Some new post-quantum Ostrowski-type integral inequalities for twice (p,q)-differentiable convex functions

Waewta Luangboon[1], Kamsing Nonlaopon[1], Jessada Tariboon[2], S.K. Ntouyas[3], and Huseyin Budak[4]

[1]Khon Kaen University Faculty of Science
[2]King Mongkut's Institute of Technology North Bangkok
[3]University of Ioannina
[4]Duzce University

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Abstract

In this paper, we establish a new (p,q)-integral identity using twice (p,q)-differentiable convex functions. Then, we use this result to derive some new post-quantum Ostrowski-type integral inequalities for twice (p,q)-differentiable convex functions. The newly established results are also proven to generalize some existing results in the area of integral inequalities of already published ones.

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Waewta Luangboon[1] | Kamsing Nonlaopon*[1] | Jessada Tariboon[2] | Sortiris K. Ntouyas[3,4] | Hüseyin Budak[5]

[1]Department of Mathematics, Khon Kaen University, Khon Kaen, Thailand
[2]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand
[3]Department of Mathematics, University of Ioannina, Ioannina, Greece
[4]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
[5]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

Correspondence
*Kamsing Nonlaopon, Department of Mathematics, Khon Kaen University, Khon Kaen, Thailand. Email: nkamsi@kku.ac.th

ABSTRACT

In this paper, we establish a new (p, q)-integral identity using twice (p, q)-differentiable convex functions. Then, we use this result to derive some new post-quantum Ostrowski-type integral inequalities for twice (p, q)-differentiable convex functions. The newly established results are also proven to generalize some existing results in the area of integral inequalities of already published ones.
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Ostrowski-type inequality, (p, q)-differentiable function, (p, q)-integral inequalities, (p, q)-calculus, Convex function

1 | INTRODUCTION

Integral inequalities are a very necessary tool in the study of applied and pure mathematics. One of the integral inequalities that many researchers have focused on significant attention is Ostrowski-type integral inequalities because it can be applied in statistics, quadrature, stochastic, probability and optimization theory, integral operator theory, and information. The classical integral inequality for the differentiable function is as follows:

Theorem 1. [1] Let $\varphi : [a_1, a_2] \to \mathbb{R}$ be a differentiable function on (a_1, a_2) whose the derivative function $\varphi : (a_1, a_2) \to \mathbb{R}$ is bounded on (a_1, a_2) and $\|\varphi'\|_\infty = \sup\limits_{t\in(a_1,a_2)} |\varphi'(t)| < \infty$. Then

$$\left|\varphi(x) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \varphi(t)\, dt\right| \leq \left[\frac{1}{4} + \frac{\left((x - \dfrac{a_1 + a_2}{2}\right)^2}{(a_2 - a_1)^2}\right](a_2 - a_1)\,\|\varphi'\|_\infty$$

for all $x \in [a_1, a_2]$.

In recent years, many researchers have focused on the Ostrowski-type integral inequalities and their applications, see[2,3,4,5,6,7,8,9,10,11,12] and the references cited therein for more details. Specifically, many researchers have worked on the Ostrowski-type integral inequalities and their applications using quantum calculus, some results can be found in[13,14,15,16,17,18,19,20,21] and the references cited therein.

Quantum calculus, also known as q-calculus, is the study of calculus without limits. The concept was revealed by renowned mathematician Euler (1707-1783), who introduced the number in q-infinite series defined by Newton. Then, in 1910, Jackson[22] defined q_{a_1}-integrals and q_{a_1}-derivatives on the interval $(0, \infty)$ extending the concept of Euler. Later, in 1966, Al-Salam[23] introduced fractional q_{a_1}-integrals and fractional q_{a_1}-derivatives. In q-calculus, we obtain q-analoques of mathematical objects that can be recaptured by taking $q \to 1$. The topic of q-calculus has been received outstanding attention from many scientists because it has numerous applications in various fields of physics and mathematics, for example, the theory of relativity, combinatorics, hypergeometric functions, orthogonal polynomials, mechanics, and number theory, see[24,25,26,27,28,29,30,31,32,33,34] and the references cited therein for more details.

In 2013, Tariboon and Ntouyas[35] presented the q_{a_1}-integrals and the q_{a_1}-derivatives on finite intervals and addressed numerous problems on q_{a_1}-analogues of classical inequalities. Recently, in 2020, Bermudo et al.[36] presented q^{a_2}-integrals and q^{a_2}-derivatives on finite intervals and also proved their some basic properties. Currently, these topics of q-calculus have been studied in various integral inequalities such as Hanh, Hermite-Hadamard, Hermite-Hadamard-like, Newton-type, Simpson-type, Fejér-type, and Ostrowski-type integral inequalities, see[37,38,39,40,41,42,43] and the references cited therein for more details.

The q-calculus generalization is called (p, q)-calculus, also known as post-quantum calculus. The (p, q)-calculus has two independent parameters that are p-number and q-number. Apparently, the q-calculus cannot be directly obtained by substituting q by q/p in q-calculus, but it can be directly obtained by taking $p = 1$ in (p, q)-calculus. Then, the classical inequalities can be gained by taking $q \to 1$. The concept of $(p, q)_{a_1}$-integrals and $(p, q)_{a_1}$-derivatives on the interval $(0, \infty)$ was first presented by Chakrabarti and Jagannathan[44] in 1991. Later, the concept of $(p, q)_{a_1}$-integrals and $(p, q)_{a_1}$-derivatives on finite intervals was presented by Tunç, and Göv[45,46] in 2016. Recently, the concept of the $(p, q)^{a_2}$-integrals and $(p, q)^{a_2}$-derivatives on the finite intervals has been presented by Vivas-Cortez et al.[47] in 2021. Currently, the topic of (p, q)-calculus has been received outstanding attention from many scientists, some new results can be found in[48,49,50,51,52,53,54,55,56] and the references cited therein.

In 2021, Ali et al.[43] introduced quantum Ostrowski-type integral inequalities for twice q-differentiable convex functions. By taking $q \to 1$, they obtain classical results on some Ostrowski-type integral inequalities for functions, whose second derivatives are h-convex functions[60]. Inspired by the above mentioned reports, we establish some new post quantum Ostrowski-type integral inequalities for twice (p, q)-differentiable convex functions to extend and generalize the results given in previous reports.

The rest of the paper is organized as follows: In Section 2, we provide some basic knowledge and definitions of (p, q)-calculus. In Section 3, post quantum Ostrowski-type integral inequalities for twice (p, q)-differentiable convex functions are presented. In Section 4, we summarize our results.

2 | PRELIMINARIES

In this section, we discuss some basic knowledge and definitions of (p, q)-calculus which will be used in our work. Throughout this paper, we assume that $0 < q < p \leq 1$ is constants and $[a_1, a_2] \subseteq \mathbb{R}$ is an interval with $a_1 < a_2$. The (p, q)-number of η is given by

$$[\eta]_{p,q} = \frac{p^\eta - q^\eta}{p - q} = p^{\eta-1} + p^{\eta-2}q + \cdots + pq^{\eta-2} + q^{\eta-1}, \quad \eta \in \mathbb{N},$$

which is a generalization of the q-analogue or q-number of η such that

$$[\eta]_q = \frac{1 - q^\eta}{1 - q} = 1 + q + \cdots + q^{\eta-2} + q^{\eta-1}, \quad \eta \in \mathbb{N},$$

see[34] for more details.

Definition 1. [45,46] For a continuous function $\varphi : [a_1, a_2] \to \mathbb{R}$, then the $(p, q)_{a_1}$-derivative on $[a_1, a_2]$ of function φ at t is defined by

$$\begin{aligned} {}_{a_1}D_{p,q}\varphi(t) &= \frac{\varphi\left(pt + (1-p)a_1\right) - \varphi\left(qt + (1-q)a_1\right)}{(p-q)(t-a_1)}, \ \ t \neq a_1, \\ {}_{a_1}D_{p,q}\varphi(a_1) &= \lim_{t \to a_1} {}_{a_1}D_{p,q}\varphi(t). \end{aligned} \tag{1}$$

The function φ is called $(p,q)_{a_1}$-differentiable function on $[a_1,a_2]$ if ${}_{a_1}D_{p,q}\varphi(t)$ exists for all $t \in [a_1,(a_2-a_1)/p+a_1]$.

Note that if $p=1$ and ${}_{a_1}D_{1,q}\varphi(t) = {}_{a_1}D_q\varphi(t)$ in (1), then (1) reduces to

$$\begin{aligned} {}_{a_1}D_q\varphi(t) &= \frac{\varphi(t)-\varphi\left(qt+(1-q)a_1\right)}{(1-q)(t-a_1)}, \quad t \neq a_1, \\ {}_{a_1}D_q\varphi(a_1) &= \lim_{t\to a_1} {}_{a_1}D_q\varphi(t), \end{aligned} \tag{2}$$

which is the well-known q_{a_1}-derivative of function φ on $[a_1,a_2]$, see[57,58] for more details.

Moreover, if $a_1=0$ and ${}_0D_q\varphi(t) = D_q\varphi(t)$ in (2), then (2) reduces to

$$\begin{aligned} D_q\varphi(t) &= \frac{\varphi(t)-\varphi(qt)}{(1-q)t}, \quad t \neq a_1, \\ D_q\varphi(a_1) &= \lim_{t\to 0} D_q\varphi(t), \end{aligned}$$

which is the well-known q-derivative of function φ on $[0,a_2]$, also called q-Jackson derivative, see[34] for more details.

Example 1. Define function $\varphi : [a_1,a_2] \to \mathbb{R}$ by $\varphi(t)=t^2+C$, where C is constant. Applying Definition 1 for $t \neq a_1$, we have

$$\begin{aligned} {}_{a_1}D_{p,q}(t^2+C) &= \frac{\left[(pt+(1-p)a_1)^2+C\right]-\left[(qt+(1-q)a_1)^2+C\right]}{(p-q)(t-a_1)} \\ &= \frac{(p+q)t^2+2a_1t[1-(p+q)]+a_1^2[(p+q)-2]}{(t-a_1)} \\ &= \frac{(p+q)(t-a_1)^2+2a_1(t-a_1)}{(t-a_1)} \\ &= [2]_{p,q}(t-a_1)+2a_1. \end{aligned}$$

Definition 2. [47] For a continuous function $\varphi : [a_1,a_2] \to \mathbb{R}$, then the $(p,q)^{a_2}$-derivative on $[a_1,a_2]$ of function φ at t is defined by

$$\begin{aligned} {}^{a_2}D_{p,q}\varphi(t) &= \frac{\varphi\left(qt+(1-q)a_2\right)-\varphi\left(pt+(1-p)a_2\right)}{(p-q)(a_2-t)}, \quad t \neq a_2, \\ {}^{a_2}D_{p,q}\varphi(a_2) &= \lim_{t\to a_2} {}^{a_2}D_{p,q}\varphi(t). \end{aligned} \tag{3}$$

The function φ is called $(p,q)^{a_2}$-differentiable function on $[a_1,a_2]$ if ${}^{a_2}D_{p,q}\varphi(t)$ exists for all $t \in [a_2-(a_2-a_1)/p, a_2]$.

Note that if $p=1$ and ${}^{a_2}D_{1,q}\varphi(t) = {}^{a_2}D_q\varphi(t)$ in (3), then (3) reduces to

$$\begin{aligned} {}^{a_2}D_q\varphi(t) &= \frac{\varphi\left(qt+(1-q)a_2\right)-\varphi(t)}{(1-q)(a_2-t)}, \quad t \neq a_2, \\ {}^{a_2}D_q\varphi(a_2) &= \lim_{t\to a_2} {}^{a_2}D_q\varphi(t), \end{aligned}$$

which is the well-known q^{a_2}-derivative of function φ on $[a_1,a_2]$, see[36,59] for more details.

Example 2. Define function $\varphi : [a_1,a_2] \to \mathbb{R}$ by $\varphi(t)=t^2+C$, where C is constant. Applying Definition 2 for $t \neq a_2$, we have

$$\begin{aligned} {}^{a_2}D_{p,q}(t^2+C) &= \frac{\left[(qt+(1-q)a_2)^2+C\right]-\left[(pt+(1-p)a_2)^2+C\right]}{(p-q)(a_2-t)} \\ &= \frac{-(p+q)t^2+2a_2t[(p+q)-1]+a_2^2[2-(p+q)]}{(a_2-t)} \\ &= \frac{-(p+q)(a_2-t)^2+2a_2(a_2-t)}{(a_2-t)} \\ &= [2]_{p,q}(t-a_2)+2a_2. \end{aligned}$$

Definition 3. [45] For a continuous function $\varphi : [a_1, a_2] \to \mathbb{R}$, then the $(p,q)_{a_1}$-integral on $[a_1, a_2]$ of function φ at t is defined by

$$\int_{a_1}^{x} \varphi(t) \ {}_{a_1}d_{p,q}t = (p-q)(x-a_1)\sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \varphi\left(\frac{q^\eta}{p^{\eta+1}}x + \left(1-\frac{q^\eta}{p^{\eta+1}}\right)a_1\right) \tag{4}$$

for $x \in [a_1, a_2]$.

The function φ is called $(p,q)_{a_1}$-integrable function on $[a_1, a_2]$ if $\int_{a_1}^{x} \varphi(t) \ {}_{a_1}d_{p,q}t$ exists for all $t \in [a_1, a_1 + p(x-a_1)]$.

Example 3. Define function $\varphi : [a_1, a_2] \to \mathbb{R}$ by $\varphi(t) = At + B$, where A and B are constants. Applying Definition 3, we have

$$\begin{aligned}\int_{a_1}^{a_2} \varphi(t) \ {}_{a_1}d_{p,q}t &= \int_{a_1}^{a_2} (At+B) \ {}_{a_1}d_{p,q}t \\ &= A(p-q)(a_2-a_1)\sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}}\left(\frac{q^\eta}{p^{\eta+1}}a_2 + \left(1-\frac{q^\eta}{p^{\eta+1}}\right)a_1\right) \\ &\quad + B(p-q)(a_2-a_1)\sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \\ &= \frac{A(a_2-a_1)(a_2-a_1(1-p-q))}{[2]_{p,q}} + B(a_2-a_1).\end{aligned}$$

Definition 4. [47] For a continuous function $\varphi : [a_1, a_2] \to \mathbb{R}$, then the $(p,q)^{a_2}$-integral on $[a_1, a_2]$ of function φ at t is defined by

$$\int_{x}^{a_2} \varphi(t) \ {}^{a_2}d_{p,q}t = (p-q)(a_2-x)\sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \varphi\left(\frac{q^\eta}{p^{\eta+1}}x + \left(1-\frac{q^\eta}{p^{\eta+1}}\right)a_2\right) \tag{5}$$

for $x \in [a_1, a_2]$.

The function φ is called $(p,q)^{a_2}$-integrable function on $[a_1, a_2]$ if $\int_{a_1}^{a_2} \varphi(t) \ {}^{a_2}d_{p,q}t$ exists for all $t \in [a_2 - p(a_2-x), a_2]$.

Example 4. Define function $\varphi : [a_1, a_2] \to \mathbb{R}$ by $\varphi(t) = At + B$, where A and B are constants. Applying Definition 4, we have

$$\begin{aligned}\int_{a_1}^{a_2} \varphi(t) \ {}^{a_2}d_{p,q}t &= \int_{a_1}^{a_2} (At+B) \ {}^{a_2}d_{p,q}t \\ &= A(p-q)(a_2-a_1)\sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}}\left(\frac{q^\eta}{p^{\eta+1}}a_1 + \left(1-\frac{q^\eta}{p^{\eta+1}}\right)a_2\right) \\ &\quad + B(p-q)(a_2-a_1)\sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \\ &= \frac{A(a_2-a_1)(a_1-a_2(1-p-q))}{[2]_{p,q}} + B(a_2-a_1).\end{aligned}$$

Lemma 1. [45] For $\alpha \in \mathbb{R} \setminus \{-1\}$, the following inequality holds:

$$\int_{a_1}^{a_2} \left(t-a_1\right)^{\alpha} \ {}_{a_1}d_{p,q}t = \frac{(a_2-a_1)^{\alpha+1}}{[\alpha+1]_{p,q}} \tag{6}$$

for $t \in [a_1, a_2]$.

Theorem 2. [46] Let $\varphi, \psi : [a_1, a_2] \to \mathbb{R}$ be continuous functions and $r > 1$ with $1/s + 1/r = 1$, then

$$\int_{a_1}^{a_2} |\varphi(t)\psi(t)| \ {}_{a_1}d_{p,q}t \leq \left(\int_{a_1}^{a_2} |\varphi(t)|^s \ {}_{a_1}d_{p,q}t\right)^{1/s}\left(\int_{a_1}^{a_2} |\psi(t)|^r \ {}_{a_1}d_{p,q}t\right)^{1/r} \tag{7}$$

for $t \in [a_1, a_2]$.

3 | MAIN RESULTS

In this section, we give some new estimates of post quantum Ostrowski-type integral inequalities for twice (p,q)-differentiable functions involving $(p,q)_{a_1}$- and $(p,q)^{a_2}$-integrals. Let $J_1 = [a_2 - p(a_2 - x), a_2]$ and $J_2 = [a_1, a_1 + p(x - a_1)]$. We start with the following (p,q)-integral identities.

Theorem 3. If $\varphi : [a_1, a_2] \to \mathbb{R}$ is a twice (p,q)-differentiable function such that ${}^{a_2}D^2_{p,q}\varphi$ and ${}_{a_1}D^2_{p,q}\varphi$ are continuous and integrable functions on J_1 and J_2, respectively. Then

$$(x-a_1)^2(a_2-x)^2\left[(a_1-x)\int_0^1 t^2\,{}_{a_1}D^2_{p,q}\varphi(tx+(1-t)a_1)\,d_{p,q}t+(x-a_2)\int_0^1 t^2\,{}^{a_2}D^2_{p,q}\varphi(tx+(1-t)a_2)\,d_{p,q}t\right]={}^{a_2}_{a_1}L_{p,q}(x), \quad (8)$$

where

$$\begin{aligned}
&{}^{a_2}_{a_1}L_{p,q}(x)\\
&=\frac{(x-a_1)(a_2-x)}{pq^3(p-q)}\Big[(x-a_1)pq\varphi(qx+(1-q)a_2)+(a_2-x)pq\varphi(qx+(1-q)a_1)-(x-a_1)(q^2+pq-p^2)\varphi(px+(1-p)a_2)\\
&\quad-(a_2-x)(q^2+pq-p^2)\varphi(px+(1-p)a_1)\Big]-\frac{[2]_{p,q}}{p^3q^3}\left[(x-a_1)^2\int_{p^2x+(1-p^2)a_2}^{a_2}\varphi(t)\,{}^{a_2}d_{p,q}t+(a_2-x)^2\int_{a_1}^{p^2x+(1-p^2)a_1}\varphi(t)\,{}_{a_1}d_{p,q}t\right].
\end{aligned}$$

Proof. Using Definition 1, we have

$$\begin{aligned}
&{}_{a_1}D^2_{p,q}\varphi(ta_2+(1-t)a_1)\\
&={}_{a_1}D_{p,q}\left({}_{a_1}D_{p,q}\varphi(ta_2+(1-t)a_1)\right)\\
&={}_{a_1}D_{p,q}\left(\frac{\varphi(pta_2+(1-pt)a_1)-\varphi(qta_2+(1-qt)a_1)}{(p-q)(a_2-a_1)t}\right)\\
&=\frac{1}{(p-q)(a_2-a_1)t}\left[\frac{\varphi(p^2ta_2+(1-p^2t)a_1)-\varphi(pqta_2+(1-pqt)a_1)}{pt(p-q)(a_2-a_1)}-\frac{\varphi(pqta_2+(1-pqt)a_1)-\varphi(q^2ta_2+(1-q^2t)a_1)}{qt(p-q)(a_2-a_1)}\right]\\
&=\frac{q\varphi(p^2ta_2+(1-p^2t)a_1)-[2]_{p,q}\varphi(pqta_2+(1-pqt)a_1)+p\varphi(q^2ta_2+(1-q^2t)a_1)}{pqt^2(p-q)^2(a_2-a_1)^2}. \qquad (9)
\end{aligned}$$

Applying (9) and Definition 3, we obtain

$$\begin{aligned}
&\int_0^1 t^2\,{}_{a_1}D^2_{p,q}\varphi(tx+(1-t)a_1)\,d_{p,q}t\\
&=\int_0^1\frac{q\varphi(p^2tx+(1-p^2t)a_1)-[2]_{p,q}\varphi(pqtx+(1-pqt)a_1)+p\varphi(q^2tx+(1-q^2t)a_1)}{pq(p-q)^2(x-a_1)^2}\,d_{p,q}t\\
&=\frac{q(p-q)(x-a_1)\sum_{\eta=0}^{\infty}\frac{q^\eta}{p^{\eta+1}}\varphi\left(p^2\frac{q^\eta}{p^{\eta+1}}x+\left(1-p^2\frac{q^\eta}{p^{\eta+1}}\right)a_1\right)}{pq(p-q)^2(x-a_1)^3}\\
&\quad-\frac{[2]_{p,q}(p-q)(x-a_1)\sum_{\eta=0}^{\infty}\frac{q^{\eta+1}}{p^{\eta+1}}\varphi\left(p\frac{q^{\eta+1}}{p^{\eta+1}}x+\left(1-p\frac{q^{\eta+1}}{p^{\eta+1}}\right)a_1\right)}{pq^2(p-q)^2(x-a_1)^3}\\
&\quad+\frac{p(p-q)(x-a_1)\sum_{\eta=0}^{\infty}\frac{q^{\eta+2}}{p^{\eta+1}}\varphi\left(\frac{q^{\eta+2}}{p^{\eta+1}}x+\left(1-\frac{q^{\eta+2}}{p^{\eta+1}}\right)a_1\right)}{pq^3(p-q)^2(x-a_1)^3}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(p-q)(x-a_1)\sum_{\eta=0}^{\infty}\frac{q^\eta}{p^{\eta+1}}\varphi\left(p^2\frac{q^\eta}{p^{\eta+1}}x+\left(1-p^2\frac{q^\eta}{p^{\eta+1}}\right)a_1\right)}{p(p-q)^2(x-a_1)^3} \\
&\quad - \frac{p[2]_{p,q}(p-q)(x-a_1)\sum_{\eta=0}^{\infty}\frac{q^{\eta+1}}{p^{\eta+2}}\varphi\left(p^2\frac{q^{\eta+1}}{p^{\eta+2}}x+\left(1-p^2\frac{q^{\eta+1}}{p^{\eta+2}}\right)a_1\right)}{pq^2(p-q)^2(x-a_1)^3} \\
&\quad + \frac{p^3(p-q)(x-a_1)\sum_{\eta=0}^{\infty}\frac{q^{\eta+2}}{p^{\eta+3}}\varphi\left(p^2\frac{q^{\eta+2}}{p^{\eta+3}}x+\left(1-p^2\frac{q^{\eta+2}}{p^{\eta+3}}\right)a_1\right)}{pq^3(p-q)^2(x-a_1)^3} \\
&= \frac{[2]_{p,q}}{p^3q^3(x-a_1)^3}\int_{a_1}^{p^2x+(1-p^2)a_1}\varphi(t)\ {}_{a_1}d_{p,q}t + \frac{(q^2+pq-p^2)\varphi(px+(1-p)a_1)}{pq^3(p-q)(x-a_1)^2} - \frac{\varphi(qx+(1-q)a_1)}{q^2(p-q)(x-a_1)^2}.
\end{aligned}
\tag{10}
$$

Using Definition 2, we have

$$
\begin{aligned}
&{}^{a_2}D_{p,q}^2\varphi(ta_1+(1-t)a_2) \\
&= {}^{a_2}D_{p,q}\left({}^{a_2}D_{p,q}\varphi(ta_1+(1-t)a_2)\right) \\
&= {}^{a_2}D_{p,q}\left(\frac{\varphi(qta_1+(1-qt)b)-\varphi(pta_1+(1-pt)a_2)}{(p-q)(a_2-a_1)t}\right) \\
&= \frac{1}{(p-q)(a_2-a_1)t}\left[\frac{\varphi(q^2ta_1+(1-q^2t)a_2)-\varphi(pqta_1+(1-pqt)a_2)}{qt(p-q)(a_2-a_1)} - \frac{\varphi(pqta_1+(1-pqt)a_2)-\varphi(p^2ta_1+(1-p^2t)a_2)}{pt(p-q)(a_2-a_1)}\right] \\
&= \frac{p\varphi(q^2ta_1+(1-q^2t)a_2)-[2]_{p,q}\varphi(pqta_1+(1-pqt)a_2)+q\varphi(p^2ta_1+(1-p^2t)a_2)}{pqt^2(p-q)^2(a_2-a_1)^2}.
\end{aligned}
\tag{11}
$$

Applying (11) and Definition 4, we obtain

$$
\begin{aligned}
&\int_0^1 t^2\ {}^{a_2}D_{p,q}^2\varphi(tx+(1-t)a_2)\ d_{p,q}t \\
&\quad = \int_0^1 \frac{p\varphi(q^2tx+(1-q^2t)a_2)-[2]_{p,q}\varphi(pqtx+(1-pqt)a_2)+q\varphi(p^2tx+(1-p^2t)a_2)}{pq(p-q)^2(a_2-x)^2}\ d_{p,q}t \\
&\quad = \frac{p(p-q)(a_2-x)\sum_{\eta=0}^{\infty}\frac{q^{\eta+2}}{p^{\eta+1}}\varphi\left(\frac{q^{\eta+2}}{p^{\eta+1}}x+\left(1-\frac{q^{\eta+2}}{p^{\eta+1}}\right)a_2\right)}{pq^3(p-q)^2(a_2-x)^3} \\
&\qquad - \frac{[2]_{p,q}(p-q)(a_2-x)\sum_{\eta=0}^{\infty}\frac{q^{\eta+1}}{p^{\eta+1}}\varphi\left(p\frac{q^{\eta+1}}{p^{\eta+1}}x+\left(1-p\frac{q^{\eta+1}}{p^{\eta+1}}\right)a_2\right)}{pq^2(p-q)^2(a_2-x)^3} \\
&\qquad + \frac{q(p-q)(a_2-x)\sum_{\eta=0}^{\infty}\frac{q^{\eta}}{p^{\eta+1}}\varphi\left(p^2\frac{q^{\eta+2}}{p^{\eta+1}}x+\left(1-p^2\frac{q^{\eta+2}}{p^{\eta+1}}\right)a_2\right)}{pq(p-q)^2(a_2-x)^3} \\
&\quad = \frac{p^3(p-q)(a_2-x)\sum_{\eta=0}^{\infty}\frac{q^{\eta+2}}{p^{\eta+3}}\varphi\left(p^2\frac{q^{\eta+2}}{p^{\eta+3}}x+\left(1-p^2\frac{q^{\eta+2}}{p^{\eta+3}}\right)a_2\right)}{pq^3(p-q)^2(a_2-x)^3}
\end{aligned}
$$

$$-\frac{p[2]_{p,q}(p-q)(a_2-x)\sum\limits_{\eta=0}^{\infty}\frac{q^{\eta+1}}{p^{\eta+2}}\varphi\left(p^2\frac{q^{\eta+1}}{p^{\eta+1}}x+\left(1-p^2\frac{q^{\eta+1}}{p^{\eta+2}}\right)a_2\right)}{pq^2(p-q)^2(a_2-x)^3}$$

$$+\frac{q(p-q)(a_2-x)\sum\limits_{\eta=0}^{\infty}\frac{q^{\eta}}{p^{\eta+1}}\varphi\left(p^2\frac{q^{\eta+2}}{p^{\eta+1}}x+\left(1-p^2\frac{q^{\eta+2}}{p^{\eta+1}}\right)a_2\right)}{pq(p-q)^2(a_2-x)^3}$$

$$=\frac{[2]_{p,q}}{p^3q^3(a_2-x)^3}\int\limits_{p^2x+(1-p^2)a_2}^{a_2}\varphi(t)\ {}^{a_2}d_{p,q}t+\frac{(q^2+pq-p^2)\varphi(px+(1-p)a_2)}{pq^3(p-q)(a_2-x)^2}-\frac{\varphi(qx+(1-q)a_2)}{q^2(p-q)(a_2-x)^2}. \tag{12}$$

By multiplying (10) and (12) by $(x-a_1)^2(a_2-x)^2(a_1-x)$ and $(x-a_1)^2(a_2-x)^2(x-a_2)$, respectively, and adding the resultant inequalities, we obtain the required identity (8). Therefore, the proof is completed. □

Remark 1. If $p=1$ in (8), then we have the following identity:

$$(x-a_1)^2(a_2-x)^2\left[(a_1-x)\int\limits_0^1 t^2\,{}_{a_1}D_q^2\varphi(tx+(1-t)a_1)\,d_qt+(x-a_2)\int\limits_0^1 t^2\,{}^{a_2}D_q^2\varphi(tx+(1-t)a_2)\,d_qt\right]={}_{a_1}^{a_2}L_q(x),$$

where

$${}_{a_1}^{a_2}L_q(x)=\frac{(x-a_1)(a_2-x)}{q^3(p-q)}\left[(x-a_1)q\varphi(qx+(1-q)a_2)+(a_2-x)q\varphi(qx+(1-q)a_1)-(q^2+q-1)(a_2-a_1)\varphi(x)\right]$$
$$-\frac{[2]_q}{q^3}\left[(x-a_1)^2\int\limits_x^{a_2}\varphi(t)\,{}^{a_2}d_qt+(a_2-x)^2\int\limits_{a_1}^{x}\varphi(t)\,{}_{a_1}d_qt\right],$$

which appeared in[43].

Theorem 4. Let $\varphi:[a_1,a_2]\to\mathbb{R}$ be a twice (p,q)-differentiable function such that ${}^{a_2}D_{p,q}^2\varphi$ and ${}_{a_1}D_{p,q}^2\varphi$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^{a_2}D_{p,q}^2\varphi|$ and $|{}_{a_1}D_{p,q}^2\varphi|$ are convex functions, then

$$\left|{}_{a_1}^{a_2}L_{p,q}(x)\right|\le(x-a_1)^2(a_2-x)^2\left[(x-a_1)\left(\frac{1}{[4]_{p,q}}\left|{}_{a_1}D_{p,q}^2\varphi(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_{a_1}D_{p,q}^2\varphi(a_1)\right|\right)\right.$$
$$\left.+\,(a_2-x)\left(\frac{1}{[4]_{p,q}}\left|{}^{a_2}D_{p,q}^2\varphi(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^{a_2}D_{p,q}^2\varphi(a_2)\right|\right)\right]. \tag{13}$$

Proof. Taking modulus of 8, applying the convexity of $|{}^{a_2}D_{p,q}^2\varphi|$ and $|{}_{a_1}D_{p,q}^2\varphi|$, and by using Lemma 1, we obtain

$$\left|{}_{a_1}^{a_2}L_{p,q}(x)\right|\le(x-a_1)^2(a_2-x)^2\left[(x-a_1)\int\limits_0^1 t^2\left|{}_{a_1}D_{p,q}^2\varphi(tx+(1-t)a_1)\right|\,d_{p,q}t+(a_2-x)\int\limits_0^1 t^2\left|{}^{a_2}D_{p,q}^2\varphi(tx+(1-t)b)\right|\,d_{p,q}t\right]$$
$$\le(x-a_1)^2(a_2-x)^2\left[(x-a_1)\int\limits_0^1 t^2\left(t\left|{}_{a_1}D_{p,q}^2\varphi(x)\right|+(1-t)\left|{}_{a_1}D_{p,q}^2\varphi(a_1)\right|\right)\,d_{p,q}t\right.$$
$$\left.+(a_2-x)\int\limits_0^1 t^2\left(t\left|{}^{a_2}D_{p,q}^2\varphi(x)\right|+(1-t)\left|{}^{a_2}D_{p,q}^2\varphi(a_2)\right|\right)\,d_{p,q}t\right]$$
$$=(x-a_1)^2(a_2-x)^2\left[(x-a_1)\left(\frac{1}{[4]_{p,q}}\left|{}_{a_1}D_{p,q}^2\varphi(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_{a_1}D_{p,q}^2\varphi(a_1)\right|\right)\right.$$
$$\left.+\,(a_2-x)\left(\frac{1}{[4]_{p,q}}\left|{}^{a_2}D_{p,q}^2\varphi(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^{a_2}D_{p,q}^2\varphi(a_2)\right|\right)\right],$$

which completes the proof. □

Corollary 1. With the assumptions of Theorem 4, if $|{}^{a_2}D^2_{p,q}\varphi|$ and $|{}_{a_1}D^2_{p,q}\varphi| \leq M$, then the following inequality holds:

$$\left|{}^{a_2}_{a_1}L_{p,q}(x)\right| \leq \frac{M(x-a_1)^2(a_2-x)^2(a_2-a_1)}{[3]_{p,q}}, \tag{14}$$

where M is constant.

Remark 2. If $p=1$ in (13), then we have the following inequality:

$$\begin{aligned}\left|{}^{a_2}_{a_1}L_q(x)\right| \leq (x-a_1)^2(a_2-x)^2 &\left[(x-a_1)\left(\frac{1}{[4]_q}\left|{}_{a_1}D^2_q\varphi(x)\right| + \frac{q^3}{[3]_q[4]_q}\left|{}_{a_1}D^2_q\varphi(a_1)\right|\right)\right.\\ &\left.+\,(a_2-x)\left(\frac{1}{[4]_q}\left|{}^{a_2}D^2_q\varphi(x)\right| + \frac{q^3}{[3]_q[4]_q}\left|{}^{a_2}D^2_q\varphi(a_2)\right|\right)\right],\end{aligned}$$

which appeared in[43].

Remark 3. If $p=1$ in (14), then we have the following inequality:

$$\left|{}^{a_2}_{a_1}L_q(x)\right| \leq \frac{M(x-a_1)^2(a_2-x)^2(a_2-a_1)}{[3]_q}, \tag{15}$$

which appeared in[43]. Moreover, if $q \to 1$ and $x=(a_1+a_2)/2$ in (15), then we obtain the following inequality:

$$\left|\varphi\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\varphi(t)dt\right| \leq \frac{M(a_2-a_1)^2}{24},$$

which appeared in[60] and it can be found in[8].

Theorem 5. Let $\varphi : [a_1,a_2] \to \mathbb{R}$ be a twice (p,q)-differentiable function on (a_1,a_2) such that ${}^{a_2}D^2_{p,q}\varphi$ and ${}_{a_1}D^2_{p,q}\varphi$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^{a_2}D^2_{p,q}\varphi|^r$ and $|{}_{a_1}D^2_{p,q}\varphi|^r$ are convex functions for $r \geq 1$, then

$$\begin{aligned}\left|{}^{a_2}_{a_1}L_{p,q}(x)\right| \leq (x-a_1)^2(a_2-x)^2\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r} &\left[(x-a_1)\left(\frac{1}{[4]_{p,q}}\left|{}_{a_1}D^2_{p,q}\varphi(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_{a_1}D^2_{p,q}\varphi(a_1)\right|^r\right)^{1/r}\right.\\ &\left.+\,(a_2-x)\left(\frac{1}{[4]_{p,q}}\left|{}^{a_2}D^2_{p,q}\varphi(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^{a_2}D^2_{p,q}\varphi(a_2)\right|^r\right)^{1/r}\right].\end{aligned} \tag{16}$$

Proof. Applying Theorem 3 and the power mean inequality, we obtain

$$\begin{aligned}\left|{}^{a_2}_{a_1}L_{p,q}(x)\right| &\leq (x-a_1)^2(a_2-x)^2\left[(x-a_1)\int_0^1 t^2\left|{}_{a_1}D^2_{p,q}\varphi(tx+(1-t)a_1)\right|\,d_{p,q}t + (a_2-x)\int_0^1 t^2\left|{}^{a_2}D^2_{p,q}\varphi(tx+(1-t)a_2)\right|\,d_{p,q}t\right]\\ &\leq (x-a_1)^2(a_2-x)^2\left[(x-a_1)\left(\int_0^1 t^2\,d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left|{}_{a_1}D^2_{p,q}\varphi(tx+(1-t)a_1)\right|^r\,d_{p,q}t\right)^{1/r}\right.\\ &\quad\left.+(a_2-x)\left(\int_0^1 t^2\,d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left|{}^{a_2}D^2_{p,q}\varphi(tx+(1-t)a_2)\right|^r\,d_{p,q}t\right)^{1/r}\right].\end{aligned}$$

Using Lemma 1 and applying the convexity of $|{}^{a_2}D^2_{p,q}\varphi|^r$ and $|{}_{a_1}D^2_{p,q}\varphi|^r$, we have

$$\left|{}_{a_1}^{a_2}L_{p,q}(x)\right| \le (x-a_1)^2(a_2-x)^2\left[(x-a_1)\left(\int_0^1 t^2\, d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left(t\left|{}_{a_1}D^2_{p,q}\varphi(x)\right|^r+(1-t)\left|{}_{a_1}D^2_{p,q}\varphi(a_1)\right|^r\right)\, d_{p,q}t\right)^{1/r}\right.$$
$$\left.+(a_2-x)\left(\int_0^1 t^2\, d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left(t\left|{}^{a_2}D^2_{p,q}\varphi(x)\right|^r+(1-t)\left|{}^{a_2}D^2_{p,q}\varphi(a_2)\right|^r\right)\, d_{p,q}t\right)^{1/r}\right]$$
$$=(x-a_1)^2(a_2-x)^2\left[(x-a_1)\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\left(\frac{1}{[4]_{p,q}}\left|{}_{a_1}D^2_{p,q}\varphi(x)\right|^r+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_{a_1}D^2_{p,q}\varphi(a_1)\right|^r\right)^{1/r}\right.$$
$$\left.+(a_2-x)\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\left(\frac{1}{[4]_{p,q}}\left|{}^{a_2}D^2_{p,q}\varphi(x)\right|^r+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^{a_2}D^2_{p,q}\varphi(a_2)\right|^r\right)^{1/r}\right],$$

which completes the proof. □

Remark 4. If $p=1$ in (16), then we have the following inequality:

$$\left|{}_{a_1}^{a_2}L_q(x)\right| \le (x-a_1)^2(a_2-x)^2\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\left[(x-a_1)\left(\frac{1}{[4]_q}\left|{}_{a_1}D^2_q\varphi(x)\right|^r+\frac{q^3}{[3]_q[4]_q}\left|{}_{a_1}D^2_q\varphi(a_1)\right|^r\right)^{1/r}\right.$$
$$\left.+(a_2-x)\left(\frac{1}{[4]_q}\left|{}^{a_2}D^2_q\varphi(x)\right|^r+\frac{q^3}{[3]_q[4]_q}\left|{}^{a_2}D^2_q\varphi(a_2)\right|^r\right)^{1/r}\right],$$

which appeared in[43].

Theorem 6. Let $\varphi : [a_1,a_2]\to\mathbb{R}$ be a twice (p,q)-differentiable function such that ${}^{a_2}D^2_{p,q}\varphi$ and ${}_{a_1}D^2_{p,q}\varphi$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^{a_2}D^2_{p,q}\varphi|^r$ and $|{}_{a_1}D^2_{p,q}\varphi|^r$ are convex functions for $r>1$ and $1/s+1/r=1$, then

$$\left|{}_{a_1}^{a_2}L_{p,q}(x)\right| \le (x-a_1)^2(a_2-x)^2\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s}\left[(x-a_1)\left(\frac{\left|{}_{a_1}D^2_q\varphi(x)\right|^r+(p+q-1)\left|{}_{a_1}D^2_q\varphi(a_1)\right|^r}{[2]_{p,q}}\right)^{1/r}\right.$$
$$\left.+(a_2-x)\left(\frac{\left|{}^{a_2}D^2_q\varphi(x)\right|^r+(p+q-1)\left|{}^{a_2}D^2_q\varphi(a_2)\right|^r}{[2]_{p,q}}\right)^{1/r}\right]. \tag{17}$$

Proof. Applying Theorem 3 and the Hölders inequality, we have

$$\left|{}_{a_1}^{a_2}L_{p,q}(x)\right| \le (x-a_1)^2(a_2-x)^2\left[(x-a_1)\int_0^1 t^2\left|{}_{a_1}D^2_{p,q}\varphi(tx+(1-t)a_1)\right|\, d_{p,q}t+(a_2-x)\int_0^1 t^2\left|{}^{a_2}D^2_{p,q}\varphi(tx+(1-t)a_2)\right|\, d_{p,q}t\right]$$
$$\le (x-a_1)^2(a_2-x)^2\left[(x-a_1)\left(\int_0^1 t^{2s}\, d_{p,q}t\right)^{1/s}\left(\int_0^1\left|{}_{a_1}D^2_{p,q}\varphi(tx+(1-t)a_1)\right|^r\, d_{p,q}t\right)^{1/r}\right.$$
$$\left.+(a_2-x)\left(\int_0^1 t^{2s}\, d_{p,q}t\right)^{1/s}\left(\int_0^1\left|{}^{a_2}D^2_{p,q}\varphi(tx+(1-t)a_2)\right|^r\, d_{p,q}t\right)^{1/r}\right].$$

Using Lemma 1 and applying the convexity of $|{}^{a_2}D^2_{p,q}\varphi|^r$ and $|{}_{a_1}D^2_{p,q}\varphi|^r$, we obtain

$$\begin{aligned}\left|{}_{a_1}^{a_2}L_{p,q}(x)\right| \le (x-a_1)^2(a_2-x)^2 &\left[(x-a_1)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\left(\int_0^1 \left(t\left|{}_{a_1}D_{p,q}^2\varphi(x)\right|^r+(1-t)\left|{}_{a_1}D_{p,q}^2\varphi(a_1)\right|^r\right)\,d_{p,q}t\right)^{1/r}\right.\\
&\left.+(a_2-x)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\left(\int_0^1 \left(t\left|{}^{a_2}D_{p,q}^2\varphi(x)\right|^r+(1-t)\left|{}^{a_2}D_{p,q}^2\varphi(a_2)\right|^r\right)\,d_{p,q}t\right)^{1/r}\right]\\
=(x-a_1)^2(a_2-x)^2&\left[(x-a_1)\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s}\left(\frac{\left|{}_{a_1}D_q^2\varphi(x)\right|^r+(p+q-1)\left|{}_{a_1}D_q^2\varphi(a_1)\right|^r}{[2]_{p,q}}\right)^{1/r}\right.\\
&\left.+(a_2-x)\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s}\left(\frac{\left|{}^{a_2}D_q^2\varphi(x)\right|^r+(p+q-1)\left|{}^{a_2}D_q^2\varphi(a_2)\right|^r}{[2]_{p,q}}\right)^{1/r}\right],\end{aligned}$$

which completes the proof. □

Remark 5. If $p=1$ in (17), then we have the following inequality:

$$\begin{aligned}\left|{}_{a_1}^{a_2}L_q(x)\right| \le{}& (x-a_1)^2(a_2-x)^2\left(\frac{1}{[2s+1]_q}\right)^{1/s}\\
&\times\left[(x-a_1)\left(\frac{\left|{}_{a_1}D_q^2\varphi(x)\right|^r+q\left|{}_{a_1}D_q^2\varphi(a_1)\right|^r}{[2]_q}\right)^{1/r}+(a_2-x)\left(\frac{\left|{}^{a_2}D_q^2\varphi(x)\right|^r+q\left|{}^{a_2}D_q^2\varphi(a_2)\right|^r}{[2]_q}\right)^{1/r}\right],\end{aligned}$$

which appeared in [43].

4 | CONCLUSIONS

In this work, we established a new (p,q)-integral identity using the second $(p,q)_{a_1}$- and $(p,q)^{a_2}$-derivatives. Then, we used this result to derive some new post quantum Ostrowski-type integral inequalities for twice (p,q)-differentiable convex functions. The main results in this study were proven to be generalizations of some previously proved results of quantum Ostrowski-type integral inequalities for twice q-differentiable convex functions. Researchers can obtain similar inequalities in future works by using (p,q)-fractional calculus.

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Author contributions

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References

1. Ostrowski A.M. Über die absolutabweichung einer differentiebaren funktion vonihrem integralmitelwert. *Comment. Math. Helv.* 1938; 10, 226-227.

2. Dragomir S.S. The Ostrowskis integral inequality for Lipschitzian mappings and applications. *Comput. Math. Appl.* 1999); 38, 33-37.

3. Cerone P., Dragomir S.S., Roumeliotis J. An inequality of Ostrowski-type for mappings whose second derivatives are bounded and applications. *East Asian Math. J.* 1999; 15, 1-9.

4. Latif M.A., Hussain S. New inequalities of Ostrowski type for co-ordinated convex functions via fractional integrals. *J. Fractional Calc. Appl.* 2012; 2(9), 1-15.

5. Sarikaya M.Z., Budak H. Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* 2017; 145,1527-1538.

6. Set E. New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* 2012; 63(7), 1147-1154.

7. Alomari M.W. A generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation. *RGMIA Res. Rep. Collect.* 2011; 14(87), 11.

8. Liu Z. Some companions of an Ostrowski type inequality and applications. *J. Inequal. Pure Appl. Math.* 2009;10(2), 52.

9. Budak H. , Sarkaya M.Z. New weighted Ostrowski type inequalities for mappings with first derivatives of bounded variation. *Transylvanian J. Math. Mech.* 2016; 8(1), 21-27.

10. Budak H., Sarikaya M.Z. A new Ostrowski type inequality for functions whose first derivatives are of bounded variation. *Moroccan J. Pure Appl. Anal.* 2016; 2(1), 1-11.

11. Budak H., Sarikaya M.Z. A companion of Ostrowski type inequalities for mappings of bounded variation and some applications. *Trans. A. Razmadze Math. Inst.* 2017; 171(2), 136-143.

12. Dragomir S.S. A companion of Ostrowskis inequality for functions of bounded variation and applications. *Int. J. Nonlinear Anal. Appl.* 2014; 5(1), 89-97.

13. Ali, M.A.; Ntouyas, S.K.; Tariboon, J. Generalization of quantum Ostrowski-type integral inequalities. *Mathematics.* 2021; 9(10), 1155.

14. Noor M.A. , Awan M.U., Noor K.I. Quantum Ostrowski inequalities for q-differentiable convex functions, *J. Math. Inequal.* 2016; 10(4), 1013-1018.

15. Aljinovi A.A. , Kovaevi D., Puljiz M., Keko A.. On Ostrowski inequality for quantum calculus. *Appl. Math. Comput.* 2021; 410, 126454.

16. Budak H., Ali M.A., Alp N., Chu Y.M. , Quantum Ostrowski type integral inequalities. *J. Math.Inequal.*, 2021.

17. Budak H., Ali M.A., Tunç, T. Quantum Ostrowski-type integral inequalities for functions of two variables. *Math. Meth. Appl. Sci.* 2021; 44, 5857-5872.

18. Kunt M., Kashuri A., Du T., Baidar A.W. Quantum Montgomery identity and quantum estimates of Ostrowski type inequalities. *AIMS Math.* 2020; 5(6), 5439-5457.

19. Aljinovi A.A., Kovaevi D., Kunt M., Puljiz M. Correction: Quantum Montgomery identity and quantum estimates of Ostrowski type inequalities. *AIMS Math.*, 2021; 6(2), 1880-1888.

20. Ali M.A., Abbas M., Budak H., Agarwal P., Murtaza G., Chu Y.M. Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. *Adv. Differ. Equ.* 2021; 2021, 25.

21. Chu, H.H., Kalsoom, H., Rashid, S., Idrees, M., Safdar, F., Chu, Y.M., Baleanu, D. Quantum analogs of Ostrowski-type inequalities for rainas function correlated with coordinated generalized ϕ-convex functions. *Symmetry.* 2020; 12(2), 308.

22. Jackson F.H. On a q-definite integrals. *Quart. J. Pure Appl. Math.* 1910; 41, 193-203.

23. Al-Salam W. Some fractional q-integrals and q-derivatives. *Proc. Edinburgh Math. Soc.* 1966; 15(2), 135-140.

24. Raychev P.P., Roussev R.P., Smirnov, Y.F. The quantum algebra $SU_q(2)$ and rotational spectra of deformed nuclei, *J. Phys. G: Nucl. Part. Phys.* 1990; 16, 137-141.

25. Exton H. *q-Hypergeometric functions and applications*, New York: Halstead Press, 1983.

26. Page D.N. Information in black hole radiation. *Phys. Rev. Lett.* 1993; 71, 3743-3746.

27. Gavrilik A.M. q-serre relations in $U_q(u_n)$ and q-deformed meson mass sum rules. *J. Phys. A: Math. Gen.*, 1994; 27, L91-L94.

28. Gauchman H. Integral inequalities in q-calculus. *J. Comput. Appl. Math.*, 2002; 47, 281-300.

29. Ernst T. A method for q-calculus. *J. Nonlinear Math. Phys.* 2003; 10, 487-525.

30. Bangerezako G. Variational q-calculus. *J. Math. Anal. Appl.* 2004; 289, 650-665.

31. Aral A., Gupta V., Agarwal R.P. *Applications of q-calculus in operator theory*, Springer: Science+Business Media, 2013.

32. Dobrogowska A., Odzijewicz A. Second order q-difference equation solvable by factorization method. *J. Comput. Appl. Math.* 2006; 193, 319-346.

33. Ismail M.E.H., Simeonov P. q-difference operators for orthogonal polynomials. *J. Comput. Appl. Math.* 2009; 233, 749-761.

34. Kac V., Cheung P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.

35. Tariboon J., Ntouyas S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.*, 2013; 2013, 282.

36. Bermudo, S., Kórus, P., Nápoles Valdés J.E. On q-Hermite-Hadamard inequalities for general convex functions. *Acta Math. Hungar.* 2020; 162, 364-374.

37. Asawasamrit S., Sudprasert C., Ntouyas S. Tariboon, J. Some result on quantum Hanh integral inequalities. *J. Inequal. Appl.* 2019; 2019, 154.

38. Prabseang, J., Nonlaopon, K., Tariboon, J. Quantum Hermite-Hadamard inequalities for double integral and q-differentiable convex functions. *J. Math. Inequal.* 2019; 13, 675686.

39. Jhanthanam S., Tariboon J., Ntouyas, S.K., Nonlaopon, K. On q-Hermite-Hadamard inequalities for differentiable convex functions. *Mathematics.* 2019; 7, 632.

40. Budak, H., Ali, M.A., Tarhanaci, M. Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions. *J. Opt. Theory Appl.* 2020; 186, 899-910.

41. Budak H., Erden S., Ali, M.A. Simpson and Newton-type inequalities for convex functions via newly defined quantum integrals. *Math. Meth. Appl. Sci.* 2020; 44, 378-390.

42. Yang,W. Some new Fejér type inequalities via quantum calculus on finite intervals. *ScienceAsia.* 2017; 43, 123-134.

43. Ali, M.A., Budak, H., Akkurt, A., Chu Y.M. Quantum Ostrowski-type inequalities for twice quantum differentiable functions in quantum calculus. *Open Math.* 2021; 19, 440-449.

44. Chakrabarti R., Jagannathan R.A. (p, q)-oscillator realization of two-paramenter quantum algebras. *J. Phys. A Math. Gen.* 1991; 24, L711-L718.

45. Tunç M., Göv E. (p, q)-Integral inequalities. *RGMIA Res. Rep. Coll.*, 2016; 19(97), 1-13.

46. Tunç M., Göv E. Some integral inequalities via (p, q)-calculus on finite intervals. *RGMIA Res. Rep. Coll.* 2016; 19(95), 1-12.

47. Vivas-Cortez M., Ali M.A., Budak H., Kalsoom H., Agarwal P. Some new HermiteHadamard and related inequalities for convex functions via (p, q)-Integral. *Entropy* 2021; 23(7), 828.

48. Prabseang J., Nonlaopon K., Tariboon J. (p, q)-Hermite-Hadamard inequalities for double integral and (p, q)-differentiablen convex functions. *Axioms.* 2019; 8, 68.

49. Soontharanon J., Sitthiwirattham T. Fractional (p, q)-calculus. *Adv. Differ. Equ.* 2020; 2020, 35.

50. Luangboon W., Nonlaopon K., Tariboon J., Ntouyas S.K. Simpson- and Newton-type inequalities for convex functions via (p, q)-calculus. *Mathematics.* 2021; 9, 1338.

51. Duran U., Acikgoz M., Esi A., Araci S. A note on the (p, q)-Hermite polynomials. *Appl. Math. Inf. Sci.* 2018, 12, 227-231.

52. Kalsoom H., Amer M., Junjua M.-u.-D., Hussain S., Shahzadi G. Some (p, q)-estimates of Hermite-Hadamard-type inequalities for coordinated convex and quasi-convex functions, *Mathematics.* 2019; 7, 683.

53. Araci S., Duran U., Acikgoz M., Srivastava H.M. A certain (p, q)-derivative operator and associated divided differences. *J. Inequal. Appl.* 2016; 2016, 301.

54. Sadjang P.N. On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas. *Results Math.* 2018; 73, 1-21.

55. Nasiruzzaman M., Mukheimer A., Mursaleen M. Some opial-type integral inequalities via (p, q)-calculus. *J. Inequal. Appl.* 2019; 2019, 295.

56. Luangboon W.,Nonlaopon K., Tariboon J., Ntouyas S.K. On Simpson type inequalities for generalized strongly preinvex functions via (p, q)-calculus and applications. *AIMS Math.* 2021; 6(9), 9236-9261.

57. Sudsutad W., Ntouyas S.K., Tariboon J. Quantum integral inequalities for convex functions. *J. Math. Inequal.* 2015; 9, 781-793.

58. Tariboon J., Ntouyas S.K. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* 2014; 2014, 121.

59. Ali M.A., Budak H., Abbas M., Chu M.Y. Quantum Hermite-Hadamard-type inequalities for functions with convex absolute values of second q^b-derivatives. *Adv. Differ. Equ.* 2021; 2021, 7.

60. Meftah B., Boukerrioua K. Some new Ostrowski-type inequalities for functions whose second derivative are h-convex via Riemann-Liouville fractional. *Malaya J. Matematik* 2014; 2 (4), 445-459.

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