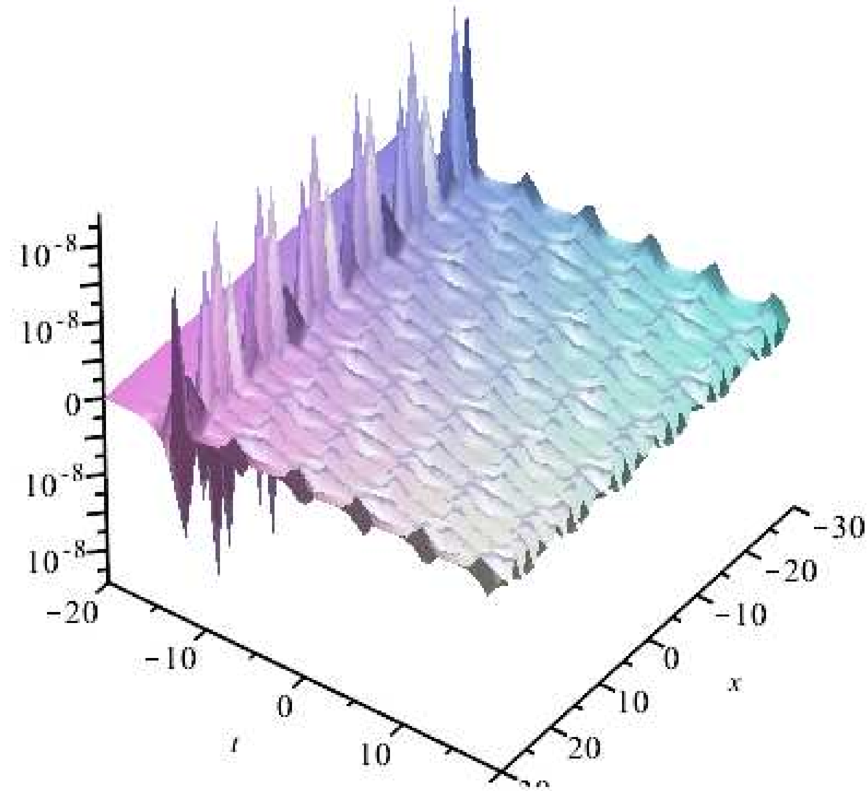

008
006
004
002
0
002
0004
0006
0008
−20
0
t
−30
−10
10
x

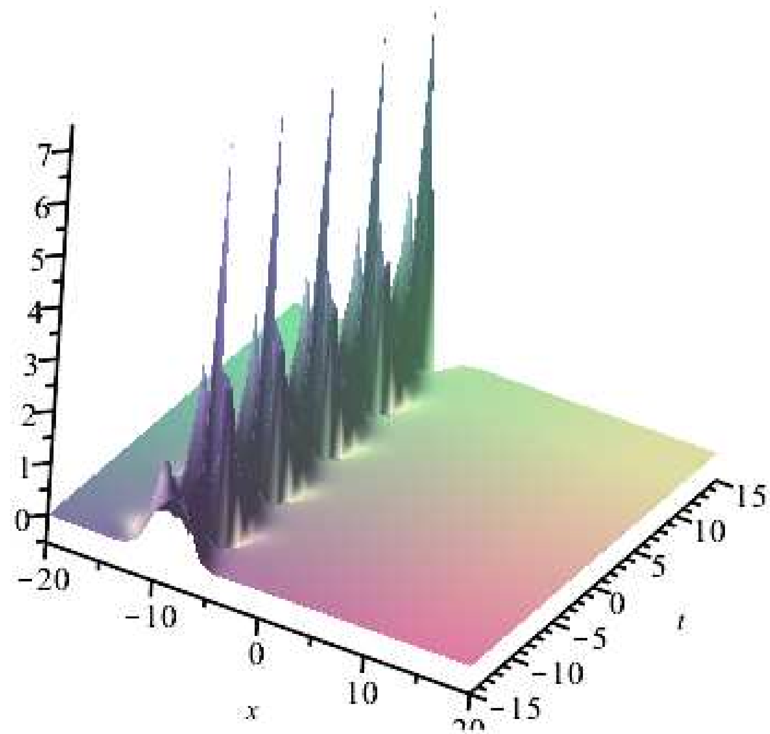


5
4
3
2
1
0
−1
−2
30
20
10
0
−10
−20
−30
−20
−10
0
10
t

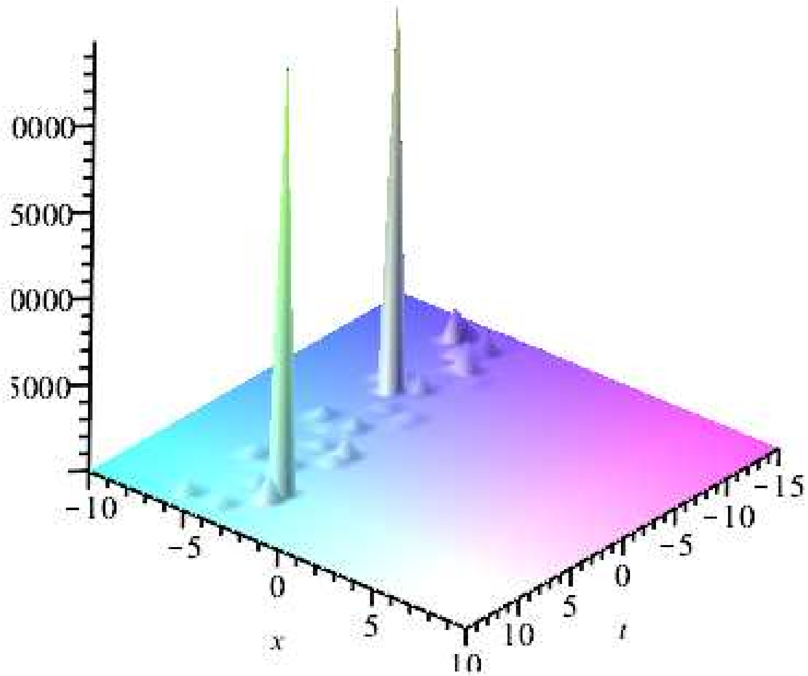


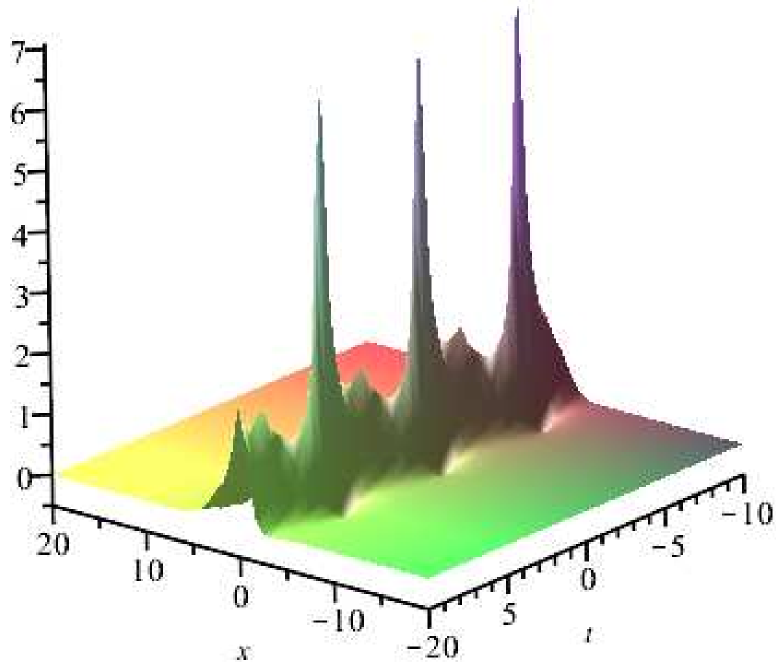
0015
0010
0005
0
0005
0010
0015
−20
−10
0
10
t
30
20
10
0
−10
−20
x


10^-8
10^-8
0
10^-8
10^-8
−20
−10
0
10
t
−30
−20
−10
0
10
20
x

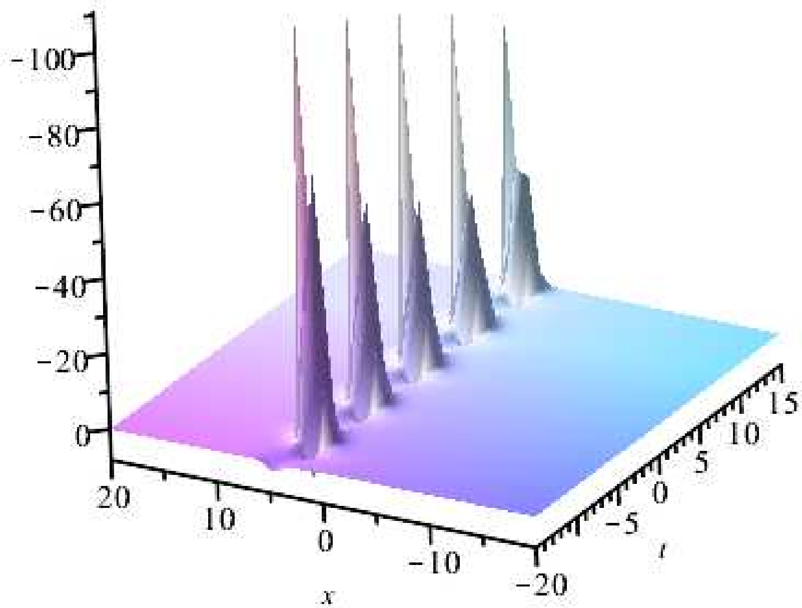


7
6
5
4
3
2
1
0
−20
−10
0
10
20
x
15
10
5
0
−5
−10
−15
t

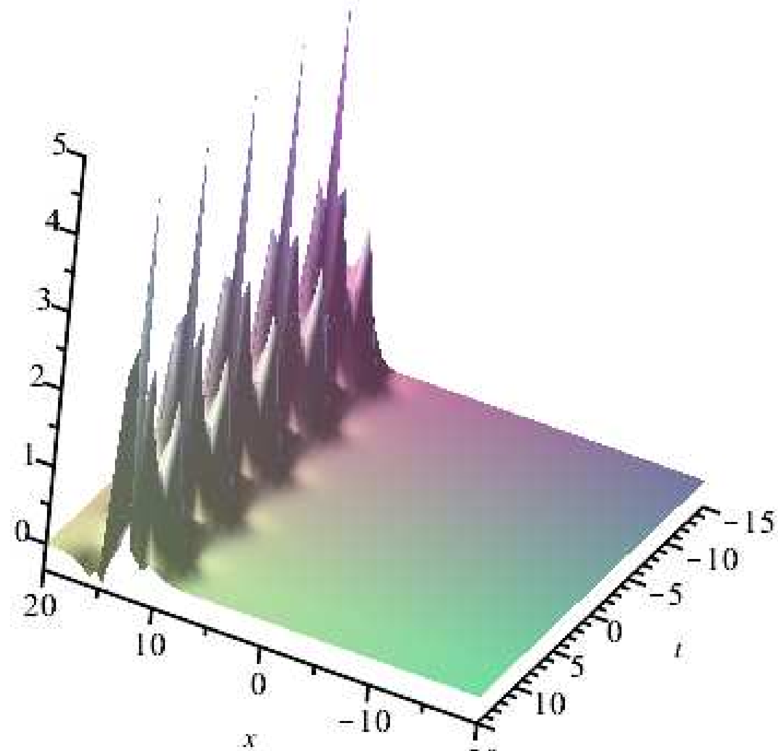

0000
5000
0000
5000
−10
−5
0
5
10
x
−15
−10
−5
0
5
10
t

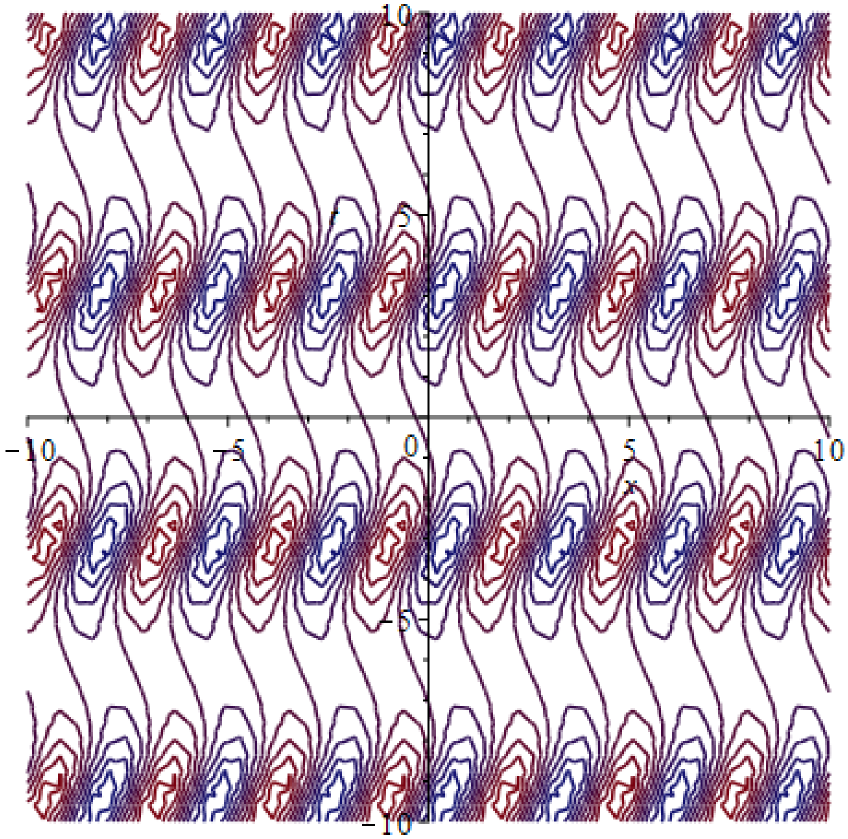


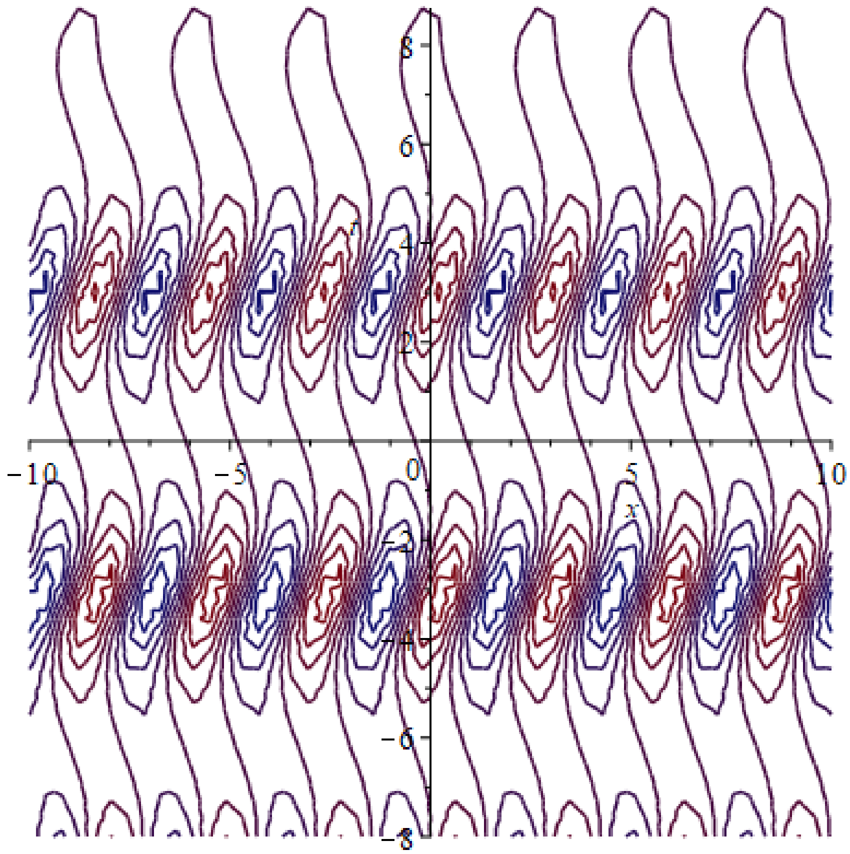
0
1
2
3
4
5
6
7
20
10
0
−10
−20
x
−10
−5
0
5
t

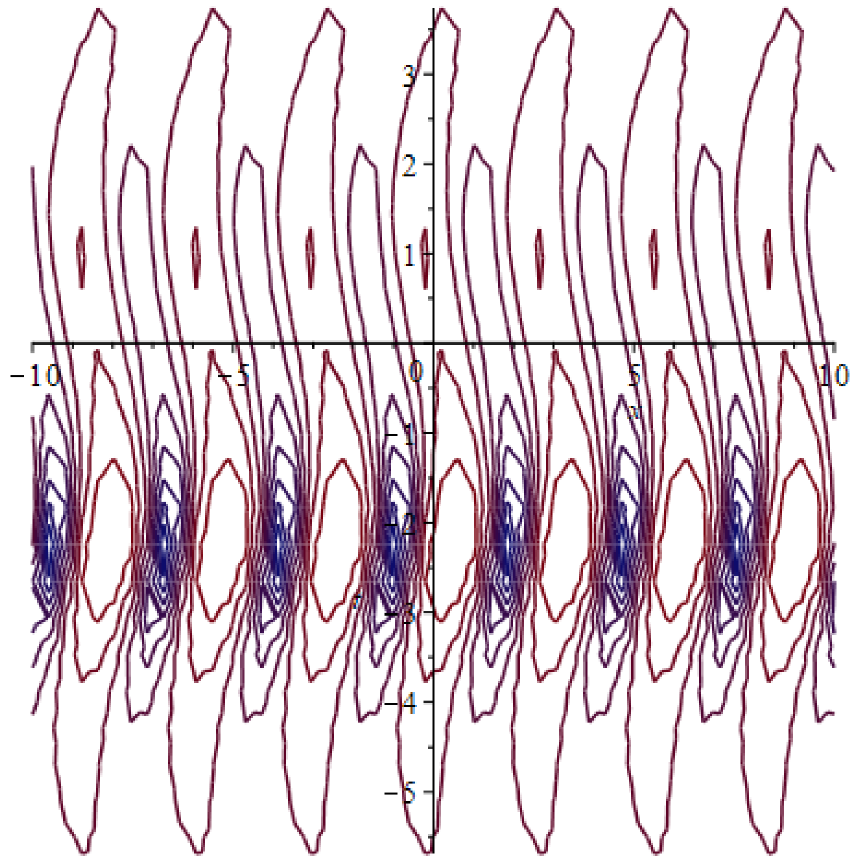


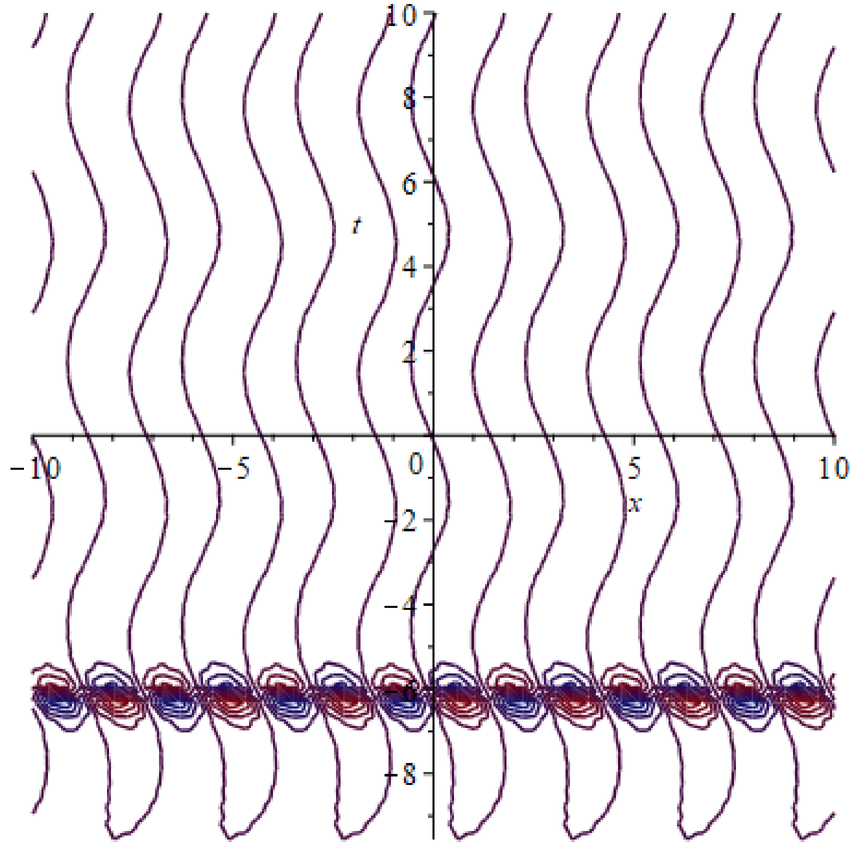
-100
-80
-60
-40
-20
0
20
10
0
-10
-20
x
-5
0
5
10
15
t

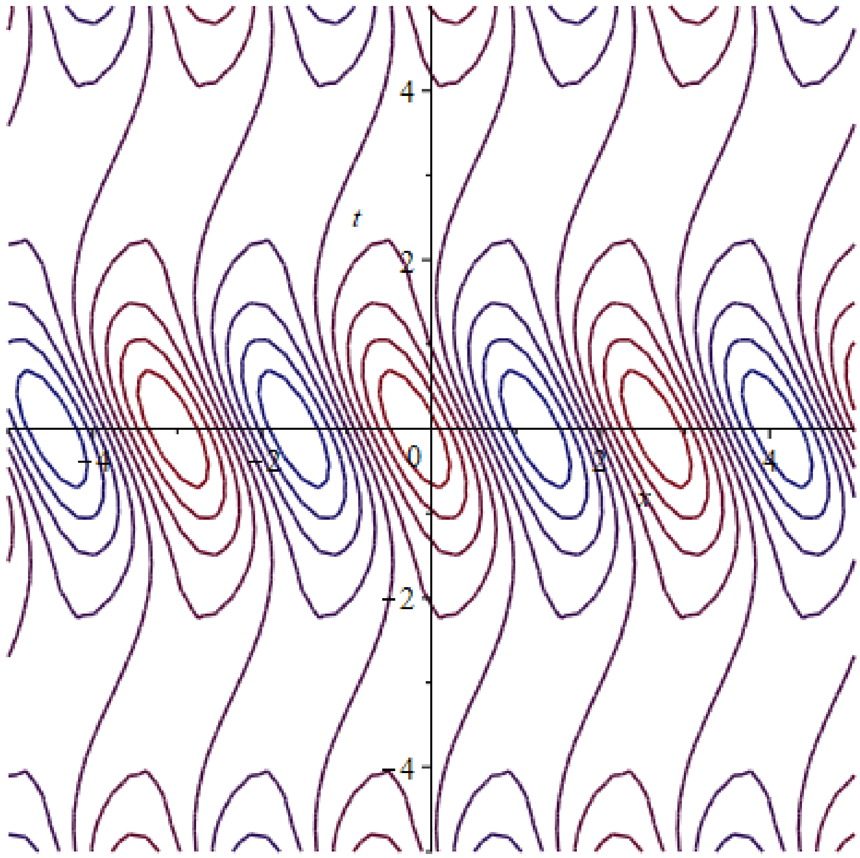

5
4
3
2
1
0
20
10
0
−10
x
−15
−10
−5
0
5
10
t

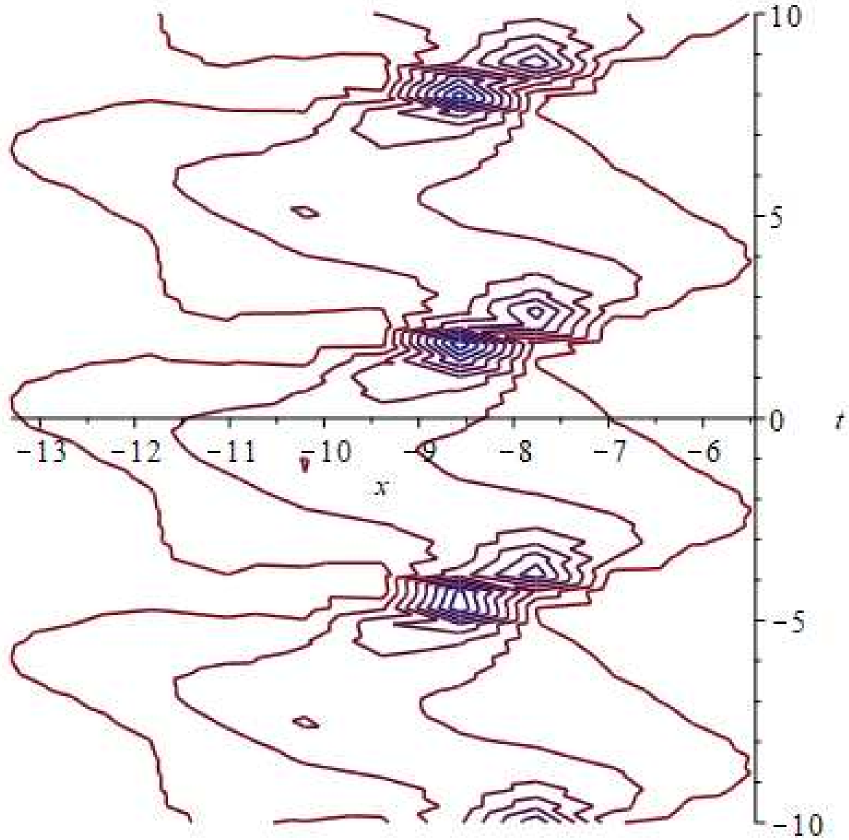

10
5
−10
−5
0
5
10
x
−5
−10


8
6
4
2
−10
−5
0
5
10
x
−2
−4
−6
−8

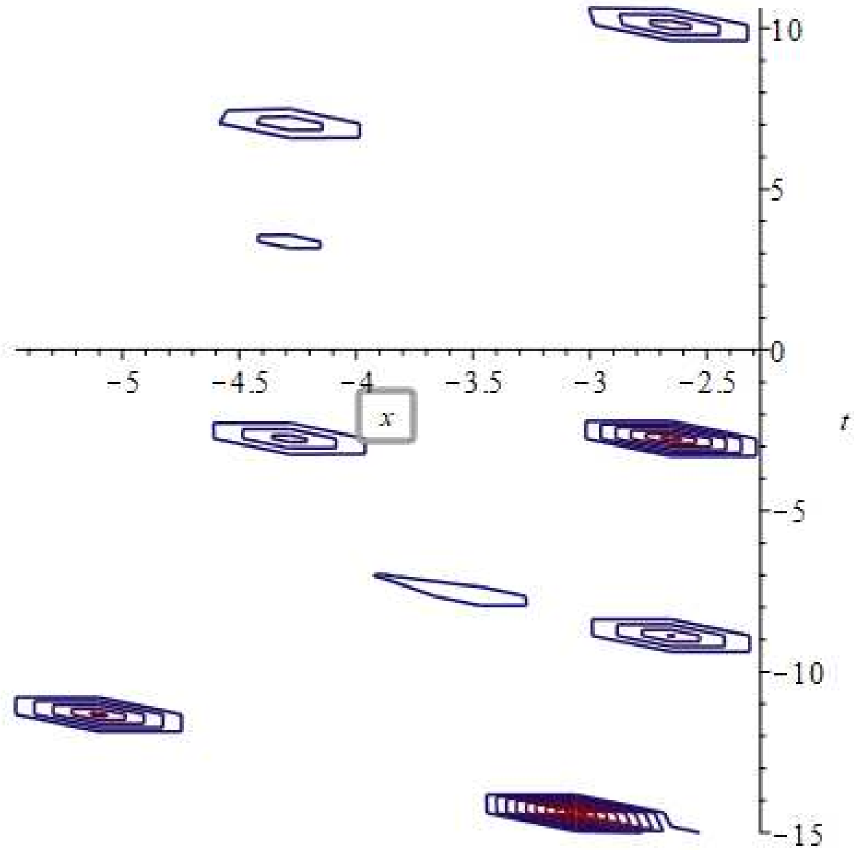

3
2
1
−10
−5
0
5
10
−1
−2
−3
−4
−5

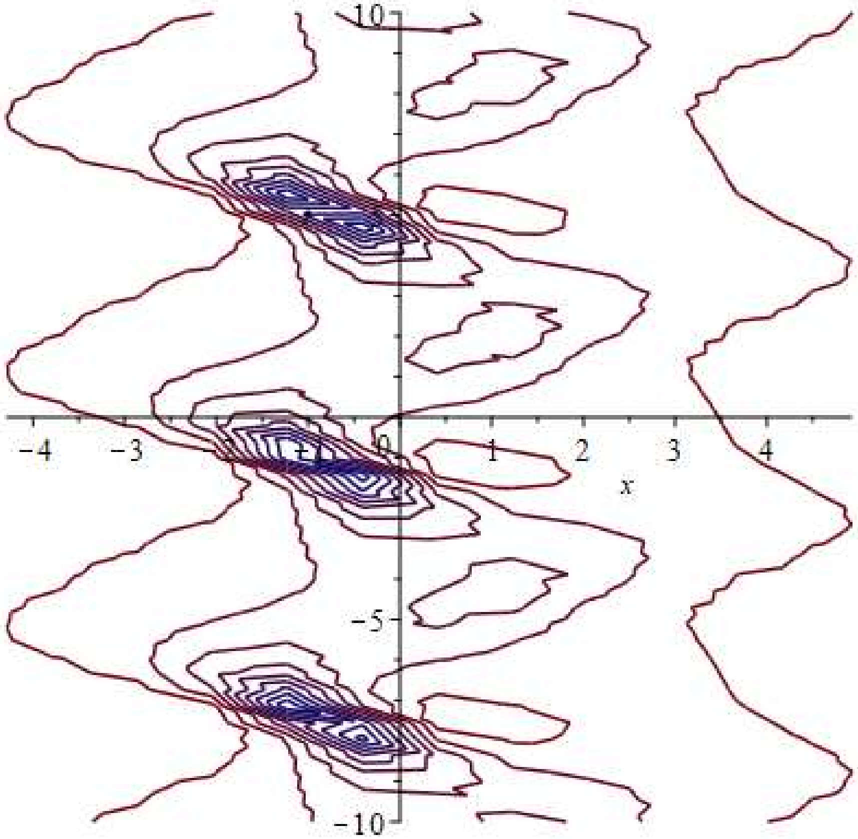

10
8
6
4
2
t
−10
−5
0
5
10
x
−2
−4
−6
−8

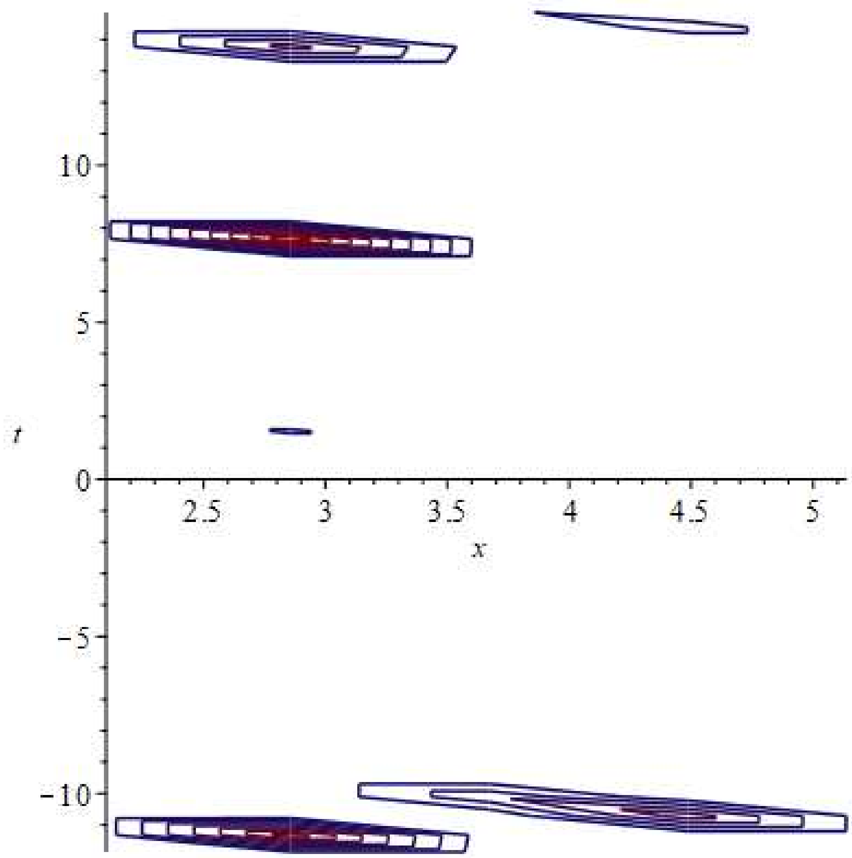

4
2
t
−4
−2
0
2
4
x
−2
−4



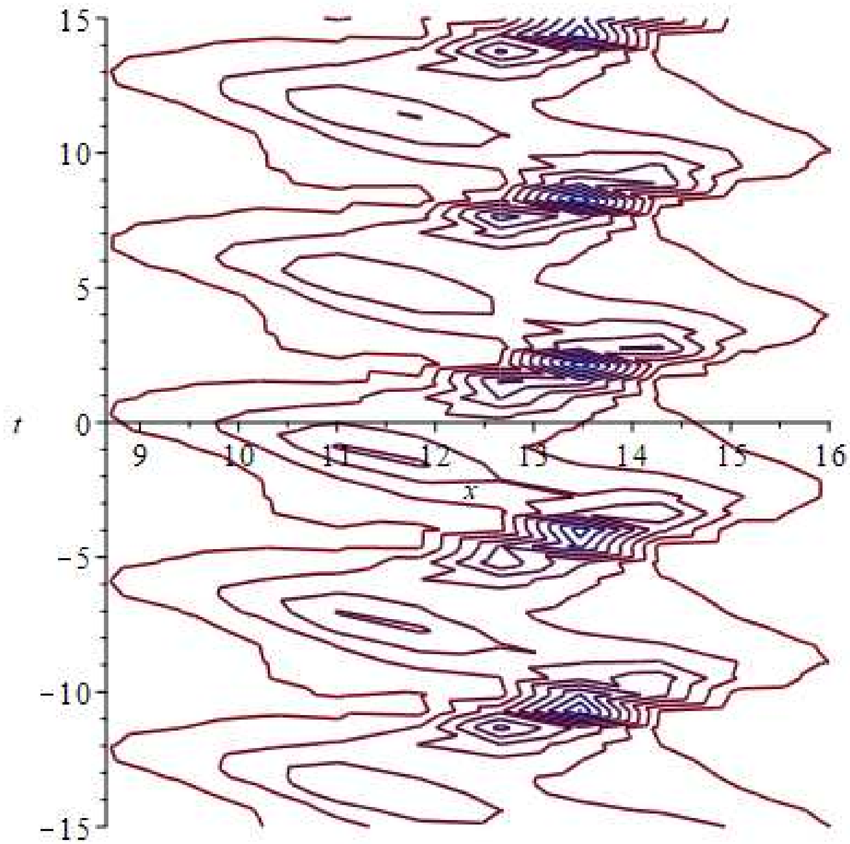
10
5
0
t
−5
−10
−13
−12
−11
−10
−9
−8
−7
−6
x

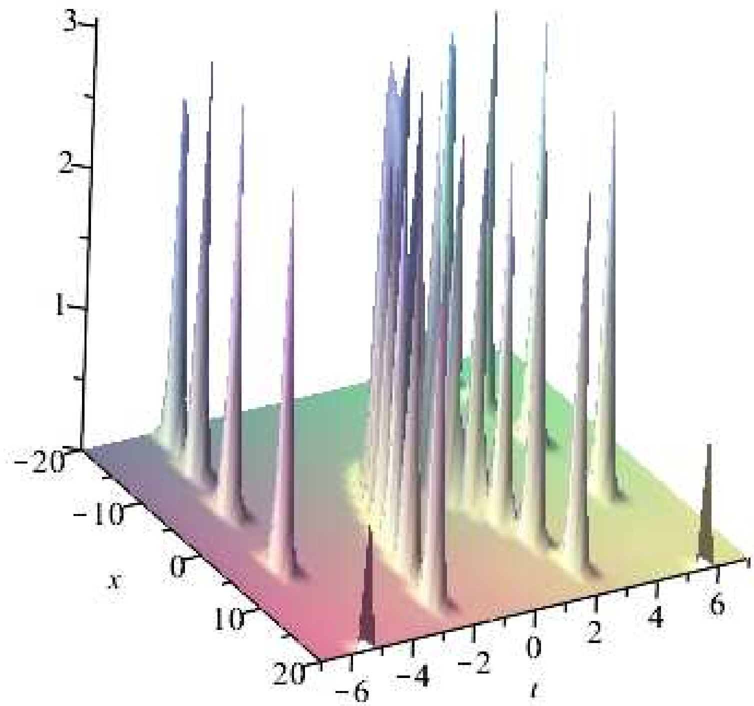

10
5
0
-5
-10
-15
-5
-4.5
-4
-3.5
-3
-2.5
x
t


10
−4
−3
−2
−1
0
1
2
3
4
x
−5
−10

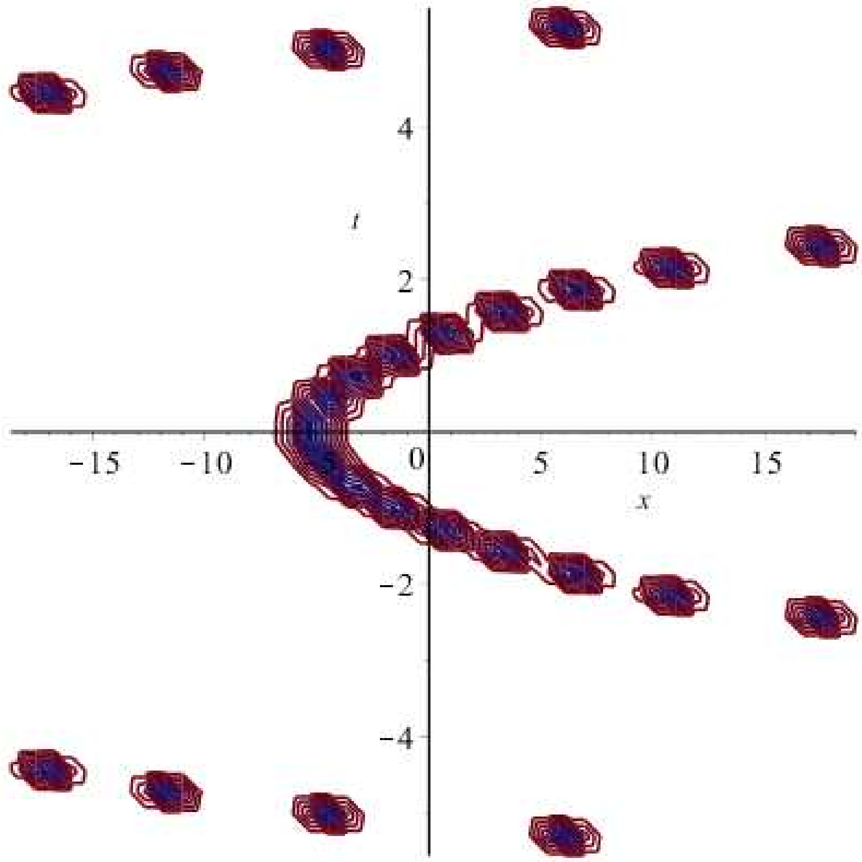


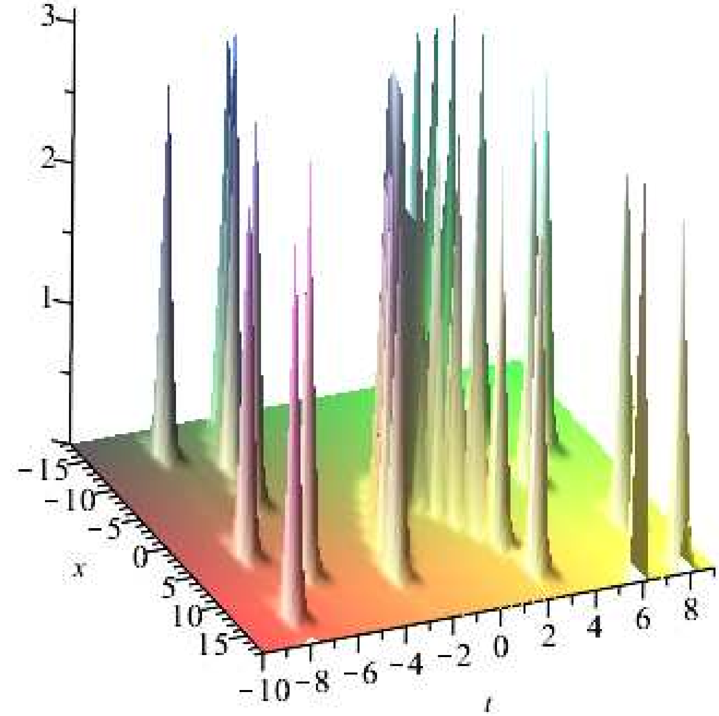
10
5
t
0
2.5
3
3.5
4
4.5
5
x
−5
−10


15
10
5
0
-5
-10
-15
t
9
10
11
12
13
14
15
16
x

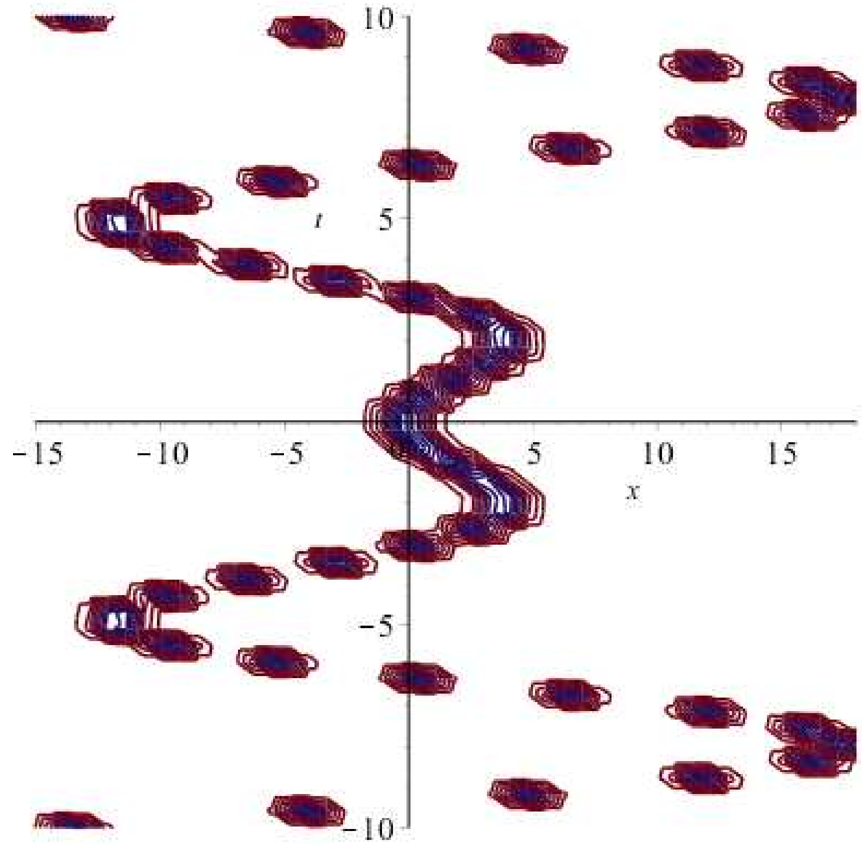


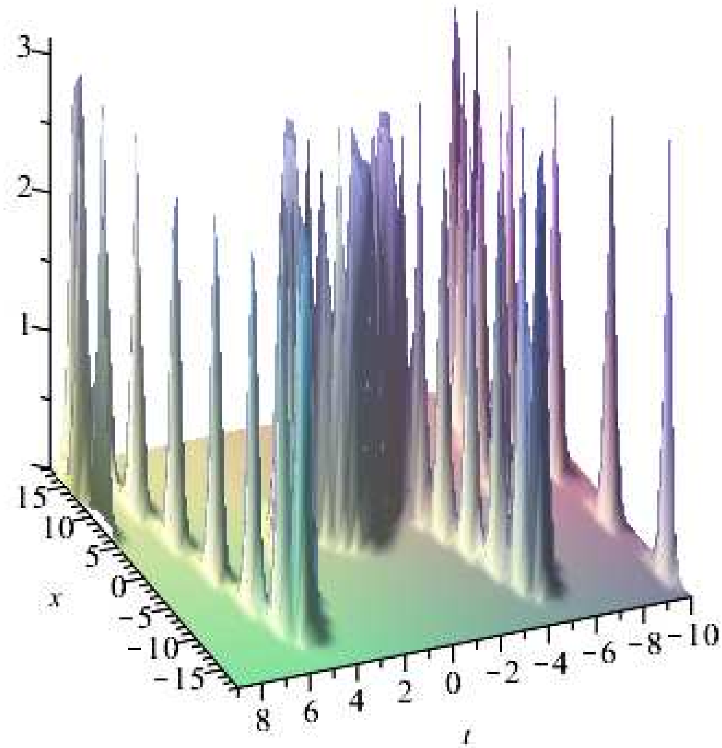
3
2
1
−20
−10
0
10
20
x
−6
−4
−2
0
2
4
6
t

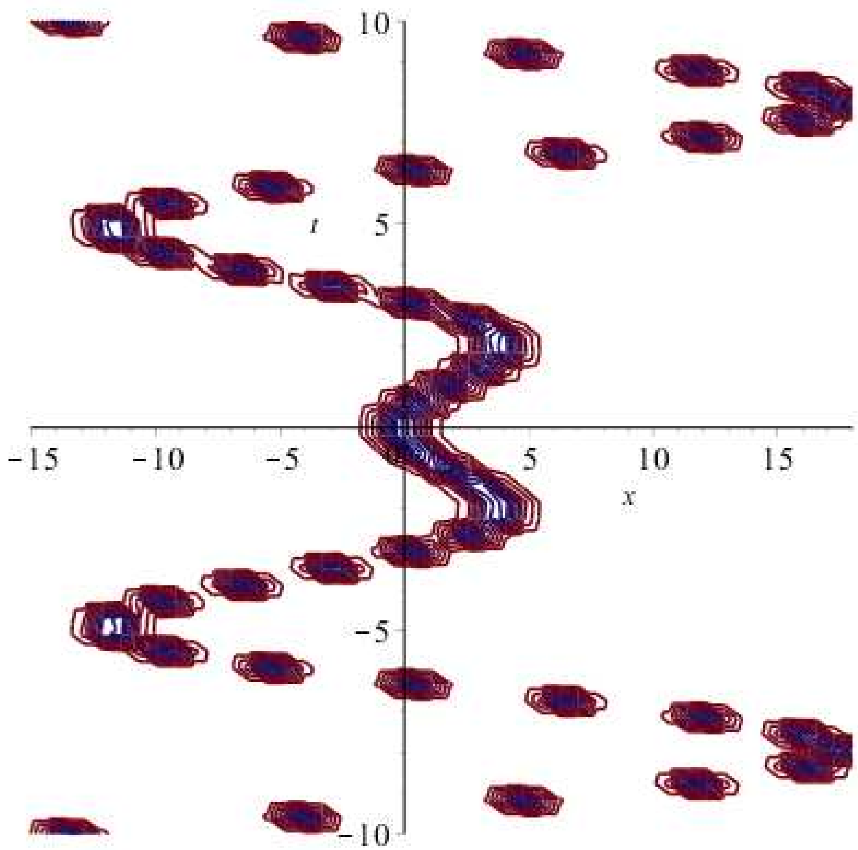

4
2
−2
−4
t
−15
−10
0
5
10
15
x

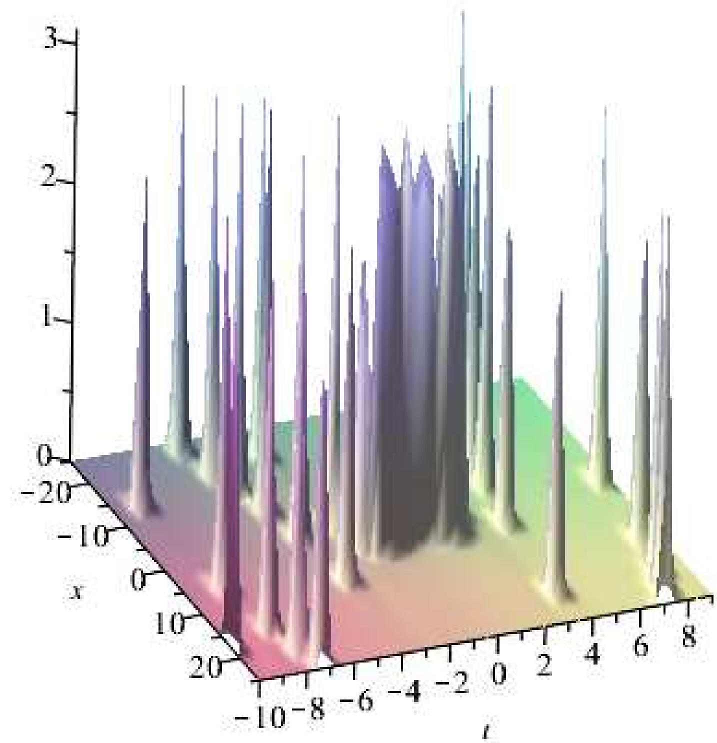


3
2
1
-15
-10
-5
0
5
10
15
x
-10
-8
-6
-4
-2
0
2
4
6
8
t

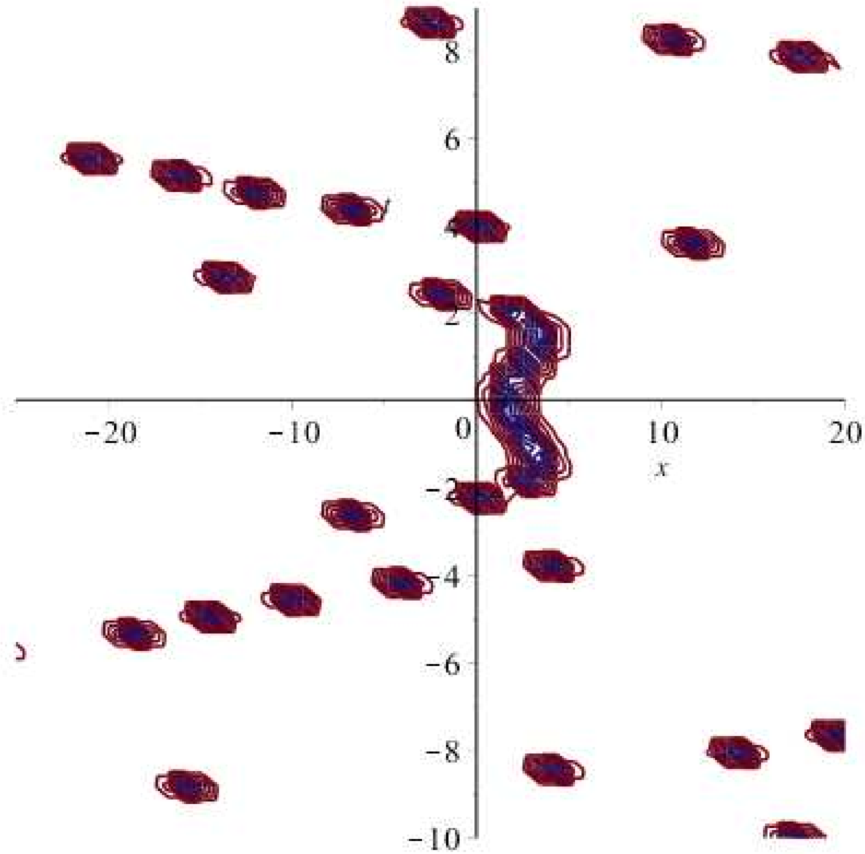

10
5
t
−15
−10
−5
5
10
15
x
−5
−10

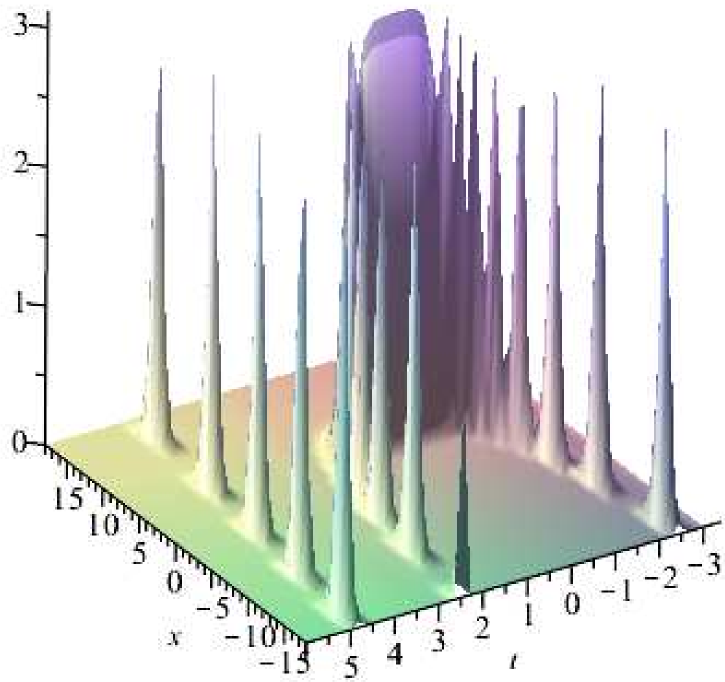

3
2
1
15
10
5
0
−5
−10
−15
x
8
6
4
2
0
−2
−4
−6
−8
−10
t


10
5
t
−5
−10
−15
−10
−5
5
10
15
x

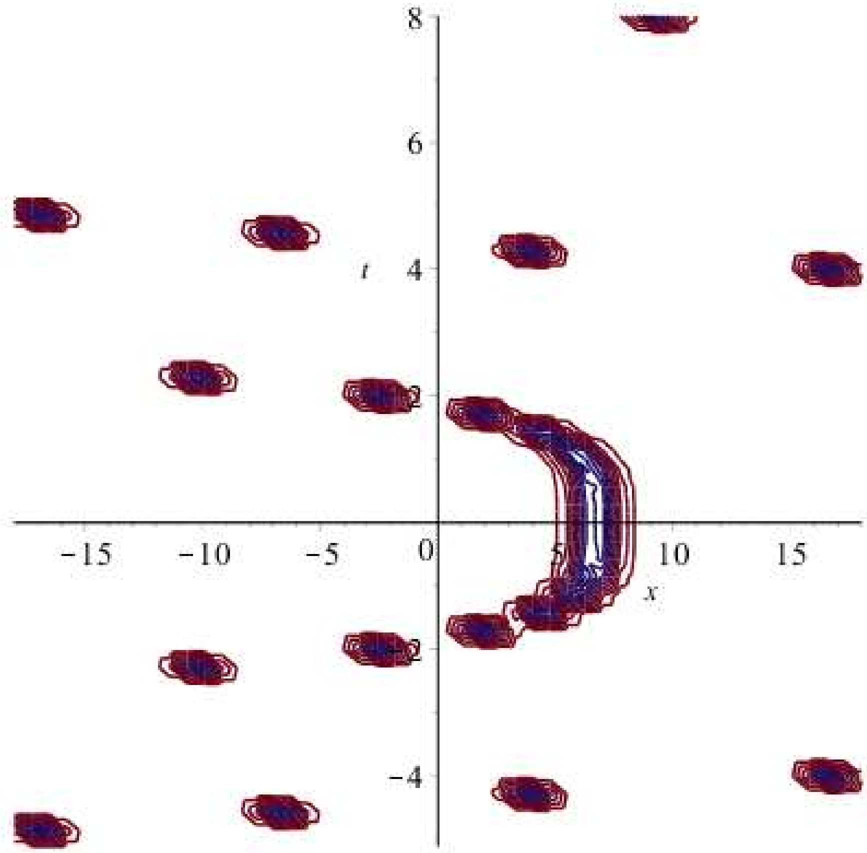


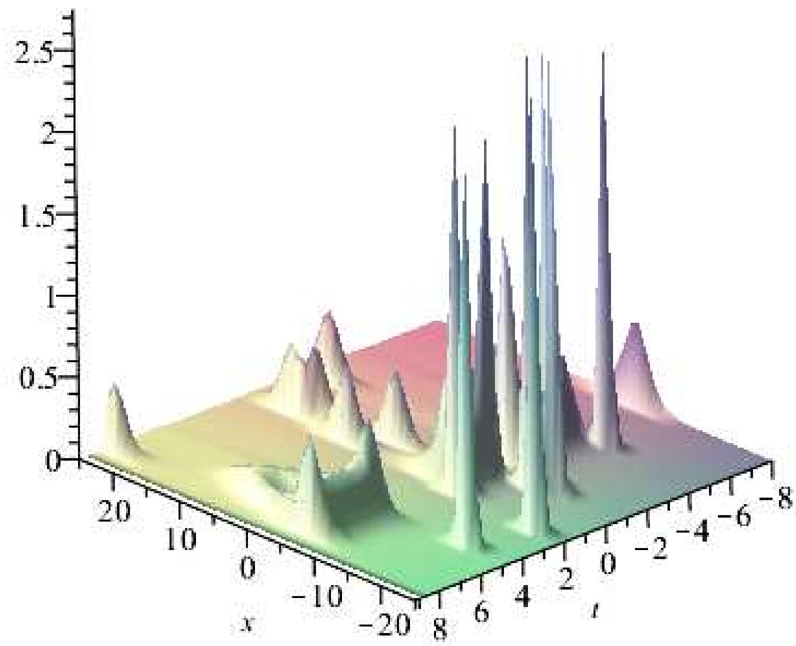
3
2
1
0
−20
−10
0
10
20
x
−10
−8
−6
−4
−2
0
2
4
6
8
t


8
6
4
2
−2
−4
−6
−8
−10
−20
−10
0
10
20
x

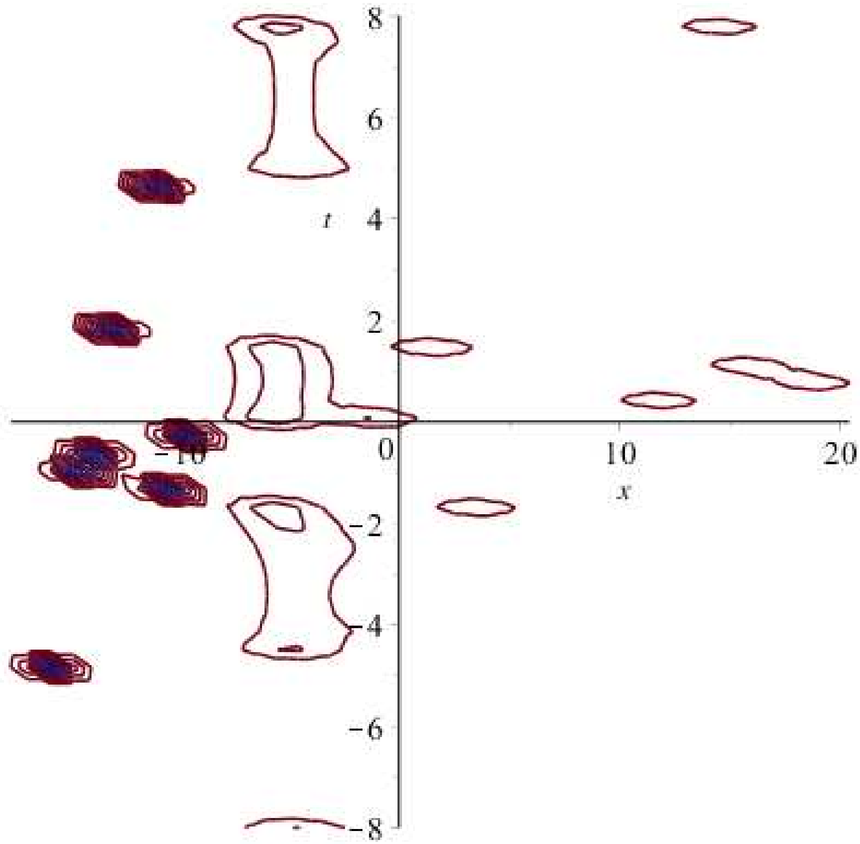


3
2
1
0
15
10
5
0
−5
−10
−15
x
5
4
3
2
1
0
−1
−2
−3
t

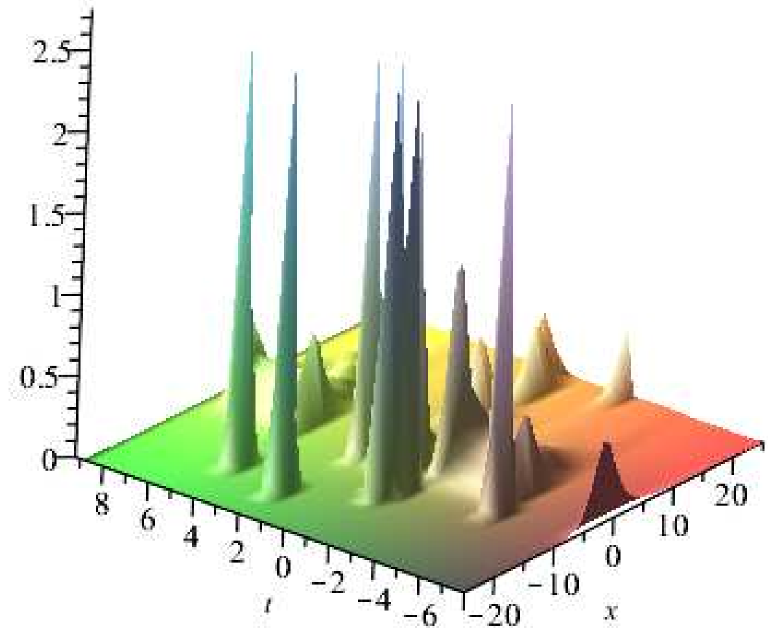

8
6
4
t
2
−15
−10
−5
0
5
10
15
x
2
−4

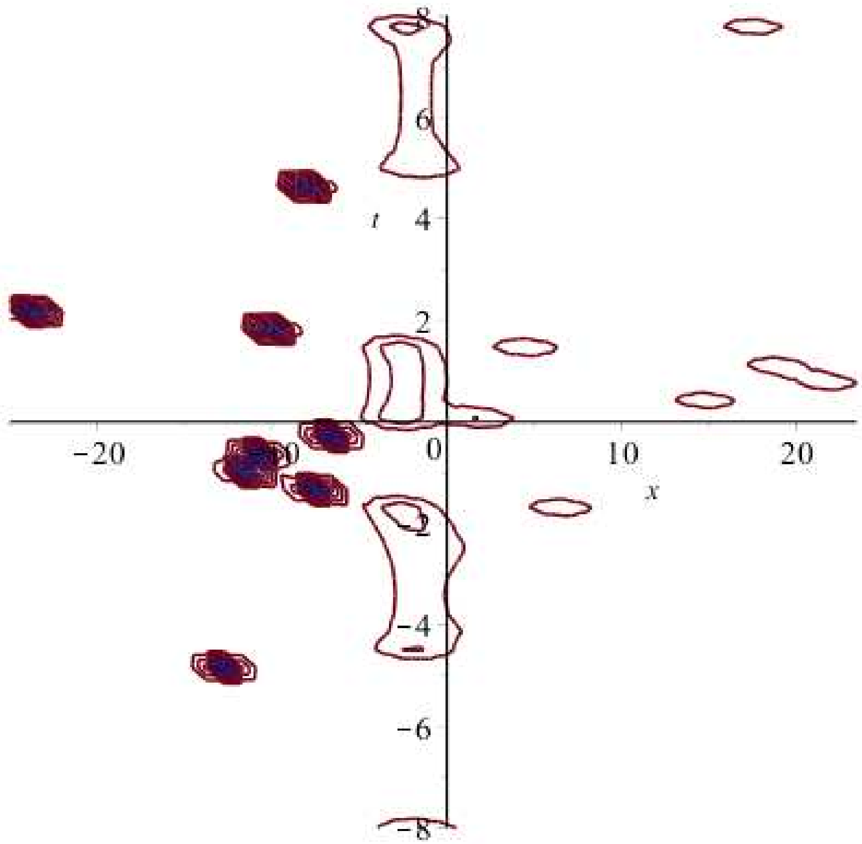


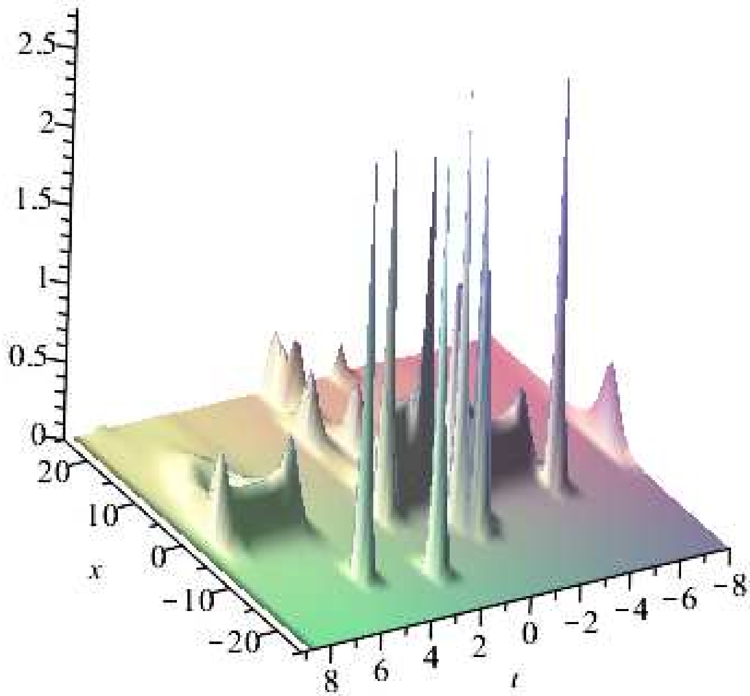
2.5
2
1.5
1
0.5
0
20
10
0
−10
−20
x
8
6
4
2
0
−2
−4
−6
−8
t

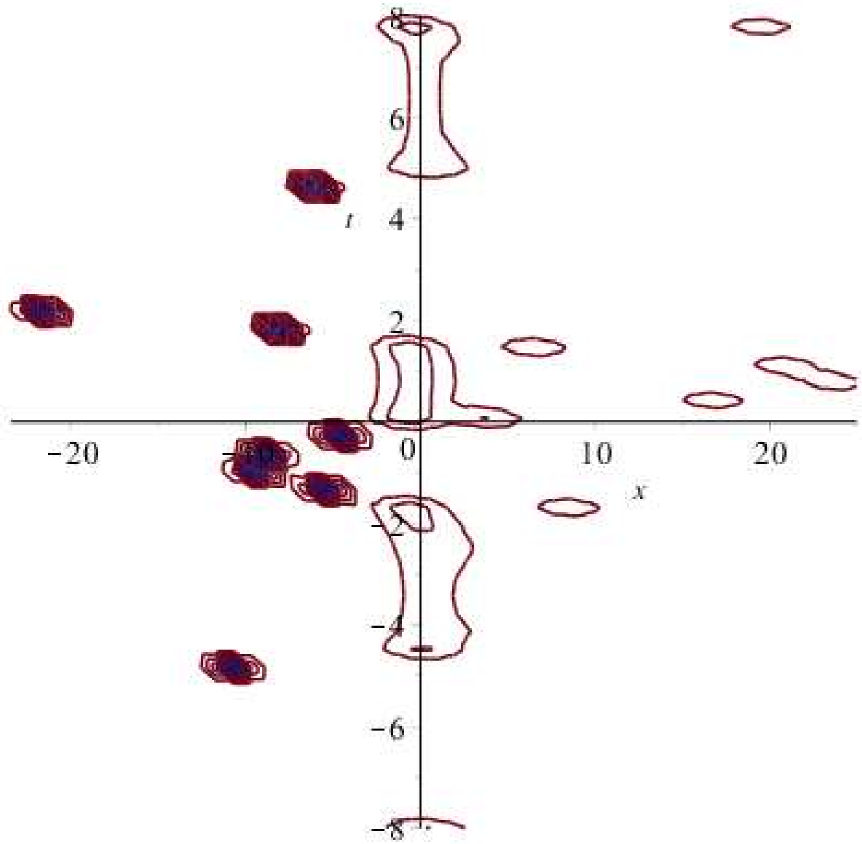


8
6
t
4
2
−10
0
10
20
x
−2
−4
−6
−8

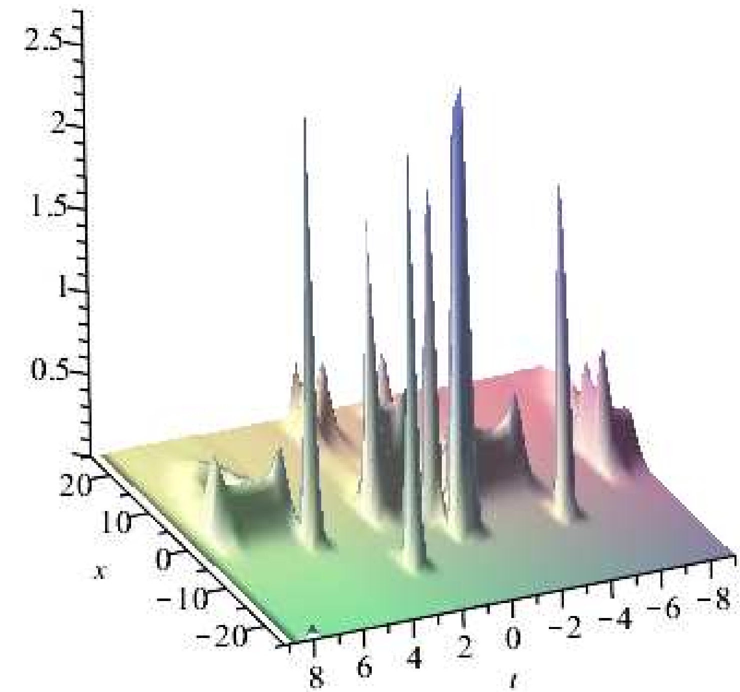

2.5
2
1.5
1
0.5
0
8
6
4
2
0
−2
−4
−6
t
−20
−10
0
10
20
x

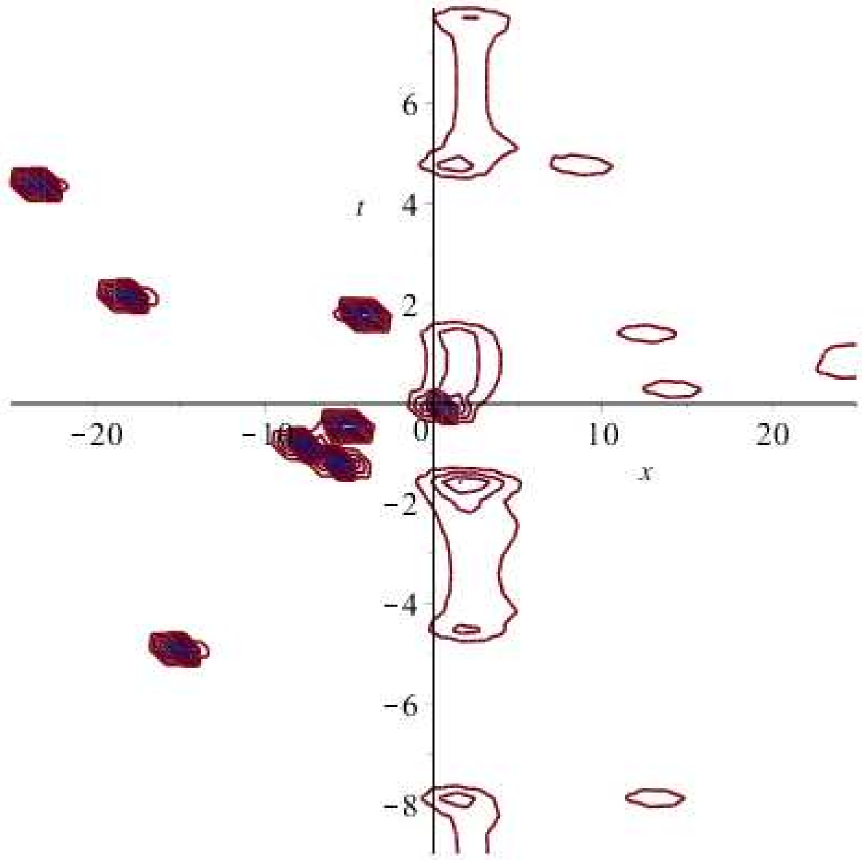

8
6
t
4
2
−20
−10
0
10
20
x
−2
−4
−6
−8

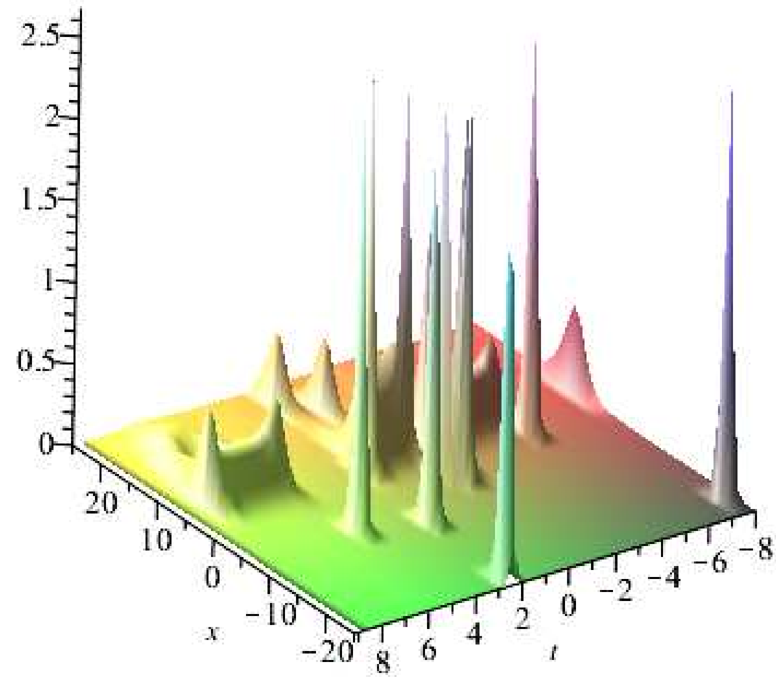

2.5
2
1.5
1
0.5
0
20
10
0
x
−10
−20
8
6
4
2
0
−2
−4
−6
−8
t

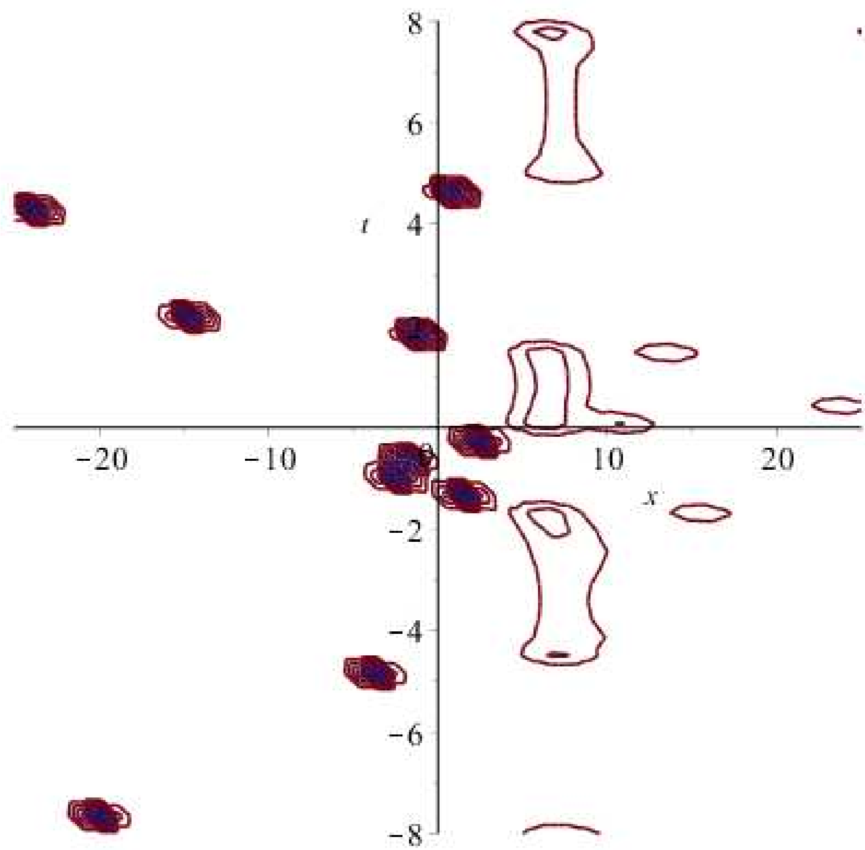


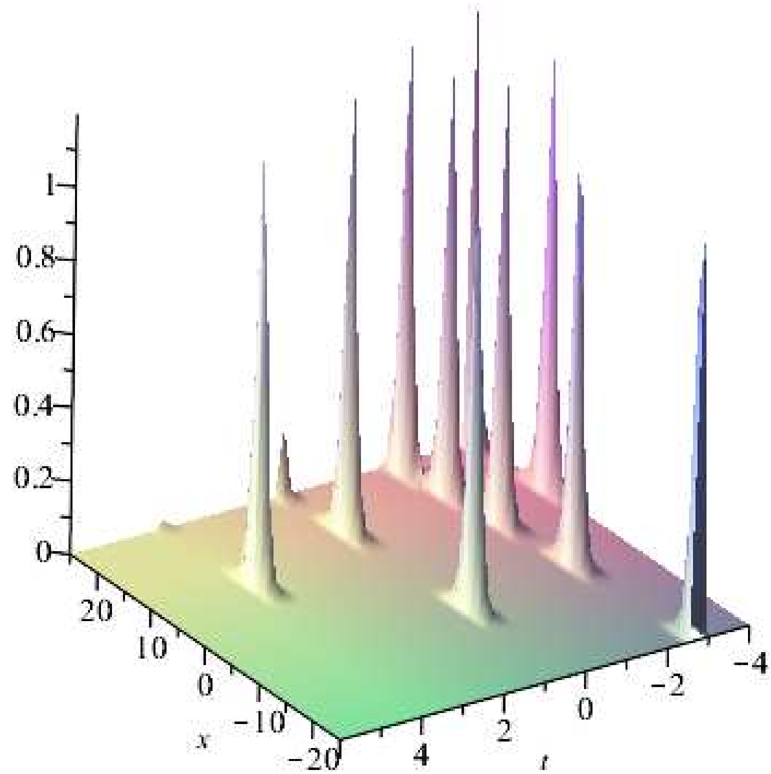
8
6
4
t
2
−20
−10
0
10
20
x
−2
−4
−6
−8


2.5
2
1.5
1
0.5
20
10
0
−10
−20
x
8
6
4
2
0
−2
−4
−6
−8
t

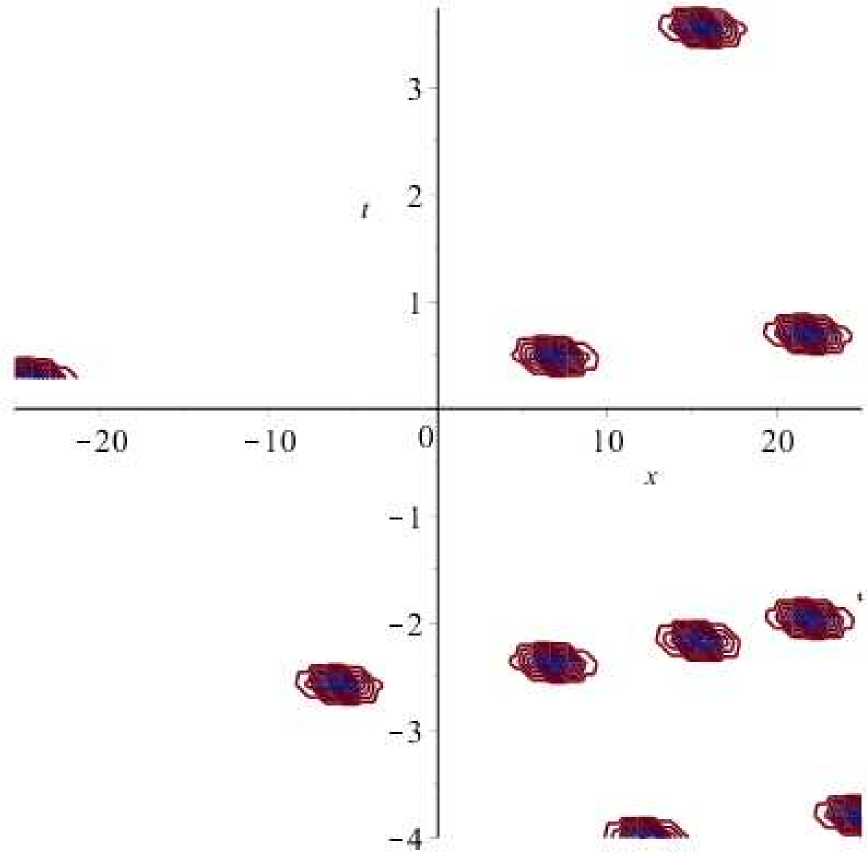

t
6
4
2
-2
-4
-6
-8
-20
-10
0
10
20
x

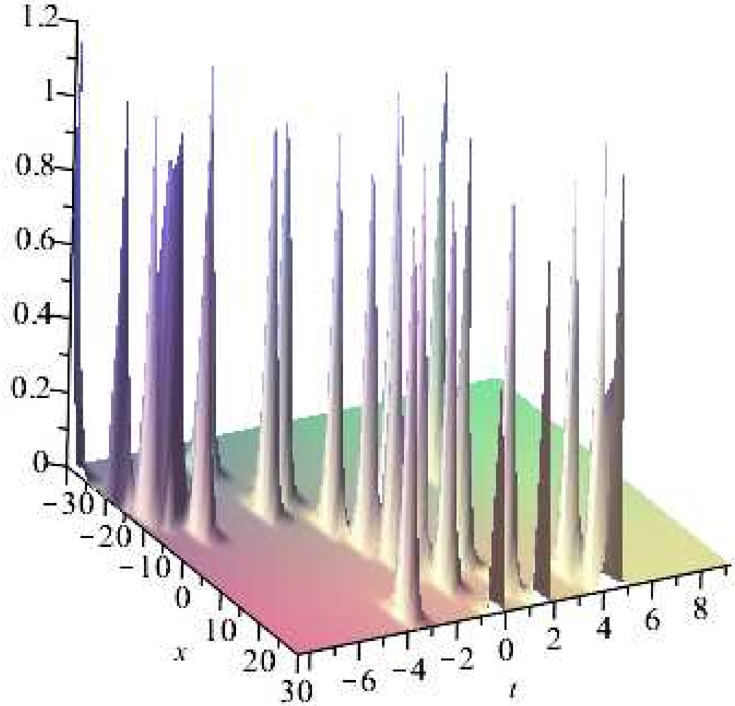

2.5
2
1.5
1
0.5
0
20
10
0
−10
−20
x
8
6
4
2
0
−2
−4
−6
−8
t


8
6
4
t
−2
−4
−6
−8
−20
−10
0
10
20
x

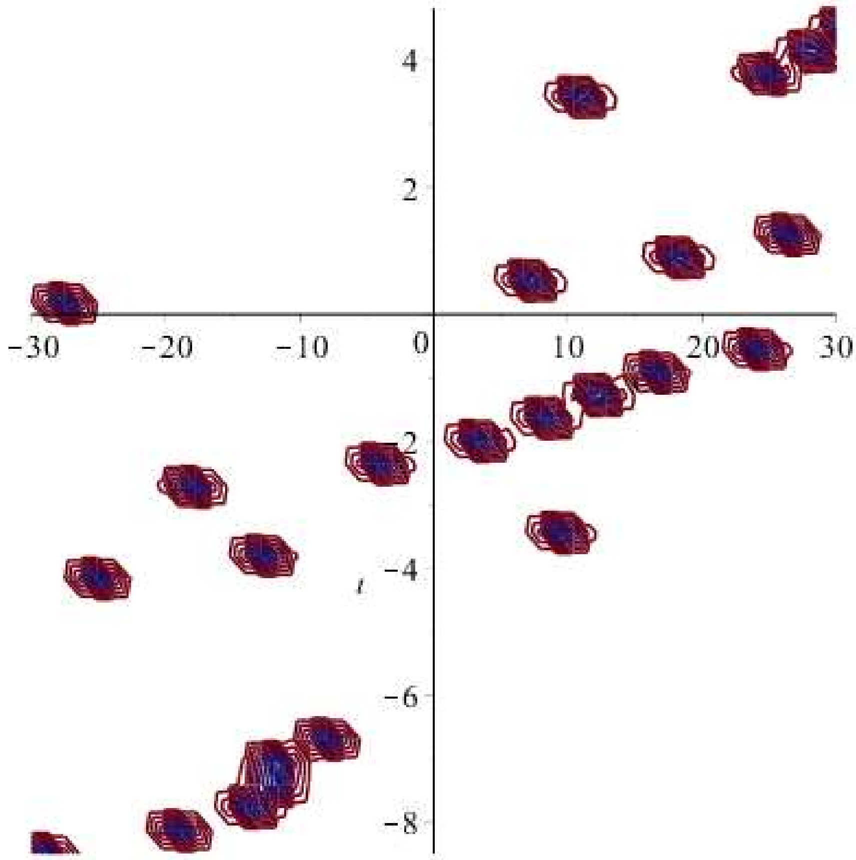


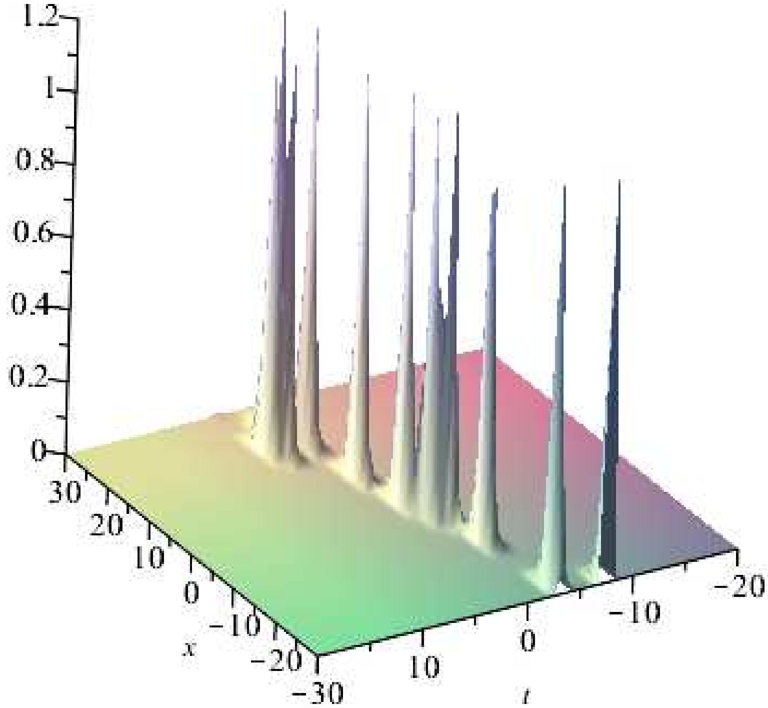
1
0.8
0.6
0.4
0.2
0
20
10
0
−10
−20
x
4
2
0
−2
−4
t


t
x
3
2
1
0
−1
−2
−3
−4
−20
−10
10
20

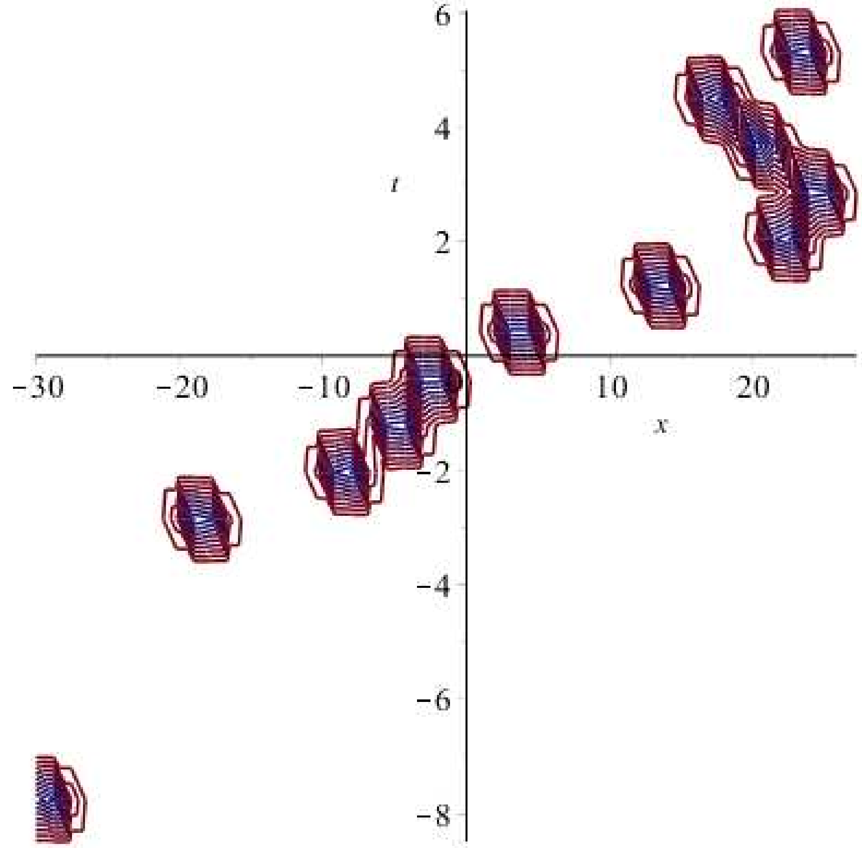


1.2
1
0.8
0.6
0.4
0.2
0
−30
−20
−10
0
10
20
30
x
−6
−4
−2
0
2
4
6
8
t

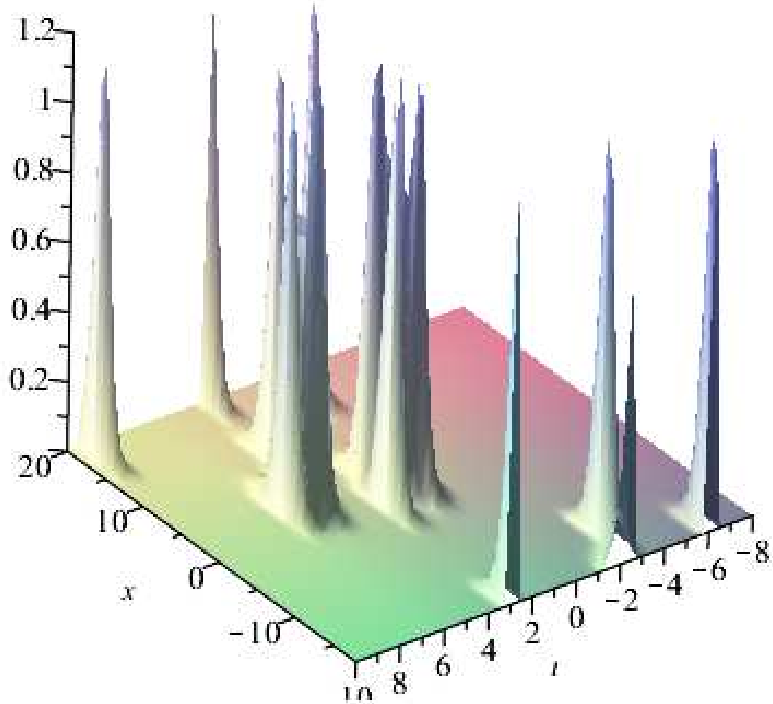

4
2
−30
−20
−10
0
10
20
30
−2
−4
−6
−8



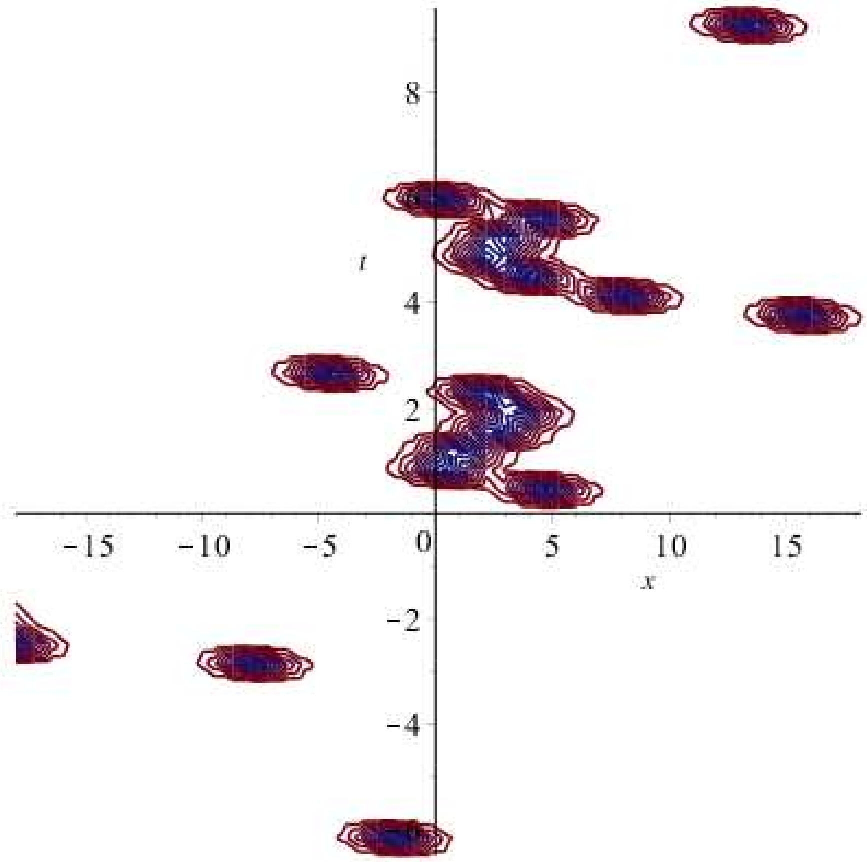
1.2
1
0.8
0.6
0.4
0.2
0
30
20
10
0
−10
−20
x
−30
10
0
−10
−20
t



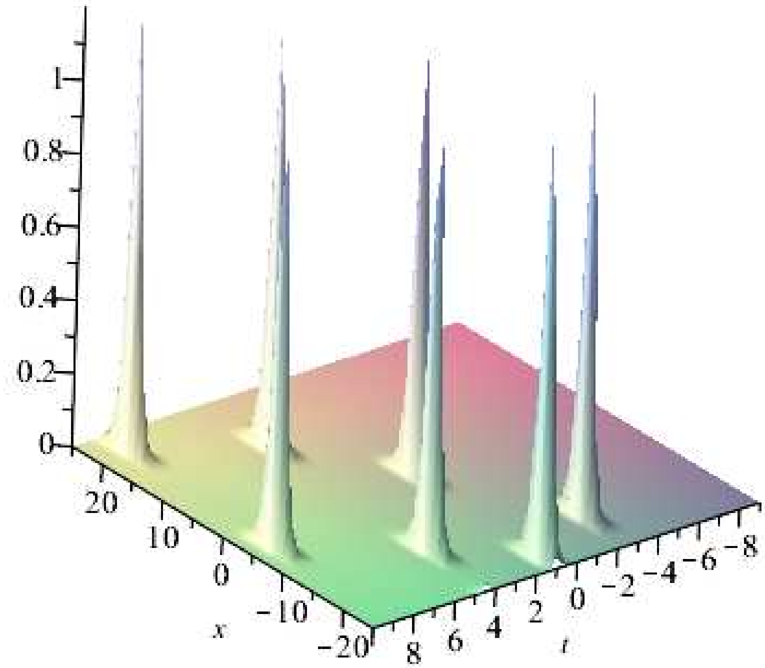
6
4
2
−2
−4
−6
−8
t
−30
−20
−10
10
20
x

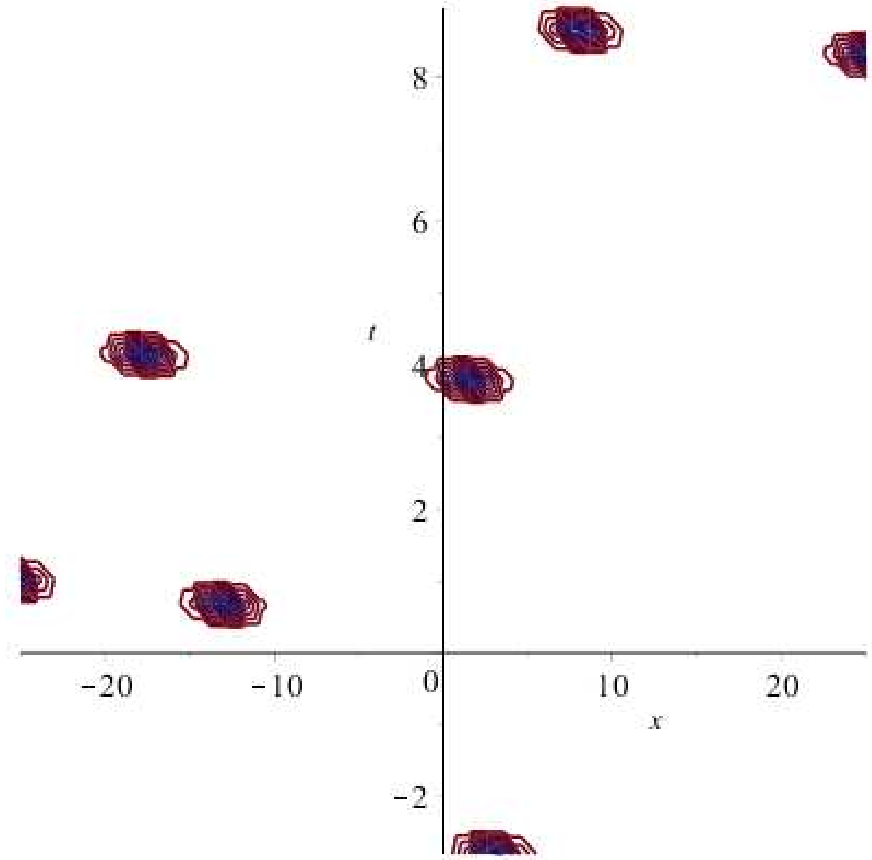


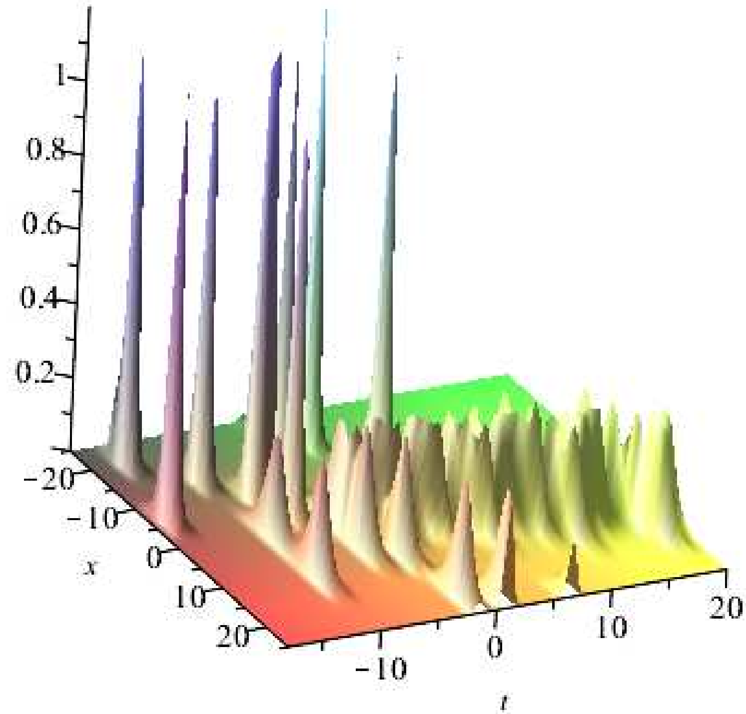
1.2
1
0.8
0.6
0.4
0.2
20
10
0
−10
x
8
6
4
2
0
−2
−4
−6
−8
t

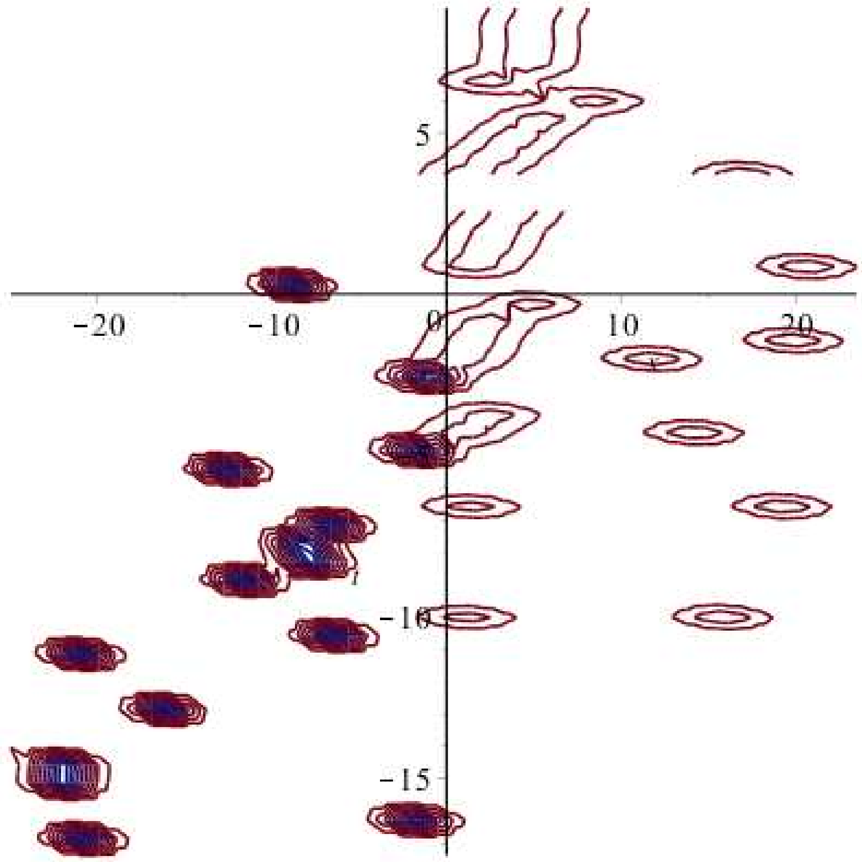


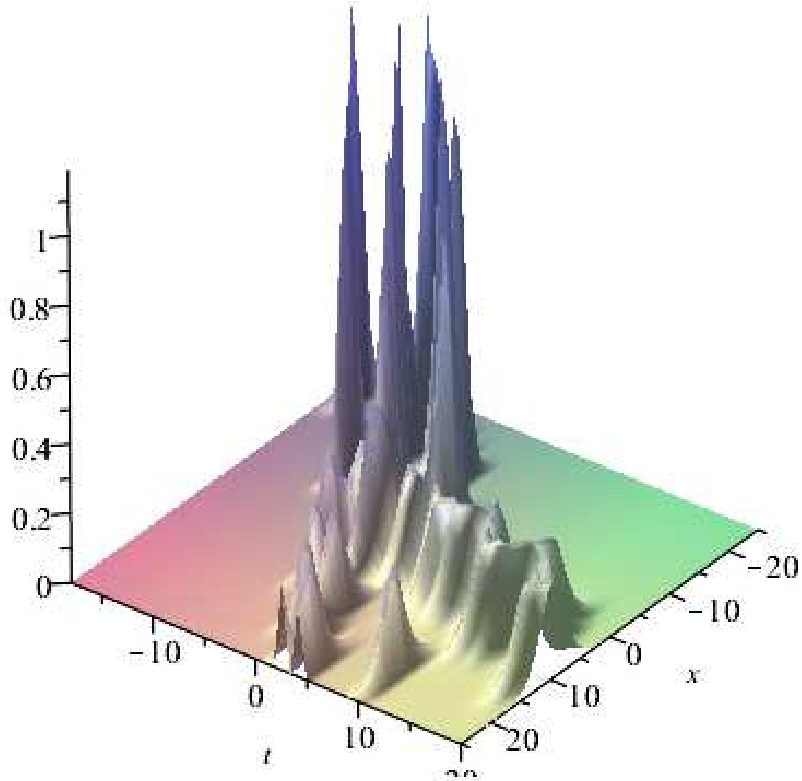
8
t
4
2
−15
−10
−5
0
5
10
15
x
−2
−4

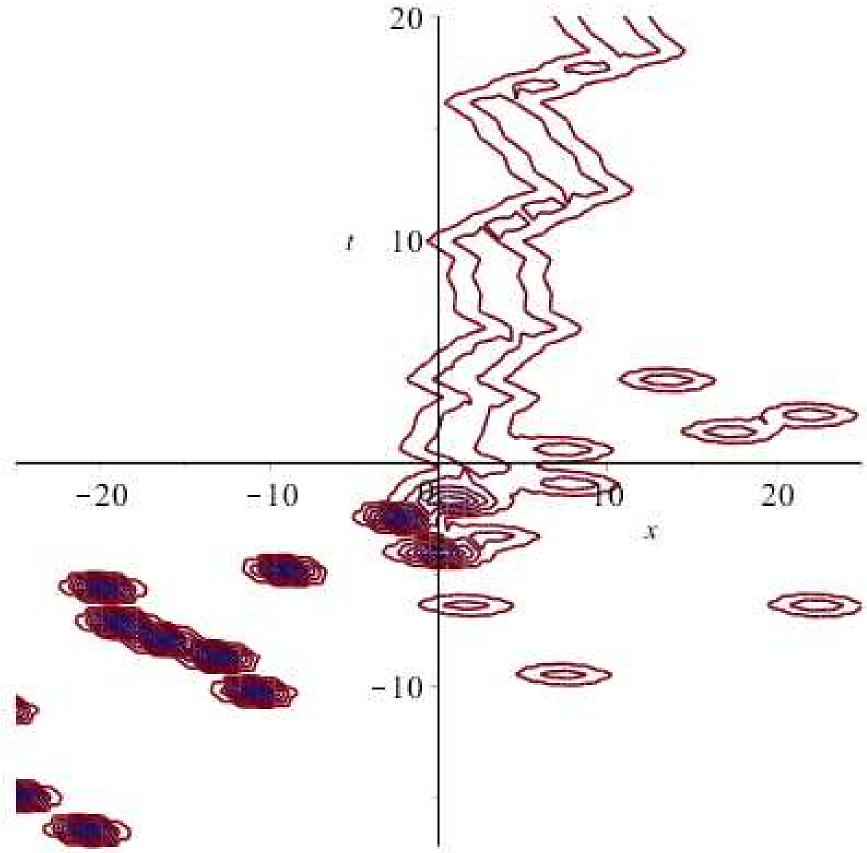

1
0.8
0.6
0.4
0.2
0
20
10
0
−10
−20
x
8
6
4
2
0
−2
−4
−6
−8
t

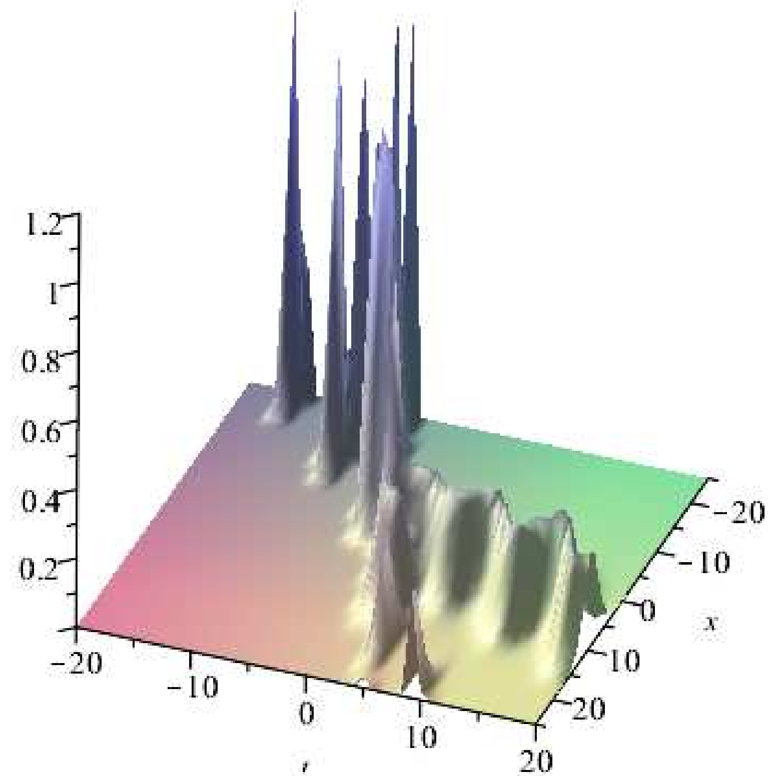

8
6
t
4
2
−2
−20
−10
0
10
20
x

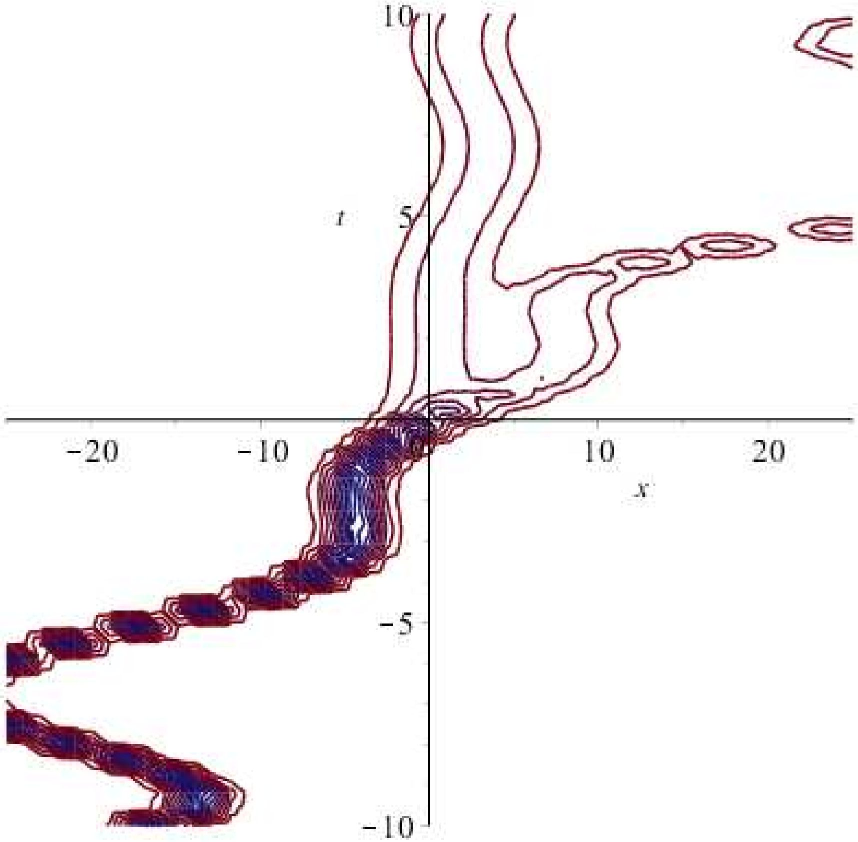

1
0.8
0.6
0.4
0.2
−20
−10
0
10
20
x
−10
0
10
20
t


5
−20
−10
0
10
20
−10
−15

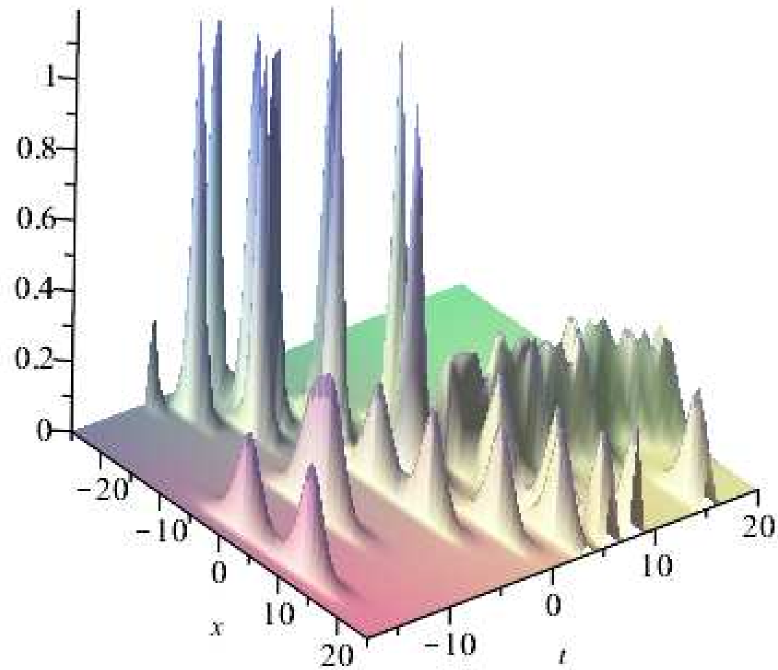

1
0.8
0.6
0.4
0.2
0
−10
0
10
t
−20
−10
0
10
20
x

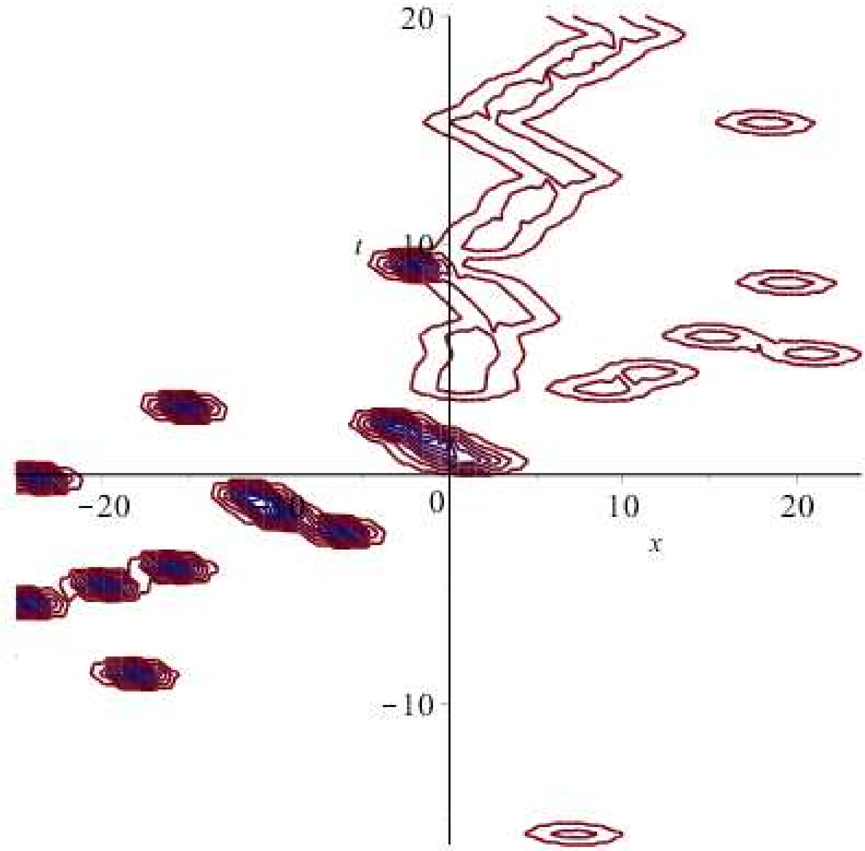

20
10
t
−20
−10
0
10
20
x
−10

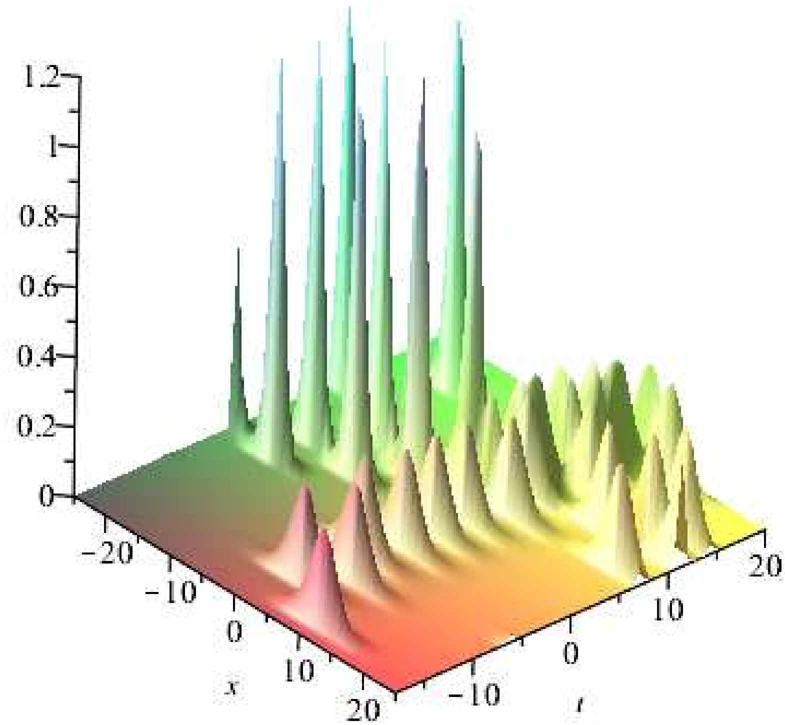

1.2
1
0.8
0.6
0.4
0.2
−20
−10
0
10
20
t
−20
−10
0
10
20
x

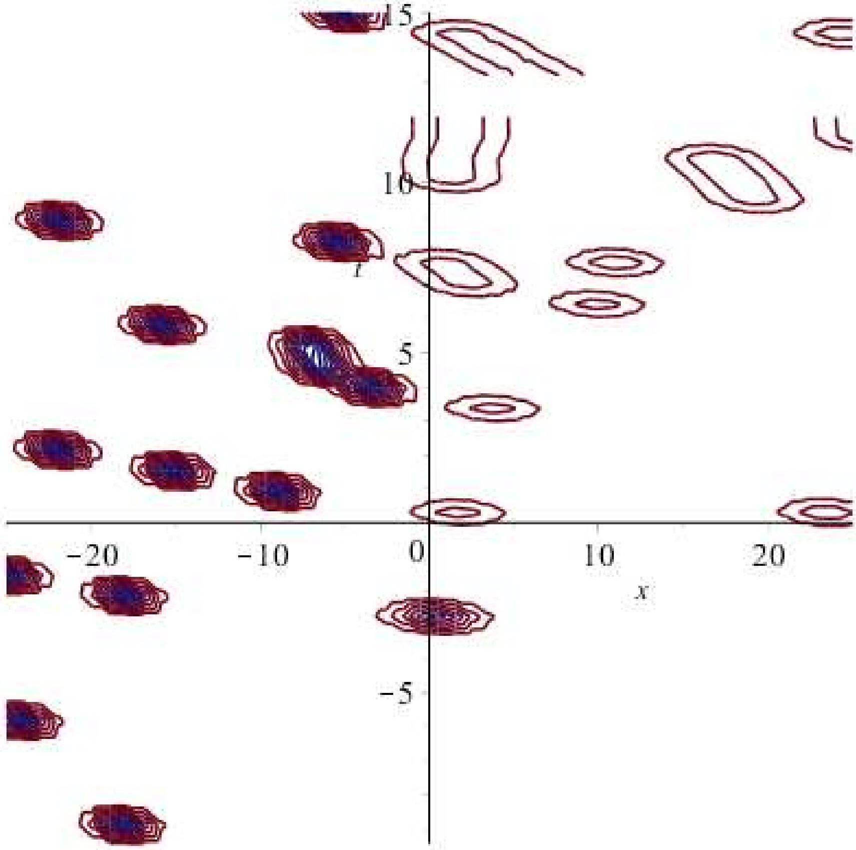


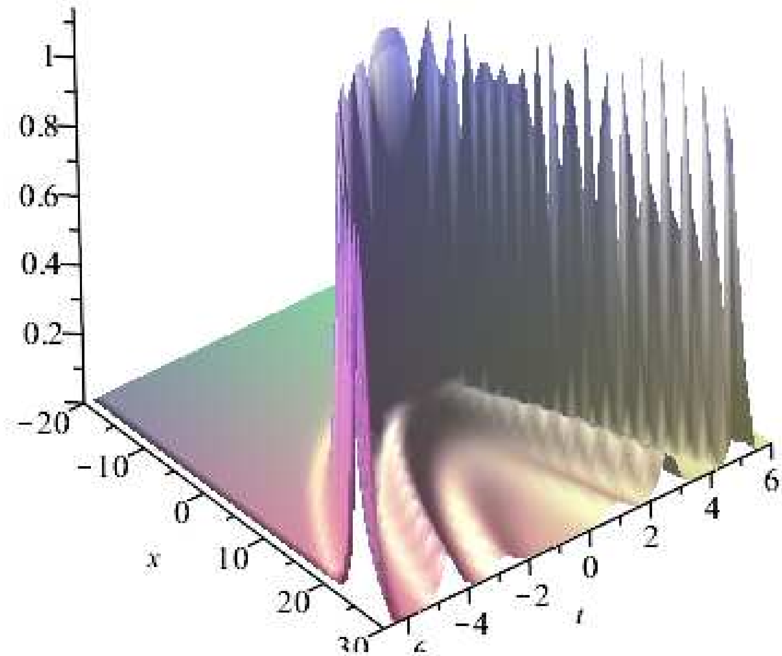
10
5
t
−20
−10
10
20
x
−5
−10


1
0.8
0.6
0.4
0.2
0
−20
−10
0
10
20
x
−10
0
10
20
t

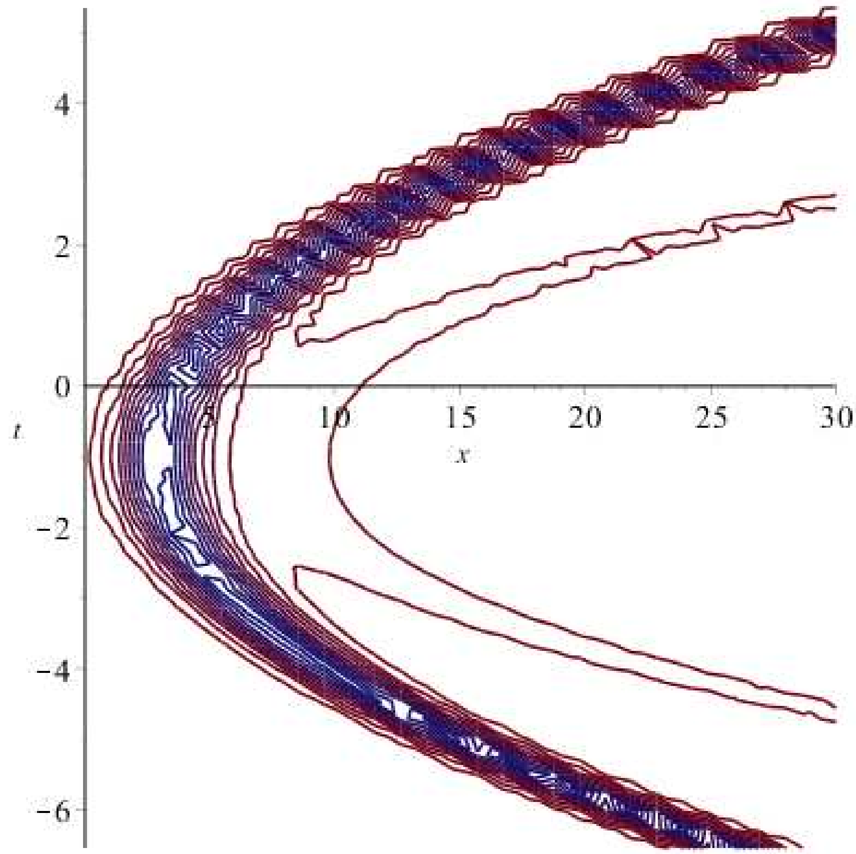

20
t
10
−20
0
10
20
x
−10

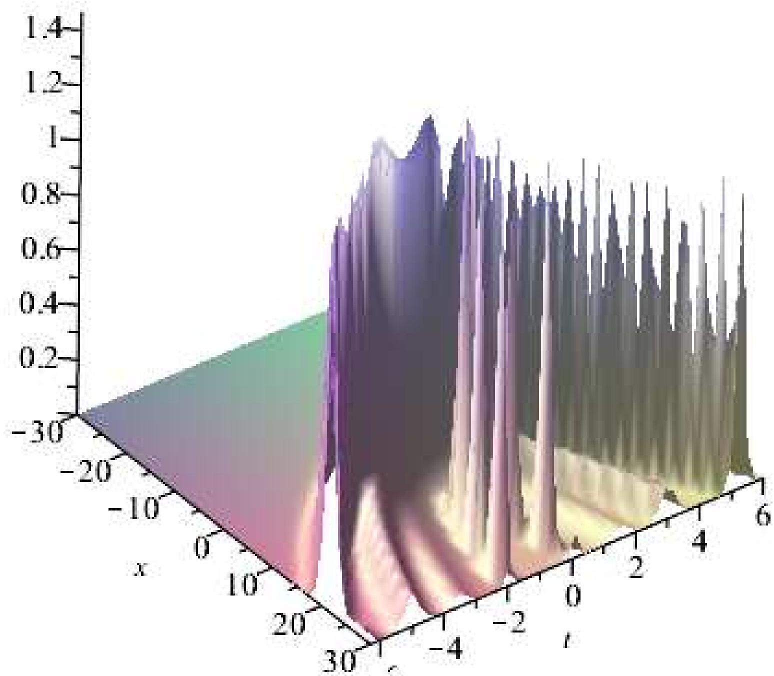

1.2
1
0.8
0.6
0.4
0.2
0
−20
−10
0
10
20
x
−10
0
10
20
t

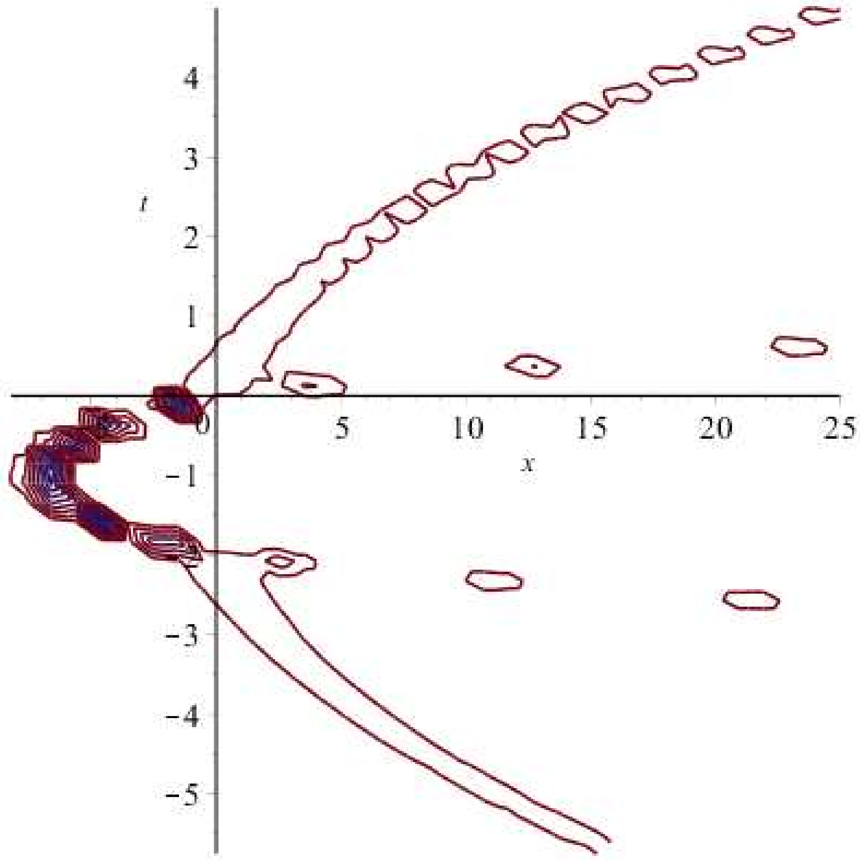

15
10
t
5
−5
−20
−10
0
10
20
x



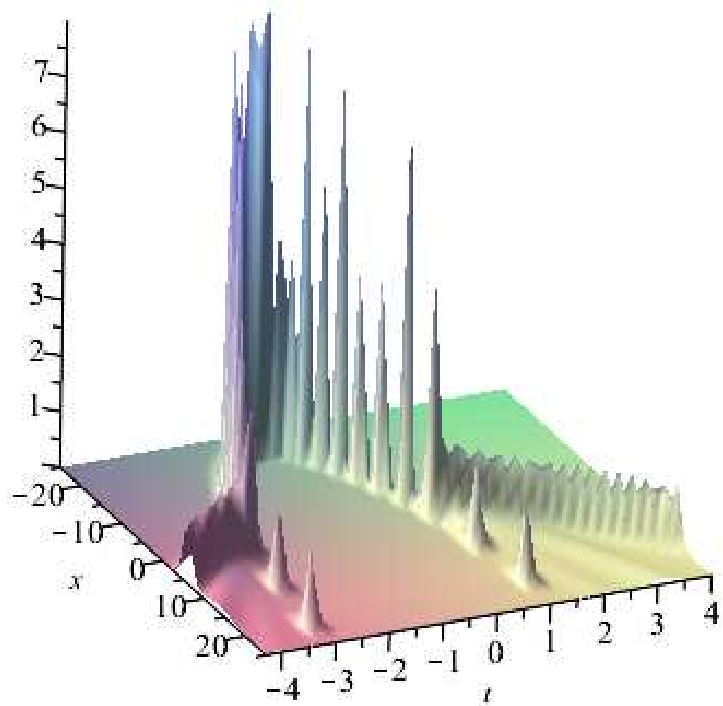
1
0.8
0.6
0.4
0.2
−20
−10
0
10
20
30
x
−4
−2
0
2
4
6
t


4
2
0
−2
−4
−6
t
5
10
15
20
25
30
x

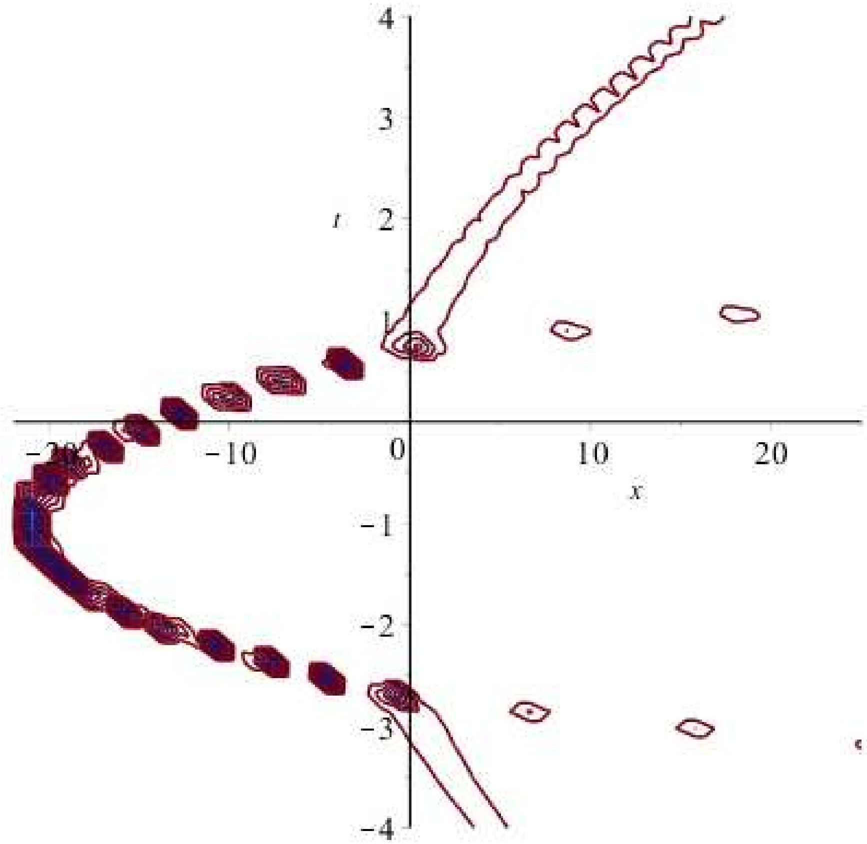

1.4
1.2
1
0.8
0.6
0.4
0.2
−30
−20
−10
0
x
10
20
30
−4
−2
0
2
4
6
t

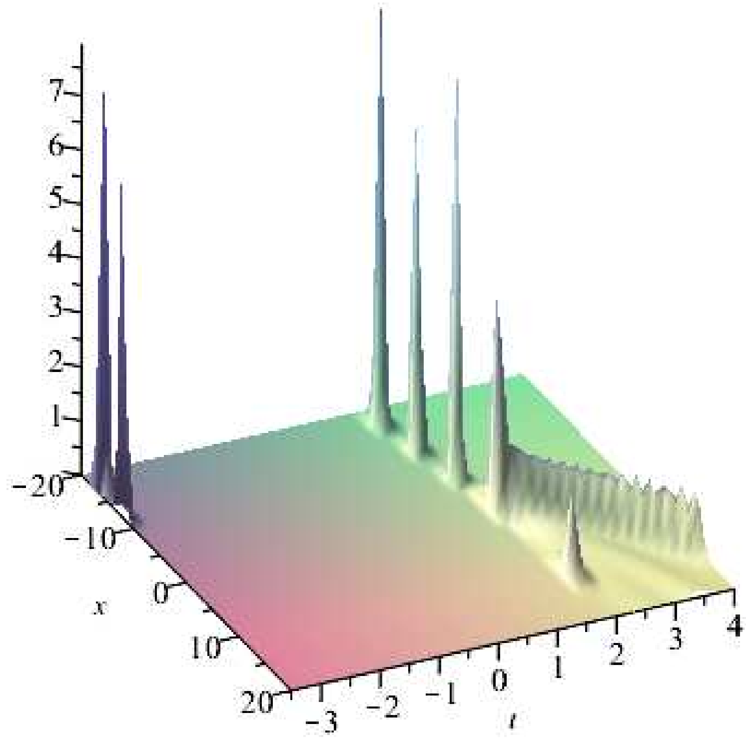


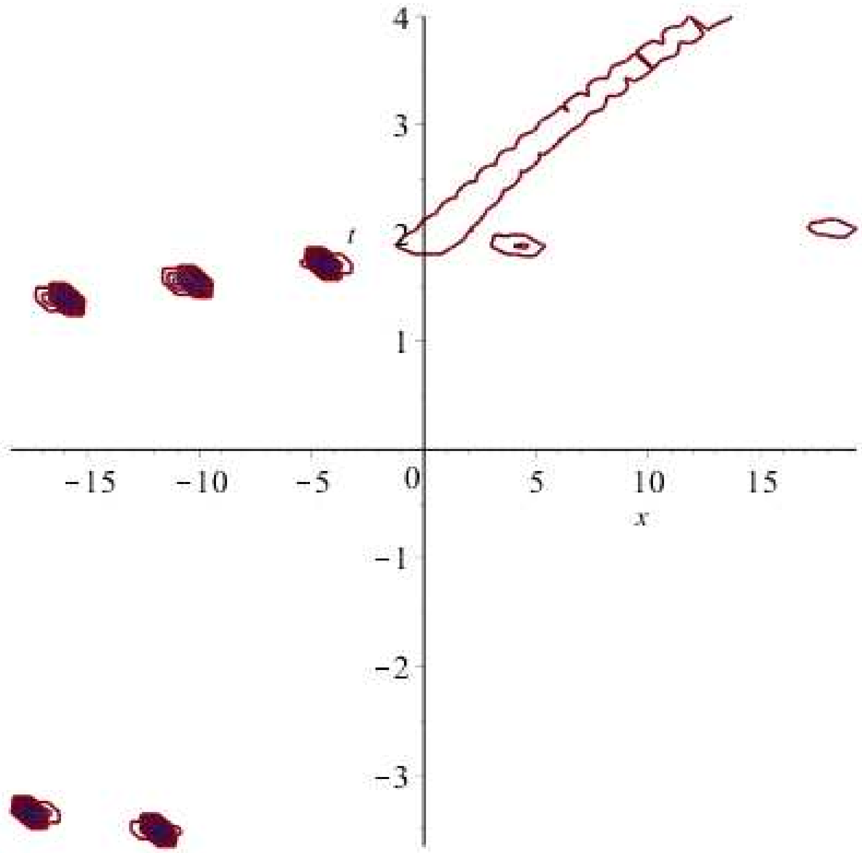
4
3
t
2
1
0
5
10
15
20
25
x
−1
−2
−3
−4
−5



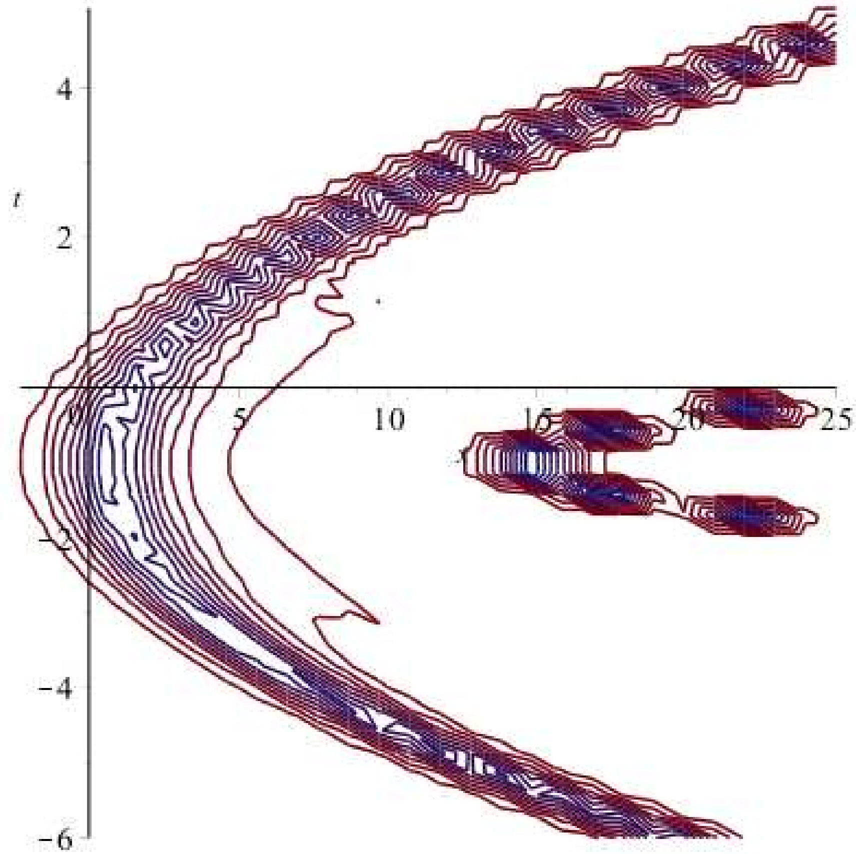
7
6
5
4
3
2
1
−20
−10
x
0
10
20
−4
−3
−2
−1
0
1
2
3
4
t


4
3
t
2
1
−20
−10
0
10
20
x
−1
−2
−3
−4


7
6
5
4
3
2
1
−20
−10
0
10
20
x
−3
−2
−1
0
1
2
3
4
t



4
3
2
1
t
−15
−10
−5
0
5
10
15
x
−1
−2
−3



4
t
2
5
10
15
20
25
-2
-4
-6

Received ** ** ****; | Revised ** ** ****; | Accepted ** ** ****

DOI: xxx/xxxx

ARTICLE TYPE

Study on generalized variable coefficient fifth-order KdV equation based on higher order dispersion term †

Zhao Zhen[1] | Pang Jing*[1,2]

[1]Inner Mongolia University of Technology, Hohhot, Inner Mongolia, China
[2]Inner Mongolia Key Laboratory of Vital Data Statistical Analysis Theory and Neural Network Modeling, Hohhot, Inner Mongolia, China

Correspondence
*Pang Jing, Inner Mongolia University of Technology, New District, Hohhot, Inner Mongolia. Email: pang$_j$@$imut.edu.cn$

Present Address
Inner Mongolia University of Technology, New District, Hohhot, Inner Mongolia

Summary

Nonlinear partial differential equations with higher order dispersion terms play an important role in dynamics research. In this paper, the fifth order KdV equation with high order dispersion term is studied and discussed. Firstly, the bilinear form of the fifth order KdV equation with high order dispersion term is derived by Hirota bilinear form. Then, the combined test function of the positive quartic function, quadratic function, exponential function and the interaction solution of the hyperbolic function of the fifth order KdV equation with variable coefficients is constructed, and the resonance multi-soliton test function of the equation is constructed by using the linear superposition principle.By means of mathematical symbol calculation, the interaction solution between high-order Lump solution and periodic cross kink solution of the fifth order KdV equation with variable coefficients and its resonance multi-solitons are solved.And by observing its corresponding graph analysis of its physical phenomenon.

KEYWORDS:
Higher order dispersion term, fifth-order KdV equation with variable coefficients, Hirota bilinear, interaction solutions, resonant multisolitons

1 | INTRODUCTION

Nonlinear partial differential equations (PDEs) have become very popular in natural science and social science, especially in plasma physics[1], ocean dynamics, lattice dynamics and fluid dynamics, etc. The most famous of the KdV equations in PDEs is the KdV equation So far, many scholars have done a lot of research on nonlinear constant coefficient and variable coefficient KdV equation, and put forward a variety of effective methods. For example: Backlund transform method[2−3], Hirota bilinear method[4−6], (G/G')-expansion method [7], linear superposition principle [8], Wronskian[9], Self similar transformation[10].

At the same time, it is found that the physical linearity described by each type of KdV equation is different. For example, in reference[11], the propagation characteristics of isolated waves in the ocean are described, and the effects of dissipation term and perturbation term contained in the KdV equation on the elastic collision of two isolated waves in the ocean are given. In reference[12], unstable drift waves in plasma physics are described. In reference[13], the propagation of solitons in non-uniform propagation media in quantum mechanics, nonlinear mechanics and other fields is described. In reference[14], isolated waves with large amplitude in atmosphere and ocean are described. In reference[15], the propagation depth and width of small amplitude surface waves in large channels and channels are slowly changed by simulation to keep vorticity not disappearing. After more in-depth research on PDEs, scholars found that the coefficient of dissipation term of KdV equation with variable

†Study on generalized variable coefficient fifth-order KdV equation based on higher order dispersion term

0 **Abbreviations:**Higher order dispersion term; fifth-order KdV equation with variable coefficients; Hirota bilinear; interaction solutions; resonant multisolitons

coefficient changes with time and space, and the equation with variable coefficient can better describe the physical phenomenon and properties behind it. Therefore, in recent years, the study of KdV equation with variable coefficient also increases, and it is found that the method used in the study of constant coefficient can also be used in the study of KdV equation with variable coefficient, and some progress has been made. For example:[16]The author uses the Pfaffians form to express the multi-soliton solution of the KdV equation with variable coefficients, and further analyzes the properties of the understood dynamics. [17]The author constructs two sets of exact periodic and soliton solutions of the fifth order KdV equation with variable coefficients and linear damping terms using the improved sine-cosine method. [18]The author first obtained the n-soliton solution of the KdV equation with variable coefficients in (2+1) dimension by using the Bell polynomial method, and analyzed the effects of soliton fission, fission and rear-end collision on the coefficients. Then the Backlund transformation of the equation was obtained, and the periodic wave solution of the equation was obtained by using the Riemannian function method.

This paper mainly studies the generalized variable coefficient Kortewey-de Vries(KdV) equation with higher order dissipative terms. The form of the generalized variable coefficient KdV equation is

$$u_t + a(t)uu_x + b(t)u_{xxx} + c(t)u^2u_x + d(t)u_xu_{2x} + e(t)uu_{xxx} + f(t)u_{xxxxx} + g(t)u_x\partial_x^{-1}u_y + h(t)u_{xxy} + k(t)uu_y = 0. \tag{1}$$

or

$$\begin{aligned}
&u_t + a(t)uu_x + b(t)u_{xxx} + c(t)u^2u_x + d(t)u_xu_{2x} + e(t)uu_{xxx} + f(t)u_{xxxxx} + g(t)u_xv + h(t)v_{xxx} + k(t)uv_x = 0,\\
&u_y = v_x.
\end{aligned}$$

where $u = u(x, y, t)$ in Eq.(1) is the function space of x, y, t. And $a(t)$, $c(t)$, $d(t)$, $e(t)$, $g(t)$, $k(t)$ are nonlinear term of time t, $b(t)$, $h(t)$ for linear dispersion of time t, $f(t)$ for higher-order dispersion on time t. Eq.(1) tension waves on a gravitational surface at a fluid interface, where surface tension, gravity, and fluid inertia are affected.

The amplitude, shape and velocity of the solitons correspond to the initial state one by one in elastic collision. While the solitons interact with each other, one soliton can be split after collision, and two or more solitons can be split into inelastic collision. The physical phenomenon of inelastic collision is called soliton fission, and conversely, it is called soliton fusion. The variable coefficients of $f(t)$ and $l(t)$ in Eq.(1) can lead to fusion and fission of solitons.

In this paper, bilinear method and linear superposition principle are used to solve the generalized KdV equation with high order dispersive term respectively, and the law of the high order dispersive term changing with time is studied. In the second part, the bilinear form of the fifth order KdV equation with variable coefficients is obtained by Hirota bilinear method. In the third part, the interaction solution of high-order Lump solution and periodic cross kink solution of the equation is obtained by constructing positive quartic function, quadratic function and the combination of exponential function and hyperbolic function. In the fourth part, the test function of the fifth order KdV equation with bilinear variable coefficients is constructed by means of the principle of linear superposition. Mathematical software was used to carry out symbolic calculation, and several groups of interaction solutions between high-order Lump solution and periodic cross kink solution and resonance multi-solitons were obtained, complementing the research on the exact solution of generalized variable coefficient fifth-order KdV equation. Finally, the conclusion is given in section 5.

2 | FIFTH ORDER KDV EQUATION WITH VARIABLE COEFFICIENTS

1. When $c(t) = \delta$, $d(t) = \gamma$, $e(t) = \beta$, $f(t) = \alpha$, $a(t) = g(t) = h(t) = k(t) = 0$, Eq.(1) transforms into the generalized fifth order KdV equation[19] with constant coefficients.

$$u_t + \alpha u_{xxxxx} + \beta uu_{xxx} + \gamma u_xu_{xx} + \delta u^2u_x = 0, \tag{2}$$

Using ansatz and Jacobi elliptic function expansion method, the elliptic cosine wave solution of the equation is constructed.
2. When $a(t) = b(t) = g(t) = h(t) = k(t) = 0$, Eq.(1) transforms into Variable coefficient sawada-kotera equation[20].

$$u_t + f(t)u^2u_x + g(t)u_xu_{x^2} + h(t)uu_{x^3} + k(t)u_{x^5} = 0, \tag{3}$$

The auto-Backlund transform, soliton solution and random soliton solution are obtained by using Hermite transform in Kondrativ distribution space.

3. When $k(t) = 4h(t) = 2g(t)$, $a(t) = 6b(t)$, $c(t) = d(t) = e(t) = f(t) = 0$, Eq.(1) is transformed into a generalized 2+1 dimensional equation[21]

$$u_t - h_1(4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy}) - h_2(6uu_x + u_{xxx}) = 0, \tag{4}$$

The bilinear form, bilinear Backlund transformation, Lax pair and Darboux transformation are constructed by using binary Bell polynomials. The equation is simplified into integrable equation, and the infinite conservation law of the equation is obtained by using binary Bell polynomials.

In this part, the Hirota bilinear method is used to obtain the bilinear form of Eq.(1), and then the form of solution is constructed to obtain the interaction solution of the equation. First use the transformation of related variables

$$u = 2(\ln g)_{xx}, \tag{5}$$

Convert Eq.(1) to the bilinear equation as follows

$$\begin{aligned} B_{KdV}(g) &= (D_xD_t + b(t)D_x^4 + f(t)D_x^6 + h(t)D_x^3D_y)g\cdot g\\ &= -g_tg_x + gg_{xt} + 3b(t)g_{xx}^2 - 4b(t)g_xg_{xxx} - 10f(t)g_{xxx}^2 + b(t)gg_{xxxx} + 15f(t)g_{xx}g_{xxxx}\\ &\quad -6f(t)g_xg_{xxxxx} + f(t)gg_{xxxxxx} + h(t)gg_{xxxy} - 3h(t)g_xg_{xxy} + 3h(t)g_{xx}g_{xy} - h(t)g_{xxx}g_y\\ &= 0. \end{aligned} \tag{6}$$

where $g = g(x, y, t)$, D-operator is defined by reference[22] :

$$D_x^mD_y^nD_t^k fg = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'})^n(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^k f(x, y, t)g(x', y', t')|_{x'=x,y'=y,t'=t}.$$

3 | INTERACTION SOLUTION BETWEEN HIGH-ORDER LUMP SOLUTION AND PERIODIC CROSS KINK SOLUTION

In order to obtain the interaction solution between the high-order Lump solution and the periodic cross kinking solution of the generalized KdV equation of the fifth order with variable coefficients, the following positive quartic function, quadratic function and the test function of the combination of exponential function and hyperbolic function are assumed

$$g = e^{-p_1\xi_1} + k_1e^{p_1\xi_2} + k_2\cos(p_2\xi_2) + k_3\cosh(\xi_3) + k_4\sinh(\xi_4) + k_5\sin(\xi_5) + a_{21}. \tag{7}$$

where

$$\begin{aligned} \xi_1 &= a_1x + a_2y + a_3(t) + a_4,\\ \xi_2 &= a_5x + a_6y + a_7(t) + a_8,\\ \xi_3 &= a_9x + a_{10}y + a_{11}(t) + a_{12},\\ \xi_4 &= a_{13}x + a_{14}y + a_{15}(t) + a_{16},\\ \xi_5 &= a_{17}x + a_{18}y + a_{19}(t) + a_{20}, \end{aligned}$$

where $a_i(2 \le i \le 21)$, $k_j(1 \le j \le 5)$ are constant. A set of algebraic equations about $a_i(1 \le i \le 21$, and $i \ne 3, 7, 9, 11, 15, 19)$ are obtained by substituting Eq.(7) into Eq.(1) and making the coefficients of x and y zero. By solving this set of overdetermined equations, we can find the following sets of rich solutions.

Case 1:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_9 = 0, a_{11}(t) = \int_a^t a_{17}^2a_{10}h(s)\,\mathrm{d}s, a_{13} = 0, p_1 = 0,\\ a_{15}(t) = \int_a^t a_{17}^2a_{14}h(s)\,\mathrm{d}s, a_{19}(t) = \int_a^t 4a_{17}^2f(s)\,\mathrm{d}s, k_1 = 0, k_2 = 0, f(t) = f(t), h(t) = h(t), \end{cases} \tag{8}$$

a_2,a_4,a_6,a_8,a_{10},a_{12},a_{14},a_{16},a_{18},k_3,k_4,k_5,p_2 are all constants. a_{17} is a constant that is not equal to zero, and the relation between the dispersion term of Eq.(1) and the higher-order dispersion term is $b(t) = \frac{5a_{17}^3f(t)-a_{18}h(t)}{a_{17}}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_1(x,y,t)=\frac{\Phi_1}{\Psi_1} \tag{9}$$

where

$$\begin{aligned}\Phi_1 =& -2a_{17}^2k_5(\sin(a_{19}(t)+a_{17}x+a_{18}y+a_{20})(k_4\sinh\left(a_{15}(t)+a_{14}y+a_{16}\right)+\\ & k_3\cosh\left(a_{11}(t)+a_{10}y+a_{12}\right)+a_{21}+1)+k_5),\\ \Psi_1 =& 2k_5\sin\left(a_{19}(t)+a_{17}x+a_{18}y+a_{20}\right)(k_4\sinh\left(a_{15}(t)+a_{14}y+a_{16}\right)+\\ & k_3\cosh(a_{11}(t)+a_{10}y+a_{12})+a_{21}+1)+2\left(a_{21}+1\right)k_4\sinh(a_{15}(t)+a_{14}y+a_{16})+\\ & k_3^2\cosh\left(a_{11}(t)+a_{10}y+a_{12}\right)^2+k_4^2\cosh\left(a_{15}(t)+a_{14}y+a_{16}\right)^2+\left(a_{21}+1\right)^2\\ & +2k_3\cosh(a_{11}(t)+a_{10}y+a_{12})(k_4\sinh\left(a_{15}(t)+a_{14}y+a_{16}\right)+a_{21}+1)-k_4^2+k_5^2\\ & -k_5^2\cos\left(a_{19}(t)+a_{17}x+a_{18}y+a_{20}\right)^2.\end{aligned}$$

Case 2:

$$\begin{cases} a_1=0Ca, a_3(t)=\int_a^t a_{17}^2a_2h(s)\,\mathrm{d}s, a_5=0, a_7(t)=a_7(t), a_9=0, a_{11}(t)=\int_a^t a_{17}^2a_{10}h(s)\,\mathrm{d}s, a_{13}=0,\\ a_{15}(t)=\int_a^t a_{17}^2a_{14}h(s)\,\mathrm{d}s, a_{19}(t)=\int_a^t 4a_{17}^5f(s)\,\mathrm{d}s, k_1=0, k_2=0, f(t)=f(t), h(t)=h(t),\end{cases} \tag{10}$$

a_2,a_4,a_6,a_8,a_{10},a_{12},a_{14},a_{16},a_{18},$k_i(3\le i\le 5)$,p_1,p_2 are all constants. a_{17} is a constant that is not equal to zero, and the relation between the dispersion term of Eq.(1) and the higher-order dispersion term is $b(t)=\frac{5a_{17}^3f(t)-a_{18}h(t)}{a_{17}}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_2(x,y,t)=\frac{\Phi_2}{\Psi_2}, \tag{11}$$

where

$$\begin{aligned}\Phi_2 =& -2a_{17}^2k_5(\sin(a_{19}(t)+a_{17}x+a_{18}y+a_{20})(k_4\sinh(a_{15}(t)+a_{14}y+a_{16})+\\ & k_3\cosh(a_{11}(t)+a_{10}y+a_{12})+a_{21})+e^{-p_1(a_3(t)+a_2y+a_4)}\sin(a_{19}(t)+a_{17}x+a_{18}y+a_{20})+k_5),\\ \Psi_2 =& e^{-p_1(a_3(t)+a_2y+a_4)}(2k_5\sin(a_{19}(t)+a_{17}x+a_{18}y+a_{20})+2k_4\sinh(a_{15}(t)+a_{14}y+a_{16})+\\ & 2k_3\cosh(a_{11}(t)+a_{10}y+a_{12})+2a_{21})-k_5^2\cos(a_{19}(t)+a_{17}x+a_{18}y+a_{20})^2+\\ & \sin(a_{19}(t)+a_{17}x+a_{18}y+a_{20})(2k_4k_5\sinh(a_{15}(t)+a_{14}y+a_{16})+a_{21}^2-k_4^2+k_5^2\\ & +2k_3k_5\cosh(a_{11}(t)+a_{10}y+a_{12})+2a_{21}k_5)+2a_{21}k_4\sinh(a_{15}(t)+a_{14}y+a_{16})+\\ & k_3^2\cosh(a_{11}(t)+a_{10}y+a_{12})^2+k_4^2\cosh(a_{15}(t)+a_{14}y+a_{16})^2+\cosh(a_{11}(t)+a_{10}y+a_{12})\\ & (2k_4k_3\sinh(a_{15}(t)+a_{14}y+a_{16})+2a_{21}k_3)+e^{-2p_1(a_3(t)+a_2y+a_4)}.\end{aligned}$$

Case 3:

$$\begin{cases} a_1=0, a_3(t)=a_3(t)a_5=0, a_7(t)=a_7(t), a_9=0, a_{11}(t)=-\int_a^t a_{13}^2a_{10}h(s)\,\mathrm{d}s, k_1=0, k_2=0,\\ a_{15}(t)=4\int_a^t a_{13}^5f(s)\,\mathrm{d}s, a_{17}=0, a_{19}(t)=-\int_a^t a_{13}^2a_{18}h(s)\,\mathrm{d}s, p_1=0, f(t)=f(t), h(t)=h(t),\end{cases} \tag{12}$$

a_2,a_4,a_6,a_8,a_{10},a_{12},a_{13},a_{14},a_{16},a_{18},$k_i(3\le i\le 5)$,p_2 are all constants. a_{13} is a constant that is not equal to zero, and the relation between the dispersion term of Eq.(1) and the higher-order dispersion term is $b(t)=-\frac{5a_{13}^3f(t)+a_{14}h(t)}{a_{13}}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_3(x,y,t)=\frac{\Phi_3}{\Psi_3}, \tag{13}$$

where

$$\begin{aligned}\Phi_3 =& 2a_{13}^2k_4(k_4\sinh\left(a_{15}(t)+a_{13}x+a_{14}y+a_{16}\right)^2-k_4\cosh(a_{15}(t)+a_{13}x+a_{14}y+a_{16})^2+\\ & \sinh(a_{15}(t)+a_{13}x+a_{14}y+a_{16})(k_5\sin(a_{19}(t)+a_{18}y+a_{20})+k_3\cosh(a_{11}(t)+a_{10}y+\\ & a_{12})+a_{21}+1)),\\ \Psi_3 =& (k_4\sinh(a_{15}(t)+a_{13}x+a_{14}y+a_{16})+k_5\sin(a_{19}(t)+a_{18}y+a_{20})+\\ & k_3\cosh(a_{11}(t)+a_{10}y+a_{12})+a_{21}+1)^2.\end{aligned}$$

Case 4:

$$\begin{cases} a_1 = 0, a_3(t) = -\int_a^t a_{13}^2 a_2 h(s)\,\mathrm{d}s, a_5 = 0, a_9 = 0, a_{11}(t) = -\int_a^t a_{13}^2 a_{10} h(s)\,\mathrm{d}s, k_1 = 0, \\ a_{15}(t) = 4\int_a^t a_{13}^5 f(s)\,\mathrm{d}s, a_{17} = 0, a_{19}(t) = -\int_a^t a_{13}^2 a_{18} h(s)\,\mathrm{d}s, k_2 = 0, f(t) = f(t), h(t) = h(t), \end{cases} \tag{14}$$

a_2,a_4,a_6,a_8,a_{10},a_{12},a_{13},a_{14},a_{16},a_{18},$k_i(3 \leq i \leq 5)$,p_1,p_2 are all constants. a_{13} is a constant that is not equal to zero, and the relation between the dispersion term of Eq.(1) and the higher-order dispersion term is $b(t) = -\frac{5a_{13}^3 f(t)+a_{14}h(t)}{a_{13}}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_4(x, y, t) = \frac{\Phi_4}{\Psi_4}, \tag{15}$$

where

$$\begin{aligned}\Phi_4 = {} & 2a_{13}^2 k_4(k_4 \sinh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right)^2 - k_4 \cosh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right)^2 \\ & + \sinh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right)(k_5 \sin\left(a_{19}(t) + a_{18}y + a_{20}\right) + k_3 \cosh(a_{11}(t) + a_{10}y \\ & + a_{12}) + a_{21}) + e^{-p_1\left(a_3(t)+a_2y+a_4\right)} \sinh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right)), \\ \Psi_4 = {} & (k_4 \sinh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right) + k_5 \sin\left(a_{19}(t) + a_{18}y + a_{20}\right) + \\ & k_3 \cosh\left(a_{11}(t) + a_{10}y + a_{12}\right) + e^{-p_1\left(a_3(t)+a_2y+a_4\right)} + a_{21})^2.\end{aligned}$$

Case 5:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), a_9 = 0, a_{10} = 0, a_{11}(t) = 0, a_{14} = \frac{a_{13}a_{18}}{a_{17}}, a_{15}(t) = 0, \\ a_{19}(t) = 0, k_1 = 0, k_2 = 0, p_1 = 0, f(t) = 0, h(t) = h(t), \end{cases} \tag{16}$$

a_2,a_4,a_6,a_8,a_{12},a_{13},a_{16},a_{18},a_{20},a_{21},$k_i(3 \leq i \leq 5)$,p_2 are all constants. a_{17} is not equal to zero for the constant, Eq.(1) the high order dispersion is zero, the relationship between a linear dispersion $b(t) = -\frac{a_{18}h(t)}{a_{17}}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_5(x, y, t) = \frac{\Phi_5}{\Psi_5}, \tag{17}$$

where

$$\begin{aligned}\Phi_5 = {} & -2a_{17}^2 k_5\left(k_3 \cosh\left(a_{11}(t) + a_{12}\right) + a_{21} + 1\right) \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) - \\ & 2k_4 \sinh\left(\frac{a_{17}a_{15}(t) + a_{17}\left(a_{13}x + a_{16}\right) + a_{13}a_{18}y}{a_{17}}\right)(\left(a_{17}^2 - a_{13}^2\right) k_5 \sin(a_{19}(t) + a_{17}x + \\ & a_{18}y + a_{20}) - a_{13}^2(k_3 \cosh\left(a_{11}(t) + a_{12}\right) + a_{21} + 1)) - 4a_{13}a_{17}k_4k_5 \cos(a_{19}(t) \\ & 2a_{17}^2k_5^2 - 2a_{13}^2k_4^2 + a_{17}x + a_{18}y + a_{20}) \cosh(\frac{a_{17}a_{15}(t) + a_{17}\left(a_{13}x + a_{16}\right) + a_{13}a_{18}y}{a_{17}}), \\ \Psi_5 = {} & +2k_5\left(k_3 \cosh\left(a_{11}(t) + a_{12}\right) + a_{21} + 1\right) \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) + \\ & 2k_4 \sinh\left(\frac{a_{17}a_{15}(t) + a_{17}\left(a_{13}x + a_{16}\right) + a_{13}a_{18}y}{a_{17}}\right)(k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) + \\ & k_3 \cosh\left(a_{11}(t) + a_{12}\right) + a_{21} + 1) + 2\left(a_{21} + 1\right) k_3 \cosh\left(a_{11}(t) + a_{12}\right) + \\ & k_3^2 \cosh(a_{11}(t) + a_{12})^2 + (a_{21} + 1)^2 - k_4^2 + k_5^2 - k_5^2 \cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2 \\ & + k_4^2 \cosh(\frac{a_{17}a_{15}(t) + a_{17}(a_{13}x + a_{16}) + a_{13}a_{18}y}{a_{17}})^2.\end{aligned}$$

Case 6:

$$\begin{cases} a_1 = 0, a_2 = 0, a_3(t) = 0, a_5 = 0, a_7(t) = a_7(t), a_9 = 0, a_{10} = 0, a_{11}(t) = 0, \\ a_{14} = \frac{a_{13}a_{18}}{a_{17}}, a_{15}(t) = 0, a_{19}(t) = 0, k_1 = 0, k_2 = 0, f(t) = 0, h(t) = h(t), \end{cases} \tag{18}$$

a_4, a_6, a_8, a_{12}, a_{13}, a_{16}, a_{18}, a_{20}, a_{21}, $k_i(3 \le i \le 5)$, p_1,p_2 are all constants. a_{17} is not equal to zero for the constant, Eq.(1) the high order dispersion is zero, the relationship between a linear dispersion $b(t) = -\frac{a_{18}h(t)}{a_{17}}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_6(x, y, t) = \frac{\Phi_6}{\Psi_6}, \tag{19}$$

where

$$\begin{aligned}\Phi_6 = &-2a_{17}^2k_5\left(k_3\cosh\left(a_{11}(t)+a_{12}\right)+e^{-p_1\left(a_3(t)+a_4\right)}+a_{21}\right)\sin\left(a_{19}(t)+a_{17}x+a_{18}y+a_{20}\right)\\ &+2k_4\sinh\left(\frac{a_{17}a_{15}(t)+a_{17}\left(a_{13}x+a_{16}\right)+a_{13}a_{18}y}{a_{17}}\right)(a_{13}^2(k_3\cosh(a_{11}(t)+a_{12})+\\ &e^{-p_1(a_3(t)+a_4)}+a_{21})+(a_{13}^2-a_{17}^2)k_5\sin(a_{19}(t)+a_{17}x+a_{18}y+a_{20}))-4a_{13}a_{17}k_4k_5\\ &\cos\left(a_{19}(t)+a_{17}x+a_{18}y+a_{20}\right)\cosh\left(\frac{a_{17}a_{15}(t)+a_{17}\left(a_{13}x+a_{16}\right)+a_{13}a_{18}y}{a_{17}}\right)\\ &-2a_{17}^2k_5^2-2a_{13}^2k_4^2,\\ \Psi_6 = &\,2k_5\left(k_3\cosh\left(a_{11}(t)+a_{12}\right)+e^{-p_1\left(a_3(t)+a_4\right)}+a_{21}\right)\sin\left(a_{19}(t)+a_{17}x+a_{18}y+a_{20}\right)+\\ &2k_4\sinh\left(\frac{a_{17}a_{15}(t)+a_{17}\left(a_{13}x+a_{16}\right)+a_{13}a_{18}y}{a_{17}}\right)(k_5\sin\left(a_{19}(t)+a_{17}x+a_{18}y+a_{20}\right)\\ &+k_3\cosh(a_{11}(t)+a_{12})+e^{-p_1(a_3(t)+a_4)}+a_{21})+e^{-p_1(a_3(t)+a_4)}(2k_3\cosh(a_{11}(t)+a_{12})\\ &+2a_{21})-k_5^2\cos(a_{19}(t)+a_{17}x+a_{18}y+a_{20})^2+2a_{21}k_3\cosh(a_{11}(t)+a_{12})+\\ &k_3^2\cosh(a_{11}(t)+a_{12})^2+e^{-2p_1(a_3(t)+a_4)}+a_{21}^2-k_4^2+k_5^2\\ &+k_4^2\cosh(\frac{a_{17}a_{15}(t)+a_{17}(a_{13}x+a_{16})+a_{13}a_{18}y}{a_{17}})^2.\end{aligned}$$

Case 7:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), a_9 = 0, a_{11}(t) = 0, a_{15}(t) = 0,\\ a_{19}(t) = 0, k_1 = 0, k_2 = 0, p_1 = 0, f(t) = 0, h(t) = 0, b(t) = 0,\end{cases} \tag{20}$$

a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{13}, a_{14}, a_{16}, a_{18}, a_{20}, a_{21}, $k_i(3 \le i \le 5)$, p_2 are all constants. Both the dispersion term and the higher-order dispersion term of Eq.(1) are zero. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_7(x, y, t) = \frac{\Phi_7}{\Psi_7}, \tag{21}$$

where

$$\begin{aligned}\Phi_7 = &-4a_{13}a_{17}k_4k_5\cos(a_{19}(t)+a_{17}x+a_{18}y+a_{20})\cosh(a_{15}(t)+a_{13}x+a_{14}y+a_{16})-\\ &2a_{17}^2k_5(k_3\cosh(a_{11}(t)+a_{10}y+a_{12})+a_{21}+1)\sin(a_{19}(t)+a_{17}x+a_{18}y+a_{20})-x\\ &+a_{14}y+a_{16})-2k_4\sinh(a_{15}(t)+a_{13}x+a_{14}y+a_{16})(\left(a_{17}^2-a_{13}^2\right)k_5\sin(a_{19}(t)+\\ &a_{17}x+a_{18}y+a_{20})-a_{13}^2\left(k_3\cosh\left(a_{11}(t)+a_{10}y+a_{12}\right)+a_{21}+1\right))-2a_{17}^2k_5^2-2a_{13}^2k_4^2,\\ \Psi_7 = &-k_5^2\cos\left(a_{19}(t)+a_{17}x+a_{18}y+a_{20}\right)^2+k_4^2\cosh\left(a_{15}(t)+a_{13}x+a_{14}y+a_{16}\right)^2+\\ &2k_5\left(k_3\cosh\left(a_{11}(t)+a_{10}y+a_{12}\right)+a_{21}+1\right)\sin\left(a_{19}(t)+a_{17}x+a_{18}y+a_{20}\right)+\\ &2k_4\sinh\left(a_{15}(t)+a_{13}x+a_{14}y+a_{16}\right)(k_5\sin\left(a_{19}(t)+a_{17}x+a_{18}y+a_{20}\right)+\\ &k_3\cosh(a_{11}(t)+a_{10}y+a_{12})+a_{21}+1)+2\left(a_{21}+1\right)k_3\cosh\left(a_{11}(t)+a_{10}y+a_{12}\right)\\ &+k_3^2\cosh\left(a_{11}(t)+a_{10}y+a_{12}\right)^2+\left(a_{21}+1\right)^2-k_4^2+k_5^2.\end{aligned}$$

Case 8:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), a_9 = 0, a_{11}(t) = 0,\\ a_{15}(t) = 0, a_{19}(t) = 0, k_1 = 0, k_2 = 0, f(t) = 0, h(t) = 0, b(t) = 0,\end{cases} \tag{22}$$

a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{13}, a_{14}, a_{16}, a_{17}, a_{18}, a_{20}, a_{21}, $k_i(3 \leq i \leq 5)$, p_1, p_2 are all constants. Both the dispersion term and the higher-order dispersion term of Eq.(1) are zero. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_8(x, y, t) = \frac{\Phi_8}{\Psi_8}, \tag{23}$$

where

$$\begin{aligned}
\Phi_8 &= e^{-p_1\left(a_3(t)+a_2y+a_4\right)}(2a_{13}^2k_4 \sinh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right) - 2a_{17}^2k_5 \sin(a_{19}(t) \\
&\quad +a_{17}x + a_{18}y + a_{20})) - 4a_{13}a_{17}k_4k_5 \cos\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) \cosh(a_{15}(t) \\
&\quad +a_{13}x + a_{14}y + a_{16}) - 2a_{17}^2k_5\left(k_3 \cosh\left(a_{11}(t) + a_{10}y + a_{12}\right) + a_{21}\right) \sin(a_{19}(t) + \\
&\quad a_{17}x + a_{18}y + a_{20}) + 2k_4 \sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})(\left(a_{13}^2 - a_{17}^2\right) k_5 \sin(a_{19}(t) + \\
&\quad a_{17}x + a_{18}y + a_{20}) + a_{13}^2(k_3 \cosh\left(a_{11}(t) + a_{10}y + a_{12}\right) + a_{21})) - 2a_{17}^2k_5^2 - 2a_{13}^2k_4^2, \\
\Psi_8 &= e^{-p_1\left(a_3(t)+a_2y+a_4\right)}(2k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) + 2k_4 \sinh(a_{15}(t) + a_{13}x \\
&\quad +a_{14}y + a_{16}) + 2k_3 \cosh\left(a_{11}(t) + a_{10}y + a_{12}\right) + 2a_{21}) - k_5^2 \cos(a_{19}(t) + a_{17}x + \\
&\quad a_{18}y + a_{20})^2 + k_4^2 \cosh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})^2 + (2k_3k_5 \cosh(a_{11}(t) + a_{10}y + \\
&\quad a_{12}) + 2a_{21}k_5) \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) + \sinh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right) \\
&\quad \left(2k_5k_4 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) + 2k_3k_4 \cosh\left(a_{11}(t) + a_{10}y + a_{12}\right) + 2a_{21}k_4\right) \\
&\quad +2a_{21}k_3 \cosh\left(a_{11}(t) + a_{10}y + a_{12}\right) + k_3^2 \cosh\left(a_{11}(t) + a_{10}y + a_{12}\right)^2 + e^{-2p_1\left(a_3(t)+a_2y+a_4\right)} \\
&\quad +a_{21}^2 - k_4^2 + k_5^2.
\end{aligned}$$

Case 9:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), a_{11}(t) = 0, a_{13} = 0, a_{15}(t) = a_{15}(t), \\ a_{19}(t) = 0, k_1 = 0, k_2 = 0, p_1 = 0, f(t) = 0, h(t) = 0, b(t) = 0, \end{cases} \tag{24}$$

a_2, a_4, a_6, a_8, a_9, a_{10}, a_{12}, a_{14}, a_{16}, a_{17}, a_{18}, a_{20}, a_{21}, $k_i(3 \leq i \leq 5)$, p_2 are all constants. Both the dispersion term and the higher-order dispersion term of Eq.(1) are zero. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_9(x, y, t) = \frac{\Phi_9}{\Psi_9}, \tag{25}$$

where

$$\begin{aligned}
\Phi_9 &= -2\left(a_{21} + 1\right) a_{17}^2k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) - 4a_9a_{17}k_3k_5 \cos(a_{19}(t) + a_{17}x \\
&\quad +a_{18}y + a_{20}) \sinh\left(a_{11}(t) + a_9x + a_{10}y + a_{12}\right) + 2k_3 \cosh\left(a_{11}(t) + a_9x + a_{10}y + a_{12}\right) \\
&\quad \left(\left(a_9^2 - a_{17}^2\right) k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) + \left(a_{21} + 1\right) a_9^2\right) - 2a_{17}^2k_5^2 + 2a_9^2k_3^2, \\
\Psi_9 &= 2\left(a_{21} + 1\right) k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) - k_5^2 \cos\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right)^2 \\
&\quad +k_3^2 \cosh\left(a_{11}(t) + a_9x + a_{10}y + a_{12}\right)^2 + 2k_3 \cosh\left(a_{11}(t) + a_9x + a_{10}y + a_{12}\right) \\
&\quad \left(k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) + a_{21} + 1\right) + \left(a_{21} + 1\right)^2 + k_5^2.
\end{aligned}$$

Case 10:

$$\begin{cases} a_1 = 0, a_2 = 0, a_3(t) = 0, a_5 = 0, a_{10} = \frac{a_9a_{18}}{a_{17}}, a_{11}(t) = 0, a_{13} = 0, \\ a_{15}(t) = a_{15}(t), a_{19}(t) = 0, k_1 = 0, k_2 = 0, k_4 = 0, f(t) = 0, h(t) = h(t), \end{cases} \tag{26}$$

a_4, a_6, a_8, a_9, a_{12}, a_{14}, a_{16}, a_{17}, a_{18}, a_{20}, a_{21}, k_3, k_5, p_1, p_2 are all constants. a_{17} is not equal to zero for the constant, Eq.(1) the high order dispersion is zero, the relationship between a linear dispersion $b(t) = -\frac{a_{18}h(t)}{a_{17}}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_{10}(x, y, t) = \frac{\Phi_{10}}{\Psi_{10}}, \tag{27}$$

where

$$
\begin{aligned}
\Phi_{10} = & -2a_{17}^2 k_5 \left(e^{-p_1\left(a_3(t)+a_4\right)} + a_{21}\right) \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) - \\
& 2k_3 \cosh\left(\frac{a_{17}a_{11}(t) + a_{17}\left(a_9 x + a_{12}\right) + a_9 a_{18} y}{a_{17}}\right)\left(\left(a_{17}^2 - a_9^2\right) k_5 \sin(a_{19}(t) + a_{17}x\right. \\
& \left.+a_{18}y + a_{20}) - a_9^2(e^{-p_1\left(a_3(t)+a_4\right)} + a_{21})\right) - 4a_9 a_{17} k_3 k_5 \cos\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) \\
& \sinh\left(\frac{a_{17}a_{11}(t) + a_{17}\left(a_9 x + a_{12}\right) + a_9 a_{18} y}{a_{17}}\right) - 2a_{17}^2 k_5^2 + 2a_9^2 k_3^2, \\
\Psi_{10} = & \, 2k_5 \left(e^{-p_1\left(a_3(t)+a_4\right)} + a_{21}\right) \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) + \\
& 2k_3 \cosh\left(\frac{a_{17}a_{11}(t) + a_{17}\left(a_9 x + a_{12}\right) + a_9 a_{18} y}{a_{17}}\right)\left(k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right)\right. \\
& \left.+e^{-p_1\left(a_3(t)+a_4\right)} + a_{21}\right) - k_5^2 \cos\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right)^2 + \\
& k_3^2 \cosh\left(\frac{a_{17}a_{11}(t) + a_{17}\left(a_9 x + a_{12}\right) + a_9 a_{18} y}{a_{17}}\right)^2 + 2a_{21} e^{-p_1\left(a_3(t)+a_4\right)} + e^{-2p_1\left(a_3(t)+a_4\right)} \\
& +a_{21}^2 + k_5^2.
\end{aligned}
$$

Case 11:

$$
\begin{cases}
a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), a_{10} = \frac{a_9 a_{18}}{a_{17}}, a_{11}(t) = 0, a_{13} = 0, \\
a_{15}(t) = a_{15}(t), a_{19}(t) = 0, k_1 = 0, k_2 = 0, k_4 = 0, p_1 = 0, f(t) = 0, h(t) = h(t),
\end{cases} \tag{28}
$$

a_2, a_4, a_6, a_8, a_9, a_{12}, a_{14}, a_{16}, a_{17}, a_{18}, a_{20}, a_{21}, k_3, k_5, p_2 are all constants. a_{17} is not equal to zero for the constant, Eq.(1) the high order dispersion is zero, the relationship between a linear dispersion $b(t) = -\frac{a_{18}h(t)}{a_{17}}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$
u_{11}(x, y, t) = \frac{\Phi_{11}}{\Psi_{11}}, \tag{29}
$$

where

$$
\begin{aligned}
\Phi_{11} = & -2\left(a_{21} + 1\right) a_{17}^2 k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) - 4a_9 a_{17} k_3 k_5 \cos(a_{19}(t) + a_{17}x \\
& +a_{18}y + a_{20}) \sinh\left(\frac{a_{17}a_{11}(t) + a_{17}\left(a_9 x + a_{12}\right) + a_9 a_{18} y}{a_{17}}\right) + \\
& 2k_3 \cosh\left(\frac{a_{17}a_{11}(t) + a_{17}\left(a_9 x + a_{12}\right) + a_9 a_{18} y}{a_{17}}\right)\left(\left(a_9^2 - a_{17}^2\right) k_5 \sin(a_{19}(t) + a_{17}x + \right. \\
& \left. a_{18}y + a_{20}) + \left(a_{21} + 1\right) a_9^2\right) - 2a_{17}^2 k_5^2 + 2a_9^2 k_3^2, \\
\Psi_{11} = & \, 2\left(a_{21} + 1\right) k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right) - k_5^2 \cos\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right)^2 \\
& +k_3^2 \cosh\left(\frac{a_{17}a_{11}(t) + a_{17}\left(a_9 x + a_{12}\right) + a_9 a_{18} y}{a_{17}}\right)^2 + \\
& 2k_3 \cosh\left(\frac{a_{17}a_{11}(t) + a_{17}\left(a_9 x + a_{12}\right) + a_9 a_{18} y}{a_{17}}\right)\left(k_5 \sin\left(a_{19}(t) + a_{17}x + a_{18}y + a_{20}\right)\right. \\
& \left.+a_{21} + 1\right) + \left(a_{21} + 1\right)^2 + k_5^2.
\end{aligned}
$$

Case 12:

$$
\begin{cases}
a_1 = 0, a_3(t) = -\int_a^t a_9^2 a_2 h(s)\,\mathrm{d}s, a_5 = 0, a_7(t) = a_7(t), a_{11}(t) = 4\int_a^t a_9^5 f(s)\,\mathrm{d}s, a_{13} = 0, a_{17} = 0, \\
a_{15}(t) = a_{15}(t), a_{19}(t) = -\int_a^t a_9^2 a_{18} h(s)\,\mathrm{d}s, k_1 = 0, k_2 = 0, k_4 = 0, p_1 = 0, f(t) = 0, h(t) = h(t),
\end{cases} \tag{30}
$$

a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{18}, a_{20}, a_{21}, k_3, k_5, p_2 are all constants. a_9 is a constant that is not equal to zero, and the relation between the dispersion term of Eq.(1) and the higher-order dispersion term is $b(t) = -\frac{2a_9^3 f(t)+a_{10}h(t)}{a_9}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_{12}(x,y,t) = \frac{\Phi_{12}}{\Psi_{12}}, \tag{31}$$

where

$$\begin{aligned}
\Phi_{12} &= 2a_9^2 k_3(\left(k_5 \sin\left(a_{19}(t) + a_{18}y + a_{20}\right) + a_{21}\right)\cosh\left(a_{11}(t) + a_9 x + a_{10}y + a_{12}\right) + \\
&\quad e^{-p_1(a_3(t)+a_2y+a_4)}\cosh\left(a_{11}(t) + a_9 x + a_{10}y + a_{12}\right) + k_3), \\
\Psi_{12} &= e^{-p_1\left(a_3(t)+a_2y+a_4\right)}(2k_3\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12}) + 2k_5\sin(a_{19}(t) + a_{18}y \\
&\quad + a_{20}) + 2a_{21}) + k_3^2\cosh\left(a_{11}(t) + a_9x + a_{10}y + a_{12}\right)^2 + (2k_5k_3\sin\left(a_{19}(t) + a_{18}y + a_{20}\right) \\
&\quad + 2a_{21}k_3)\cosh\left(a_{11}(t) + a_9x + a_{10}y + a_{12}\right) + 2a_{21}k_5\sin(a_{19}(t) + a_{18}y + \\
&\quad a_{20}) - k_5^2\cos\left(a_{19}(t) + a_{18}y + a_{20}\right)^2 + e^{-2p_1\left(a_3(t)+a_2y+a_4\right)} + a_{21}^2 + k_5^2.
\end{aligned}$$

Case 13:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), a_{11}(t) = 4\int_a^t a_9^5 f(s)\,\mathrm{d}s, a_{13} = 0, a_{15}(t) = a_{15}(t), \\ a_{17} = 0, a_{19}(t) = -\int_a^t a_9^2 a_{18} h(s)\,\mathrm{d}s, k_1 = 0, k_2 = 0, k_4 = 0, p_1 = 0, f(t) = 0, h(t) = h(t), \end{cases} \tag{32}$$

a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{18}, a_{20}, a_{21}, k_3, k_5, p_2 are all constants. a_9 is a constant that is not equal to zero, and the relation between the dispersion term of Eq.(1) and the higher-order dispersion term is $b(t) = -\frac{5a_9^3 f(t)+a_{10}h(t)}{a_9}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_{13}(x,y,t) = \frac{\Phi_{13}}{\Psi_{13}}, \tag{33}$$

where

$$\begin{aligned}
\Phi_{13} &= 2a_9^2 k_3\left(\left(k_5\sin\left(a_{19}(t) + a_{18}y + a_{20}\right) + a_{21} + 1\right)\cosh\left(a_{11}(t) + a_9x + a_{10}y + a_{12}\right) + k_3\right), \\
\Psi_{13} &= k_3^2\cosh\left(a_{11}(t) + a_9x + a_{10}y + a_{12}\right)^2 + 2k_3\left(k_5\sin\left(a_{19}(t) + a_{18}y + a_{20}\right) + a_{21} + 1\right) \\
&\quad \cosh\left(a_{11}(t) + a_9x + a_{10}y + a_{12}\right) + 2\left(a_{21} + 1\right)k_5\sin\left(a_{19}(t) + a_{18}y + a_{20}\right) - \\
&\quad k_5^2\cos\left(a_{19}(t) + a_{18}y + a_{20}\right)^2 + \left(a_{21} + 1\right)^2 + k_5^2.
\end{aligned}$$

Case 14:

$$\begin{cases} a_1 = 0, a_3(t) = -\int_a^t a_{13}^2 a_2 h(s)\,\mathrm{d}s, a_5 = 0, a_7(t) = a_7(t), a_9 = 0, a_{11}(t) = -\int_a^t a_{13}^2 a_{10} h(s)\,\mathrm{d}s, \\ a_{13} = 0, a_{15}(t) = 4\int_a^t a_{13}^5 a_2 f(s)\,\mathrm{d}s, a_{17} = 0, a_{19}(t) = a_{19}(t), k_1 = 0, k_2 = 0, k_5 = 0, \\ f(t) = 0, h(t) = h(t), \end{cases} \tag{34}$$

a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{18}, a_{20}, a_{21}, k_3, k_4, p_1, p_2 are all constants. a_{13} is a constant that is not equal to zero, and the relation between the dispersion term of Eq.(1) and the higher-order dispersion term is $b(t) = -\frac{5a_{13}^3 f(t)+a_{14}h(t)}{a_{13}}$. The exact solution of Eq.(1) can be obtained by transformation (5).

$$u_{14}(x,y,t) = \frac{\Phi_{14}}{\Psi_{14}}, \tag{35}$$

where

$$\begin{aligned}
\Phi_{14} &= 2a_{13}^2 k_4(\left(k_3\cosh\left(a_{11}(t) + a_{10}y + a_{12}\right) + a_{21}\right)\sinh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right) + \\
&\quad e^{-p_1(a_3(t)+a_2y+a_4)}\sinh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right) - k_4), \\
\Psi_{14} &= (k_4\sinh\left(a_{15}(t) + a_{13}x + a_{14}y + a_{16}\right) + k_3\cosh\left(a_{11}(t) + a_{10}y + a_{12}\right) + \\
&\quad e^{-p_1\left(a_3(t)+a_2y+a_4\right)} + a_{21})^2.
\end{aligned}$$

The above 14 groups of equations can be divided into three categories:

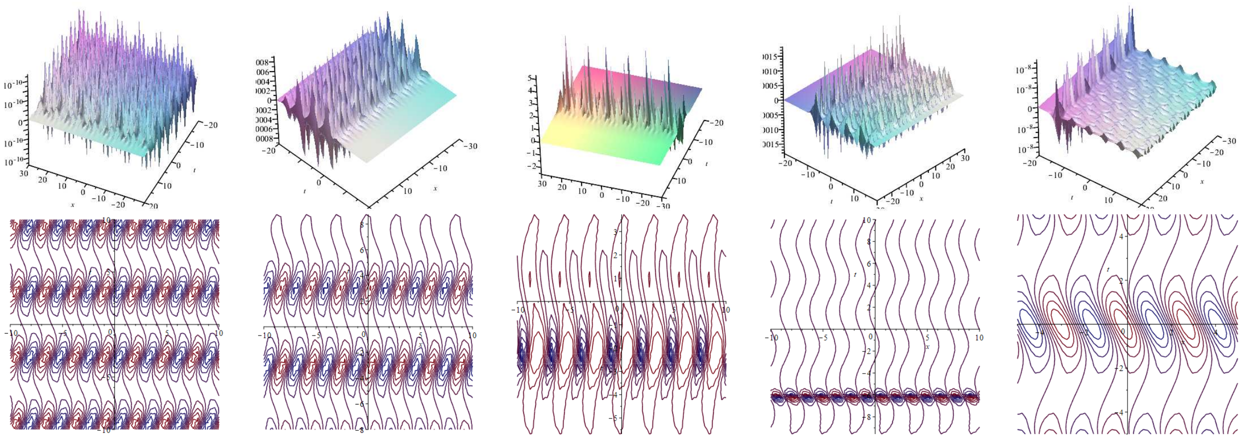
The first type of:$u_1(x,y,t)$, $u_2(x,y,t)$, $u_3(x,y,t)$, $u_4(x,y,t)$, $u_{13}(x,y,t)$ can obtain the relationship between the higher order dispersion term and the linear dispersion term.

The second type ofThe higher order dispersion terms of $u_7(x,y,t)$, $u_8(x,y,t)$, $u_9(x,y,t)$ are zero, and the relation between linear dispersion terms can be obtained.

The third type of:$u_5(x,y,t)$, $u_6(x,y,t)$, $u_{10}(x,y,t)$, $u_{11}(x,y,t)$, $u_{12}(x,y,t)$, $u_{14}(x,y,t)$ of high-order dispersion and linear dispersion is zero.

According to the above three cases, the corresponding types of images are drawn by mathematical software for analysis.

When $y = 10$, $y = -5$, $y = 0$, $y = 4$, $y = -10$, the equation of the physical properties and dynamic structure, as shown in Fig.1. The parameters of $u_1(x,y,t)$ are selected as the 3D image and contour map of $a_2 - 1$, $a_3(t) = \sin(t)$, $a_4 = -2.3$, $a_9 = 1$, $a_{10} = 1$, $a_{11} = t$, $a_{12} = 1$, $a_{14} = 2.3$, $a_{15}(t) = \cos(t)$, $a_{16} = 1$, $a_{17} = 2.2$, $a_{18} = 1.5$, $a_{19}(t) = \sin(t)$, $a_{20} = 1.3$, $a_{21} = -1.2$.

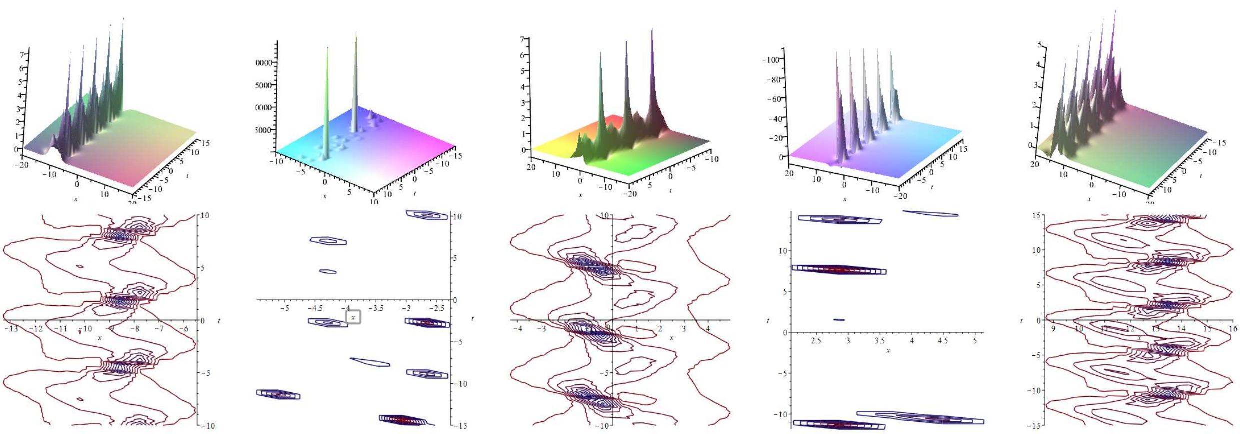


(a) y=10 (b) y=4 (c) y=0 (d) y=-5 (f) y=-10

Fig.1 When $y = 10$, $y = 4$, $y = 0$, $y = -5$, $y = -10$, $u_1(x,y,t)$ interaction solution of three-dimensional and contour figures.

It can be observed from Fig.1 that when different parameter values are selected for spatial variable Y and time variable t, the image is an unsmooth solitary spiky wave, and its maximum peaks are all equal. Fig.1(a) to Fig.2(c) Solitary wave behavior is gradually moving in the direction of $t \to +\infty$, and the maximum wave peak is upward. This solitary wave behavior of plasma is bright wave.

When $y = 10$, $y = 4$, $y = -1$, $y = -5$, $y = -16$, the equation of the physical properties and dynamic structure, as shown in Fig.2. The parameters of $u_5(x,y,t)$ are selected as the 3D image and contour map of $a_3(t) = \sin(t)$, $a_9 = 1.2$, $a_{10} = 1$, $a_{11}(t) = \cos(t)$, $a_{12} = 1$, $a_{17} = 0.5$, $a_{18} = -0.06$, $a_{19}(t) = \sin(t)$, $a_{20} = 1$, $a_{21} = -1.5$.



(a) y=10 (b) y=4 (c) y=0 (d) y=-5 (f) y=-16

Fig.2 When $y = 10$, $y = 4$, $y = 0$, $y = -5$, $y = -16$, $u_5(x,y,t)$ interaction solution of three-dimensional and contour figures.

By observing the Fig.2 can be dynamic images were observed in $x-t$ flat level, higher order Lump cross kink wave solutions and periodic solutions of interaction force from strong to weak to strong, and solitary wave solution, are mutual inelastic collision between amplitude and shape all happen very big change, peak shape isolated sharp wave tapering and reach maximum peak.

4 | RESONANT MULTISOLITON SOLUTION

Finally, applying the principle of linear superposition, the bilinear generalized variable coefficient KdV equation is assumed to have the following solution form

$$g = \sum_{i=0}^{N} \varepsilon_i \widehat{g}_i = \sum_{i=0}^{N} \varepsilon_i \exp(\varrho_i). \tag{36}$$

where $\widehat{g}_i = \exp(\varrho_i) = \exp(\xi_i x + \lambda_i y + \tau_i t)$, $1 \le i \le N$,and ξ_i, λ_i, τ_i are all constants.Substituting the above Eq.(36) into the bilinear Eq.(6), using the bilinear equation identities, we can obtain

$$P(D_x, D_y, D_t) = \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j P(\xi_i - \xi_j, \lambda_i - \lambda_j, \tau_i - \tau_j) \exp(\varrho_i + \varrho_j). \tag{37}$$

So we can draw the double linear equation solution of g if and only if $P(\xi_i - \xi_j, \lambda_i - \lambda_j, \tau_i - \tau_j) \exp(\varrho_i + \varrho_j) = 0$.
According to the above linear superposition principle, the polynomial corresponding to bilinear Eq.(6) is

$$P(x, y, t) = xt + b(t)x^4 + f(t)x^6 + h(t)x^3 y. \tag{38}$$

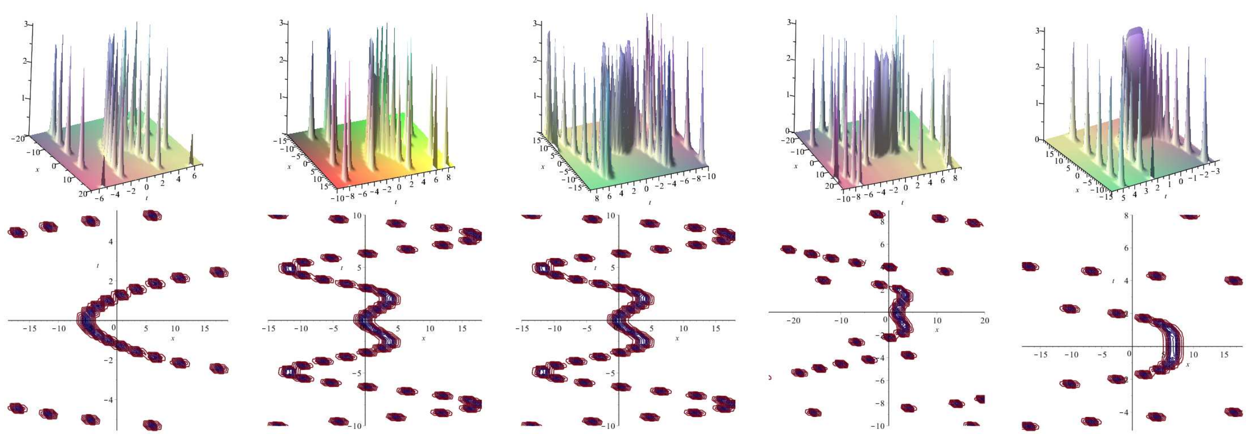
Substituting Eq.(38) into Eq.(37), we get

$$(\xi_i - \xi_j)(\tau_i - \tau_j) + b(t)(\xi_i - \xi_j)^4 + f(t)(\xi_i - \xi_j)^6 + h(t)(\xi_i - \xi_j)^3(\lambda_i - \lambda_j) = 0. \tag{39}$$

By simplifying Eq.(39), we will get the following sets of solutions. We choose two cases as examples:
Case 1: $\xi_i = \xi_i, \lambda_i = -\frac{b(t)}{h(t)}\xi_i, \tau_i = -f(t)\xi_i^5$. The resonant multisolitons of Eq.(1) can be obtained by transformation (5)

$$u_{15}(x, y, t) = 2\frac{\sum_{i=1}^{N} \xi_i^2 e^{\xi_i x - \frac{b(t)\xi_i y}{h(t)} - f(t)\xi_i^5 t}}{\sum_{i=1}^{N} e^{\xi_i x - \frac{b(t)\xi_i y}{h(t)} - f(t)\xi_i^5 t}} - 2\frac{(\sum_{i=1}^{N} \xi_i e^{\xi_i x - \frac{b(t)\xi_i y}{h(t)} - f(t)\xi_i^5 t})^2}{(\sum_{i=1}^{N} e^{\xi_i x - \frac{b(t)\xi_i y}{h(t)} - f(t)\xi_i^5 t})^2} \tag{40}$$

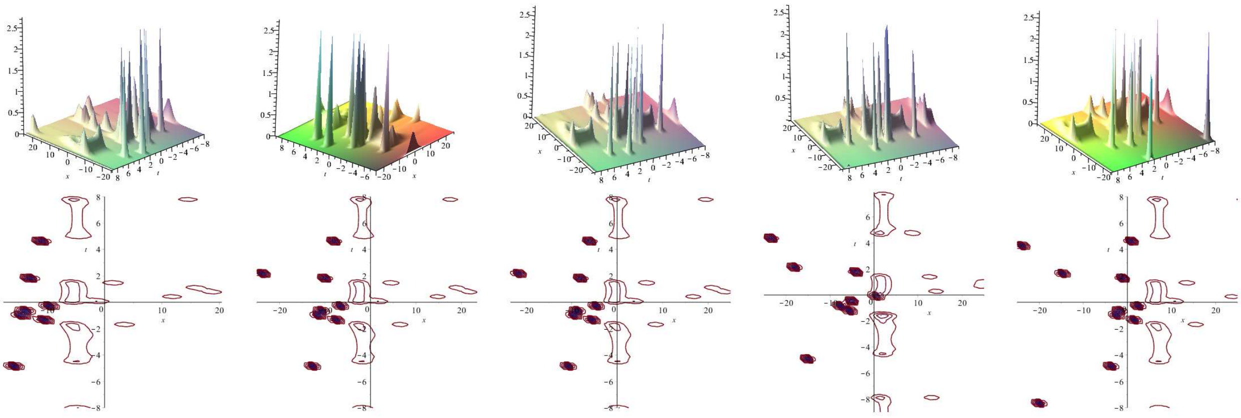
In case 1, N=3,6, and $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$ equation when the physical properties of structure and dynamics, as shown in figure 3 and Fig.4, when $u_{15}(x, y, t)$ parameters selection $N = 3$, $f(t) = \sin t$, $h(t) = \tan t$, $b(t) = t$, $\xi_1 = -1.3$, $\xi_3 = 1.1$ and $N = 6$, $f(t) = \cos t$, $h(t) = \sin t$, $b(t) =$, $\xi_1 = 1$, $\xi_2 = 1.2$, $\xi_3 = 1.1$, $\xi_4 = 0.5$, $\xi_5 = -0.25$, $\xi_6 = 2.1$ three-dimensional figure and contour map.



(a) y=-5 (b) y=-2 (c) y=0 (d) y=2 (f) y=7

Fig.3 When $N = 3$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, $u_{15}(x, y, t)$ interaction solution of three-dimensional and contour figures.

When $N = 3, 6$, we can observe that the ratio of linear dispersion terms and the higher-order dispersion terms increase with the increase of $t \to +\infty$. As can be seen from the contour diagram in Fig.3, the initial enlarged $\varepsilon W \varepsilon$ image gradually shrinks and then gradually expands, and the isolated peak wave gradually increases and then decreases with the ratio of linear dispersion terms and the transformation of higher-order dispersion terms. The dynamic behavior is always bright. It can be observed from the contour line in Fig.4 that peak-like isolated sharp waves move slowly toward $t \to +\infty$ with the increase of the ratio of linear dispersion terms and the increase of higher-order dispersion terms, and elastic collision occurs. The amplitude and maximum peak value of isolated waves remain unchanged.



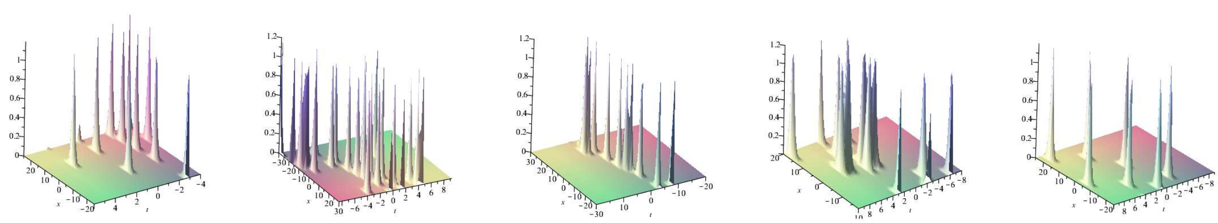
(a) y=-5 (b) y=-2 (c) y=0 (d) y=2 (f) y=7

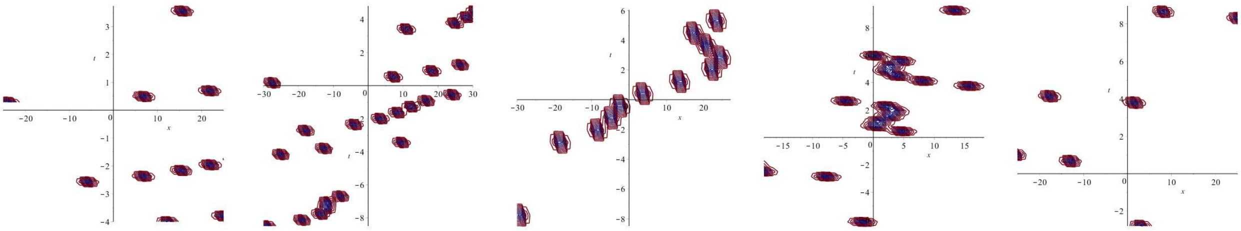
Fig.4 When $N = 6$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, $u_{15}(x, y, t)$ interaction solution of three-dimensional and contour figures

Case 2:$\xi_i = \xi_i$, $\lambda_i = (2 - \frac{f(t)}{h(t)})\xi_i^3$, $\tau_i = -(2 + b(t))\xi_i^3$.The resonant multisolitons of Eq.(1) can be obtained by transformation (5)

$$u_{16}(x, y, t) = 2\frac{\sum_{i=1}^{N} \xi_i^2 e^{\xi_i x + 2\xi_i^3 y - \frac{f(t)\xi_i^3 y}{h(t)} - 2\xi_i^3 t - b(t)\xi_i^3 t}}{\sum_{i=1}^{N} e^{\xi_i x + 2\xi_i^3 y - \frac{f(t)\xi_i^3 y}{h(t)} - 2\xi_i^3 t - b(t)\xi_i^3 t}} - 2\frac{(\sum_{i=1}^{N} \xi_i e^{\xi_i x + 2\xi_i^3 y - \frac{f(t)\xi_i^3 y}{h(t)} - 2\xi_i^3 t - b(t)\xi_i^3 t})^2}{(\sum_{i=1}^{N} e^{\xi_i x + 2\xi_i^3 y - \frac{f(t)\xi_i^3 y}{h(t)} - 2\xi_i^3 t - b(t)\xi_i^3 t})^2} \tag{41}$$

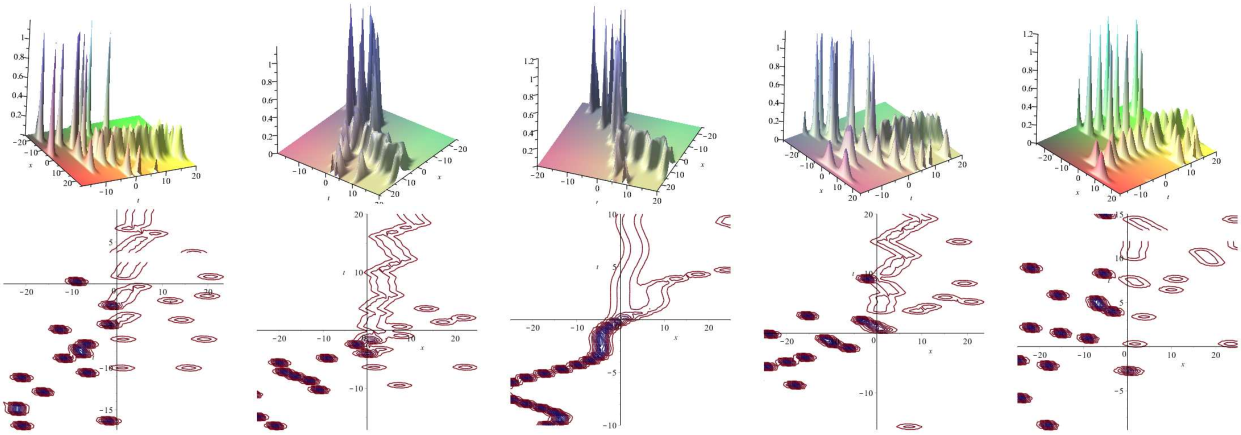
In case 2,N=2,3,6, and $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$ equation when the physical properties of structure and dynamics, as shown in Fig.5 and Fig.6, when $u_{16}(x, y, t)$ parameters selection $N = 2$, $f(t) = \cos t$, $h(t) = \sin t$, $b(t) = \sin t$, $\xi_1 = 0.25$, $\xi_2 = 1.8$ and $N = 3$, $f(t) = \cos t$, $h(t) = \sin t$, $b(t) = \cos t$, $\xi_1 = \frac{1}{2}$, $\xi_2 = -0.25$, $\xi_3 = 1.3$ and $N = 6$, $f(t) = \cos t$, $h(t) = \sin t$, $b(t) = \cos t$, $\xi_1 = \frac{1}{2}$, $\xi_2 = -0.25$, $\xi_3 = 1.3$, $\xi_4 = 0.5$, $\xi_5 = -0.25$, $\xi_6 = 2.1$ three-dimensional figure and contour map.





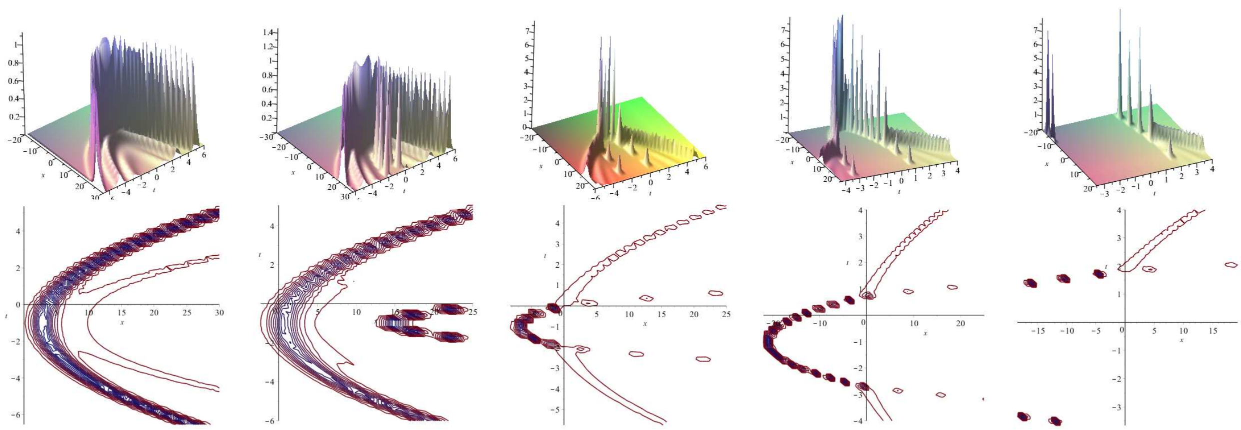
(a) y=-5 (b) y=-2 (c) y=0 (d) y=2 (f) y=7

Fig.5 When $N = 2$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, $u_{16}(x, y, t)$ interaction solution of three-dimensional and contour figures.



(a) y=-5 (b) y=-2 (c) y=0 (d) y=2 (f) y=7

Fig.6 When $N = 3$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, $u_{16}(x, y, t)$ interaction solution of three-dimensional and contour figures.



(a) y=-5 (b) y=-2 (c) y=0 (d) y=2 (f) y=7

Fig.7 When $N = 6$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, $u_{16}(x, y, t)$ interaction solution of three-dimensional and contour figures.

When $N = 2, 3, 6$, we can observe that the ratio of linear dispersion terms and the higher-order dispersion terms decrease with the increase of $t \to +\infty$. It can be observed from the contour diagram in Fig.5 and Fig.7 that the soliton gradually decreases from the initial concentrated distribution to x-t with a fast speed. The peak isolated sharp wave also gradually decreases with the ratio of linear dispersion terms and the transformation of higher-order dispersion terms, and inelastic collisions occur between multiple solitons.

5 | CONCLUSION

In this paper, the interaction solutions and resonant multisolitons of the generalized variable coefficient KdV equation with higher order dispersion terms are studied. Contain nonlinear partial differential equation of higher order dispersion in some complicated cases play a very important role, this makes the study of this equation is very important, first of all, we through the proper variable transformation makes five order variable coefficient KdV equations into linear equations, then structure are four function, quadratic function, exponential function and hyperbolic function of interaction in the form of a solution, Several kinds of interaction solutions of KdV equation with fifth order variable coefficients are obtained by using the constructed auxiliary functions. In addition, we also solved the resonance multi-soliton solution of the equation through the principle of linear superposition. When the values of N are different, the images formed are compared and analyzed. This paper further improves the study of the fifth order KdV equation with variable coefficients.

Author contributions

ZhenZhao: Reviewed literature, provided innovation points, and wrote papers JingPang: To provide ideas for paper writing, the overall framework of the paper, looking for journals

Financial disclosure

None reported.

Conflict of interest

The authors declare no potential conflict of interests.

SUPPORTING INFORMATION

The following supporting information is available as part of the online article:

References

1. Huete C, Velikovich AL, Martínez-Ruiz D, Calvo-Rivera A. Stability of expanding accretion shocks for an arbitrary equation of state. *Journal of Fluid Mechanics.* 2021;927.
2. Hosseini , Kamyar S, Majid MM, Ma WX, Hammouch Z. ;.
3. Liu P, Cheng J, Ren B, Yang JR. Bäcklund transformations, consistent Riccati expansion solvability, and soliton-cnoidal interaction wave solutions of Kadomtsev-Petviashvili equation. *Chinese Physics B.* 2020;29.
4. Wang QX, Xie JQ, Zhang ZY, Wang LQ. Bilinear immersed finite volume element method for solving matrix coefficient elliptic interface problems with non-homogeneous jump conditions. *Computers and Mathematics with Applications.* 2021;86:1-15.
5. Hoque MF, Roshid HP, Alshammari FS. ;.
6. Hua YF, Guo BL, Ma WX, Lü X. ;.
7. Berndt M, Breil J, Galera S, Kucharik M, Maire PH, Shashkov M. Two-step hybrid conservative remapping for multimaterial arbitrary Lagrangian-Eulerian methods. *Comput Phys.* 2011;230(17):6664–6687.
8. Ma WX, Fan EG. Linear superposition principle applying to Hirota bilinear equations. *Computers Mathematics with Applications.* 2011;61(4):950–959.

9. Freeman NC, Nimmo JJC. Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations:the Wronskian technique. *Phusics Letters.* 1983;95:1–3.

10. Zhang HM, Cai C, Fu XJ. Self-Similar Transformation and Vertex Configurations of the Octagonal Ammann-Beenker Tiling. *Chinese Physics Letters.* 2018;35(6):46–49.

11. Li J, Gu XF, Yu T, et al. Simulation of nonlinear interaction of isolated waves in ocean based on variable coefficient Korteweg-de Vries equation. *Transactions of Oceanology and Limnology.* 2011;1:1–12.

12. Cohen BI, Krommes JA, Tang WM, Rosenbluth MN. Non-linear saturation OF THE dissipative trapped-ion mode by mode coupling. *Nuclear Fusion.* 1976;16(6):971–992.

13. Yuan P, Deng WP. Optical Soliton Solutions for the Fifth-order Variable-coefficient Korteweg-de Vries Equation. *Journal of Sichuan University of Science.* 2016;29(5):97–100.

14. Xu GQ. Painlevé integrability of a generalized fifth-order KdV equation with variable coefficients: Exact solutions and their interactions. *Chinese Physics B.* 2013;22(5):1–8.

15. Wazwaz AM. Two new integrable modified KdV equations, of third-and fifth-order, with variable coefficients: multiple real and multiple complex soliton solutions. *Waves in Random and Complex Media.* 2019;5(31):867–878.

16. Zhaoa HQ. Soliton propagation and collision in a variable-coefficient coupled Korteweg-de Vries equation. *European Physical Journal B.* 2012;9(85):1–6.

17. Triki H, Wazwaz AM. Traveling wave solutions for fifth-order KdV type equations with time-dependent coefficients. *Communications in Nonlinear Science and Numerical Simulation.* 2014;19(3):404–408.

18. Wang XR, Zhang XE, Zhang Y, Dong HH. *The Interactions of N-Soliton Solutions for the Generalized 2+1-Dimensional Variable-Coefficient Fifth-Order KdV Equation.* 2015.

19. Khater AH, Hassan MM, Temsah RS. Cnoidal wave solutions for a class of fifth-order KdV equations. *Mathematics and Computers in Simulation.* 2005;70(4):221–226.

20. Chen B, Xie YC. An auto-Backlund transformation and exact solutions of stochastic Wick-type Sawada-Kotera equations. *Chaos, Solitons and Fractals.* 2005;23(1):243–248.

21. Wang YH, Chen Y. Binary Bell polynomial manipulations on the integrability of a generalized (2+1)-dimensional Korteweg-de Vries equation. *Journal of Mathematical Analysis and Applications.* 2013;400(2):624–634.

22. Wang HY, Li CX, Zhao JX, Yu GF). *The Direct Method in Soliton Theory*. Beijing: Tsinghua University Press; 2008. ISBN 9787302173786.

How to cite this article: Zhao Z., J. Pang (—-), , , —-;–: *–*.