Inverse scattering transforms of the inhomogeneous fifth-order defocusing nonlinear Schrodinger equation with zero boundary conditions and nonzero boundary conditions: bound-state solitons and rogue waves

Jinjin Mao[1], Shoufu Tian[2], Tian Zhou Xu[1], and Lin-Fei Shi[1]

[1]Beijing Institute of Technology
[2]China University of Mining and Technology

April 16, 2024

Abstract

The present work studies the inverse scattering transformation (IST) of the inhomogeneous fifth-order defocusing nonlinear Schrodinger (ifoNLS) equation with zero boundary conditions (ZBCs) and non-zero boundary conditions (NZBCs). Firstly, the bound-state (BS) solitons of the ifoNLS equation with ZBCs are derived by generalization of the residue theorem and the Laurent's series for the first time. Then combining with the robust IST, the matrix Riemann-Hilbert (RH) problem of the ifoNLS equation with NZBCs are revealed. Based on the resulting RH problem, a new higher-order rogue wave (RW) solution of the ifoNLS equation are found by the modified Darboux transformation. Finally, some corresponding graphs are given by selecting appropriate parameters to further discuss the unreported dynamic behavior of the BS solitons and RW solutions, which have not been reported before.

Hosted file

`Our manscript.tex` available at https://authorea.com/users/737929/articles/712549-inverse-scattering-transforms-of-the-inhomogeneous-fifth-order-defocusing-nonlinear-schrodinger-equation-with-zero-boundary-conditions-and-nonzero-boundary-conditions-bound-state-solitons-and-rogue-waves

figures/2-1/2-1-eps-converted-to.pdf

figures/2-2/2-2-eps-converted-to.pdf

figures/2-3/2-3-eps-converted-to.pdf

figures/2-4/2-4-eps-converted-to.pdf

figures/5-1/5-1-eps-converted-to.pdf

figures/5b/5b-eps-converted-to.pdf

figures/5c/5c-eps-converted-to.pdf

figures/6-1/6-1-eps-converted-to.pdf

figures/6-2/6-2-eps-converted-to.pdf

figures/6-3/6-3-eps-converted-to.pdf

figures/7-1/7-1-eps-converted-to.pdf

figures/7-2/7-2-eps-converted-to.pdf

figures/7-3/7-3-eps-converted-to.pdf

figures/8-1/8-1-eps-converted-to.pdf

figures/8-2/8-2-eps-converted-to.pdf

figures/8-3/8-3-eps-converted-to.pdf

figures/9-1/9-1-eps-converted-to.pdf

figures/9-2/9-2-eps-converted-to.pdf

figures/9-3/9-3-eps-converted-to.pdf

figures/10-1/10-1-eps-converted-to.pdf

figures/10-2/10-2-eps-converted-to.pdf

figures/10-3/10-3-eps-converted-to.pdf

Noname manuscript No.
(will be inserted by the editor)

Inverse scattering transforms of the inhomogeneous fifth-order defocusing nonlinear Schrödinger equation with zero boundary conditions and nonzero boundary conditions: bound-state solitons and rogue waves

Jin-Jin Mao · Shou-Fu Tian · Tian-Zhou Xu · Lin-Fei Shi

Received: date / Accepted: date

Abstract The present work studies the inverse scattering transformation (IST) of the inhomogeneous fifth-order defocusing nonlinear Schrödinger (ifoNLS) equation with zero boundary conditions (ZBCs) and non-zero boundary conditions (NZBCs). Firstly, the bound-state (BS) solitons of the ifoNLS equation with ZBCs are derived by generalization of the residue theorem and the Laurent's series for the first time. Then combining with the robust IST, the matrix Riemann-Hilbert (RH) problem of the ifoNLS equation with NZBCs are revealed. Based on the resulting RH problem, a new higher-order rogue wave (RW) solution of the ifoNLS equation are found by the modified Darboux transformation. Finally, some corresponding graphs are given by selecting appropriate parameters to further discuss the unreported dynamic behavior of the BS solitons and RW solutions, which have not been reported before.

This work was supported by the National Natural Science Foundation of China under Grant No. 11975306, the Natural Science Foundation of Jiangsu Province under Grant No. BK20181351, the Six Talent Peaks Project in Jiangsu Province under Grant No. JY-059, and the Fundamental Research Fund for the Central Universities under the Grant Nos. 2019ZDPY07 and 2019QNA35.

J. J. Mao (✉)
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, People's Republic of China
E-mail: mao-jinjin0806@163.com, 3120205703@bit.edu.cn

S.F. Tian
School of Mathematics and Institute of Mathematical Physics, China University of Mining and Technology, Xuzhou 221116, People's Republic of China
E-mail: sftian@cumt.edu.cn and shoufu2006@126.com

T.Z. Xu
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, People's Republic of China
E-mail: xutianzhou@bit.edu.cn

L.F. Shi
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, People's Republic of China
E-mail: 3120205702@bit.edu.cn

(Some figures in this article are in colour only in the electronic version)

1 Introduction

As one of the current research focuses, the nonlinear evolution (NLE) equations with zero boundary conditions (ZBCs) and nonzero boundary conditions (NZBCs) has brought to the forefront of nonlinear systems in the past decades [1]−[4]. Many methods have been provided to find the solutions of NLE equations. In 1967, Gardner et al. first proposed the inverse scattering transformation (IST) and applied it to the KdV equation[5], which was the most effective method to solve the initial value problem of the integrable system in soliton theory. The classical IST methods are generally studied on the basis of Gel'fand-Levitan-Marchenkoo integral equation. Subsequently, Zakharov et al. appropriately simplified the IST method by issuing Riemann-Hilbert (RH) formula [6], and used this method to obtain the soliton solutions of many NLE equations [7]−[11]. Thereby, the research on the RH formula made important progress in the field of integrable systems, and it is still a hot topic today[13]−[22].

In recent years, the study of bound-state (BS) solitons and rogue wave (RW) solutions based on the RH method has drawn much attention, thereby more and more BS solitons and RW solutions of the NLE have been found. In 1972, Zakharov and Shabat derived the multiple poles soliton solutions of the focused nonlinear Schrödinger (NLS) equation [23]. Thereafter the increasing multiple-poles solitons of various nonlinear integrable equations have been solved, for example, the modified KdV equation [24], the sine-Gordon (sG) equation [25], Sasa-Satsuma (SS) equation [26], Wadati-Konno-Ichikawa (WKI) equation [27], the complex modified KdV equation [28]. In addition, they discussed the asymptotic for multiple pole solitons [29,30]. Most importantly, using the original IST method to solve the BS solitons requires a large amount of calculation [24,25], and some relatively complex constraints need to be solved. However, the RH problem with multipl poles can be directly expressed by employing the residue theorem and Laurent's series [27,28], which not only simplifies the calculation, but also obtains the BS solitons. Subsequently, Bilman and Miller found that the robust IST can be applied to solve the higher-order RW solutions of the focusing NLS equation [31]. Simultaneously, this method is used to solve respiratory wave solutions, rational W-type soliton solutions and so on. Afterwards, the robust IST is used to solve more nonlinear integrable models of RWs, such as the fifth-order NLS equation [32], the sixth-order NLS equation [33], the Hirota equation [34], the quartic NLS equation [35], and the generalized NLS equation [36].

In this paper, we mainly study the inhomogeneous fifth-order NLS (ifoNLS) equation [37]

$$\begin{aligned} &iq_t - i\epsilon q_{xxxxx} - 10i\epsilon|q|^2 q_{xxx} - 20i\epsilon q_x q^* q_{xx} - 30i\epsilon|q|^4 q_x \\ &- 10i\epsilon(|q_x|^2 q)_x + q_{xx} + 2q|q|^2 - iq_x = 0, \end{aligned} \tag{1}$$

where $q = q(x,t)$ represents the complex functions of x and t, ϵ is the perturbation parameter, and superscript $*$ is the complex conjugate. In 2015, Chen first constructed the generalized Darboux transformation (DT) of the ifoNLS equation (1), and then obtained the RW solutions based on the generalized DT [37]. In 2019, Feng et al. studied the determinant representation of the N-fold DT based on Lax pair. Moreover, the higher-order solitary wave, breather wave (BW) and RW solutions the ifoNLS equation (1) are obtained by using the N-fold DT [38]. In 2020, Yang et al. discussed the ifoNLS equation (1) with NZBCs in detail. For the inverse scattering problem, For the inverse scattering problem, they discussed simple zeros and double zeros cases of scattering coefficients respectively, and further obtained their exact solutions [39]. However, the BS solitons of the equation (1) with zero boundary and the RW of the equation (1) with non-zero boundary have not been analyzed. Therefore, we will use the RH problem to obtain the BS solitons of the ifoNLS equation (1) with ZBCs. Then the RH problem of the equation (1) with NZBCs is discussed. Finally, based on the obtained RH problem and the DT method, the RW solution of the equation (1) with NZBCs is obtained.

The ifoNLS equation (1) satisfies the following Lax pairs

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi, \tag{2}$$

where

$$\begin{aligned} U &= -i\lambda\sigma_3 + Q,\\ V &= -16i\lambda^5\epsilon\sigma_3 + 16\lambda^4\epsilon Q - 8i\lambda^3\epsilon(Q^2 + Q_x)\sigma_3 + 4\lambda^2\epsilon(2Q^3 + Q_xQ - QQ_x - Q_{xx})\\ &\quad - 2i\lambda^2\sigma_3 - i\lambda\sigma_3 + 2\lambda Q - 2i\lambda\epsilon(3Q^4 + 6Q^2Q_x + Q_x^2 - QQ_{xx} - Q_{xx}Q - Q_{xxx})\sigma_3\\ &\quad - i(Q^2 + Q_x)\sigma_3 + Q + \epsilon(6Q^5 - 6Q^3Q_x + 6Q^2Q_xQ - 6Q_xQQ_x - 4Q_x^2Q\\ &\quad - 2QQ_{xx}Q - 8Q^2Q_{xx} - Q_xQ_{xx} + Q_{xx}Q_x - Q_{xxx}Q + QQ_{xxx} + Q_{xxxx}), \end{aligned} \tag{3}$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{4}$$

The structure of this paper is as follows: In Sec. 2, we construct the RH problem of the ifoNLS equation (1) with ZBCs, and then derive the BS soliton with one higher-order pole. In Sec. 3, we construct the RH problem of the ifoNLS equation (1) with NZBCs by means of robust IST. Then combined with the modified DT method to solve RH problem, the exact BW and RW solutions of the ifoNLS equation (1) with NZBCs are obtained. In the last section, we give some conclusions.

2 The IST with ZBCs and BS solution

In this section, we will study the BS soliton $q(x,t)$ of the ifoNLS equation (1) with ZBCs through infinity under the following conditions

$$\lim_{x\to\pm\infty} q(x,t) = 0. \tag{5}$$

The next moment we will express the IST and BS soliton of equation (1) with ZBCs through the research of RH problem.

2.1 The structure of the RH problem with ZBCs

Let $x \to \pm\infty$, we can rewrite the Lax pair (2) into the following form

$$\Psi_x = U_0\Psi = -i\lambda\sigma_3\Psi, \qquad \Psi_t = V_0\Psi = (16\lambda^4\epsilon + 2\lambda + 1)U_0\Psi, \tag{6}$$

which satisfies the fundamental matrix solutions $\Psi_\pm^{fd}(x,t,\lambda)$, given by

$$\begin{aligned}&\Psi_\pm^{fd}(x,t,\lambda) = e^{-i\Xi(x,t,\lambda)\sigma_3},\\ &\Xi(x,t,\lambda) = \lambda[x + (16\lambda^4\epsilon + 2\lambda + 1)t].\end{aligned} \tag{7}$$

We get the following Jost solutions $\Psi_\pm(x,t,\lambda)$

$$\Psi_\pm(x,t,\lambda) \to e^{-i\Xi(x,t,\lambda)\sigma_3}, \qquad as \qquad x \to \pm\infty. \tag{8}$$

Then, the modified Jost solutions $\mu_\pm(x,t,\lambda)$ are expressed as

$$\mu_\pm(x,t,\lambda) = \Psi_\pm(x,t,\lambda)e^{i\Xi(x,t,\lambda)\sigma_3}, \tag{9}$$

which results in $\mu_\pm(x,t,\lambda) \to \mathbb{I}$ as $x \to \pm\infty$, and satisfy the following Volterra integral equations

$$\begin{aligned}\mu_-(x,t,\lambda) &= \mathbb{I} + \int_{-\infty}^{x} e^{i\lambda\sigma_3(\xi-x)}Q(y,t)\mu_-(y,t,\lambda)e^{-i\lambda\sigma_3(\xi-x)}\mathrm{d}\xi,\\ \mu_+(x,t,\lambda) &= \mathbb{I} - \int_{x}^{+\infty} e^{i\lambda\sigma_3(\xi-x)}Q(y,t)\mu_+(y,t,\lambda)e^{-i\lambda\sigma_3(\xi-x)}\mathrm{d}\xi.\end{aligned} \tag{10}$$

Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C}|\mathrm{Im}\lambda > 0\}$, $\mathbb{C}_- = \{\lambda \in \mathbb{C}|\mathrm{Im}\lambda < 0\}$. Obviously the columns $\mu_{-,1}$ and $\mu_{+,2}$ are analytic in $\mathbb{C}_+$, and continuously extends to $\mathbb{C}_+ \bigcup \mathbb{R}$. The columns $\mu_{+,1}$ and $\mu_{-,2}$ are analyzed in $\mathbb{C}_-$, and continuously extended to $\mathbb{C}_- \bigcup \mathbb{R}$.

These Jost solutions $\Psi_+(x,t,\lambda)$ and $\Psi_-(x,t,\lambda)$ are both solutions representing Lax pair (2). Therefore, $\Psi_+(x,t,\lambda)$ and $\Psi_-(x,t,\lambda)$ can be connected by the constant scattering matrix $S(\lambda) = (s_{ij}(\lambda))_{2\times2}$ in the following form

$$\Psi_+(x,t,\lambda) = \Psi_-(x,t,\lambda)S(\lambda), \qquad \lambda \in \mathbb{R}, \tag{11}$$

or

$$\mu_+(x,t,\lambda) = \mu_-(x,t,\lambda)e^{-i\Xi(x,t,\lambda)\sigma_3}S(\lambda)e^{i\Xi(x,t,\lambda)\sigma_3}, \qquad \lambda \in \mathbb{R}, \tag{12}$$

where $\mu_\pm(x,t,\lambda)$ and $S(\lambda)$ have the following symmetries

$$\mu_\pm(x,t,\lambda) = \sigma_2\mu_\pm^*(x,t,\lambda^*)\sigma_2, \qquad S(\lambda) = \sigma_2S^*(\lambda^*)\sigma_2, \tag{13}$$

with $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Furthermore, we can obtain

$$s_{11}(\lambda) = s_{22}^*(\lambda^*), \qquad s_{12}(\lambda) = -s_{21}^*(\lambda^*). \tag{14}$$

The scattering coefficient can be showed as the Wronskians determinant form

$$\begin{aligned} s_{11}(\lambda) &= W(\Psi_{+,1}, \Psi_{-,2}), \qquad s_{12}(\lambda) = W(\Psi_{+,2}, \Psi_{-,2}), \\ s_{21}(\lambda) &= W(\Psi_{-,1}, \Psi_{+,1}), \qquad s_{22}(\lambda) = W(\Psi_{-,1}, \Psi_{+,2}), \end{aligned} \tag{15}$$

we can get that s_{11} can be analytic continuation to $\mathbb{C}_-$, and similarly s_{22} can also be analyzed in $\mathbb{C}_+$. In addition, $s_{11}, s_{22} \to 1$ as $\lambda \to \infty$ in $\mathbb{C}_-, \mathbb{C}_+$, respectively.

Next, we will construct the RH problem of the inverse spectral problem. First, we need to consider the sectionally meromorphic matrices

$$M(x,t,\lambda) = \begin{cases} \left[\mu_{-,1}, \dfrac{\mu_{+,2}}{s_{22}}\right], & \lambda \in \mathbb{C}_+, \\ \left[\dfrac{\mu_{+,1}}{s_{11}}, \mu_{-,2}\right], & \lambda \in \mathbb{C}_-. \end{cases} \tag{16}$$

Then, we obtain the following RH problem

Theorem 1. $M(x,t,\lambda)$ solve the following RH problem

$$\begin{cases} M(x,t,\lambda) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\ M_+(x,t,\lambda) = M_-(x,t,\lambda) J(x,t,\lambda), \qquad \lambda \in \mathbb{R}, \\ M(x,t,\lambda) \to \mathbb{I}, \qquad \lambda \to \infty, \end{cases} \tag{17}$$

where the jump matrix $J(x,t,\lambda)$ is

$$J(x,t,\lambda) = \begin{pmatrix} 1 & r(\lambda) e^{-2i\Xi(x,t,\lambda)} \\ r^*(\lambda^*) e^{2i\Xi(x,t,\lambda)} & 1 + |r(\lambda)|^2 \end{pmatrix}, \tag{18}$$

with $r(\lambda) = \frac{s_{12}}{s_{22}}$.

From equations (13) and (14), we get $M_+(\lambda) = \sigma_2 M_-^*(\lambda^*)\sigma_2$. Taking

$$M(x,t,\lambda) = \mathbb{I} + \frac{1}{\lambda} M^{(1)}(x,t,\lambda) + O(\frac{1}{\lambda^2}), \qquad \lambda \to \infty, \tag{19}$$

then the potential $q(x,t)$ of the ifoNLS equation (1) with ZBCs is given by is given by the following formula

$$q(x,t) = 2iM_{12}^{(1)}(x,t,\lambda) = \lim_{\lambda\to\infty} 2i\lambda M_{12}(x,t,\lambda). \tag{20}$$

2.2 BS soliton with one higher-order pole

Generally, we assume that there are exactly discrete spectral points λ satisfying $s_{22}(\lambda) = 0$ in $\mathbb{C}_+$ and those discrete spectral point λ^* satisfying $s_{11}(\lambda^*) = 0$ in $\mathbb{C}_-$. Without thinking simple poles, we assume that $s_{22}(\lambda)$ has N higher-order poles $\lambda_n, n = 1,2,3,\cdots,N$ in $\mathbb{C}_+$, which means

$$\begin{aligned} s_{22}(\lambda) &= (\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}(\lambda-\lambda_3)^{n_3} \times \cdots \times (\lambda-\lambda_N)^{n_N} s_{22}^{(0)}(\lambda), \\ s_{11}(\lambda^*) &= (\lambda-\lambda_1^*)^{n_1}(\lambda-\lambda_2^*)^{n_2}(\lambda-\lambda_3^*)^{n_3} \times \cdots \times (\lambda-\lambda_N^*)^{n_N} s_{11}^{(0)}(\lambda^*), \end{aligned} \tag{21}$$

where $s_{22}^{(0)}(\lambda) = s_{11}^{(0)}(\lambda^*) \neq 0$ for all $\lambda \in \mathbb{C}_+$. According to the symmetric relation (14) of the scattering matrix, we get the following results

$$s_{22}(\lambda_n) = s_{11}(\lambda_n^*) = 0. \tag{22}$$

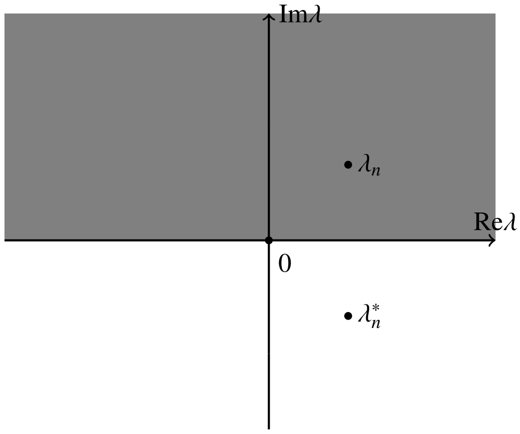


Figure 1. Depicts the discrete spectrum and the contours of the RH problem on complex λ-plane, $\mathbb{C}_+$ (gray) and region $\mathbb{C}_-$ (white).

Therefore, the relevant discrete spectrum point set is represented as

$$\Upsilon = \{\lambda_n, \lambda_n^*\}_{n=1}^{N}, \tag{23}$$

and its distributions are shown in Figure 1.

We obtain the explicit soliton solutions of the ifoNLS equation (1) with ZBCs by considering the reflectionless potential, i.e. $r(\lambda) = 0$. In this section, we will discuss the case of one higher-order pole. This means that $s_{22}(\lambda)$ has one Nth order zero point on the upper half plane, i.e. $s_{22}(\lambda) = (\lambda - \lambda_0)^N s_{22}^{(0)}(\lambda)$ (Im$\lambda > 0, N > 1, s_{22}^{(0)}(\lambda_0) \neq 0$). Similarly, we obtain that $M_{11}(x,t,\lambda)$ has an Nth order pole at $\lambda = \lambda_0^*$, and $M_{12}(x,t,\lambda)$ has one Nth order pole at $\lambda = \lambda_0$. Based on the normalization condition for matrix $M(x,t,\lambda)$, we write RH problem as follows

$$M_{11}(x,t,\lambda) = 1 + \sum_{n=1}^{N} \frac{F_n(x,t)}{(\lambda - \lambda_0^*)^n}, \qquad M_{12}(x,t,\lambda) = \sum_{n=1}^{N} \frac{G_n(x,t)}{(\lambda - \lambda_0)^n}. \tag{24}$$

Simultaneously, defining

$$\begin{aligned}
&e^{-2i\Xi(x,t,\lambda)} = \sum_{s=0}^{+\infty} f_s(x,t)(\lambda - \lambda_0)^s, && e^{2i\Xi(x,t,\lambda)} = \sum_{s=0}^{+\infty} f_s^*(x,t)(\lambda - \lambda_0^*)^s,\\
&M_{11}(x,t,\lambda) = \sum_{s=0}^{+\infty} \zeta_s(x,t)(\lambda - \lambda_0)^s, && M_{12}(x,t,\lambda) = \sum_{s=0}^{+\infty} \xi_s(x,t)(\lambda - \lambda_0^*)^s,\\
&r(\lambda) = r_0(\lambda) + \sum_{m=1}^{N} \frac{r_m}{(\lambda - \lambda_0)^m}, && r^*(\lambda^*) = r_0^*(\lambda^*) + \sum_{m=1}^{N} \frac{r_m^*}{(\lambda - \lambda_0^*)^m},
\end{aligned} \tag{25}$$

where

$$
\begin{aligned}
&f_s(x,t) = \lim_{\lambda\to\lambda_0} \frac{1}{s!}\frac{\partial^s}{\partial\lambda^s} e^{-2i\Xi(x,t,\lambda)}, \qquad \zeta_s(x,t) = \lim_{\lambda\to\lambda_0} \frac{1}{s!}\frac{\partial^s}{\partial\lambda^s} M_{11}(x,t,\lambda),\\
&\xi_s(x,t) = \lim_{\lambda\to\lambda_0^*} \frac{1}{s!}\frac{\partial^s}{\partial\lambda^s} M_{12}(x,t,\lambda), \qquad r_m = \lim_{\lambda\to\lambda_0} \frac{1}{(N-m)!}\frac{\partial^{N-m}}{\partial\lambda^{N-m}}[(\lambda-\lambda_0)^N r(\lambda)],\\
&s = 0,1,2,3,\cdots, \qquad m = 1,2,3,4,\cdots,N,
\end{aligned} \tag{26}
$$

and $r_0(\lambda)$ means analytic on the upper half plane.

On the basis of the Theorem 1, (24) and (25), then we collect the correlation coefficients of $(\lambda-\lambda_0)^{-n}$ and $(\lambda-\lambda_0^*)^{-n}$ to get

$$
\begin{aligned}
F_n(x,t) &= -\sum_{m=n}^{N}\sum_{s=0}^{m-n} r_m^* f_{m-n-s}^*(x,t)\xi_s(x,t),\\
G_n(x,t) &= \sum_{m=n}^{N}\sum_{s=0}^{m-n} r_m f_{m-n-s}(x,t)\zeta_s(x,t),
\end{aligned} \tag{27}
$$

with $n = 1,2,3,\cdots,N$.

Similarly, putting (24) into (26), we get following results

$$
\begin{aligned}
\xi_s(x,t) &= \sum_{l=1}^{N}\binom{l+s-1}{s}\frac{(-1)^s G_l(x,t)}{(\lambda_0^*-\lambda_0)^{l+s}}, \qquad s = 0,1,2,\cdots,\\
\zeta_s(x,t) &= \begin{cases} 1 + \displaystyle\sum_{l=1}^{N}\frac{F_l(x,t)}{(\lambda_0-\lambda_0^*)^l}, & s = 0,\\ \displaystyle\sum_{l=1}^{N}\binom{l+s-1}{s}\frac{(-1)^s F_l(x,t)}{(\lambda_0-\lambda_0^*)^{l+s}}, & s = 1,2,3,\cdots. \end{cases}
\end{aligned} \tag{28}
$$

Putting equations (28) into equations (27), we can obtain

$$
\begin{aligned}
F_n(x,t) &= -\sum_{m=n}^{N}\sum_{s=0}^{m-n}\sum_{l=1}^{N}\binom{l+s-1}{s}\frac{(-1)^s r_m^* f_{m-n-s}^*(x,t) G_l}{(\lambda_0^*-\lambda_0)^{l+s}},\\
G_n(x,t) &= \sum_{m=n}^{N} r_m f_{m-n}(x,t)\\
&\quad + \sum_{m=n}^{N}\sum_{s=0}^{m-n}\sum_{l=1}^{N}\binom{l+s-1}{s}\frac{(-1)^s r_m f_{m-n-s}(x,t) F_l}{(\lambda_0-\lambda_0^*)^{l+s}},\\
n &= 1,2,3,\cdots,N.
\end{aligned} \tag{29}
$$

Subsequently, we can conclude the following theorem

Theorem 2. Based on the ZBCs at infinity given in (5), the Nth order BS soliton of the ifoNLS equation (1) is

$$
q(x,t) = 2i\left(\frac{\det(\mathbb{I}+\Omega^*\Omega+|\eta\rangle\langle\Upsilon_0|)}{\det(\mathbb{I}+\mathit{\Omega}^*\mathit{\Omega})} - 1\right), \tag{30}
$$

where $\langle Y_0| = [1,0,0,\cdots,0]_{1\times N}$ and

$$\begin{aligned}
&\Omega = [\Omega_{nl}]_{N\times N} = \left[-\sum_{m=n}^{N}\sum_{s=0}^{m-n}\begin{pmatrix} l+s-1 \\ s \end{pmatrix}\frac{(-1)^s r_m^* f_{m-n-s}^*(x,t)}{(\lambda_0^*-\lambda_0)^{l+s}}\right]_{N\times N},\\
&|\eta\rangle = [\eta_1,\eta_2,\eta_3,\cdots,\eta_N]^\top, \qquad \eta_n = \sum_{m=n}^{N} r_m f_{m-n}(x,t),\\
&n,l = 1,2,3,\cdots,N.
\end{aligned} \tag{31}$$

Proof First, we introduce

$$|F\rangle = [F_1,F_2,F_3,\cdots,F_N]^\top, \qquad |G\rangle = [G_1,G_2,G_3,\cdots,G_N]^\top, \tag{32}$$

where the superscript "⊤" means transposition. Subsequently, we can transform equations (29) into the following form

$$|F\rangle = \Omega|G\rangle, \qquad |G\rangle = |\eta\rangle - \Omega^*|F\rangle. \tag{33}$$

Then, we can obtain

$$|F\rangle = \Omega(\mathbb{I}+\Omega^*\Omega)^{-1}|\eta\rangle, \qquad |G\rangle = (\mathbb{I}+\Omega^*\Omega)^{-1}|\eta\rangle. \tag{34}$$

Putting equations (34) into equations (24), we have

$$\begin{aligned}
M_{11}(x,t,\lambda) &= 1+\sum_{n=1}^{N}\frac{F_n(x,t)}{(\lambda-\lambda_0^*)^n} = \frac{\det(\mathbb{I}+\Omega^*\Omega+|\eta\rangle\langle \mathrm{Y}(\lambda)|\Omega)}{\det(\mathbb{I}+\Omega^*\Omega)},\\
M_{12}(x,t,\lambda) &= \sum_{n=1}^{N}\frac{G_n(x,t)}{(\lambda-\lambda_0)^n} = \frac{\det(\mathbb{I}+\Omega^*\Omega+|\eta\rangle\langle \mathrm{Y}^*(\lambda^*)|)}{\det(\mathbb{I}+\Omega^*\Omega)} - 1,
\end{aligned} \tag{35}$$

where $\langle Y(\lambda)| = \left[\frac{1}{(\lambda-\lambda_0^*)},\frac{1}{(\lambda-\lambda_0^*)^2},\cdots,\frac{1}{(\lambda-\lambda_0^*)^N}\right]$. According to the expression (20), the Theorem 2.2 is finally proved. □

When $N=2$, $\lambda=\lambda_0$ is the second-order zero point of s_{22}, then $r(\lambda)$ rewrite it as follows

$$r(\lambda) = r_0(\lambda) + \frac{r_1}{\lambda-\lambda_0} + \frac{r_2}{(\lambda-\lambda_0)^2}, \tag{36}$$

and Ω_{11}, Ω_{12}, Ω_{21} and Ω_{22} are expressed as

$$\begin{aligned}
\Omega_{11} &= -\frac{r_1^* f_0^*}{\lambda_0^*-\lambda_0} - \frac{r_2^* f_1^*}{\lambda_0^*-\lambda_0} + \frac{r_2^* f_0^*}{(\lambda_0^*-\lambda_0)^2},\\
\Omega_{12} &= -\frac{r_1^* f_0^*}{(\lambda_0^*-\lambda_0)^2} - \frac{r_2^* f_1^*}{(\lambda_0^*-\lambda_0)^2} + \frac{2r_2^* f_0^*}{(\lambda_0^*-\lambda_0)^3},\\
\Omega_{21} &= -\frac{r_2^* f_0^*}{\lambda_0^*-\lambda_0}, \qquad \Omega_{22} = -\frac{r_2^* f_0^*}{(\lambda_0^*-\lambda_0)^2},
\end{aligned} \tag{37}$$

$|\eta\rangle$ is a column vector in the following form

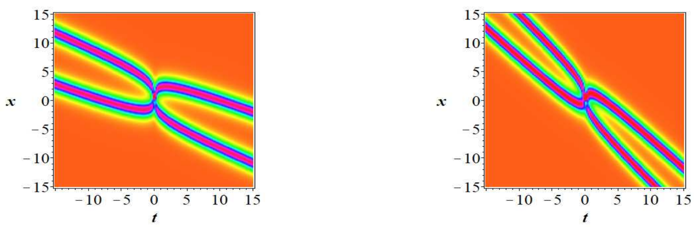
$$\eta_1 = r_1 f_0 + r_2 f_1, \qquad\qquad \eta_2 = r_2 f_0, \tag{38}$$

and $\langle Y_0| = [1, 0]$. Furthermore, on the basis of Theorem 2, let $r_1 = r_2 = 1$, $\lambda_0 = a+bi$, we obtain that the second-order BS soliton solution of the ifoNLS equation (1) is

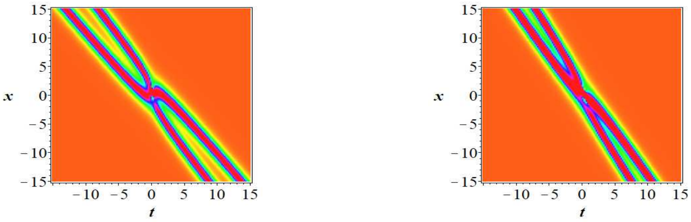
$$q(x,t) = -\frac{32ib^3\left(\varpi_1 e^{-2i(a+bi)\varpi_2} + \varpi_3 e^{\varpi_4}\right)}{\left(\varpi_5 + e^{4b\varpi_6}\right)e^{4b\varpi_6} + 258b^8}, \tag{39}$$

where

$$\begin{aligned}
\varpi_1 = & \Big(2560\, ia^4b^5\epsilon - 15360\, ia^2b^7\epsilon + 2560\, ib^9\epsilon - 10240\, a^3b^6\epsilon + 10240\, ab^8\epsilon + 128\, iab^5 \\
& +32\, ib^5 - 128\, b^6\Big)\, t + 32\, ib^5 x - 16\, b^5, \\
\varpi_2 = & \; 16\, b^4 t\epsilon - 64\, iab^3 t\epsilon - 96\, a^2b^2 t\epsilon + 64\, ia^3 bt\epsilon + 16\, a^4 t\epsilon + 2\, ibt + 2\, at + t + x, \\
\varpi_3 = & \Big(-160\, ia^4 b\epsilon + 960\, ia^2b^3\epsilon - 160\, ib^5\epsilon - 640\, a^3b^2\epsilon + 640\, ab^4\epsilon - 8\, iab - 2\, ib - 8\, b^2\Big)\, t \\
& - 2\, ibx + 2\, i - b, \\
\varpi_4 = & \Big(-32\, ia^5\epsilon + 320\, ia^3b^2\epsilon - 160\, iab^4\epsilon + 480\, a^4 b\epsilon - 960\, a^2b^3\epsilon + 96\, b^5\epsilon - 4\, ia^2 + 4\, ib^2 \\
& -2\, ia + 24\, ab + 6\, b)\, t - 2\, iax + 6\, bx, \\
\varpi_5 = & \Big(1638400\, a^8b^6\epsilon^2 + 6553600\, a^6b^8\epsilon^2 + 9830400\, a^4b^{10}\epsilon^2 + 6553600\, a^2b^{12}\epsilon^2 \\
& +1638400\, b^{14}\epsilon^2 + 163840\, a^5b^6\epsilon - 327680\, a^3b^8\epsilon - 491520\, ab^{10}\epsilon + 40960\, a^4b^6\epsilon \\
& -245760\, a^2b^8\epsilon + 40960\, b^{10}\epsilon + 4096\, a^2b^6 + 4096\, b^8 + 2048\, ab^6 + 256\, b^6\Big)\, t^2 \\
& + \Big(40960\, a^4b^6\epsilon - 245760\, a^2b^8\epsilon + 40960\, b^{10}\epsilon + 2048\, ab^6 + 512\, b^6\Big)\, xt \\
& +(81920\, a^3b^7\epsilon - 81920\, ab^9\epsilon - 20480\, a^4b^5\epsilon + 122880\, a^2b^7\epsilon - 20480\, b^9\epsilon + 1024\, b^7 \\
& -1024\, ab^5 - 256\, b^5\Big)\, t + 256\, b^6x^2 + 64\, b^6 - 256\, b^5x + 96\, b^4, \\
\varpi_6 = & \Big(80\, a^4\epsilon - 160\, a^2b^2\epsilon + 16\, b^4\epsilon + 4\, a + 1\Big)\, t + x.
\end{aligned} \tag{40}$$



(a) **(b)**



(**c**) (**d**)

Figure 2. The density plots of the BS soliton solutions (39) for the ifoNLS equation (1) with the parameters $a = \frac{1}{3}$, $b = \frac{2}{3}$ and (a) $\epsilon = \frac{1}{2}$; (b) $\epsilon = \frac{1}{3}$; (c) $\epsilon = \frac{1}{4}$; (d) $\epsilon = \frac{1}{6}$.

The BS soliton solution (39) means the interaction between two soliton solutions, in which the high peak is caused by the interaction of two solitons with related eigenvalues. The relevant evolution process for the solutions (39) at different coefficient ϵ are counseled in Figs. 2. We can find that the change of parameter ϵ affects the phase for the two solitons in Figs. 2.

3 The IST with NZBCs and RW

In this section, we will study the RW solution $q(x,t)$ of the ifoNLS equation (1) with NZBCs through infinity under the following conditions

$$\lim_{x\to\pm\infty} q(x,t) = Be^{i(\omega x+\nu t)}, \tag{41}$$

where ω and $B > 0$ expressions all real constants, $\nu = (30B^4\omega - 20B^2\omega^3 + \omega^5)\epsilon + 2B^2 - \omega^2 + \omega$.

3.1 The structure of the RH problem with NZBCs

According to the robust IST, we obtain the RH problem of the ifoNLS equation (1) with NZBCs. For any time t, as $x \to \pm\infty$, $q(x,t)$ will tend to plane wave $Be^{i(\omega x+\nu t)}$. Based on the gauge transformation $\Psi(x,t) = e^{i(\omega x+\nu t)\frac{\sigma_3}{2}}\psi(x,t)$, Lax pair (2) will be converted to

$$\psi_x = X\psi, \qquad \psi_t = T\psi, \tag{42}$$

where

$$X = -i(\lambda + \frac{\omega}{2})\sigma_3 + Q_1, \qquad Q_1 = \begin{pmatrix} 0 & qe^{-i(\omega x+\nu t)} \\ -q^* e^{i(\omega x+\nu t)} & 0 \end{pmatrix},$$
$$T = e^{-i(\omega x+\nu t)\frac{\sigma_3}{2}}(V - \frac{i\nu}{2}\sigma_3)e^{i(\omega x+\nu t)\frac{\sigma_3}{2}}. \tag{43}$$

Using the NZBCs (41), we can rewrite the above Lax pair (42) into the following form

$$\psi_{\pm x} = X_\pm \psi_\pm, \qquad \psi_{\pm t} = T_\pm \psi_\pm, \tag{44}$$

where

$$X_{\pm} = -i(\lambda + \frac{\omega}{2})\sigma_3 + Q_0, \qquad Q_0 = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix},$$
$$T_{\pm} = \delta(\lambda)X_{\pm} = \Big[16\,\lambda^4\epsilon - 8\,\lambda^3\omega\,\epsilon + \left(-8\,B^2\epsilon + 4\,\omega^2\epsilon\right)\lambda^2 + \left(12\,B^2\omega\,\epsilon - 2\,\omega^3\epsilon + 2\right)\lambda$$
$$+6\,B^4\epsilon - 12\,B^2\omega^2\epsilon + \omega^4\epsilon - \omega + 1\Big]X_{\pm}. \tag{45}$$

According to the Lax pair (44), we can get the following solution

$$\psi_{\pm}(\lambda, x, t) = Y(\lambda)e^{-i\theta(\lambda,x,t)\sigma_3}, \qquad Y(\lambda) = n(\lambda)\begin{pmatrix} 1 & \frac{i(\lambda+\frac{\omega}{2}-\rho)}{B} \\ \frac{i(\lambda+\frac{\omega}{2}-\rho)}{B} & 1 \end{pmatrix}, \tag{46}$$

where

$$n(\lambda)^2 = \frac{\lambda + \frac{\omega}{2} + \rho}{2\rho}, \qquad \theta(\lambda, x, t) = \rho(\lambda)[x + \delta(\lambda)t], \tag{47}$$

and $\rho(\lambda)^2 = (\lambda + \frac{\omega}{2})^2 + B^2$, which is a two-sheeted Riemann surface for λ with branch points being $\lambda = -\frac{\omega}{2} \pm iB$, $\rho(\lambda) = \lambda + O(\lambda^{-1})$, and $\det(Y(\lambda)) = 1$ for $\lambda \neq -\frac{\omega}{2} \pm iB$. The branch cut of $\rho(\lambda)$ is $\eta = \eta_- \cup \eta_+$ with $\eta_- = \left[-iB - \frac{\omega}{2}, -\frac{\omega}{2}\right]$ and $\eta_+ = \left[iB - \frac{\omega}{2}, -\frac{\omega}{2}\right]$, and η is oriented upward. (see Fig.3)

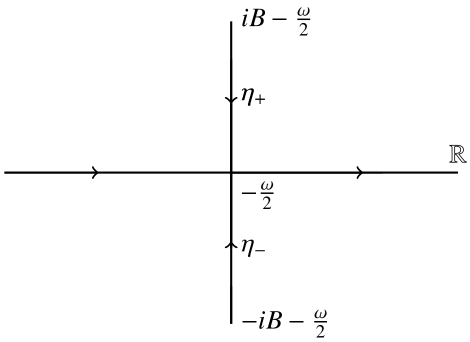


Figure 3. The contour $\Sigma_0 = \mathbb{R} \cup \eta$ of the basic RH problem.

We assume that $\Phi_{\pm}(\lambda, x, t)$ are also the solution of the Lax pair (44) and satisfies the asymptotic conditions $\Phi_{\pm}(\lambda, x, t) \to \psi_{\pm}(\lambda, x, t)$ as $x \to \pm\infty$. Then, taking transformation

$$\mu_{\pm}(\lambda, x, t) = \Phi_{\pm}(\lambda, x, t)e^{i\theta(\lambda,x,t)\sigma_3}, \tag{48}$$

with

$$\mu_{\pm}(\lambda, x, t) \to Y(\lambda), \qquad x \to \pm\infty. \tag{49}$$

Then we can immediately calculate that $\mu_{\pm}$ satisfies the following expression

$$(Y^{-1}\mu_{\pm})_x - i\rho\left[Y^{-1}\mu_{\pm}, \sigma_3\right] = Y^{-1}(Q_1 - Q_0)\mu_{\pm},$$
$$(Y^{-1}\mu_{\pm})_t - i\rho\delta\left[Y^{-1}\mu_{\pm}, \sigma_3\right] = Y^{-1}(T - T_{\pm})\mu_{\pm}, \tag{50}$$

and satisfy the following Volterra integral equations

$$\mu_-(x,t,\lambda) = Y + \int_{-\infty}^{x} Ye^{i\rho\sigma_3(\xi-x)}\left[Y^{-1}(Q_1-Q_0)\mu_-(\xi,t,\lambda)\right]e^{-i\rho\sigma_3(\xi-x)}\mathrm{d}\xi,$$
$$\mu_+(x,t,\lambda) = Y - \int_{x}^{+\infty} Ye^{i\rho\sigma_3(\xi-x)}\left[Y^{-1}(Q_1-Q_0)\mu_+(\xi,t,\lambda)\right]e^{-i\rho\sigma_3(\xi-x)}\mathrm{d}\xi. \quad (51)$$

Firstly, we will use the robust IST proposed by Bilman and Miller to construct RH problem [31]. Let we assume $q(x,t) - Be^{(\omega x+\nu t)} \in L^1(\mathbb{R})$ and $\mu_\pm = [\mu_{\pm 1}, \mu_{\pm 2}]$. Since μ_{-1} contains the exponential function $e^{-2i\rho(\xi-x)}$, it can be verified that μ_{-1} is analytical on $\mathbb{C}_+ \setminus \eta_+$ (where $\mathbb{C}_+ = \{\lambda : \mathrm{Im}\lambda > 0\}$). Using the similar method, we can find that μ_{-2} is analytical on $\mathbb{C}_- \setminus \eta_-$ (where $\mathbb{C}_- = \{\lambda : \mathrm{Im}\lambda < 0\}$). In summary, μ_{-1} and μ_{+2} are analytically continuous to $\mathbb{C}_+ \setminus \eta_+$, while μ_{+1} and μ_{-2} are analytically continuous to $\mathbb{C}_- \setminus \eta_-$.

Since $\Phi_\pm(\lambda,x,t)$ satisfy the Lax pair (42) for $\lambda \in \Sigma_0 \setminus \{-\frac{\omega}{2} \pm iB\}$, we can give the scattering relation by scattering matrix $S(\lambda)$

$$\Phi_+(\lambda,x,t) = \Phi_-(\lambda,x,t)S(\lambda), \qquad \lambda \in \Sigma_0 \setminus \{-\frac{\omega}{2} \pm iB\}, \quad (52)$$

the scattering matrix $S(\lambda)$ is shown as

$$S(\lambda) = \begin{pmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{pmatrix}, \qquad \det(S(\lambda)) = 1, \quad (53)$$

where $S_{11}(\lambda) = S_{22}^*(\lambda^*)$, $S_{12}(\lambda) = -S_{21}^*(\lambda^*)$. Furthermore, the Beals-Coifman (BC) simultaneous solution of the Lax pair (42) is obtained

$$\phi^{BC}(x,t,\lambda) = \begin{cases} \left[\Phi_{-,1}(\lambda,x,t), \dfrac{\Phi_{+,2}(\lambda,x,t)}{S_{22}(\lambda)}\right], & \lambda \in \mathbb{C}_+ \setminus \eta_+, \\ \left[\dfrac{\Phi_{+,1}(\lambda,x,t)}{S_{11}(\lambda)}, \Phi_{-,2}(\lambda,x,t)\right], & \lambda \in \mathbb{C}_- \setminus \eta_-. \end{cases} \quad (54)$$

Let $M^{BC}(x,t,\lambda) = \phi^{BC}(x,t,\lambda)e^{i\theta\sigma_3}$, the jumping curve for $M^{BC}(x,t,\lambda)$ is $\mathbb{R} \cup \eta$. According to the similar calculation shown in [40,41], we derived another similar solution of the Lax pair (42) for the smaller λ (here we assume that ε). In order that this solution has no singularities, we define

$$\phi(x,t,\lambda) = \begin{cases} \phi^{BC}(x,t,\lambda), & \lambda \in D_+ \cup D_-, \\ \phi^{in}(x,t,\lambda), & \lambda \in D_0, \end{cases} \quad (55)$$

where $\phi^{BC}(x,t,\lambda)$ means the BC simultaneous solution. $\phi^{in}(x,t,\lambda)$ represents a complete function, which is redefined as $\phi(x,t,\lambda)\phi(L,0,\lambda)^{-1}$. D_0 represents an open disk with a boundary of $\Sigma_+ \cup \Sigma_-$ and radius of ε. It is worth noting that we choose the appropriate ε to further make the scattering data $S_{11}(\lambda)$, $S_{22}(\lambda)$ are not equal to zero outside the disk. Concurrently, the branch cut η is included in this disk. In addition, the related domains $D_\pm = \{\lambda \in \mathbb{C} :|\ \lambda\ |\geq \varepsilon, \mathrm{Im}\lambda \gtrless 0\}$ and $\Sigma = (-\infty,-\varepsilon] \cup [\varepsilon,+\infty) \cup \Sigma_+ \cup \Sigma_-$

are shown in Fig.4. Set $M(x,t,\lambda) = \phi(x,t,\lambda)e^{i\theta\sigma_3}$, then the RH problem of the ifoNLS equation (1) with NZBCs are

Theorem 3. $M(x,t,\lambda)$ solves the following RH problem

$$\begin{cases} M(x,t,\lambda) \text{ is analytic in } \mathbb{C}\setminus\{\Sigma\cup\eta\}, \\ M_+(x,t,\lambda) = \begin{cases} M_-(x,t,\lambda)e^{-i\theta\sigma_3}J(x,t,\lambda)e^{i\theta\sigma_3}, & \lambda\in\Sigma, \\ M_-(x,t,\lambda)e^{2i\rho_+(\lambda)[x+\delta(\lambda)t]\sigma_3}, & \lambda\in\eta, \end{cases} \\ M(x,t,\lambda)\to\mathbb{I}, \quad \lambda\to\infty, \end{cases} \tag{56}$$

where the jump matrix $J(x,t,\lambda)$ is

$$J(x,t,\lambda) = \begin{cases} \left[\Phi_{-,1}(L,0,\lambda), \dfrac{\Phi_{+,2}(L,0,\lambda)}{S_{22}(\lambda)}\right], & \lambda\in\Sigma_+, \\ \left[\dfrac{\Phi_{+,1}(L,0,\lambda)}{S_{11}(\lambda)}, \Phi_{-,2}(L,0,\lambda)\right], & \lambda\in\Sigma_-, \\ \begin{bmatrix} 1 & R(\lambda) \\ R^*(\lambda^*) & 1+|R(\lambda)|^2 \end{bmatrix}, & \lambda\in(-\infty,-\varepsilon]\cup[\varepsilon,+\infty), \end{cases} \tag{57}$$

with L is a fixed real number, $R(\lambda) = \frac{S_{12}(\lambda)}{S_{22}(\lambda)}$ and the corresponding contour is shown in Figure .4. Then, we can deduce that the solution of the ifoNLS equation (1) is

$$q(x,t) = \lim_{\lambda\to\infty} 2i\lambda M_{12}(x,t,\lambda)e^{i(\omega x+\nu t)}. \tag{58}$$

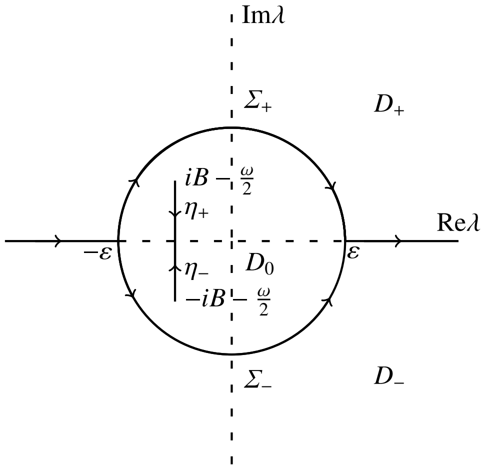


Figure 4. Definitions of the regions $D_\pm$, D_0 and $\Sigma_\pm$, η.

3.2 RW of the ifoNLS equation

In this section, we will use the modified DT for the Theorem 1 to obtain the higher-order RW of the ifoNLS equation (1). We make the following specification transformation

$$\widetilde{\phi}(x,t,\lambda)=\begin{cases}\mathbf{T}(x,t,\lambda)\phi(x,t,\lambda), & \lambda\in D_+\cup D_-,\\ \mathbf{T}(x,t,\lambda)\phi(x,t,\lambda)\mathbf{T}(L,0,\lambda)^{-1}, & \lambda\in D_0,\end{cases} \tag{59}$$

where $\phi(x,t,\lambda)$ satisfies the above Lax pair (42) and obtain $\phi(L,0,\lambda)=\mathbb{I}$ for $\lambda\in D_0$. The $\mathbf{T}$ is expressed as

$$\mathbf{T}(x,t,\lambda)=\mathbb{I}+\frac{\mathbf{H}(x,t)}{\lambda-\varsigma}+\frac{\mathbf{Y}(x,t)}{\lambda-\varsigma^*}, \tag{60}$$

for any point $\lambda\in D_0$ with $\mathbf{H}(x,t)$ and $\mathbf{Y}(x,t)$ being written as

$$\begin{aligned}\mathbf{H}(x,t)&=\frac{4\beta^2(1-\vartheta^*(x,t))\mathbf{s}(x,t)\mathbf{s}^\top(x,t)\sigma_2+2i\beta\mathcal{N}(x,t)\sigma_2\mathbf{s}^*(x,t)\mathbf{s}^\top(x,t)\sigma_2}{4\beta^2|1-\vartheta(x,t)|^2+\mathcal{N}^2(x,t)},\\ \mathbf{Y}(x,t)&=\frac{4\beta^2(\vartheta(x,t)-1)\sigma_2\mathbf{s}^*(x,t)\mathbf{s}^\dagger(x,t)-2i\beta\mathcal{N}(x,t)\mathbf{s}(x,t)\mathbf{s}^\dagger(x,t)}{4\beta^2|1-\vartheta(x,t)|^2+\mathcal{N}^2(x,t)},\end{aligned} \tag{61}$$

where $\beta=\mathrm{Im}(\varsigma)$, $\mathbf{s}(x,t)=\phi(x,t)\mathbf{c}$, $\mathcal{N}(x,t)=\mathbf{s}^\dagger(x,t)\mathbf{s}(x,t)$, $\vartheta(x,t)=\mathbf{s}^\top(x,t)\sigma_2\mathbf{s}'(x,t)$ and $\mathbf{c}=(c_1,c_2)^\top$ means an arbitrary column vector. Then the homologous jump condition of matrix $\widetilde{M}(x,t,\lambda)=\widetilde{\phi}(x,t,\lambda)e^{i\rho\sigma_3}$ changes at $\lambda\in\Sigma_+\cup\Sigma_-$, and the homologous jump matrix $J(\lambda)$ is rewritten as

$$\widetilde{J}(x,t,\lambda)=\begin{cases}\mathbf{T}(L,0,\lambda)J(x,t,\lambda), & \lambda\in\Sigma_+,\\ J(x,t,\lambda)\mathbf{T}(L,0,\lambda)^{-1}, & \lambda\in\Sigma_-.\end{cases} \tag{62}$$

Then, we can get the potential function $\widetilde{q}(x,t)$ from the new RH problem $\widetilde{M}(x,t,\lambda)$, namely

$$\widetilde{q}(x,t)=\lim_{\lambda\to\infty}2i\lambda\widetilde{M}_{12}(x,t,\lambda)e^{i(\omega x+\nu t)}=q(x,t)+2i(\mathbf{H}_{12}-\mathbf{H}_{21}^*)e^{i(\omega x+\nu t)}. \tag{63}$$

Furthermore, when we change $\mathbf{c}$ into the form $\varepsilon^{-1}\mathbf{c}_\infty$, where $\mathbf{c}_\infty\in\mathbb{C}^2\setminus\{0\}$ represents a fixed vector, and lead ε to 0, the matrix $\mathbf{T}(x,t,\lambda)$ is also represented as a limit process, that is

$$\mathbf{T}_\infty(x,t,\lambda)=\mathbb{I}+\frac{\mathbf{H}_\infty(x,t)}{\lambda-\varsigma}+\frac{\mathbf{Y}_\infty(x,t)}{\lambda-\varsigma^*}, \tag{64}$$

where $\mathbf{H}_\infty(x,t)=\lim_{\varepsilon\to0}\mathbf{H}(x,t)$ and $\mathbf{Y}_\infty(x,t)=\lim_{\varepsilon\to0}\mathbf{Y}(x,t)$.

Given the vector $\mathbf{s}(x,t)$, we can solve the ifoNLS equation (1) in combination with the DT. We regard the background eigenvector matrix $\phi_{bg}(x,t,\lambda)=\psi_\pm(x,t,\lambda)$, $\phi_{bg}^{in}(x,t,\lambda)=\phi_{bg}(x,t,\lambda)\phi_{bg}(0,0,\lambda)^{-1}$ represents the basic solutions, then we have the following results

$$\phi_{bg}^{in}(x,t,\lambda)=(x+\delta(\lambda)t)\frac{\sin(\theta(\lambda,x,t))}{\theta(\lambda,x,t)}X_\pm+\cos(\theta(\lambda,x,t))\mathbb{I}. \tag{65}$$

Furthermore, we can get

$$\mathbf{s}(x,t,\varsigma)=\phi_{bg}^{in}(x,t,\varsigma)\mathbf{c}=i\iota(x,t,\varsigma)\left[-(\varsigma+\frac{\omega}{2})\sigma_3\mathbf{c}+B\sigma_2\mathbf{c}\right]+\chi(x,t,\varsigma)\mathbf{c}, \tag{66}$$

where

$$\iota(x,t,\varsigma)=(x+\delta(\varsigma)t)\frac{\sin(\theta(\varsigma,x,t))}{\theta(\varsigma,x,t)}, \qquad \chi(x,t,\varsigma)=\cos(\theta(\varsigma,x,t)), \tag{67}$$

and

$$\begin{aligned}\mathcal{N}(x,t)=&\mathbf{s}^{\dagger}(x,t)\mathbf{s}(x,t)=\left[|\iota|^2(\varsigma+\frac{\omega}{2})(\varsigma^*+\frac{\omega}{2})+|\chi|^2+|\iota|^2B^2\right]\mathbf{c}^{\dagger}\mathbf{c}\\&+\left[i\iota^*\chi(\varsigma^*+\frac{\omega}{2})-i\iota\chi^*(\varsigma+\frac{\omega}{2})\right]\mathbf{c}^{\dagger}\sigma_3\mathbf{c}+(i\iota B\chi^*-i\iota^*B\chi)\mathbf{c}^{\dagger}\sigma_2\mathbf{c}\\&+i|\iota|^2(\varsigma^*-\varsigma)B\mathbf{c}^{\dagger}\sigma_1\mathbf{c},\end{aligned} \tag{68}$$

$$\begin{aligned}\vartheta(x,t)=&\mathbf{s}^{\top}(x,t)\sigma_2\mathbf{s}'(x,t)=\left[\iota'\chi(\varsigma+\frac{\omega}{2})-\iota\chi'(\varsigma+\frac{\omega}{2})+\iota\chi\right]\mathbf{c}^{\top}\sigma_1\mathbf{c}+\left[\iota\iota'(\varsigma+\frac{\omega}{2})^2\right.\\&\left.+\iota^2(\varsigma+\frac{\omega}{2})+\iota\iota'B^2+\chi\chi'\right]\mathbf{c}^{\top}\sigma_2\mathbf{c}+(i\iota'\chi B-i\iota\chi'B)\mathbf{c}^{\top}\mathbf{c}-\iota^2B\mathbf{c}^{\top}\sigma_3\mathbf{c}.\end{aligned} \tag{69}$$

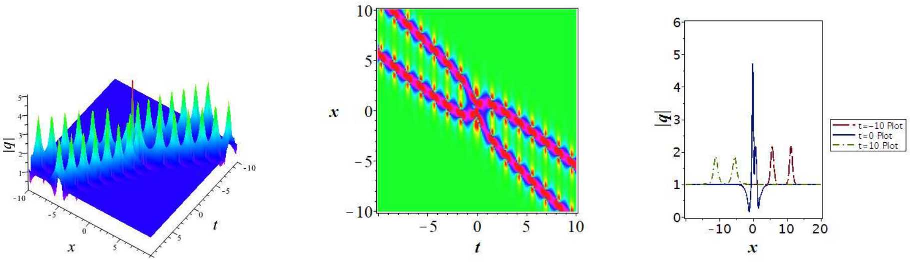
Finally, the solutions of the ifoNLS equation (1) are

$$\begin{aligned}\widetilde{q}(x,t)&=\left[B+2i(\mathbf{H}_{12}-\mathbf{H}_{21}^*)\right]e^{i(\omega x+\nu t)}\\&=\left(B+\frac{8\beta^2[(1-\vartheta^*)s_1^2-(1-\vartheta)s_2^{*2}]+8\beta\mathcal{N}s_1s_2^*}{4\beta^2|1-\vartheta|^2+\mathcal{N}^2}\right)e^{i(\omega x+\nu t)},\end{aligned} \tag{70}$$

where $\mathbf{s}=(s_1,s_2)^{\top}$, $\mathcal{N}$ and ϑ are given in expressions (66), (68) and (69). Furthermore, setting $\mathbf{c}=\mathbf{c}_{\infty}\varepsilon^{-1}$ with $\varepsilon\to 0$, the solution (70) can be rewritten as follows

$$\widetilde{q}_{\infty}(x,t)=\left[B-\frac{8\beta^2(\vartheta_{\infty}^*s_{\infty1}^2-\vartheta_{\infty}s_{\infty2}^{*2})-8\beta\mathcal{N}_{\infty}s_{\infty1}s_{\infty2}^*}{4\beta^2|\vartheta_{\infty}|^2+\mathcal{N}_{\infty}^2}\right]e^{i(\omega x+\nu t)}, \tag{71}$$

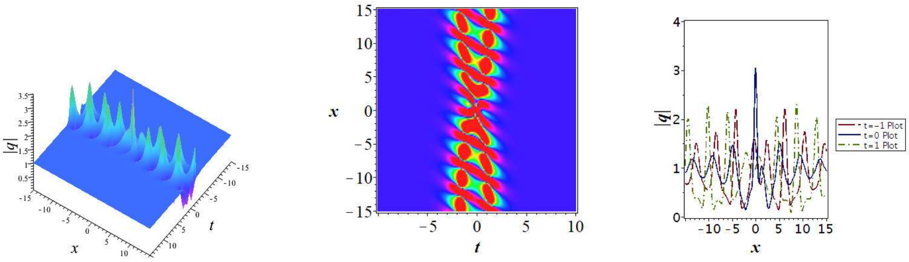
where $\mathbf{s}_{\infty}$, $\mathcal{N}_{\infty}$ and ϑ_{∞} are given in expressions (66), (68) and (69) with $\mathbf{c}$ substituted by $\mathbf{c}_{\infty}$, respectively.



(a) (b) (c)

Figure 5. The temporal-spatial periodic BW solutions (70) for the ifoNLS equation (1) with the parameters $B=1$, $\omega=\frac{1}{10}$, $\epsilon=0.0005$, $c_1=i$, $c_2=i+1$, $\ell=\frac{4}{3}$. (a) Three dimensional plot; (b) The density plot; (c) The wave propagation along the x-axis with $t=-10$ (long-dashed line), $t=0$ (solid line), $t=10$ (dash-dotted line).

According to the spectral analysis theorem, when selecting different ς, the properties of the corresponding solutions will change. When $\varsigma=-\frac{\omega}{2}+i\ell B$ with $|\ell|>1$, it becomes the temporal-spatial periodic BW, which can be verified by Fig. 5. But when$|\ell| < 1$, it becomes a spatial periodic BW and can be verified from Fig. 6.



(a) (b) (c)

Figure 6. The spatial periodic BW solutions (70) for the ifoNLS equation (1) with the parameters $B = 1$, $\omega = \frac{1}{10}$, $\epsilon = 0.005$, $c_1 = i$, $c_2 = i+1$, $\ell = \frac{2}{3}$. (a) Three dimensional plot; (b) The density plot; (c) The wave propagation along the x-axis with $t = -1$ (long-dashed line), $t = 0$ (solid line), $t = 1$ (dash-dotted line).

We obtain the RW of the ifoNLS equation (1) by making $\varsigma = -\frac{\omega}{2} \pm iB$. For convenience, we only calculate the case of $\varsigma = -\frac{\omega}{2} + iB$ ($\varsigma = -\frac{\omega}{2} - iB$ can perform similar calculations). Furthermore, we obtain

$$\mathbf{s}(x,t) = \begin{pmatrix} c_1 + B(c_1+c_2)(x+\delta(\varsigma)t) \\ c_2 - B(c_1+c_2)(x+\delta(\varsigma)t) \end{pmatrix}, \tag{72}$$

$$\mathbf{s}^{'}(x,t) = \begin{pmatrix} -\dfrac{1}{3}iB\left[B(x+\delta t)^3 + 3i(x+\delta t)^{'}\right](c_1+c_2) - ic_1(x+\delta t)\left[B(x+\delta t)+1\right] \\ \dfrac{1}{3}iB\left[B(x+\delta t)^3 + 3i(x+\delta t)^{'}\right](c_1+c_2) - ic_2(x+\delta t)\left[B(x+\delta t)-1\right] \end{pmatrix}, \tag{73}$$

$$\mathcal{N} = |c_1|^2 + |c_2|^2 + 2\mathrm{Re}\{B(c_1-c_2)^*(c_1+c_2)(x+\delta t)\} + 2B^2|c_1+c_2|^2|x+\delta t|^2, \tag{74}$$

$$\vartheta = -\frac{1}{3}B[2B(x+\delta t)^3 - 3i(x+\delta t)^{'}](c_1+c_2)^2 + 2c_1c_2(x+\delta t) - B(x+\delta t)^2(c_1^2-c_2^2). \tag{75}$$

Then we can find that the first-order RW can be deduced at $c_1 + c_2 = 0$.

(a) For $c_1 = -c_2 = 1$, we obtain the first-order RW solution as (see Fig.7)

$$\widetilde{q}(x,t) = \left(B + \frac{4B^2[(x+\delta t)^* - (x+\delta t)] - 4B}{B^2\left[1 + 2(x+\delta t) + 2(x+\delta t)^* + 4|x+\delta t|^2\right] + 1}\right)e^{i(\omega x + \nu t)}. \tag{76}$$

(b) For $\mathbf{c}_\infty = (c, -c)^\top$, we obtain the first-order RW solution with its (see Fig.8)

$$\widetilde{q}(x,t) = \left(\frac{B\left[4B^2|x+\delta t|^2 + 4B(x+\delta t)^* - 4B(x+\delta t) - 3\right]}{4B^2|x+\delta t|^2 + 1}\right)e^{i(\omega x + \nu t)}. \tag{77}$$

(c) For $c_1 + c_2 \neq 0$ e.g. $c_1 = c + \frac{\varpi}{2}$, $c_2 = -c + \frac{\varpi}{2}$ with $c \in \mathbb{C} \setminus \{0\}$ and $\varpi \ll 1$. Let $x = \frac{\bar{x}}{|\varpi|}$, $t = \frac{\bar{t}}{|\varpi|}$, if $(\bar{x}, \bar{t}) \in \mathbb{R}^2$ is also fixed, and then we obtain

$$\mathbf{s}(x,t) = \begin{pmatrix} c + e^{i\arg(\varpi)} B(\bar{x} + \delta\bar{t}) \\ -c - e^{i\arg(\varpi)} B(\bar{x} + \delta\bar{t}) \end{pmatrix} + O(\varpi), \tag{78}$$

$$\mathcal{N} = 2|c|^2 + 4\mathrm{Re}\{Bc^* e^{i\arg(\varpi)}(\bar{x} + \delta\bar{t})\} + 2B^2(\bar{x}^2 + |\delta|^2\bar{t}^2) + O(\varpi), \tag{79}$$

$$\vartheta = \frac{1}{|\varpi|}\left[-\frac{2}{3}B^2 e^{2i\arg(\varpi)}(\bar{x} + \delta\bar{t})^3 - 2cBe^{i\arg(\varpi)}(\bar{x} + \delta\bar{t})^2 - 2c^2(\bar{x} + \delta\bar{t})\right] + O(1). \tag{80}$$

Further, we can get $\widetilde{q}(x,t) \approx 1$, unless the leading term in ϑ proportional to $\frac{1}{|\varpi|}$ is cancelled, these terms will form a cubic equation of $\mathfrak{n} = \bar{x} + \delta\bar{t}$, and its three roots are $\mathfrak{n} = 0$ and $\frac{1}{2}c(B\varpi)^{-1}(-3 \pm i\sqrt{3})$, respectively (see Fig.9).

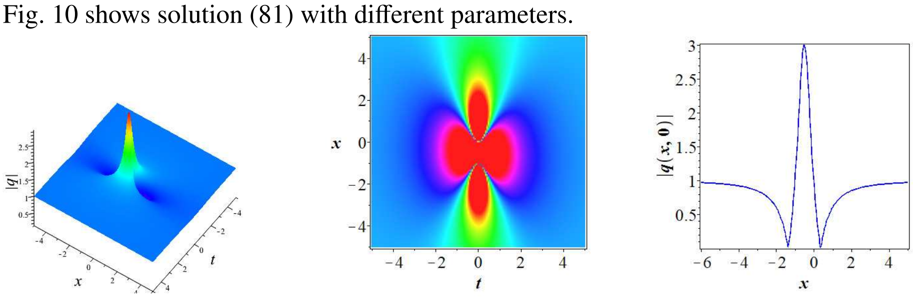
(d) For $\mathbf{c}_\infty = (1,1)^\top$, we can obtain the second-order RW

$$\widetilde{q}(x,t) = \left(B - 12B\frac{\mathfrak{P}}{\mathfrak{Q}}\right)e^{i(\omega x + \nu t)}, \tag{81}$$

where

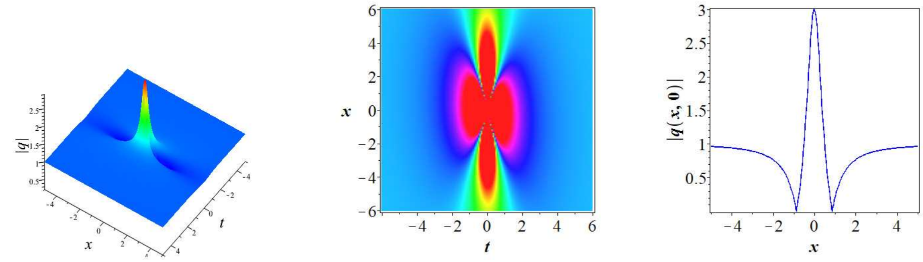
$$\begin{aligned}
\mathfrak{P} =& 16B^5(x+\delta t)^3(x+\delta t)^{*2} - 16B^5(x+\delta t)^2(x+\delta t)^{*3} + 24iB^3(x+\delta t)'(x+\delta t)^* \\
& - 6iB^2(x+\delta t)^{*'} - 16B^4(x+\delta t)^3(x+\delta t)^* + 48B^4(x+\delta t)^2(x+\delta t)^{*2} - 6iB^2(x+\delta t)' \\
& - 16B^4(x+\delta t)(x+\delta t)^{*3} - 24iB^3(x+\delta t)(x+\delta t)^{*'} - 12B^3(x+\delta t)^2(x+\delta t)^* \\
& + 12B^3(x+\delta t)(x+\delta t)^{*2} + 4B^3(x+\delta t)^3 - 4B^3(x+\delta t)^{*3} - 24iB^4(x+\delta t)^2(x+\delta t)^{*'} \\
& - 24iB^4(x+\delta t)^{*2}(x+\delta t)' + 24B^2(x+\delta t)(x+\delta t)^* - 9B(x+\delta t) + 9B(x+\delta t)^* - 3, \\
\mathfrak{Q} =& 64B^6(x+\delta t)^3(x+\delta t)^{*3} + 96iB^5(x+\delta t)^3(x+\delta t)^{*'} - 96iB^5(x+\delta t)^{*3}(x+\delta t)' \\
& - 48B^4(x+\delta t)^3(x+\delta t)^* + 144B^4(x+\delta t)^2(x+\delta t)^{*2} - 48B^4(x+\delta t)(x+\delta t)^{*3} \\
& + 144B^4(x+\delta t)'(x+\delta t)^{*'} - 72iB^3(x+\delta t)(x+\delta t)^{*'} + 72iB^3(x+\delta t)^*(x+\delta t)' \\
& + 108B^2(x+\delta t)(x+\delta t)^* + 9.
\end{aligned} \tag{82}$$

Fig. 10 shows solution (81) with different parameters.



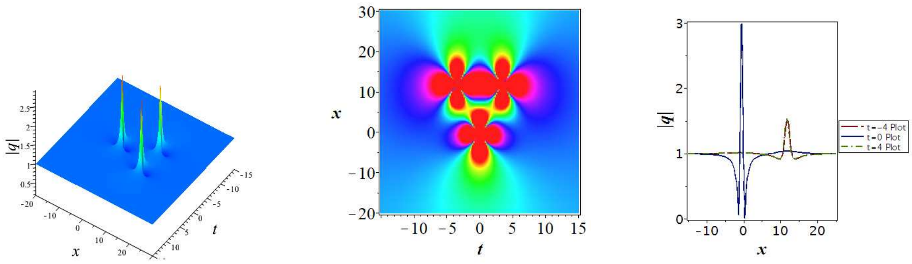
(a) **(b)** **(c)**

Figure 7. The first-order RW solutions (76) for the ifoNLS equation (1) with the parameters $B = 1$, $\omega = \frac{1}{2}$, $\epsilon = 0.005$, $c_1 = 1$, $c_2 = -1$. (a) Three dimensional plot; (b) The density plot; (c) The wave propagation along the x-axis with $t = 0$.



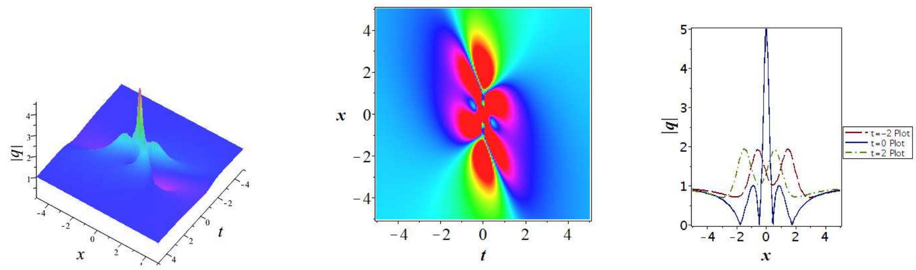
(**a**) (**b**) (**c**)

Figure 8. The first-order RW solutions (77) for the ifoNLS equation (1) with the parameters $B = 1$, $\omega = \frac{3}{5}$, $\epsilon = 0.05$, $c_1 = 1$, $c_2 = -1$. (a) Three dimensional plot; (b) The density plot; (c) The wave propagation along the x-axis with $t = 0$.



(**a**) (**b**) (**c**)

Figure 9. The first-order RW solutions (76) for the ifoNLS equation (1) with the parameters $B = 1$, $\omega = \frac{1}{2}$, $\epsilon = \frac{1}{1000}$, $c_1 = \cos(\frac{1}{2})$, $c_2 = -1$. (a) Three dimensional plot; (b) The density plot; (c) The wave propagation along the x-axis with $t = -4$ (long-dashed line), $t = 0$ (solid line), $t = 4$ (dash-dotted line).



(**a**) (**b**) (**c**)

Figure 10. The second-order RW solutions (81) for the ifoNLS equation (1) with the parameters $B = 1$, $\omega = \frac{1}{20}$, $\epsilon = 0.0005$, $c_1 = 1$, $c_2 = 1$. (a) Three dimensional plot; (b) The density plot; (c) The wave propagation along the x-axis with $t = -2$ (long-dashed line), $t = 0$ (solid line), $t = 2$ (dash-dotted line).

4 Conclusions

The present work studied the BS soliton and RW solutions of the ifoNLS equation (1) with ZBCs and NZBCs by the RH problem. In this context, the RH problem of the ifoNLS equation (1) is constructed, and the Nth order BS solitons of the equation (1) with ZBCs are obtained by the Laurent's series and the residue theorem. Also, some dynamic behaviors of the second-order BS soliton solution were analyzed for the equation (1) in the form of images. It is manifested that parameters can change the shape and size between the two waves (Fig. 2). In the meantime, the RH problem of the ifoNLS equation (1) with NZBCs are constructed by robust IST (Sec. 3). Then the solution of the ifoNLS equation (1) obtained via a one-fold DT. The graphs of the temporal-spatial periodic BWs and the spatial periodic BWs were drawn, which revealed that parameter ς had a certain influence on the BW solution. Finally, the first-order and the second-order RW were obtained by modulating parameters in equation (1).

Although the exact solutions of the equation (1) with NZBCs is derived in Ref. [39], and the RW solutions of the equation is studied by DT in Ref. [38]. However, in this paper, we mainly study the BS solitons of the equation (1) with ZBCs and the RW solutions of the equation with NZBCs, which makes the obtained solutions have more extensive significance and richer content. In addition, the proposed method in this paper can be further extended to identify some other nonlinear systems, and the method can be optimized to improve the results in future.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant No. 11975306, the Natural Science Foundation of Jiangsu Province under Grant No. BK20181351, the Six Talent Peaks Project in Jiangsu Province under Grant No. JY-059, and the Fundamental Research Fund for the Central Universities under the Grant Nos. 2019ZDPY07 and 2019QNA35.

References

1. G. W. Bluman,S. Kumei, Symmetries and differential equations. Springer-Verlag, New York, (1989).
2. V. B. Matveev, M. A. Salle, Darboux transformation and solitons. Berlin: Springer, (1991).
3. R. Hirota, Direct methods in soliton theory. Berlin: Springer, (2004).
4. M. J. Ablowitz, P. A. Clarkson, solitons, nonlinear evolution equations and inverse scattering. Cambridge: Cambridge University Press, (1991).
5. C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, Method for solving the Korteweg-de Vries equation. Phys. Rev. Lett. **19** (1967) 1095.
6. V. E. Zakharov, S. V. Manakov, S. P. Novikov, L. P. Pitaevskii, The theory of solitons: The inverse scattering method. New York: Consultants Bureau, (1984).
7. X. G. Geng, J. P. Wu, Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation, Wave Motion, **60** (2016) 62-72.
8. J. P. Wu, X. G. Geng, Inverse scattering transform and soliton classification of the coupled modified Korteweg-de Vries equation, Commun. Nonlinear Sci. Numer. Simul. **53** (2017) 83-93.

9. W. X. Ma, Riemann-Hilbert problems of a six-component mKdV system and its soliton solutions, Acta Math. Sci. Ser. **39** (2019) 509-523.
10. G. Q. Zhang, Z. Y. Yan, Inverse scattering transforms and soliton solutions of focusing and defocusing nonlocal mKdV equations with non-zero boundary conditions, Phys. D, **402** (2020) 132170.
11. Y. Zhao, E. G. Fan, Inverse scattering transformation for the Fokas-Lenells equation with nonzero boundary conditions, arXiv preprint arXiv:1912.12400, (2019)
12. Y. L. Yang, E. G. Fan, Riemann-Hilbert approach to the modified nonlinear Schrödinger equation with non-vanishing asymptotic boundary conditions. Phys. D, **417** (2021) 132811.
13. W. X. Ma, Riemann-Hilbert problems and N-soliton solutions for a coupled mKdV system. J. Geom. Phys. **132** (2018) 45-54.
14. B. L. Guo, L. M. Ling, Riemann-Hilbert approach and N-soliton formula for coupled derivative Schrödinger equation. J. Math. Phys. **53** (2012) 133-3966.
15. N. Liu, B. L. Guo, Painlevé-type asymptotics of an extended modified KdV equation in transition regions. J. Differ. Equations, **280** (2021) 203-235.
16. X. G. Geng, K. D. Wang, M. M. Chen, Long-time asymptotics for the spin-1 Gross-Pitaevskii equation. Commun. Math. Phys. **382** (2021) 585-611 .
17. J. Xu, E. G. Fan, Long-time asymptotics for the Fokas-Lenells equation with decaying initial value problem: without solitons. J. Differ. Equations, **259** (2015) 1098-1148.
18. D. S. Wang, B. L. Guo, X. L. Wang, Long-time asymptotics of the focusing Kundu-Eckhaus equation with nonzero boundary conditions. J. Diff. Equations, **266(9)** (2019) 5209-5253.
19. R. Jenkins, K. D. McLaughlin, The semiclassical limit of focusing NLS for a family of square barrier initial data. Comm. Pure Appl. Math. **67** (2014) 246-320.
20. P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation. Ann. Math. **137** (1993) 295-368.
21. G. Biondini, D. Mantzavinos, Long-time asymptotics for the focusing nonlinear Schrödinger equation with nonzero boundary conditions at infinity and asymptotic stage of modulational instability. Comm. Pure Appl. Math. LXX **70(12)** (2017) 2300-2365.
22. S. F. Tian, Initial-boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method. J. Differ. Equations **262(1)** (2017) 506-558.
23. V. E. Zakharov, A. Shabat, Exact theory of two-dimensional self-focusing and one dimensional self-modulation of waves in nonlinear media. Sov. Phys. JETP, **34(1)** (1972) 62-69.
24. M. Wadati, K. Ohkuma, Multiple pole solutions of the modified Korteweg-de Vries equation. J. Phys. Soc. Japan, **51(6)** (1981) 2029-2035.
25. H. Tsuru, M. Wadati, The multiple pole solutions of the sine-Gordon equation. J. Phys. Soc. Japan, **9** (1984) 2908-2921.
26. L. M. Ling, The algebraic representation for high order solution of Sasa-Satsuma equation. Discrete Continuous Dynamical Systems-S. **9** (2016) 6.
27. Y. S. Zhang, J. G. Rao, Y. Cheng, J. S. He, Riemann-Hilbert method for the Wadati-Konno-Ichikawa equation: N simple poles and one higher-order pole. Physica D **399** (2019) 173-185.
28. Y. S. Zhang, X. X. Tao, S. W. Xu, The bound-state soliton solutions of the complex modified KdV equation. Inverse Problems. **36(6)** (2020) 065003.
29. D. Bilman, R. Buckingham, Large-order asymptotics for multiple-pole solitons of the focusing nonlinear Schrödinger equation. J. Nonlinear Sci. **29** (2019) 2185-229.
30. X. E. Zhang, L. M. Ling, Asymptotic analysis of high order solitons for the Hirota equation. Phys. D. **426** (2021) 132982.
31. D. Bilman, P. Miller, A robust inverse scattering transform for the focusing nonlinear Schrödinger equation. Comm. Pure. Appl. Math. LXXII **72(8)** (2019) 1722-1805.
32. S. Y. Chen, Z. Y. Yan, The higher-order nonlinear Schrödinger equation with non-zero boundary conditions: robust inverse scattering transform, breathers, and rogons. Phys. Lett. A **383** (2019) 125906.
33. W. Q. Peng, Y. Chen, Inverse scattering transforms for the sixth-order nonlinear Schrö dinger equation with zero/nonzero boundary conditions: Bound-state soliton and rogue wave, arXiv preprint arXiv:2103.14296, (2021).
34. S. Y. Chen, Z. Y. Yan, The Hirota equation: Darboux transform of the Riemann-Hilbert problem and higher-order rogue waves. Appl. Math. Lett. **95** (2019) 65-71.
35. N. Liu, B. L. Guo, Solitons and rogue waves of the quartic nonlinear Schrödinger equation by Riemann-Hilbert approach. Nonlinear Dyn. **100** (2020) 629-646.
36. X. E. Zhang, Y. Chen, Inverse scattering transformation for generalized nonlinear Schrödinger equation. Appl. Math. Lett. **98** (2019) 306-313.

37. Y. N. Chen, Rogue wave solutions for an inhomogeneous fifth-order nonlinear Schrödinger equation from Heisenberg ferromagnetism. J. Progress. Res. Math, **4(2)** (2015) 328-338.
38. L. L. Feng, S. F. Tian, T. T. Zhang, Solitary wave, breather wave and rogue wave solutions of an inhomogeneous fifth-order nonlinear Schrödinger equation from Heisenberg ferromagnetism. Rocky Mountain J. Math. **49(1)** (2019) 29-45.
39. J. J. Yang, S. F. Tian, Z. Q. Li, Riemann-Hilbert approach to the inhomogeneous fifth-order nonlinear Schrödinger equation with non-vanishing boundary conditions. arXiv preprint arXiv:2001.08597, (2020).
40. P. Defit, X. Zhou, Direct and inverse scattering on the line with arbitrary singularities. Comm. Pure Appl. Math. **44** (1991) 485-533.
41. X. Zhou, Direct and inverse scattering transformations with arbitray spectral singularities. Comm. Pure Appl. Math. **42** (1989) 895-938.