Multiplicity of Homoclinic solutions for fractional discrete \$p-\$Laplacian equations

Jiabin Zuo[1], Tahar Bouali[2], Rafik GUEFAIFIA[3], and Libo Yang[4]

[1]Guangzhou University
[2]University of Larbi Tebessi-Informatics and systemes
[3]Larbie tebessi University, Tebessa, Algeria.
[4]Huaiyin Institute of Technology

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Abstract

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Multiplicity of Homoclinic solutions for fractional discrete $p-$Laplacian equations

Jiabin Zuo[1] Tahar Bouali [2] Rafik Guefaifia[2] Libo Yang[3*]

[1]School of Mathematics and Information Science, Guangzhou University, Guangzhou, 510006, China.

[2]Laboratory of Mathematics, Informatics and systemes (LAMIS), University of Larbi Tebessi, 12000 Tebessa, Algeria .

[3]Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an,223003, China .

Abstract

In this study, we investigate the existence and multiplicity of solutions for a fractional discrete $p-$ Laplacian equation on $\mathbb{Z}$, via the mountain pass lemma and Ekeland's variational principle. Under suitable hypotheses on functions V and f, we prove that this equation admits at least two nonnegative and nontrivial homoclinic solutions when the real parameter $\lambda > 0$ is large enough.

Keywords: Fractional discrete $p-$Laplace equation, Mountain pass lemma, Homoclinic solutions, Ekeland's variational principle, Multiplicity of solutions.

Mathematics Subject Classification (2010): 35J60, 35R11, 35K05, 49M25.

1 Introduction and main result

In recent years, lots of researchers pay their attentions on the problem of the second order difference equation

$$-\triangle_T u(j) + V(j)u(j) = f(j,\ u(j)) \text{ in } \mathbb{Z} \tag{1.1}$$

where $T > 0$ is a real number, $\mathbb{Z}$ is the set of all integers, the function $V : \mathbb{Z} \to [0,\ \infty)$, $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$, and $-\triangle_T u(j)$ is the discrete Laplace operator, defined as

$$-\triangle_T u(j) = \frac{1}{T^2}[u((j+1)T) - 2u(jT) + u((j-1)T)],$$

for any $u : \mathbb{Z} \to \mathbb{R}$. This equation can be regards as the discrete version of the famous Schrödinger equation. In addition, homoclinic orbits play a very important role in studying the dynamics of discrete Schrödinger equations. There are lots of literatures about second-order difference equations and homoclinic orbits, we just collect some papers; see, for example, [1, 5, 12, 14, 18, 19, 23]. Especially, Agarwal, Perera and O'Regan in [1] considered the existence of solutions for second order difference equations like (1.1) by using variational methods for the frist time.

On the other side, the fractional Laplacian and related problems are all hot topics in recent years, the study of nonlocal problems has been received an increasing amount of attentions. For more details on fractional Laplacian and fractional Sobolev Spaces, see [10]. Nonlocal fractional problems appear in many fields, such as quantum mechanics, anomalous diffusion, finance, optimization and

*Corresponding author: yanglibo80@126.com.(L.B. Yang).

Co-authors: zuojiabin88@163.com (J.B. Zuo); rafik.guefaifia@univ-tebessa.dz (R. Guefaifia);tahahr.bouali@univ-tebessa.dz (T. Bouali).

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game theory, see [2, 5] and the references therein. For more applications of fractional operators, we refer to [3, 4, 9, 13, 21, 23, 24, 27, 28] and the references therein.

Recently, Ciaurri *et al.* in [7] considered the following equation:

$$(-\triangle_T)^s u = f, \tag{1.2}$$

where $s \in (0,1)$, $(-\triangle_T)^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\triangle T} u(j) - u(j)) \frac{dt}{t^{1+2s}}$ is the discrete fractional Laplace operator, Γ denote the Gamma function, and let $v(t,j) = e^{t\triangle T} u(j)$, which is the solution of the problem as follow:

$$\begin{cases} \partial_t v(t,j) = \triangle_T u(j), \text{ in } \mathbb{Z}_T \times (0,\ \infty), \\ v(0,j) = u(j), \text{ on } \mathbb{Z}_T, \end{cases}$$

where $\mathbb{Z}_T = \{Tj : j \in \mathbb{Z}\}$.

By the Theorem 1.1 of [7], for any $u \in \mathcal{L}_s$,

$$(-\triangle_T)^s u(j) = \sum_{k\in\mathbb{Z}, k\neq j} (u(j) - u(k)) \mathcal{K}_s^T (j-k),$$

where

$$\mathcal{L}_s = \{u : \mathbb{Z}_T \to \mathbb{R} | \sum_{k\in\mathbb{Z}} \frac{|u(k)|}{(1+|k|)^{1+2s}} < \infty\}$$

and

$$\mathcal{K}_s^T(k) = \frac{4^s \Gamma(1/2+s)}{\sqrt{\pi}|\Gamma(-s)|} \cdot \frac{\Gamma(|k|-s)}{T^{2s}\Gamma(|k|+1+s)}$$

for any $k \in \mathbb{Z}\backslash\{0\}$ and $\mathcal{K}_S^T(0) = 0$.

Meanwhile, if u is bounded, An interesting result is that $\lim_{s\to 1^-} (-\triangle_T)^s u(j) = -\triangle_T u(j)$. In particular, [7] stated that the solutions of the fractional Laplace equation $(-\triangle)^s u = f$ in $\mathbb{R}$ can be approximated by the solutions to equation (1.2).

In [26], Xiang and Zhang first inverstigated the equation

$$\begin{cases} (-\triangle_1)^s u(k) + V(k)\,|u(k)| = \lambda f(k,u(k)), \text{ for k} \in\ \mathbb{Z} \\ u(k) \to 0, \qquad\qquad \text{as} \quad |k| \to \infty. \end{cases} \tag{*}$$

where $s \in (0,1)$, $V : \mathbb{Z} \to (0,\ \infty)$, $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is a continuous function with respect to the second variable and satisfies asymptotically linear growth at infinity. Under some suitable hypotheses, two solutions were obtained by using variational principle.

Usually, the solutions of the continuous fractional Problems can be approximated by the solutions of the discrete fractional Laplacian equations. For the nonlocality and singularity of discrete fractional Laplace operator, numerical analysis is difficult for this type equations, see for example [11] and the reference cited therein.

Motivated by the above literatures, in this paper, we investigate the existence of homoclinic solutions of a class of difference equation driven by the fractional discrete $p-$Laplace operator on $\mathbb{Z}$. Precisely speaking, we study

$$\begin{cases} (-\triangle_T)_p^s u(k) + V(k)\,|u(k)|^{p-2}\, u(k) = \lambda f(k,u(k)), \text{ for k} \in\ \mathbb{Z} \\ u(k) \to 0, \qquad\qquad \text{as} \quad |k| \to \infty, \end{cases} \tag{1.4}$$

where $T > 0$ is a real number, $s \in (0,1)$, $1 < p < \infty$, $\mathbb{Z}$ denote the set of whole integers, the potential function $V : \mathbb{Z} \to (0,\ \infty)$ and the nonlinearity $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and $(-\triangle_T)^s_p$ is the fractional discrete $p-$Laplace operator, which is defined as

$$(-\triangle_T)^s_p u(j) = \sum_{m \in Z, m \neq j} |u(j) - u(m)|^{p-2}(u(j) - u(m)) K^T_{s,p}(j - m)\ ,$$

for any $j \in \mathbb{Z}$, $u \in \mathcal{L}_{p,s}$, $\mathcal{L}_{P,s} = \{u : \mathbb{Z}_T \to \mathbb{R} | \sum_{k \in \mathbb{Z}} \frac{|u(k)|}{(1+|k|)^{1+ps}} < \infty\}$.

Here,$K_{s,p}$ is the discrete kernel, satisfies the following expression:

there exist constants $0 < c_{s,p} \leq C_{s,p} < \infty$ such that

$$\begin{cases} \frac{c_{s,p}}{T^{ps}|j|^{1+ps}} \leq K^T_{s,p}(j) \leq \frac{C_{s,p}}{T^{ps}|j|^{1+ps}}, \text{ for any } j \in \mathbb{Z}\backslash\{0\} \\ K_{s,p}(0) = 0. \end{cases} \tag{1.5}$$

When $T = 1$, we have

$$(-\triangle_1)^s_p u(j) = 2 \sum_{m \in Z, m \neq j} |u(j) - u(m)|^{p-2}(u(j) - u(m)) K_{s,p}(j - m)\ .$$

Meanwhile, when $p = 2$, the fractional discrete $p-$Laplace operator $(-\triangle_T)^s_p u$ is consistence with the usual fractional discrete laplace operator $(-\triangle_T)^s u$, and when $p = 2$ and $d = 1$,then Eq. (1.4) reduces to equation $(*)$.

As usual, a function $u : \mathbb{Z} \to \mathbb{R}$ is a homoclinic solution of Eq. (1.4) if $u(k) \to 0$ as $|k| \to \infty$.

Next, we give the hypotheses which will be used in this paper. we suppose that $V : \mathbb{Z} \to (0,\ \infty)$ is a continuous function, fulfills

(V) For all $k \in \mathbb{Z}$, there exists a positive constant V_0 such that $V(k) \geq V_0 > 0$. And $V(k) \to \infty$ as $|k| \to \infty$.

The continuous function $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, satisfies

(f_1) $\lim\limits_{t \to 0} \frac{f(k,t)}{t^{p-1}} = 0$ uniformly for all $k \in \mathbb{Z}$;

(f_2) For all $T > 0$, $\sup\limits_{|t| \leq T} |F(\cdot, t)| \in \ell^1$, where $\ell^1 := \{u : \mathbb{Z} \to \mathbb{R} | \sum_{j \in \mathbb{Z}} |u(j)| < \infty\}$, and $F(k,t) = \int_0^1 f(k,t)\, d\tau$;

(f_3) $\lim \sup\limits_{|t| \to \infty} \frac{F(k,t)}{t^p} \leq 0$ uniformly for all $k \in \mathbb{Z}$;

(f_4) $F(h_0, b_0) > 0$ for same $h_0 \in \mathbb{Z}$ and $b_0 \in \mathbb{R}\backslash\{0\}$.

When $1 < q < 2 < p < \infty$, a simple example of f, fulfilling $(f_1) - (f_4)$ is

$$f(k,t) = \begin{cases} |t|^{p-2}\, t\ , \text{if } |t| \leq 1 \\ |t|^{q-2}\, t\ , \text{if } |t| > 1. \end{cases}$$

Set

$$\lambda^* = \frac{|b_0|^p \left(pC_{s,p} \sum_{m \neq h_0} \frac{1}{|h_0 - m|^{1+ps}} + V(h_0)\right)}{pF(h_0, b_0)}.$$

Theorem 1.1. *If conditions (V) and $(f_1) - (f_4)$ hold. Then for any $\lambda > \lambda^*$, Eq. (1.4) has at least two nontrivial and nonnegative homoclinic solutions.*

To our best knowledge, Theorem 1.1 is the first result based on variational methods to study the existence of solutions for fractional discrete $p-$Laplacian equation.

More precisely, in this paper, when positive constant λ is big enough, we prove the existence of two nontrivial nonnegative homoclinc solutions of Eq.(1.4) by using the mountain pass theorem and Ekeland's variational principle. However, at present, it is still an open problem for all $\lambda > 0$, which can be one of our further research directions.

The paper is organized as follows. In Sect. 2, we present a variational framework to Eq. (1.4) and some preliminary results. In Sect. 3, by means of critical point theory, we obtain two distinct nontrivial and nonnegative homoclinc solutions for Eq (1.4).

2 Variational setting and preliminaries

In this section, we first recall some basic definitions, which can be found in [8, 13, 26]. Then we give the variational setting to Eq. (1.4) and discuss its properties. For any $1 \leq v < \infty$, ℓ^v is defined as

$$\ell^v := \{u : \mathbb{Z} \to \mathbb{R} | \sum_{j\in\mathbb{Z}} |u(j)|^v < \infty\}$$

with the norm

$$\|u\|_v = (\sum_{j\in\mathbb{Z}} |u(j)|^v)^{1/v}.$$

Set

$$\|u\|_\infty := \sup_{j\in\mathbb{Z}} |u(j)| < \infty.$$

Define

$$\ell^\infty = \{u : \mathbb{Z} \to \mathbb{R} : \|u\|_\infty < \infty\}.$$

Then $(\ell^v,\ \|\cdot\|_v)$ and $(\ell^\infty,\ \|\cdot\|_\infty)$ are Banach spaces; see [8]. Clearly, $\ell^{v_1} \subset \ell^{v_2}$ if $1 \leq v_1 \leq v_2 \leq \infty$. From now on, we shortly denote by $\|\cdot\|_v$ the norm of ℓ^v for all $v \in [1,\ \infty]$.

For interval $I \subset \mathbb{R}$, we define

$$\ell_I^v := \{u : I \to \mathbb{R} : \sum_{j\in I} |u(j)|^v < \infty\}.$$

Define

$$W = \{u : \mathbb{Z} \to \mathbb{R} : \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |u(j) - u(k)|^p K_{sp}(j-k) + \sum_{j\in\mathbb{Z}} V(j)|u(j)|^p < \infty\}.$$

with the norm

$$\|u\|_W = \left([u]_{s,p}^p + \sum_{j\in\mathbb{Z}} V(j)|u(j)|^p\right)^{1/p},$$

where

$$[u]_{s,p} := \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |u(j) - u(k)|^p K_{s,p}(j-m)\right)^{1/p}.$$

Lemma 2.1. *: If $u \in \ell^p$, then $[u]_{s,p} < \infty$. Moreover there exists $C(s,p) > 0$, such that $[u]_{s,p} \le C\|u\|_p$ for all $u \in \ell^p$.*

Proof. : Let $u \in \ell^p$. Then

$$[u]_{s,p}^p = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |u(j) - u(k)|^p K_{sp}(j-m)$$

$$\begin{aligned}
&\le 2^{p-1}C_{s,p}\sum_{j\in\mathbb{Z}}\sum_{k=j}\frac{|u(j)|^p + |u(k)|^p}{|j-k|^{1+ps}}\\
&= 2^{p-1}C_{s,p}\sum_{k\neq 0}\frac{|u(0)|^p + |u(k)|^p}{|k|^{1+ps}} + 2^{p-1}C_{s,p}(\sum_{j=0}\sum_{k\neq 0}\frac{|u(j)|^p}{|k|^{1+ps}}) + 2^{p-1}C_{s,p}(\sum_{j=0}\sum_{k\neq 0}\frac{|u(k+j)|^p}{|k|^{1+ps}})\\
&= 2^{p-1}C_{s,p}\sum_{k\neq 0}\frac{|u(0)|^p + |u(k)|^p}{|k|^{1+ps}} + 2^{p-1}C_{s,p}(\sum_{k\neq 0}\sum_{j=0}\frac{|u(j)|^p}{|k|^{1+ps}}) + 2^{p-1}C_{s,p}(\sum_{k\neq 0}\sum_{j\neq 0}\frac{|u(k+j)|^p}{|k|^{1+ps}})\\
&\le 3\cdot 2^{p-1}C_{s,p}\sum_{k\neq 0}\frac{1}{|k|^{1+ps}}\sum_{j\in\mathbb{Z}}|u(j)|^p\\
&= C^p\sum_{j\in\mathbb{Z}}|u(j)|^p,
\end{aligned}$$

where $0 < C^p = 3\cdot 2^{p-1}C_{sp}\sum_{k\neq 0}\frac{1}{|k|^{1+ps}} < \infty$. Therefore, the proof is complete. □

Lemma 2.2. *:The norm*

$$\|u\| := \left(\sum_{j\in\mathbb{Z}} V(j)|u(j)|^p\right)^{1/p}$$

is an equivalent norm of W. Moreover $(W, \|\cdot\|_W)$ is a Banach space.

Proof. : The proof is similar to [25], for completeness, we give its details. Using assumption (V) and Lemma 2.1, we have

$$\sum_{j\in\mathbb{Z}} V(j)|u(j)|^p \le \|u\|_W^p \le C\sum_{j\in\mathbb{Z}}|u(j)|^p + \sum_{j\in\mathbb{Z}} V(j)|u(j)|^p$$

$$\le C\frac{1}{V_0}\sum_{j\in\mathbb{Z}} V(j)|u(j)|^p + \sum_{j\in\mathbb{Z}} V(j)|u(j)|^p$$

$$= C\sum_{j\in\mathbb{Z}} V(j)|u(j)|^p,$$

which leads to $\|u\| = (\sum_{j\in\mathbb{Z}} V(j)|u(j)|^p)^{1/p}$ being an equivalent norm of W. Finally, we show that $(W, \|\,\|_W)$ is complete. Let $\{v_n\}_n$ be a Cauchy sequence in W. Observe that

$$\|u\|_p \le V_0^{-\frac{1}{p}}\|u\|$$

for all $u \in W$. Then $\{v_n\}_n$ is also a Cauchy sequence in ℓ^p. By the completeness of ℓ^p, there exists $u \in \ell^p$ such that $u_n \to u$ in ℓ^p. Furthermore, Lemma 2.1 and assumption (V) imply that $u_n \to u$ strongly in W as $n \to \infty$. Thus, we conclude the proof. □

Moreover, we have the following compactness result:

Lemma 2.3. *If condition (V) holds, then the embedding $W \hookrightarrow \ell^{\nu}$ is compact for any $p \leq \nu < \infty$.*

Proof. The proof is similar to papers [13, 26].

First we show that the result holds for the case $\nu = p$. According to the assumption (V), we have $\|u\|_p \leq V_0^{-\frac{1}{p}}\|u\|$ for all $u \in W$. Clearly, the embedding $W \to \ell^p$ is continuous.

Next we prove that $W \to \ell^p$ is compact. For $\{u_n\}_n \subset W$, we assume that there exists $D > 0$ such that $\|u_n\|_W^p \leq D$ for all $n \in \mathbb{N}$. Since W is a reflexive Banach space, there exist a subsequence of $\{u\}_n$ still denoted by $\{u_n\}_n$ and a function $u \in W$ such that $u_n \rightharpoonup u$ in W. By assumption (V), for any $\epsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that for all $|j| > j_0$

$$V(j) > \frac{1 + D}{\epsilon}.$$

Set $I = [-j_0, j_0]$ and define

$$W_I := \{u : I \to \mathbb{R} : \sum_{j \in I} \sum_{j \neq k \in I} |u(j) - u(k)|^p K_{s,p}(j - k) + \sum_{j \in I} V(j)|u(j)|^p < \infty\}.$$

Observe that the dimension of W_I is finite. Then $\{u_n\}_n$ is a bounded sequence in W_I, thanks to $\{u_n\}_n$ is bounded in ℓ_I^p. Thus, up to a subsequence we may assume that, $u_n \to u$ on I. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\sum_{j \in I} |u_n(j) - u(j)|^p \leq \frac{\delta}{1 + D}.$$

Then, for all $n > n_0$,

$$\begin{aligned}
\sum_{j \in \mathbb{Z}} |u_n(j) - u(j)|^p &< \frac{\epsilon}{1 + C} + \frac{\epsilon}{1 + C} \sum_{|j| > j_0} V(j)|u_n(j) - u(j)|^p \\
&\leq \frac{\epsilon}{1 + C}(1 + \|u_n\|_W^p) \leq \delta.
\end{aligned}$$

Thus, we deduce that $u_n \to u$ in ℓ^p. Now we consider the case $\nu > p$. Note that

$$\|u(j)\|_\infty \leq (\sum_{j \in \mathbb{Z}} |u(j)|^p)^{1/p}$$

for all $u \in \ell^p$. Then

$$\begin{aligned}
(\sum_{j \in \mathbb{Z}} |u(j)|^\nu)^{1/\nu} &= \|u\|_\infty (\sum_{j \in \mathbb{Z}} (\frac{|u(j)|}{\|u\|_\infty})^p)^{1/\nu} \\
&\leq \|u\|_\infty (\sum_{j \in \mathbb{Z}} (\frac{|u(j)|}{\|u\|_\infty})^p)^{1/\nu} \\
&= \|u\|\infty^{1-\frac{p}{\nu}} (\sum_{j \in \mathbb{Z}} |u(j)|^p)^{1/\nu} \\
&\leq \|u\|_p^{1-\frac{p}{V}} \|u\|_p^{\frac{p}{V}} \\
&= \|u\|_p
\end{aligned}$$

for all $u \in \ell^p \backslash \{0\}$. Thus,

$$\|u\|_\nu \leq \|u\|_p$$

for all $u \in \ell^p$. This inequality together with the result of the case $\nu = p$ leads to the proof. $\square$

In order to obtain some properties of energy functional associated with Eq. (1.4), the following result is needed.

Lemma 2.4. *Assume that U is a compact subset of W. Then for any $\epsilon > 0$ there is a $j_0 \in \mathbb{N}$ such that*

$$\left[\sum_{|j|>j_0} V(j)|u(j)|^p\right]^{1/p} < \epsilon, for\ any\ u \in U.$$

Proof. We prove it by contradiction, suppose that there exist $\epsilon > 0$ and a sequence $\{u_n\} \subseteq U$ such that

$$[\sum_{|j|>n} V(j)|u_n(j)|^p]^{1/p} > \epsilon \text{ for all } n \in \mathbb{N}.$$

Due to the compactness of U, passing to a subsequence we may assume that $u_n \to u$ in W for some $u \in U$. Thus, there exists $n_0 \in \mathbb{N}$, such that $\|u_n - u\| < \frac{\epsilon}{2}$ for any $n \geq n_0$, moreover, there exists $j_1 \in \mathbb{N}$ such that

$$[\sum_{|j|>j_1} V(j)|u(j)|^p]^{1/p} < \frac{\epsilon}{2}.$$

Recall the classical Minkowski inequality:

$$\left[\sum_{i=1}^m |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^m |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^m |y_i|^p\right]^{\frac{1}{p}} \text{ for all } m \in \mathbb{Z}, x_1...x_m.y_1...y_m \in \mathbb{R}. \tag{2.1}$$

By (2.1), we have

$$\begin{aligned}
\epsilon &< [\sum_{|j|>n} V(j)|u_n(j) - u(j)|^p]^{1/p} \leq [\sum_{|j|>n} V(j)|u_n(j)|^p]^{1/p} + [\sum_{|j|>n} V(j)|u(j)|^p]^{1/p} \\
&\leq \|u_n - u\| + \frac{\epsilon}{2} \\
&< \epsilon,
\end{aligned}$$

which is a contradiction. We get the proof. $\square$

For each $u \in W$, we define the associated energy functional with Eq. (1.4) as

$$I_\lambda(u) = \Psi(u) - \lambda J(u)\,,$$

where

$$\Psi(u) = \frac{1}{p}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |u(j) - u(m)|^p K_{s,p}(j-m) + \frac{1}{p}\sum_{j\in\mathbb{Z}} V(j)|u(j)|^p$$

and

$$J(u) = \sum_{j\in\mathbb{Z}} F(j, u(j))\,.$$

Lemma 2.5. *If V satisfies (V), then Ψ is well-defined, of class $C^1(W,\mathbb{R})$ and*

$$\langle \Psi'(u),\ v\rangle = \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}(j-m) + \sum_{j\in\mathbb{Z}} V(j)|u(j)|^{p-2}u(j)v(j)$$

for all $u, v \in W$.

Proof. According to lemma 2.1, the functional Ψ is well-defined on W. Fix $u, v \in W$. We first prove that

$$\lim_{t\to 0^+}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \frac{|u(j)+tv(j)-u(m)-tv(m)|^p - |u(j)-u(m)|^p}{p}K_{s,p}(j-m) = \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}(j-m). \tag{2.2}$$

Choose $C > 0$ such that $\|u\|_W, \|v\|_W \le C$. For any $\epsilon > 0$ there exists $h_1 \in \mathbb{N}$ such that

$$\sum_{|j|>h}\sum_{|m|>h} |u(j)-u(m)|^p K_{s,p}(j-m))^{\frac{1}{p}} < \epsilon \tag{2.3}$$

for all $h > h_1$. Indeed, for any $h \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{|j|>h}\sum_{|m|>h} |u(j)-u(m)|^p K_{s,p}(j-m) &\le C_{s,p}2^{p-1}\sum_{|j|>h}\sum_{|m|>h, m\neq j}\frac{(|u(j)|^p+|u(m)|^p)}{|j-m|^{1+ps}}\\ &\le 2^pC_{s,p}\sum_{|j|>h}\sum_{|m|>h, m=j}\frac{|u(j)|^p}{|j-m|^{1+ps}}\\ &\le 2^pC_{s,p}(\sum_{k\neq 0}\frac{1}{|k|^{1+ps}})\sum_{|j|>h}|u(j)|^p.\end{aligned}$$

It follows from $u \in W$ that (2.3) holds.

For $h \in \mathbb{N}$, if $|j| \le h$ and $|m| > 2h$, then $|j-m| \ge |m|-|j| \ge |m|-h > \frac{|m|}{2}$. Thus, there exists $h_2 \in \mathbb{N}$ such that

$$(\sum_{|j|\le h}\sum_{|m|>2h} |u(j)-u(m)|^p K_{s,p})^{\frac{1}{p}} < \epsilon \tag{2.4}$$

for all $h > h_2$. Fix $h > \max\{h_1.h_2\}$. Clearly, there exists $t_0 \in (0,1)$ such that for all $0 < t < t_0$, we get

$$\begin{gathered}\textstyle\sum_{|j|\le 2h}\sum_{|m|\le 2h}|\frac{|u(j)+tv(j)-u(m)-tv(m)|^p-|u(j)-u(m)|^p}{p}\\ -|u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))|K_{s,p}(j-m)\\ <\epsilon.\end{gathered}$$

Fix $0 < t < t_0$. For $j, m \in \mathbb{Z}$, by the mean value theorem, we can choose $0 < t_{j,m} < t$ such that

$$\frac{(|u(j)+tv(j)-u(m)-tv(m)|^p - |u(j)-u(m)|^p)}{tp}K_{s,p}(j-m) = |y(j)-y(m)|^{p-2}(y(j)-y(m))(v(j)-v(m))K_{s,p}(j-m), \tag{2.5}$$

where $y(j) = u(j) + t_{j,m}v(j)$. Clearly, $y \in W$ and $\|y\|_W \le 2C$. Observe that

$$
\begin{aligned}
&|\sum_{|j|\le h}\sum_{m|>2h}|u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}| \\
&\le \sum_{|j|\le h}\sum_{m|>wh}|u(j)-u(m)|^{p-1}|v(j)-v(m)|K_{sp} \\
&\le (\sum_{|j|\le h}\sum_{m|>2h}|u(j)-u(m)|^{p}K_{s,p})^{\frac{p-1}{p}}(\sum_{|j|\le h}\sum_{m|>2h}|v(j)-v(m)|^{p}K_{s,p})^{\frac{1}{p}} \le C\epsilon. \qquad (2.6)
\end{aligned}
$$

By Holder's inequality and (2.3), (2.6),

$$
\begin{aligned}
&|\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\frac{|u(j)+tv(j)-u(m)-tv(m)|^p-|u(j)-u(m)|^p}{p}K_{s,p}(j-m) \\
&\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}(j-m)| \\
\le\ & \epsilon+\sum_{|j|\le h}\sum_{m|>h}+\sum_{|j|>h}\sum_{|m|\le h} \\
&+\sum_{|j|>h}\sum_{|m|>h}|(\phi_p(y(j)-y(m))-\phi_p(u(j)-u(m)))(v(j)-v(m))|K_{s,p}(j-m) \\
\le\ & C\epsilon+\sum_{|j|\le h}\sum_{m|>2h}+\sum_{|j|>2h|}\sum_{m|\le h} \\
&+\sum_{|j|>h}\sum_{|m|>h}|(\phi_p(y(j)-y(m))-\phi_p(u(j)-u(m)))(v(j)-v(m))|K_{s,p}(j-m) \\
\le\ & C\epsilon,
\end{aligned}
$$

where $\phi_p(\tau) := |\tau|^{p-2}\tau$ for all $\tau \in \mathbb{R}$. Thus, (2.2) holds true.

An analogous argument gives

$$\lim_{t\to 0^+}\frac{\|u+tv\|^p-\|u\|^p}{pt} = \sum_{j\in\mathbb{Z}} V(j)|u(j)|^{p-2}u(j)v(j) .$$

Thus, we get

$$
\begin{aligned}
\langle\Psi'(u),v\rangle \ =\ & \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}(j-m) \\
&+\sum_{j\in\mathbb{Z}} V(j)|u(j)|^{p-2}u(j)v(j) .
\end{aligned}
$$

Therefore, Ψ is Gâteaux differentiable in W. Finally, we prove that $\Psi' : W \to W^*$ is continuous. To this aim, we assume that $\{u_n\}_n$ is a sequence in W such that $u_n \to u$ in W as $n \to \infty$. By Lemma 2.4, for any $\epsilon > 0$, there exists $h \in \mathbb{N}$ such that

$$(\sum_{|j|>h}\sum_{|m|>h}|u_n(j)-u_n(m)|^pK_{s,p}(j-m))^{1/p} < \epsilon \text{ for all } n \in \mathbb{N}$$

and

$$(\sum_{|j|>h}\sum_{|m|>h}|u(j)-u(m)|^pK_{s,p}(j-m))^{1/p} < \epsilon.$$

In addition, there exists $n_0 \in \mathbb{N}$ such that

$$(\sum_{|j|\le 2h|}\sum_{m|\le 2h}|[\phi(u_n(j)-u_n(m))-\phi(u(j)-u(m))]K_{s,p}^{1/p'}(j-m)|^{p'})^{1/p'}<\epsilon$$

for all $n \ge n_0$, where $p' = \frac{p}{p-1}$. For any $v \in W$ with $\|v\|_W \le 1$, and for any $n \ge n_0$, by the Hölder inequality and a similar discussion to above, we deduce

$$\begin{aligned}
&|\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}[\phi(u_n(j)-u_n(m))-\phi(u(j)-u(m))](v(j)-v(m))K_{s,p}(j-m)|\\
\le\ &(\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|[\phi(u_n(j)-u_n(m))-\phi(u(j)-u(m))]K_{s,p}^{1/p'}(j-m)|^{p'})^{1/p'}\\
&\times(\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|v(j)-v(m)|^p K_{s,p}(j-m))^{1/p}\\
\le\ &C\epsilon\|v\|_W.
\end{aligned}$$

Similarly, one can show that

$$|\sum_{k\in\mathbb{Z}}V(k)(|u_n|^{p-2}u_n-|u|^{p-2}u)v|\le C\epsilon\|v\|_W$$

as $n\to\infty$. Thus,

$$\|\Psi'(u_n)-\Psi'(u)\| = \sup_{\|v\|\le 1}|\{\Psi'(u_n)-\Psi'(u),\ v\}|\to 0,$$

which means that Ψ' is continuous. Consequently, we prove that $\Psi \in C^1(W,\mathbb{R})$. □

Lemma 2.6. *If conditions (V) and (f_1) hold, then $J \in C^1(W,\mathbb{R})$ with*

$$\langle J'(u),\ v\rangle = \sum_{j\in\mathbb{Z}} f(j,u(j))v(j)$$

for all $u, v \in W$.

Proof. By (f_1) , there exists $\delta > 0$ such that $|f(j,t)| \le |t|^{p-1}$ for all $j \in \mathbb{Z}$, $|t| \le \delta$.

Integrating we have

$$|F(j,t)| \le \frac{|t|^p}{p} \text{ for all } j\in\mathbb{Z}, |t|\le\delta \tag{2.7}$$

for all $u \in W$.

There exists $h \in \mathbb{N}$ such that $|u(k)| \le \delta$ for all $j \in \mathbb{Z}$, $|j| > h$, we obtain

$$\begin{aligned}
\left|\sum_{j\in\mathbb{Z}}F(j,u(j))\right| &= \left|\sum_{|j|\le h}F(j,u(j))+\sum_{|j|>h}F(j,u(j))\right|\\
&\le \sum_{|j|\le h}|F(j,u(j))|+\frac{1}{p}\sum_{|j|>h}|u(j)|^p\\
&\le \sum_{|j|\le h}|F(j,u(j))|+\frac{1}{pV_0}\sum_{|j|>h}V(j)\,|u(j)|^p\,,
\end{aligned}$$

thus J is well defined. Now, fix $u, v \in W$. We show that

$$lim_{t\to 0^+}\frac{J(u+tv)-J(u)}{t}=\sum_{j\in\mathbb{Z}} f(j,u(j))v(j)), \tag{2.8}$$

indeed, chose $R>0$ such that $\|u\|_p, \|v\|_p \le R$. Let $\delta>0$ be such that (2.7) holds and

$$\max\{u(j), v(j)\} \le \frac{\delta}{2} \text{ for all } j\in\mathbb{Z}, |j|>h.$$

For all $\epsilon>0$, there exists $h\in\mathbb{N}$ such that

$$\sum_{|j|>h} V(j)\,|v(j)|^p < \frac{\epsilon}{6V_0(2R)^{p-1}}.$$

Moreover, we can find $t_0\in(0,1)$ such that

$$\sum_{|j|\le h}\left|\frac{F(j,u(j))+tv(j)-F(j,u(j))}{t}-f(j,u(j))v(j)\right|<\frac{\epsilon}{3}.$$

Now fix $0<t\le t_0$. For all $|j|>h$, there exists $0\le t_k\le t$ such that

$$\frac{F(j,u(j))+tv(j)-F(j,u(j))}{t}=f(j,u(j)+t_k v(j))v(j).$$

We define $w\in W$ by $w(j)=0$ for all $|j|\le h$ and $w(j)=u(j)+t_j v(j)$ for all $|j|>h$. Therefore, $\|w\|\le\|u\|+\|v\|$ and $|w(j)|\le\delta$ for all $j\in\mathbb{Z}$. Summarizing what proved above, we have

$$\begin{aligned}
\left|\frac{J(u+tv)-J(u)}{t}-\sum_{j\in\mathbb{Z}}F(j,u(j))v(j)\right| &\le \frac{\epsilon}{3}+\sum_{|j|>h}|F(j,w(j))v(j)|+\sum_{|j|>h}|F(j,u(j))v(j)|\\
&\le \frac{\epsilon}{3}+\sum_{|j|>h}|w(j)|^{p-1}\,|v(j)|+\sum_{|j|>h}|u(j)|^{p-1}\,|v(j)|\\
&\le \frac{\epsilon}{3}+\frac{1}{V_0}\left(\left[\sum_{|j|>h}|w(j)|^p\right]^{\frac{1}{q}}+\left[\sum_{|j|>h}|u(j)|^p\right]^{\frac{1}{q}}\right)\left[\left[\sum_{|j|>h}|v(j)|^p\right]^{\frac{1}{q}}\right]\\
&< \frac{\epsilon}{3}+\frac{1}{V_0}\left[(2R)^{p-1}+R^{p-1}\right]\frac{\epsilon}{6V_0(2R)^{p-1}}\\
&< \epsilon.
\end{aligned}$$

Hence, (2.8) holds. So J is Gâteaux differentiable.

Next, similar to Lemma 2.5, we can prove that $J\in C^1(W,\mathbb{R})$.

Combining Lemma 2.5 and Lemma 2.6, we know that $I_\lambda\in C^1(W,\mathbb{R})$. □

Lemma 2.7. *If conditions* (V) *and* (f_1) *hold,* $1<q<p<\infty$, *then a critical point of* I_λ *is a homoclinic solution of Eq. (1.4) for all* $\lambda>0$.

Proof. Let $u\in W$ be a critical point of I_λ, that is, $I'_\lambda(u)=0$. Then

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}(j-m)+\sum_{j\in\mathbb{Z}}V(j)|u(j)|^{p-2}u(j)v(j)\\
&=\lambda\sum_{j\in\mathbb{Z}}f(j,u(j))v(j))
\end{aligned} \tag{2.9}$$

for all $v \in W$. For each $k \in \mathbb{Z}$, we define e_k as

$$e_k(j) = \delta_{kj} := \begin{cases} l \, , j = k \\ 0, j \neq k. \end{cases}$$

Obviously, $e_k \in W$. Choosing $v = e_k$ in (2.9), we get

$$p \sum_{j=k} |u(k) - u(j)|^{p-2}(u(k) - u(j))K_{sp}(k - j) + V(k)|u(k)|^{p-2}u(k) = \lambda f(k, u(k)) \, ,$$

which means that u is a solution of (1.4). Furthermore, according to $u \in W$ and Lemma 2.3, we can easily infer that $u(k) \to 0$ as $|k| \to \infty$. Hence u is a homoclinic solution of (1.4). □

3 Proof of Theorem 1.1

In this section, we employ the general mountain pass lemma (see [22]) to prove our main result. we first verify that the functional I_λ possesses the mountain pass geometry.

Lemma 3.1. *If conditions* (V) *and* $(f_1) - (f_4)$ *hold and*

$$\lambda > \frac{|b_0|^p \left(pC_{s,p} \sum_{m\neq h_0} \frac{1}{|h_0-m|^{1+ps}} + V(h_0)\right)}{pF(h_0, b_0)},$$

then the functional I_λ fulfills the mountain pass geometry.

Proof. On the one hand, according to (f_1), for any $0 < \epsilon < \frac{V_0}{p\lambda}$ there exists $\delta > 0$ such that

$$F(j, t) \leq \frac{\epsilon}{p} |t|^p \text{ for all } |t| < \delta \text{ and } j \in \mathbb{Z}.$$

Since $\|u\|_\infty \leq \|u\|_p$, we can find $0 < \omega < |b_0|^{p-1} \, V(h_0)^{\frac{1}{p}}$ such that $\|u\|_\infty < \delta$ for all $u \in W$ with $\|u\| = \omega$. Here h_0 and b_0 come from assumption (f_4). Then

$$\begin{aligned} I_\lambda(u) &= \Psi(u) - \lambda J(u) \geq \frac{\|u\|^p}{p} - \lambda\epsilon \, \|u\|^p \\ &\geq (\frac{1}{p} - \frac{\lambda\epsilon}{V_0}) \, \|u\|^p \\ &\geq (\frac{1}{p} - \frac{\lambda\epsilon}{V_0})\omega^p \\ &> 0. \end{aligned}$$

On the other hand, set $e = b_0 e_{h_0}(j)$ and $e_{h_0}(j) = 1$ if $j = h_0; e_{h_0}(j) = 0$ if $j \neq h_0$. Then $\|e\| = |b_0|^{p-1} \, V(h_0)^{\frac{1}{p}} > \omega$ and

$$\begin{aligned} I_\lambda(e) &= \frac{1}{p} \sum_{j\in\mathbb{Z}} \sum_{m\neq j} |e(j) - e(m)|^p K_{s,p}(j - m) + \frac{1}{p} \sum_{j\in\mathbb{Z}} V(j)|e(j)|^p - \lambda \sum_{j\in\mathbb{Z}} F(j, e(j)) \\ &= \frac{|b_0|^p}{p}(p \sum_{m\neq h_0} |K_{s,p}(j - m) + V(h_0)) - \lambda F(h_0, b_0) \\ &\leq \frac{|b_0|^p}{p}(pC_{s,p} \sum_{m\neq h_0} |\frac{1}{|h_0 - m|^{1+ps}} + V(h_0)) - \lambda F(h_0, b_0) \\ &< 0 \end{aligned}$$

for all

$$\lambda > \frac{|b_0|^p \left(pC_{s,p} \sum_{m\neq h_0} \frac{1}{|h_0-m|^{1+ps}} + V(h_0)\right)}{pF(h_0,b_0)}.$$

Therefore, the functional I_λ fulfills he mountain pass geometry. □

Lemma 3.2. *If conditions (V) and* $(f_1)-(f_3)$ *hold, then for all* $\lambda > 0$*, the functional* I_λ *fulfills the* $(PS)_C$ *condition (see [13]) in* W *for all* $c \in \mathbb{R}$.

Proof. Fix $\lambda > 0$, we first show that I_λ is coercive on W, i.e. $\lim_{\|u\|\to\infty} I_\lambda(u) = +\infty$.

By condition (f_3), for all $\epsilon \in (0, \frac{V_0}{p\lambda})$, there exists $T > 0$ such that

$$F(j,t) \le \epsilon\, |t|^p \quad \text{for all} \quad j \in \mathbb{Z} \quad \text{and} \quad |t| > T.$$

Again by (f_2), there exists $\theta \in \ell^1$ such that

$$|F(j,t)| \le \theta(j) \quad \text{for all} \quad j \in \mathbb{Z} \quad \text{and} \quad |t| \le T.$$

For all $u \in W$, we have

$$\begin{aligned} I_\lambda(u) =&\Psi(u) - \lambda J(u) \ge \frac{\|u\|^p}{p} - \lambda \sum_{|u(j)|\le T} F(j,u(j)) - \lambda \sum_{|u(j)|>T} F(j,u(j)) \\ \ge& \frac{\|u\|^p}{p} - \lambda\, \|\theta\|_1 - \lambda\epsilon\, \|u\|_p^p \\ \ge& (\frac{1}{p} - \frac{\lambda\epsilon}{V_0})\, \|u\|^p - \lambda\, \|\theta\|_1\,, \end{aligned} \tag{3.1}$$

which implies that coerciveness is true.

Next we prove that I_λ fulfills $(PS)c$ condition. Let $\{u_n\}_n$ be a sequence in W such that $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ in W^*. We know that $\{u_n\}_n$ is bounded due to the coercivity of I_λ. Thus, according to Lemma 2.3, there exists a subsequence of$\{u_n\}_n$, still denoted by $\{u_n\}_n$, such that $u_n \rightharpoonup u$ in W and $u_n \to u$ in ℓ^p. Then

$$\lim_{n\to\infty} \langle I'_\lambda(u_n - I'_\lambda(u), u_n - u)\rangle = 0. \tag{3.2}$$

Similar to Lemma 2.6, it is obvious that

$$\lim_{n\to\infty} \sum_{j\in\mathbb{Z}} (f(j,u_n(j) - f(j,u(j))(u_n(j) - u(j)) = 0.$$

Combining (3.2), we know that $\|u_n - u\| \to 0$, i.e., $u_n \to u$ in W . □

The proof of Theorem 1.1: By Lemmas 3.1-3.2 and mountain pass lemma, we have that for all

$$\lambda > \frac{|b_0|^p\, (pC_{s,p} \sum_{m\neq h_0} \frac{1}{|h_0-m|^{1+2s}} + V(h_0))}{pF(h_0,b_0)},$$

there exists a sequence $\{u_n\}_n \subset W$ such that

$$I_\lambda(u_n) \to c_\lambda > 0 \text{ and } I'_\lambda(u_n) \to 0, \text{as } n \to \infty,$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I_\lambda(\gamma(t))$$

and $\Gamma = \{\gamma \in ([0,1], X) : \gamma(0) = 1, \gamma(1) = e\}$.

So there exists a subsequence of $\{u_n\}_n$ (still denoted by $\{u_n\}_n$) such that $u_n \to u_\lambda^{(1)}$ strongly in W. Furthermore, $I_\lambda(u_\lambda^{(1)}) = \alpha \ge 0$ and $I'_\lambda(u_\lambda^{(1)}) = 0$. Hence, Lemma 2.7 implies that $u_\lambda^{(1)}$ is a homoclinic solution of (1.4).

Next we prove that Eq (1.4) has another homoclinic solution. Choose $\omega \in \mathbb{R}$ such that $I_\lambda(e) < \omega < 0$, where e is given by Lemma 3.1. Set

$$M = \{u \in W : I_\lambda(u) \le \omega\}.$$

It is clear that $M \ne 0$. It follows from (3.1) that M is a bounded subset in W.

Now we infer that I_λ is bounded below on M. If not, we suppose that there exists a sequence $\{u_n\}_n \subset M$ such that

$$\lim_{n\to\infty} I_\lambda(u_n) = -\infty. \tag{3.3}$$

Since $\{u_n\}_n$ is bounded, up to a subsequence, we have $u_n \rightharpoonup u$ in W and $u_n \to u$ in ℓ^p. Similar to Lemma 2.6, we know that J is continuous in ℓ^p. We obtain that Ψ is weakly lower semi-continuous in W thanks to the convexity of Ψ. Thus,

$$\lim_{n\to\infty}\inf\ I_\lambda(u_n) \ge I_\lambda(u) > -\infty,$$

which contradicts (3.3). So we can define

$$c_\lambda^{\sim} = \inf\{I_\lambda(u) : u \in M\} = \inf_W I_\lambda(u).$$

Then $c_\lambda^{\sim} < 0$ for all $\lambda > 0$. On basis of Lemma 3.1 and the Ekeland variational principle, applied in M, there exists a sequence $\{u_n\}_n$ such that

$$c_\lambda^{\sim} \le I_\lambda(u_n) \le c_\lambda^{\sim} + \frac{1}{n} \tag{3.4}$$

and

$$I_\lambda(\upsilon) \ge\ I_\lambda(u_n) - \frac{\|u_n - \upsilon\|}{n} \tag{3.5}$$

for all $\upsilon \in M$.

It is clear that $\{u_n\}_n$ is a $(PS)_{c_\lambda^{\sim}}$ sequence for the functional I_λ. Similar to Lemma 3.2, there exists a subsequence of $\{u_n\}_n$ (still denoted by $\{u_n\}_n$) such that $u_n \to u_\lambda^{(2)}$ in W. So, we get a nontrivial homoclinic solution $u_\lambda^{(2)}$ of Eq. (1.4) fulfilling

$$I_\lambda(u_\lambda^{(2)}) \le \omega < 0.$$

Furthermore, we have

$$I_\lambda(u_\lambda^{(2)}) = c_\lambda^{\sim} \le \omega < 0 < \alpha < c_\lambda = I_\lambda(u_\lambda^{(1)})$$

for all

$$\lambda > \frac{|b_0|^p \left(pC_{s,p}\sum_{m\ne h_0}\frac{1}{|h_0-m|^{1+2s}} + V(h_0)\right)}{pF(h_0, b_0)}.$$

Therefore, equation (1.4) has at least two nontrivial homoclinic solutions.

Finally, we show that all critical points of the functional I_λ are nonnegative. Let $u \in W\backslash\{0\}$ be a critical point of I_λ. Then $I'_\lambda(u_n) = 0$ and $u_k \to 0$ as $|k| \to \infty$. Let $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. We have $\left\langle I'_\lambda(u_n), -u^- \right\rangle = 0$, due to $I'_\lambda(u_n) = 0$. It follows from $f(k,t) = 0$ for all $k \in \mathbb{Z}$, $t \le 0$ that

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |u(j) - u(m)|^{p-2}(-u^-(j) + u^-(m))K_{s,p}(j-m) + \sum_{j\in\mathbb{Z}} V(j)|u(j)|^{p-2}(-u^-(j)) = 0,$$

which implies that

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |u(j) - u(m)|^{p-2}(-u^-(j) + u^-(m))K_{s,p}(j-m) \le 0.$$

We know that for all $j, m \in \mathbb{Z}$,

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |u^-(j) - u^-(m)|^p K_{s,p}(j-m) \le 0,$$

which means that $u^-(j) = u^-(m)$ for all $j, m \in \mathbb{Z}$. Thus, there exists $C \ge 0$ such that $u^-(k) \equiv C$ for any $k \in \mathbb{Z}$. By virtue of $u_k \to 0$ and $u^-(k) \le |u(k)|$, we get that $C = 0$. Hence, we infer that $u^-(k) = 0$, which ends the proof.

References

[1] Agarwal, R.P., Perera, K., O'Regan, D.: Multiple positive solutions of singular and nonsingular discrete problems via variational methods. Nonlinear Anal. 58, 69-73 (2004).

[2] Applebaum, D.: Lévy processesfrom probability to finance quantum groups. Not. Am. Math. Soc. 51, 1336-1347 (2004).

[3] Autuori, G., Pucci, P.: Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$. J. Differential Equations. 255, 2340-2362 (2013).

[4] Bilel, M., Rafik, G., Salah, B., Tahar, B.: Existence of positive solutions for a new class of nonlocal p(x)-Kirchhoff elliptic systems via sub-super solutions concept. Applied Sciences. 20, 132-14 (2018).

[5] Cabada, A., Iannizzotto, A., Tersian, S.: Multiple solutions for discrete boundary value problems. J. Math. Anal. Appl. 356, 418-428 (2009).

[6] Caffarelli, L.: Nonlocal diffusions, drifts and games. Nonlinear Partial Differ. Equ. 7, 37-52 (2012).

[7] Ciaurri,Ó., Roncal, L., Stinga, P.R., Torrea, J.L., Varona, J.L.: Nonlocal discrete diffusion equation and the fractional discrete Laplacian, regularity and applications. Adv. Math. 330, 688-738 (2018).

[8] Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V.: Banach Space Theory. Springer Berlin (2011).

[9] Fiscella, A., Pucci, P., Zhang, B.: p-fractional Hardy – Schrodinger – Kirhhoff systems with critical nonlinearities. Adv. Nonlinear Anal. 8, 1111-1131 (2019).

[10] Fiscella, A., Servadei, R., Valdinoci, E.: Density properties for fractional Sobolev spaces. Ann. Acad. Sci. Fenn., Math. 40, 235-253 (2015).

[11] Huang, Y., Oberman, A.: Numerical methods for the fractional Laplacian: afinite difference-quadrature approach. SIAM J. Numer. Anal.52, 3056-3084 (2014).

[12] Iannizzotto, A., Rȧdulescu, V.: Positive homoclinic solutions for the discrete p-Laplacian with a coercive weight function. Differ. Integral Equ. 27, 35-44 (2014).

[13] Iannizzotto, A., Tersian, S.A.: Multiple homoclinic solutions for the discrete p-Laplacian via critical point theory. J. Math. Anal. Appl. 403, 173-182 (2013).

[14] Izydorek, M., Janzewsk Mawhin, Homoclinics for singular strong force Lagrangian systems. Adv. Nonlinear Anal. 9 ,644-653 (2020).

[15] Ju, X., Hu, D., Xiang, M.: The Nehari manifold method for discrete fractional p-Laplacian equations. Advances in Difference Equations. 559, 1-21 (2020).

[16] Laskin, N.: Fractional quantum mechanics and Lévy path integrals. Phys. Lett. A. 268, 298-305 (2000).

[17] Linda, M., Tahar, B., Rafik, G., Mohamad, B.: Existence of weak solutions for p-Laplacian problem with impulsive effects. Applied Sciences. 22, 128–145 (2020).

[18] Ma, M., Guo, Z.: Homoclinic orbits for second order self-adjoint difference equations. J. Math. Anal. Appl. 323,513-521 (2005).

[19] Mihailescu M., Rȧdulesu, V., Tersian, S.: Homoclinic solutions of difference equations with variable exponents.Topol. Methods Nonlinear, Anal. 38, 277-289 (2011).

[20] Molica Bisci, G., Rȧdulescu, V., Servadei, R.: Variational Methods for Nonlocal Fractional Problems. Cambridge University Press, Cambridge (2015).

[21] Salah, B., Rafik, G., Tahar, B.: Existence of Positive Solutions For a Class of Quasilinear Singular Elliptic Systems Involving Caffarelli-Kohn- Nirenberg Exponent With Sign-Changing Functions . Indian J. Pure Appl. Math, 705–715, (2018).

[22] Servadei, R., Valdinoci, E.: Mountain Pass solutions for non-local elliptic operators. J. Math. Anal. Appl. 389, 887-898 (2012).

[23] Tahar, B., Rafik, G.: Existence and uniqueness of weak solution for a nonlocal problem involving the p-Laplacian. IJPAM. 98, 11-21 (2015).

[24] Valdinoci, E.: From the longjump random walkto the fractional Laplacian. Bol. Soc. Esp. Mat. Apl. 49, 33-44 (2009).

[25] Xiang, M., Hu, D., Yang, D.: Least energy solutions for fractional Kirchhoff problems with logarithmic nonlinearity. Nonlinear. Anal. 198, 111899 (2020).

[26] Xiang, M., Zhang, B.: Homoclinic solutions for fractional discrete Laplacian equations. Nonlinear Anal. 198, 111886,1-15 (2020).

[27] Xiang, M., Zhang, B., Yang, D.: Multiplicity results for variable-order fractional Laplacian equations with variable growth. Nonlinear. Anal. 178, 190-204 (2019).

[28] Xiang, M., Rȧdulescu, V., Zhang, B.: A critical fractional Choquard Kirchhoff problem with magnetic field. Commun. Contemp Math. 21, 185004 (2019).