The scaling-law flows: An attempt at scaling-law vector calculus

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Abstract

In this paper, the scaling-law vector calculus, which is connected between the vector calculus and the scaling law in fractal geometry, is addressed based on the Leibniz derivative and Stieltjes integral for the first time. The scaling-law Gauss-Ostrogradsky-like, Stokes-like, and Green-like theorems, and Green-like identities are considered in sense of the scaling-law vector calculus. The scaling-law Navier-Stokes-like equations are obtained in detail. The obtained result is as a potentially mathematical tool proposed to develop an important way of approaching this challenge for the scaling-law flows.

The scaling-law flows: An attempt at scaling-law vector calculus

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Abstract

In this paper, the scaling-law vector calculus, which is connected between the vector calculus and the scaling law in fractal geometry, is addressed based on the Leibniz derivative and Stieltjes integral for the first time. The scaling-law Gauss-Ostrogradsky-like, Stokes-like, and Green-like theorems, and Green-like identities are considered in sense of the scaling-law vector calculus. The scaling-law Navier-Stokes-like equations are obtained in detail. The obtained result is as a potentially mathematical tool proposed to develop an important way of approaching this challenge for the scaling-law flows.

Key words: scaling law vector calculus, fractal geometry, scaling-law flow, scaling-law Navier-Stokes-like equations

1 Introduction

The classical calculus is called the Newton-Leibniz calculus, which contains the differential calculus and the integral calculus. The differential calculus was proposed by Newton in 1665 [1,2] and by Leibniz in 1684 [3]. The integral calculus was coined by Newton in 1665 [1,2] and by Leibniz in 1686 [4]. Based on the Newton-Leibniz calculus, the vector calculus, denoted by Hamilton in

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1844 [5], by Tait in 1890 [6], by Heaviside in 1893 [7], and by Gibbs in 1901[8], was applied in the fields of mechanics, hydrodynamics, and electricity [9].

The calculus with respect to monotone functions is one of the classes of the general calculus operators [10,11]. This theory consists of the differential calculus with respect to monotone function, which is called the Leibniz derivative due to Leibniz [12,13], and the integral calculus with respect to monotone function, which is called the Stieltjes integral due to Stieltjes [14]. The integral calculus with respect to monotone function was developed by Widder [15], by Horst [16] and by Stoll [17], respectively. The vector calculus with respect to monotone function was proposed in [18].

The scaling laws are the connections between the fractal geometry and measure in various complex phenomena [19,20]. The experimental evidence for the flow in the extended self-similarity scaling laws was considered in [21]. The scaling laws for the turbulent flow in the pipes were presented in [22,23]. The scaling laws for the wall-bounded shear flows were developed in [24]. The self-similarity scaling laws in turbulent flows was discussed in [25].

The scaling-law calculus, which is considered to develop the connection between the fractal geometry and calculus with respect to monotone functions, was proposed to model the scaling-law behaviors [11,26]. The Gauss [27], Ostrogradsky [28], Stokes [29] and Green [30] tasks have not been extended in the sense of the scaling-law calculus. Due to the present investigation for the scaling-law differential calculus and the scaling-law integral calculus, the scaling-law vector calculus has not been developed based on the vector calculus with respect to monotone function. Motivated by the present idea, the aim of the present paper is to propose the definitions for the scaling-law vector calculus, to present its fundamental theorems, and to suggest the potential and important applications in scaling-law flows. The structure of the paper is designed as follows. In Section 2, the general calculus operators are given. In Section 3, the theory and properties of the scaling-law vector calculus are presented. In Section 4, the scaling-law Navier-Stokes-type equations for the scaling-law flows are discussed. Finally, the conclusions are given in Section 5.

2 Preliminaries

In this section, we introduce the definitions and theorems of the general calculus operators containing the calculus with respect to monotone function and scaling-law calculus.

2.1 The calculus with respect to monotone function

Let $\varphi_\vartheta(t) = (\varphi \circ \vartheta)(t) = \varphi(\vartheta(t))$, where $\vartheta(t)$ is the monotone function, e.g., $\vartheta^{(1)}(t) = d\vartheta(t)/dt > 0$.

Let $\Lambda(\varphi)$ be the set of the continuous derivatives of the functions $\varphi(\vartheta)$ with respect to the variable ϑ in the domain $\Im$.

Let $\Xi(\vartheta)$ be the set of the continuous derivatives of the functions $\vartheta(t)$ with respect to the variable t in the domain $\aleph$.

Let us consider the set of the continuous derivatives of the composite functions, defined as follows:

$$\Re(\varphi_\vartheta) = \{\varphi_\vartheta(t) : \varphi_\vartheta(t) \in \Lambda(\varphi), \vartheta \in \Xi(\vartheta)\}.$$

2.2 The Leibniz derivative

Let $\varphi_\vartheta \in \Re(\varphi_\vartheta)$. The Leibniz derivative of the function $\varphi_\vartheta(t)$ is defined as [11,18,26]

$$D_{t,\vartheta}^{(1)}\varphi_\vartheta(t) = \frac{1}{\vartheta^{(1)}(t)}\frac{d\varphi_\vartheta(t)}{dt}. \tag{1}$$

The geometric interpretation of the Leibniz derivative is the rate of change of the functional $\varphi_\vartheta(t)$ with the function $\vartheta(t)$ in the independent variable t [11,18,26].

Let $\varphi_\vartheta \in \Re(\varphi_\vartheta)$. The total Leibniz-type differential with respect to monotone function $\vartheta(t)$ of the function $\varphi_\vartheta(t)$, denoted as $d\varphi_\vartheta(t) = d\varphi(\vartheta(t))$, is defined as

$$d\varphi_\vartheta(t) = \left(\vartheta^{(1)}(t)\, D_{t,\vartheta}^{(1)}\varphi_\vartheta(t)\right) dt. \tag{2}$$

2.3 The Stieltjes integral

Let $\Phi_\vartheta \in \Re(\Phi_\vartheta)$. The Stieltjes integral of the function $\Phi_\vartheta(t)$ in the interval $[a, b]$ is defined as [11,18,26]

$${}_aI_{b,\vartheta}^{(1)}\Phi_\vartheta(t) = \int_a^b \Phi_\vartheta(t)\,\vartheta^{(1)}(t)\,dt. \tag{3}$$

Similarly, the geometric interpretation of the Stieltjes integral is the area enclosed by the integrand function $\Phi_\vartheta(t)$ and the function $\vartheta(t)$ in the independent variable $t \in [a, b]$ [11,18,26].

Their properties are given as follows:

(O1) The chain rule for the Leibniz derivative is given as follows [18]:

$$D_{t,\vartheta}^{(1)} \Theta \{\varphi_\vartheta (t)\} = \Theta^{(1)} (\varphi) \cdot D_{t,\vartheta}^{(1)} \varphi_\vartheta (t), \tag{4}$$

where $\Theta^{(1)} (\varphi) = d\Theta (\varphi) / d\varphi$.

(O2) The change-of-variable theorem for the Stieltjes integral reads as follows [18]:

$${}_aI_{t,\vartheta}^{(1)} \left(\Theta^{(1)} (\varphi) \cdot D_{t,\vartheta}^{(1)} \varphi_\vartheta (t)\right) = \Theta \{\varphi_\vartheta (t)\} - \Theta \{\varphi_\vartheta (a)\}. \tag{5}$$

2.4 The Leibniz-type partial derivatives

Let $\Theta = \Theta (x, y, z) = \Theta (\alpha (x), \beta (y), \gamma (z))$, where $\alpha^{(1)} (x) > 0$, $\beta^{(1)} (y) > 0$ and $\gamma^{(1)} (z) > 0$.

The Leibniz-type partial derivatives of the scalar field Θ are defined as [18]

$$\partial_{x,\alpha}^{(1)} \Theta = \frac{1}{\alpha^{(1)} (x)} \frac{\partial \Theta}{\partial x}, \tag{6}$$

$$\partial_{y,\beta}^{(1)} \Theta = \frac{1}{\beta^{(1)} (y)} \frac{\partial \Theta}{\partial y} \tag{7}$$

and

$$\partial_{z,\gamma}^{(1)} \Theta = \frac{1}{\gamma^{(1)} (z)} \frac{\partial \Theta}{\partial z}, \tag{8}$$

respectively.

The total Leibniz-type differential of the scalar field Θ is defined as [18]:

$$d\Theta = \left(\alpha^{(1)} (x) \partial_{x,\alpha}^{(1)} \Theta\right) dx + \left(\beta^{(1)} (y) \partial_{y,\beta}^{(1)} \Theta\right) dy + \left(\gamma^{(1)} (z) \partial_{z,\gamma}^{(1)} \Theta\right) dz. \tag{9}$$

which leads to

$$\frac{d\Theta}{dt} = \left(\alpha^{(1)} (x) \partial_{x,\alpha}^{(1)} \Theta\right) \frac{dx}{dt} + \left(\beta^{(1)} (y) \partial_{y,\beta}^{(1)} \Theta\right) \frac{dy}{dt} + \left(\gamma^{(1)} (z) \partial_{z,\gamma}^{(1)} \Theta\right) \frac{dz}{dt}. \tag{10}$$

2.5 The scaling-law calculus

Let us consider the set of the continuous derivatives of the composite functions, defined as follows:

$$\Re\left(\varphi_{\eta}\right)=\left\{\varphi_{\vartheta}\left(t\right):\varphi_{\vartheta}\left(t\right)\in\Lambda\left(\varphi\right),\vartheta\in\Xi\left(\vartheta\right)\right\},$$

where the fractal scaling law is defined as [19,20]

$$\vartheta\left(t\right)=\lambda t^{\eta} \tag{11}$$

with the normalization constant $\lambda\geq 0$, the radius $t\geq 0$, and the scaling exponent $\eta\geq 0$.

Here, we take $-\infty<t<\infty$, $-\infty<\lambda<\infty$ and $-\infty<\eta<\infty$.

2.6 The scaling-law derivative

Let $\varphi_{\eta}\in\Re\left(\varphi_{\eta}\right)$, e.g.,$\varphi_{\eta}\left(t\right)=\left(\varphi\circ\left(\lambda t^{\eta}\right)\right)\left(t\right)=\varphi\left(\lambda t^{\eta}\right)$.

The scaling-law derivative of the function $\varphi_{\eta}\left(t\right)$ is defined as [11,26]

$$^{SL}D_{t}^{(1)}\varphi_{\eta}\left(t\right)=\frac{d\varphi_{\eta}\left(t\right)}{d\left(\lambda t^{\eta}\right)}=\frac{1}{\lambda\eta t^{\eta-1}}\frac{d\varphi_{\eta}\left(t\right)}{dt}. \tag{12}$$

The geometric interpretation of the scaling-law derivative is the rate of change of the functional $\varphi_{\eta}\left(t\right)$ with the function $\vartheta=\lambda t^{\eta}$ in the independent variable t [11,26].

Let $\varphi_{\eta}\in\Re\left(\varphi_{\eta}\right)$. The total scaling-law differential of the function $\varphi_{\eta}\left(t\right)$, denoted as $d\varphi_{\eta}\left(t\right)$, is defined as [11,26]

$$d\varphi_{\eta}\left(t\right)={}^{SL}D_{t}^{(1)}\varphi_{\eta}\left(t\right)d\left(\lambda t^{\eta}\right)=\left(\lambda\eta t^{\eta-1}{}^{SL}D_{t}^{(1)}\varphi_{\eta}\left(t\right)\right)dt. \tag{13}$$

2.7 The scaling-law integral

Let $\Phi_{\eta}\in\Re\left(\Phi_{\eta}\right)$. The scaling-law integral of the function $\Phi_{\eta}\left(t\right)$ in the interval $[a,b]$ is defined as [11,26]

$${}_{a}^{SL}I_{b}^{(1)}\Phi_{\eta}\left(t\right)=\int_{a}^{b}\Phi_{\eta}\left(t\right)d\left(\lambda t^{\eta}\right)=\int_{a}^{b}\Phi_{\eta}\left(t\right)\lambda\eta t^{\eta-1}dt. \tag{14}$$

Similarly, the geometric interpretation of the scaling-law integral is the area enclosed by the integrand function $\Phi_\eta(t)$ and the function $\vartheta(t) = \lambda t^\eta$ in the independent variable $t \in [a, b]$ [11,26].

Their properties are presented as follows:

$(P1)$ The chain rule for the scaling-law derivative is given as follows [11,26]:

$$ {}^{SL}D_t^{(1)} \Theta\left\{\varphi_\eta(t)\right\} = \Theta^{(1)}(\varphi) \cdot {}^{SL}D_t^{(1)} \varphi_\eta(t), \tag{15} $$

where $\Theta^{(1)}(\varphi) = d\Theta(\varphi)/d\varphi$.

$(P2)$ The change-of-variable theorem for the scaling-law integral can be given as follows [11,26]:

$$ {}_a^{SL}I_t^{(1)} \left(\Theta^{(1)}(\varphi) \cdot {}^{SL}D_t^{(1)} \varphi_\eta(t)\right) = \Theta\left\{\varphi_\eta(t)\right\} - \Theta\left\{\varphi_\eta(a)\right\}. \tag{16} $$

2.8 The scaling-law gradient

In order to discuss the scaling-law gradient, we consider the Cartesian-type coordinate system $\left(\lambda_1 x^{D_1}, \lambda_2 y^{D_2}, \lambda_3 z^{D_3}\right)$, which leads to the Cartesian coordinate system (x, y, z), where the scaling exponents $D_1 = D_2 = D_3 = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

In the Cartesian-type coordinate system $\left(\lambda_1 x^{D_1}, \lambda_2 y^{D_2}, \lambda_3 z^{D_3}\right)$, the scaling-law gradient is defined as

$$ \begin{aligned} &\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \\ &= i\left(\lambda_1 D_1 x^{D_1 - 1}\right) \partial_x^{(1)} + j\left(\lambda_2 D_2 y^{D_2 - 1}\right) \partial_y^{(1)} + k\left(\lambda_3 D_3 z^{D_3 - 1}\right) \partial_z^{(1)}, \end{aligned} \tag{17} $$

where i, j and k are the unit vector in the Cartesian coordinate system.

Let us consider the scaling-law scalar field, defined by:

$$ X = X\left(\lambda_1 x^{D_1}, \lambda_2 y^{D_2}, \lambda_3 z^{D_3}\right). \tag{18} $$

The scaling-law gradient of the scaling-law scalar field X is given as

$$\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X$$
$$= i\left(\lambda_1 D_1 x^{D_1-1}\right)\partial_x^{(1)} X + j\left(\lambda_2 D_2 y^{D_2-1}\right)\partial_y^{(1)} X + k\left(\lambda_3 D_3 z^{D_3-1}\right)\partial_z^{(1)} X. \tag{19}$$

From (17) and (18) we have that

$$dX$$
$$= \left(\lambda_1 D_1 x^{D_1-1}\right)\partial_x^{(1)} X dx + \left(\lambda_2 D_2 y^{D_2-1}\right)\partial_y^{(1)} X dy + \left(\lambda_3 D_3 z^{D_3-1}\right)\partial_z^{(1)} X dz$$
$$= \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X \cdot \mathbf{n} dl = \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X d\mathbf{l}, \tag{20}$$

where $\mathbf{n}$ is the unit normal to the surface, dl is the distance measured along the normal $\mathbf{n}$, and $d\mathbf{l} = \mathbf{n}dl = idx + jdy + kdz$.

The scaling-law direction derivative of the scaling-law scalar field X along the normal $\mathbf{n}$ is defined as

$$\frac{dX}{dl} = \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X \cdot \mathbf{n} = \partial_n^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X. \tag{21}$$

The scaling-law Laplace-like operator, denoted as

$$\nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} = \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}, \tag{22}$$

of the scaling-law scalar field X is defined as

$$\nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X$$
$$= \left[\left(\lambda_1 D_1 x^{D_1-1}\right)\partial_x^{(1)}\right]^2 X + \left[\left(\lambda_2 D_2 y^{D_2-1}\right)\partial_y^{(1)}\right]^2 X + \left[\left(\lambda_3 D_3 z^{D_3-1}\right)\partial_z^{(1)}\right]^2 X. \tag{23}$$

Let the scaling-law vector field, defined by:

$$\widehat{\mathrm{O}} = \widehat{\mathrm{O}}\left(\lambda_1 x^{D_1}, \lambda_2 y^{D_2}, \lambda_3 z^{D_3}\right) = \widehat{\mathrm{O}}_x i + \widehat{\mathrm{O}}_y j + \widehat{\mathrm{O}}_z k. \tag{24}$$

The scaling-law divergence of the scaling-law vector field $\widehat{\mathrm{O}}$ is defined as

$$\begin{aligned}
&\nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \widehat{\mathrm{O}} \\
&= \left(\lambda_1 D_1 x^{D_1-1}\right) \partial_x^{(1)} \widehat{\mathrm{O}}_x + \left(\lambda_2 D_2 y^{D_2-1}\right) \partial_y^{(1)} \widehat{\mathrm{O}}_y + \left(\lambda_3 D_3 z^{D_3-1}\right) \partial_z^{(1)} \widehat{\mathrm{O}}_z.
\end{aligned} \tag{25}$$

The scaling-law curl of the scaling-law vector field $\widehat{\mathrm{O}}$ is defined as

$$\begin{aligned}
&\nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \times \widehat{\mathrm{O}} \\
&= \begin{pmatrix} i & j & k \\ \left(\lambda_1 D_1 x^{D_1-1}\right) \partial_x^{(1)} & \left(\lambda_2 D_2 y^{D_2-1}\right) \partial_y^{(1)} & \left(\lambda_3 D_3 z^{D_3-1}\right) \partial_z^{(1)} \\ \widehat{\mathrm{O}}_x & \widehat{\mathrm{O}}_y & \widehat{\mathrm{O}}_z \end{pmatrix} \\
&= \begin{pmatrix} \left(\lambda_2 D_2 y^{D_2-1}\right) \partial_y^{(1)} & \left(\lambda_3 D_3 z^{D_3-1}\right) \partial_z^{(1)} \\ \widehat{\mathrm{O}}_y & \widehat{\mathrm{O}}_z \end{pmatrix} i - \begin{pmatrix} \left(\lambda_1 D_1 x^{D_1-1}\right) \partial_x^{(1)} & \left(\lambda_3 D_3 z^{D_3-1}\right) \partial_z^{(1)} \\ \widehat{\mathrm{O}}_x & \widehat{\mathrm{O}}_z \end{pmatrix} j \\
&+ \begin{pmatrix} \left(\lambda_1 D_1 x^{D_1-1}\right) \partial_x^{(1)} & \left(\lambda_2 D_2 y^{D_2-1}\right) \partial_y^{(1)} \\ \widehat{\mathrm{O}}_x & \widehat{\mathrm{O}}_y \end{pmatrix} k \\
&= \left(\left(\lambda_2 D_2 y^{D_2-1}\right) \partial_y^{(1)} \widehat{\mathrm{O}}_z - \left(\lambda_3 D_3 z^{D_3-1}\right) \partial_z^{(1)} \widehat{\mathrm{O}}_y\right) i \\
&+ \left(\left(\lambda_3 D_3 z^{D_3-1}\right) \partial_z^{(1)} \widehat{\mathrm{O}}_x - \left(\lambda_1 D_1 x^{D_1-1}\right) \partial_x^{(1)} \widehat{\mathrm{O}}_z\right) j \\
&+ \left(\left(\lambda_1 D_1 x^{D_1-1}\right) \partial_x^{(1)} \widehat{\mathrm{O}}_y - \left(\lambda_2 D_2 y^{D_2-1}\right) \partial_y^{(1)} \widehat{\mathrm{O}}_x\right) k.
\end{aligned} \tag{26}$$

The properties for the scaling-law gradient can be presented as follows:

$$\begin{aligned}
&\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \times \left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \times \widehat{\mathrm{O}} \right) \\
&= \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \widehat{\mathrm{O}} \right) - \nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \widehat{\mathrm{O}},
\end{aligned} \tag{27}$$

$$\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \times \widehat{\mathrm{O}} \right) = 0, \tag{28}$$

$$\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \times \left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X \right) = 0, \tag{29}$$

and

$$\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} (XY) = Y \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X + X \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} Y, \tag{30}$$

where $Y = Y\left(\lambda_1 x^{D_1}, \lambda_2 y^{D_2}, \lambda_3 z^{D_3}\right)$.

3 The scaling-law vector calculus

Let $\mathbf{l} = \left(\lambda_1 x^{D_1}, \lambda_2 y^{D_2}, \lambda_3 z^{D_3}\right)$ be the scaling-law vector line.

The arc length is presented as follows:

$$\begin{aligned} \ell &= \int\limits_0^\ell d\ell \\ &= \int\limits_a^b \sqrt{\left(\lambda_1 D_1 x^{D_1-1}\right)^2 \left(\tfrac{dx}{dt}\right)^2 + \left(\lambda_2 D_2 y^{D_2-1}\right)^2 \left(\tfrac{dy}{dt}\right)^2 + \left(\lambda_3 D_3 z^{D_3-1}\right)^2 \left(\tfrac{dz}{dt}\right)^2} dt, \end{aligned} \tag{31}$$

where

$$d\ell = \sqrt{\left(\lambda_1 D_1 x^{D_1-1}\right)^2 \left(\frac{dx}{dt}\right)^2 + \left(\lambda_2 D_2 y^{D_2-1}\right)^2 \left(\frac{dy}{dt}\right)^2 + \left(\lambda_3 D_3 z^{D_3-1}\right)^2 \left(\frac{dz}{dt}\right)^2} dt. \tag{32}$$

The scaling-law line integral of the scaling-law vector field $\widehat{\mathrm{O}}$ along the curve $\mathbf{l}$, denoted by Π, is defined as

$$\Pi = \int\limits_\ell \widehat{\mathrm{O}} \cdot d\mathbf{l} = \int\limits_\ell \widehat{\mathrm{O}} \cdot \mathbf{n} d\ell, \tag{33}$$

which leads to

$$\begin{aligned}\Pi &= \int_{\ell} \widehat{\mathrm{O}} \cdot d\mathbf{l} = \int_{\ell} \widehat{\mathrm{O}} \cdot \mathbf{n} d\ell \\ &= \int_{\ell} \left(\lambda_1 D_1 x^{D_1-1}\right) \widehat{\mathrm{O}}_x dx + \left(\lambda_2 D_2 y^{D_2-1}\right) \widehat{\mathrm{O}}_y dy + \left(\lambda_3 D_3 z^{D_3-1}\right) \widehat{\mathrm{O}}_z dz,\end{aligned} \tag{34}$$

where the element of the scaling-law line is

$$\begin{aligned} d\mathbf{l} &= \mathbf{n} d\ell \\ &= i\left(\lambda_1 D_1 x^{D_1-1}\right) dx + j\left(\lambda_2 D_2 y^{D_2-1}\right) dy + k\left(\lambda_3 D_3 z^{D_3-1}\right) dz \\ &= id\left(\lambda_1 x^{D_1}\right) + jd\left(\lambda_2 y^{D_2}\right) + kd\left(\lambda_3 z^{D_3}\right)\end{aligned} \tag{35}$$

with the unit vector $\mathbf{n}$ tangent to the scaling-law vector line $\mathbf{l}$.

From (34) we give

$$\Pi = \int_a^b \left[\left(\lambda_1 D_1 x^{D_1-1}\right) \widehat{\mathrm{O}}_x \frac{dx}{dt} + \left(\lambda_2 D_2 y^{D_2-1}\right) \widehat{\mathrm{O}}_y \frac{dy}{dt} + \left(\lambda_3 D_3 z^{D_3-1}\right) \widehat{\mathrm{O}}_z \frac{dz}{dt}\right] dt \tag{36}$$

since

$$\begin{aligned}&\left(\lambda_1 D_1 x^{D_1-1}\right) \widehat{\mathrm{O}}_x dx + \left(\lambda_2 D_2 y^{D_2-1}\right) \widehat{\mathrm{O}}_y dy + \left(\lambda_3 D_3 z^{D_3-1}\right) \widehat{\mathrm{O}}_z dz \\ &= \left[\left(\lambda_1 D_1 x^{D_1-1}\right) \widehat{\mathrm{O}}_x \tfrac{dx}{dt} + \left(\lambda_2 D_2 y^{D_2-1}\right) \widehat{\mathrm{O}}_y \tfrac{dy}{dt} + \left(\lambda_3 D_3 z^{D_3-1}\right) \widehat{\mathrm{O}}_z \tfrac{dz}{dt}\right] dt.\end{aligned} \tag{37}$$

Let $S = S\left(\lambda_1 x^{D_1}, \lambda_2 y^{D_2}\right)$.

The scaling-law double integral of the scaling-law scalar field X on the region S, denoted by $\mathrm{M}(X)$, is defined as

$$\begin{aligned}\mathrm{M}(X) &= \iint_S X dS \\ &= \iint_S X\left(\lambda_1 D_1 x^{D_1-1}\right)\left(\lambda_2 D_2 y^{D_2-1}\right) dxdy \\ &= \iint_S \mathrm{T} d\left(\lambda_1 x^{D_1}\right) d\left(\lambda_2 y^{D_2}\right),\end{aligned} \tag{38}$$

where $dS = \left(\lambda_1 D_1 x^{D_1-1}\right)\left(\lambda_2 D_2 y^{D_2-1}\right) dxdy = d\left(\lambda_1 x^{D_1}\right) d\left(\lambda_2 y^{D_2}\right)$ is the element of the scaling-law area.

Thus, we have that

$$\begin{aligned}
&\mathrm{M}\left(X\right) = \iint\limits_{S} XdS \\
&= \int\limits_{c}^{d} \left[\int\limits_{a}^{b} X \left(\lambda_1 D_1 x^{D_1-1}\right) dx \right] \left(\lambda_2 D_2 y^{D_2-1}\right) dy \\
&= \int\limits_{a}^{b} \left[\int\limits_{c}^{d} X \left(\lambda_2 D_2 y^{D_2-1}\right) dy \right] \left(\lambda_1 D_1 x^{D_1-1}\right) dx \\
&= \int\limits_{c}^{d} \left[\int\limits_{a}^{b} Xd\left(\lambda_1 x^{D_1}\right) \right] d\left(\lambda_2 y^{D_2}\right) = \int\limits_{a}^{b} \left[\int\limits_{c}^{d} Xd\left(\lambda_2 y^{D_2}\right) \right] d\left(\lambda_1 x^{D_1}\right),
\end{aligned} \tag{39}$$

where $x \in [a,b]$ and $y \in [c,d]$.

The scaling-law volume integral of the scaling-law scalar field X in the domain Ω is defined as

$$\begin{aligned}
&V\left(X\right) = \iiint\limits_{\Omega} XdV \\
&= \iiint\limits_{\Omega} X \left(\lambda_1 D_1 x^{D_1-1}\right) \left(\lambda_2 D_2 y^{D_2-1}\right) \left(\lambda_3 D_3 z^{D_3-1}\right) dxdydz \\
&= \iiint\limits_{\Omega} Xd\left(\lambda_1 x^{D_1}\right) d\left(\lambda_2 y^{D_2}\right) d\left(\lambda_3 z^{D_3}\right),
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
dV &= \left(\lambda_1 D_1 x^{D_1-1}\right) \left(\lambda_2 D_2 y^{D_2-1}\right) \left(\lambda_3 D_3 z^{D_3-1}\right) dxdydz \\
&= d\left(\lambda_1 x^{D_1}\right) d\left(\lambda_2 y^{D_2}\right) d\left(\lambda_3 z^{D_3}\right)
\end{aligned}$$

is the element of volume.

Thus, we have that

$$\begin{aligned}
&\iiint\limits_{\Omega} XdV = \int\limits_{f}^{g} \left[\int\limits_{c}^{d} \left(\int\limits_{a}^{b} X \left(\lambda_1 D_1 x^{D_1-1}\right) dx \right) \left(\lambda_2 D_2 y^{D_2-1}\right) dy \right] \left(\lambda_3 D_3 z^{D_3-1}\right) dz \\
&= \int\limits_{f}^{g} \left[\int\limits_{a}^{b} \left(\int\limits_{c}^{d} X \left(\lambda_2 D_2 y^{D_2-1}\right) dy \right) \left(\lambda_1 D_1 x^{D_1-1}\right) dx \right] \left(\lambda_3 D_3 z^{D_3-1}\right) dz \\
&= \int\limits_{a}^{b} \left[\int\limits_{c}^{d} \left(\int\limits_{e}^{f} X \left(\lambda_3 D_3 z^{D_3-1}\right) dz \right) \left(\lambda_2 D_2 y^{D_2-1}\right) dy \right] \left(\lambda_1 D_1 x^{D_1-1}\right) dx \\
&= \int\limits_{f}^{g} \left[\int\limits_{c}^{d} \left(\int\limits_{a}^{b} Xd\left(\lambda_1 x^{D_1}\right) \right) d\left(\lambda_2 y^{D_2}\right) \right] d\left(\lambda_3 z^{D_3}\right) \\
&= \int\limits_{f}^{g} \left[\int\limits_{a}^{b} \left(\int\limits_{c}^{d} Xd\left(\lambda_2 y^{D_2}\right) \right) d\left(\lambda_1 x^{D_1}\right) \right] d\left(\lambda_3 z^{D_3}\right) \\
&= \int\limits_{a}^{b} \left[\int\limits_{c}^{d} \left(\int\limits_{e}^{f} Xd\left(\lambda_3 z^{D_3}\right) \right) d\left(\lambda_2 y^{D_2}\right) \right] d\left(\lambda_1 x^{D_1}\right),
\end{aligned} \tag{41}$$

where $x \in [a,b]$, $y \in [c,d]$ and $z \in [f,g]$.

Let the scaling-law surface be defined by $\mathbf{S} = \mathbf{S}\left(\lambda_1 x^{D_1}, \lambda_2 y^{D_2}, \lambda_3 z^{D_3}\right)$.

The scaling-law surface integral of the scaling-law vector field $\widehat{\mathrm{O}}$ on the scaling-law surface $\partial\Omega$ of the domain Ω is defined as

$$\iint\limits_{\partial\Omega} \widehat{\mathrm{O}} \cdot \mathbf{dS} = \iint\limits_{\partial\Omega} \widehat{\mathrm{O}} \cdot \mathbf{a} dS, \tag{42}$$

where $\mathbf{a} = \mathbf{dS}/\left|\mathbf{dS}\right| = \mathbf{dS}/dS$ is the unit normal vector to the scaling-law surface $\partial\Omega$ with $dS = |\mathbf{dS}|$, and

$$\begin{aligned}
\mathbf{dS} &= id\left(\lambda_2 y^{D_2}\right) d\left(\lambda_3 z^{D_3}\right) + jd\left(\lambda_1 x^{D_1}\right) d\left(\lambda_3 z^{D_3}\right) + kd\left(\lambda_1 x^{D_1}\right) d\left(\lambda_2 y^{D_2}\right) \\
&= i\left(\lambda_2 D_2 y^{D_2-1}\right)\left(\lambda_3 D_3 z^{D_3-1}\right) dydz + j\left(\lambda_1 D_1 x^{D_1-1}\right)\left(\lambda_3 D_3 z^{D_3-1}\right) dxdz \\
&\quad + k\left(\lambda_1 D_1 x^{D_1-1}\right)\left(\lambda_2 D_2 y^{D_2-1}\right) dxdy
\end{aligned} \tag{43}$$

is the element of the scaling-law surface.

From (42) and (43) we present

$$\begin{aligned}
&\iint\limits_{\partial\Omega} \widehat{\mathrm{O}} \cdot \mathbf{dS} \\
&= \iint\limits_{\partial\Omega} \widehat{\mathrm{O}}_x d\left(\lambda_2 y^{D_2}\right) d\left(\lambda_3 z^{D_3}\right) + \widehat{\mathrm{O}}_y d\left(\lambda_1 x^{D_1}\right) d\left(\lambda_3 z^{D_3}\right) + \widehat{\mathrm{O}}_z d\left(\lambda_1 x^{D_1}\right) d\left(\lambda_2 y^{D_2}\right) \\
&= \iint\limits_{\partial\Omega} \widehat{\mathrm{O}}_x \left(\lambda_2 D_2 y^{D_2-1}\right)\left(\lambda_3 D_3 z^{D_3-1}\right) dydz + \iint\limits_{\partial\Omega} \widehat{\mathrm{O}}_y \left(\lambda_1 D_1 x^{D_1-1}\right)\left(\lambda_3 D_3 z^{D_3-1}\right) dxdz \\
&+ \iint\limits_{\partial\Omega} \widehat{\mathrm{O}}_z \left(\lambda_1 D_1 x^{D_1-1}\right)\left(\lambda_2 D_2 y^{D_2-1}\right) dxdy.
\end{aligned} \tag{44}$$

The flux of the scaling-law vector field $\widehat{\mathrm{O}}$ across the scaling-law surface $\partial\Omega$, denoted by $G\left(\widehat{\mathrm{O}}\right)$, is defined as

$$G\left(\widehat{\mathrm{O}}\right) = \oiint\limits_{\partial\Omega} \widehat{\mathrm{O}} \cdot \mathbf{dS}. \tag{45}$$

The scaling-law divergence of the scaling-law vector field $\widehat{\mathrm{O}}$ is defined as

$$\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \widehat{\mathrm{O}} = \lim_{\Delta V_m \to 0} \frac{1}{\Delta V_m} \oiint\limits_{\Delta\partial\Omega_m} \widehat{\mathrm{O}} \cdot \mathbf{dS}, \tag{46}$$

where the scaling-law volume V is divided into a large number of small sub-volumes ΔV_m with the scaling-law surfaces $\Delta\Omega_m$, and $\mathbf{dS}$ is the element of the scaling-law surface $\partial\Omega$ bounding the solid Ω.

Here, (17) is equal to (46) in the Cartesian-type coordinate system.

The scaling-law curl of the scaling-law vector field $\widehat{\mathrm{O}}$ is defined as

$$(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \times \widehat{\mathrm{O}}) \cdot \mathbf{n} = \lim_{\Delta S_m \to 0} \frac{1}{\Delta S_m} \oint\limits_{\Delta \ell_m} \widehat{\mathrm{O}} \cdot d\mathbf{l}, \tag{47}$$

where $d\mathbf{l}$ is the element of the scaling-law vector line, ΔS_m is a small scaling-law surface element perpendicular to $\mathbf{n}$, $\Delta \ell_m$ is the closed curve of the boundary of ΔS_m, and $\mathbf{n}$ is oriented in a positive sense.

Here, (18) is (47) in the Cartesian-type coordinate system.

From (46) we present the scaling-law Gauss-Ostrogradsky-like theorem for the scaling-law vector calculus as follows.

Let us consider that

$$\begin{aligned}
&\oiint\limits_{\partial\Omega} \widehat{\mathrm{O}} \cdot \mathbf{n} dS = \oiint\limits_{\partial\Omega} \widehat{\mathrm{O}} \cdot \mathbf{dS} \\
&= \oiint\limits_{\partial\Omega} \widehat{\mathrm{O}}_x d\left(\lambda_2 y^{D_2}\right) d\left(\lambda_3 z^{D_3}\right) + \widehat{\mathrm{O}}_y d\left(\lambda_1 x^{D_1}\right) d\left(\lambda_3 z^{D_3}\right) + \widehat{\mathrm{O}}_z d\left(\lambda_1 x^{D_1}\right) d\left(\lambda_2 y^{D_2}\right) \\
&= \oiint\limits_{\partial\Omega} \widehat{\mathrm{O}}_x \left(\lambda_2 D_2 y^{D_2-1}\right)\left(\lambda_3 D_3 z^{D_3-1}\right) dydz + \oiint\limits_{\partial\Omega} \widehat{\mathrm{O}}_y \left(\lambda_1 D_1 x^{D_1-1}\right)\left(\lambda_3 D_3 z^{D_3-1}\right) dxdz \\
&+ \oiint\limits_{\partial\Omega} \widehat{\mathrm{O}}_z \left(\lambda_1 D_1 x^{D_1-1}\right)\left(\lambda_2 D_2 y^{D_2-1}\right) dxdy.
\end{aligned} \tag{48}$$

The scaling-law Gauss-Ostrogradsky-like theorem for the scaling-law vector calculus states that

$$\oiiint\limits_{\Omega} \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \widehat{\mathrm{O}} dV = \oiint\limits_{\partial\Omega} \widehat{\mathrm{O}} \cdot \mathbf{a} dS \tag{49}$$

or

$$\oiiint\limits_{\Omega} \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \widehat{\mathrm{O}} dV = \oiint\limits_{\partial\Omega} \widehat{\mathrm{O}} \cdot \mathbf{dS}. \tag{50}$$

When $D_1 = D_2 = D_3 = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$, (49) becomes the Gauss-Ostrogradsky theorem, proposed by Gauss in 1813 [27] and by Ostrogradsky in 1828 [28].

From (47) we present the scaling-law Stokes-like theorem for the scaling-law vector calculus as follows.

We now consider that

$$\begin{aligned}
&\oint_{\ell} \widehat{\mathrm{O}} \cdot \mathbf{dl} \\
&= \oint_{\ell} \widehat{\mathrm{O}}_x \left(\lambda_1 D_1 x^{D_1-1}\right) dx + \widehat{\mathrm{O}}_y \left(\lambda_2 D_2 y^{D_2-1}\right) dy + \widehat{\mathrm{O}}_z \left(\lambda_3 D_3 z^{D_3-1}\right) dz \\
&= \oint_{\ell} \widehat{\mathrm{O}}_x d\left(\lambda_1 x^{D_1}\right) + \widehat{\mathrm{O}}_y d\left(\lambda_2 y^{D_2}\right) + \widehat{\mathrm{O}}_z d\left(\lambda_3 z^{D_3}\right).
\end{aligned} \tag{51}$$

The scaling-law Stokes-like theorem for the scaling-law vector calculus states that

$$\oiint_{\partial\Omega} \left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \times \widehat{\mathrm{O}} \right) \cdot \mathbf{a} dS = \oint_{\ell} \widehat{\mathrm{O}} \cdot \mathbf{dl} \tag{52}$$

or

$$\oiint_{\partial\Omega} \left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \times \widehat{\mathrm{O}} \right) \cdot \mathbf{dS} = \oint_{\ell} \widehat{\mathrm{O}} \cdot \mathbf{dl}. \tag{53}$$

Here, when $D_1 = D_2 = D_3 = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$, (52) is the Stokes theorem, proposed by Stokes in 1845 [29].

With use of (52) and (53), we present the scaling-law Green-like theorem and identities for the scaling-law vector calculus as follows.

The scaling-law Green-like theorem for the scaling-law vector calculus states

$$\begin{aligned}
&\oint_{\ell} \left(\lambda_1 D_1 x^{D_1-1}\right) \widehat{\mathrm{O}}_x dx + \left(\lambda_2 D_2 y^{D_2-1}\right) \widehat{\mathrm{O}}_y dy \\
&= \iint_{S} \left(\left(\lambda_1 D_1 x^{D_1-1}\right) {}^{SL}\partial_x^{(1)} \widehat{\mathrm{O}}_y - \left(\lambda_2 D_2 y^{D_2-1}\right) {}^{SL}\partial_y^{(1)} \widehat{\mathrm{O}}_x \right) \left(\lambda_1 D_1 x^{D_1-1}\right) \left(\lambda_2 D_2 y^{D_2-1}\right) dxdy,
\end{aligned} \tag{54}$$

where S is the domain bounded by the scaling-law contour ℓ.

The scaling-law Green-like identity of first type via scaling-law vector calculus states that

$$\iiint_{\Omega} \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \left(X \nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} Y + \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} Y \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X \right) dV$$
$$= \oiint_{\partial\Omega} X \partial_u^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} Y dS. \tag{55}$$

The scaling-law Green-like identity of second type via scaling-law vector calculus states that

$$\iiint_{\Omega} \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \left(X \nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} Y - Y \nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X \right) dV$$
$$= \oiint_{\partial\Omega} \left(X \partial_u^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} Y - Y \partial_u^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} X \right) dS. \tag{56}$$

Here, the Green theorem and identities, proposed by Green in 1828 [30], are the special cases of the Green-like theorem and identities when $D_1 = D_2 = D_3 = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

4 On the Navier-Stokes-type equations of the scaling-law flow

Let us consider the coordinate system, defined as

$$\left(\lambda_0 t^{D_0}, \lambda_1 x^{D_1}, \lambda_2 y^{D_2}, \lambda_3 z^{D_3}\right) = \lambda_0 t^{D_0} + i\lambda_1 x^{D_1} + j\lambda_2 y^{D_2} + k\lambda_3 z^{D_3}, \tag{57}$$

where i, j and k are the unit vector, and $\lambda_0 t^{D_0}$ is the fractal scaling law [31] with the normalization constant $\lambda_0 \geq 0$, the time $t \geq 0$, and the scaling exponent $-\infty < D_0 < \infty$.

Let $\Xi = \Xi\left(\lambda_0 t^{D_0}, \lambda_1 x^{D_1}, \lambda_2 y^{D_2}, \lambda_3 z^{D_3}\right)$ be the scaling-law scalar fluid field.

The total scaling-law differential of the scaling-law scalar field is given as follows:

$$d\Xi = \left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}\Xi dt + \left(\lambda_1 D_1 x^{D_1-1}\right)\partial_x^{(1)}\Xi dx + \left(\lambda_2 D_2 y^{D_2-1}\right)\partial_y^{(1)}\Xi dy$$
$$+\left(\lambda_3 D_3 z^{D_3-1}\right)\partial_z^{(1)}\Xi dz, \tag{58}$$

which leads to

$$\tfrac{d\Xi}{dt} = \left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}\Xi + \left(\lambda_1 D_1 x^{D_1-1}\right)\partial_x^{(1)}\Xi \tfrac{dx}{dt} + \left(\lambda_2 D_2 y^{D_2-1}\right)\partial_y^{(1)}\Xi \tfrac{dy}{dt}$$
$$+\left(\lambda_3 D_3 z^{D_3-1}\right)\partial_z^{(1)}\Xi \tfrac{dz}{dt}. \tag{59}$$

From (59) the material scaling-law derivative of the scaling-law fluid density Ξ is defined as

$$\frac{D\Xi}{Dt} = \left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}\Xi + \upsilon\cdot\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\Xi, \tag{60}$$

where $\upsilon = (\partial x/\partial t, \partial y/\partial t, \partial z/\partial t) = i\upsilon_x + j\upsilon_y + k\upsilon_z$ are denoted as the velocity vector.

When $D_0 = D_1 = D_2 = D_3 = 1$ and $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1$, (58) is the Euler notation of the material derivative [32], and (60) is the Stokes notation of the material derivative [33,34].

From (60) the transport theorem for the scaling-law flow can be given as follows:

$$\frac{D}{Dt}\iiint\limits_{\Omega(t)} \Xi dV = \iiint\limits_{\Omega(t)} \left[\left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}\Xi + \upsilon\cdot\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\Xi\right] dV, \tag{61}$$

which, by using (49), yields that

$$\frac{D}{Dt}\iiint\limits_{\Omega(t)} \Xi dV = \oiint\limits_{\Omega(t)} \left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}\Xi dV + \oiint\limits_{\partial\Omega(t)} \Xi\upsilon\cdot\mathbf{dS} \tag{62}$$

since

$$\iiint\limits_{\Omega(t)} \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \Xi dV = \oiint\limits_{\partial\Omega(t)} \Xi \left(\upsilon \cdot \mathbf{a}\right) dS = \oiint\limits_{\partial\Omega(t)} \Xi \upsilon \cdot \mathbf{dS}, \tag{63}$$

where $\partial\Omega\left(t\right)$ is the surface of $\Omega\left(t\right)$, $\mathbf{a}$ is the unit normal to the scaling-law surface, and υ is the velocity vector.

Taking $D_0 = D_1 = D_2 = D_3 = 1$ and $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1$, we obtain the Reynolds transport theorem [35].

The conservation of the mass of the scaling-law flow is given as

$$\left(\lambda_0 D_0 t^{D_0 - 1}\right) \partial_t^{(1)} \rho + \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \rho = 0 \tag{64}$$

or

$$\left(\lambda_0 D_0 t^{D_0 - 1}\right) \partial_t^{(1)} \rho + \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \left(\upsilon \rho\right) = 0 \tag{65}$$

because

$$\frac{D}{Dt} \iiint\limits_{\Omega(t)} \rho dV = \iiint\limits_{\Omega(t)} \left[\left(\lambda_0 D_0 t^{D_0 - 1}\right) \partial_t^{(1)} \rho + \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \rho \right] dV, \tag{66}$$

which is derived from the mass of the scaling-law flow, defined as

$$\mathrm{M} = \iiint\limits_{\Omega(t)} \rho dV \tag{67}$$

where ρ and M are the density and mass of the scaling-law flow, respectively.

Here, for $D_0 = D_1 = D_2 = D_3 = 1$ and $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1$, (64) is the conservation of the mass [32].

The Cauchy-type strain tensor for the scaling-law flow, denoted by ϖ, is defined as

$$\varpi = \frac{1}{2}\left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \upsilon + \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\right). \tag{68}$$

From $D_0 = D_1 = D_2 = D_3 = 1$ and $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1$, (68) becomes the Cauchy strain tensor [36], and can be applied to describe the power-law strain [37].

The scaling-law Stokes-type strain tensor for the scaling-law flow, denoted byω, is defined as

$$\omega = \frac{1}{2}\left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \upsilon - \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\right). \tag{69}$$

The scaling-law Stokes-type velocity gradient tensor for the scaling-law flow, denoted by $\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \upsilon$, is presented as follows:

$$\begin{aligned} &\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \upsilon = \omega + \Lambda \\ &= \tfrac{1}{2}\left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \upsilon + \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\right) \\ &+\tfrac{1}{2}\left(\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \upsilon - \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\right). \end{aligned} \tag{70}$$

The stress tensor for the scaling-law flow, denoted by $\mathbf{U}$, is defined as

$$\mathbf{U} = -p\mathbf{I} + 2\mu\mathbf{h}, \tag{71}$$

where μ is the shear moduli of the viscosity, and $\mathbf{I}$ is the unit tensor.

Here, (69) and (70) are the generalized cases of the Stokes strain tensor and Stokes velocity gradient tensor [33], when $D_0 = D_1 = D_2 = D_3 = 1$ and $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1$.

The conservation of the momentums for the scaling-law flow is given as follows:

$$\frac{D}{Dt}\iiint\limits_{\Omega(t)} \rho \upsilon dV = \iiint\limits_{\Omega(t)} \mathbf{W} dV + \oiint\limits_{\mathbf{S}(t)} \mathbf{U} \cdot \mathbf{dS} \tag{72}$$

where $\mathbf{W}$ represents the specific body force.

Therefore, we have that

$$\left(\lambda_0 D_0 t^{D_0-1}\right) \partial_t^{(1)} (\rho \upsilon) + \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} (\rho \upsilon) = \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \mathbf{U} + \mathbf{W} \tag{73}$$

since

$$\begin{aligned} &\iiint\limits_{\Omega(t)} \left(\left(\lambda_0 D_0 t^{D_0-1}\right) \partial_t^{(1)} (\rho \upsilon) + \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} (\rho \upsilon) \right) dV \\ &= \iiint\limits_{\Omega(t)} \left(\mathbf{W} + \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \mathbf{U} \right) dV, \end{aligned} \tag{74}$$

where

$$\frac{D}{Dt}\iiint\limits_{\Omega(t)} \rho \upsilon dV = \iiint\limits_{\Omega(t)} \left[\left(\lambda_0 D_0 t^{D_0-1}\right) \partial_t^{(1)} (\rho \upsilon) + \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} (\rho \upsilon) \right] dV \tag{75}$$

and

$$\oiint_{\mathbf{S}(t)} \mathbf{U} \cdot \mathbf{dS} = \iiint_{\Omega(t)} \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \mathbf{U} dV. \tag{76}$$

From (71) we have

$$\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \mathbf{U} = -\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} p + \mu \nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \upsilon \tag{77}$$

such that

$$\begin{aligned} &\oiint_{\mathbf{S}(t)} \mathbf{U} \cdot \mathbf{dS} = \iiint_{\Omega(t)} \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \mathbf{U} dV \\ &= -\iiint_{\Omega(t)} \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} p dV + \iiint_{\Omega(t)} \mu \nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \upsilon dV. \end{aligned} \tag{78}$$

It follows from (78) that

$$\begin{aligned} &\frac{D}{Dt} \iiint_{\Omega(t)} (\rho \upsilon) dV = \iiint_{\Omega(t)} \mathbf{W} dV + \oiint_{\mathbf{S}(t)} \mathbf{U} \cdot \mathbf{dS} \\ &= -\iiint_{\Omega(t)} \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} p dV + \iiint_{\Omega(t)} \mu \nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \upsilon dV + \iiint_{\Omega(t)} \mathbf{W} dV. \end{aligned} \tag{79}$$

From (66) we obtain

$$
\begin{aligned}
&\frac{D}{Dt}\iiint\limits_{\Omega(t)}(\rho\upsilon)dV \\
&= \iiint\limits_{\Omega(t)}\left[\left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}(\rho\upsilon) + \upsilon\cdot\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}(\rho\upsilon)\right]dV \\
&= -\iiint\limits_{\Omega(t)}\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} p dV + \iiint\limits_{\Omega(t)}\mu\nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\upsilon dV + \iiint\limits_{\Omega(t)}\mathbf{W}dV \\
&= \iiint\limits_{\Omega(t)}\left[-\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} p + \mu\nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\upsilon + \mathbf{W}\right]dV.
\end{aligned}
\tag{80}
$$

Therefore, from (66) we present

$$
\begin{aligned}
&\iiint\limits_{\Omega(t)}\left[\left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}(\rho\upsilon) + \upsilon\cdot\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}(\rho\upsilon)\right]dV \\
&= \iiint\limits_{\Omega(t)}\left[-\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} p + \mu\nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\upsilon + \mathbf{W}\right]dV,
\end{aligned}
\tag{81}
$$

which leads to

$$
\begin{aligned}
&\left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}(\rho\upsilon) + \upsilon\cdot\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}(\rho\upsilon) = -\nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} p\;{}_{l} \\
&+\mu\nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}}\upsilon + \mathbf{W}.
\end{aligned}
\tag{82}
$$

In view of (82), we obtain

$$\rho\left(\left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}\upsilon+\upsilon\cdot\nabla^{\begin{pmatrix}D_1,D_2,D_3\\ \lambda_1,\lambda_2,\lambda_3\end{pmatrix}}\upsilon\right)$$
$$=-\nabla^{(\alpha,\beta,\gamma)}p+\mu\nabla^{\begin{pmatrix}2D_1,2D_2,2D_3\\ \lambda_1,\lambda_2,\lambda_3\end{pmatrix}}\upsilon+\mathbf{W},\tag{83}$$

which yields that

$$\left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}\upsilon+\upsilon\cdot\nabla^{\begin{pmatrix}D_1,D_2,D_3\\ \lambda_1,\lambda_2,\lambda_3\end{pmatrix}}\upsilon$$
$$=-\tfrac{1}{\rho}\nabla^{(\alpha,\beta,\gamma)}p+\tfrac{\mu}{\rho}\nabla^{\begin{pmatrix}2D_1,2D_2,2D_3\\ \lambda_1,\lambda_2,\lambda_3\end{pmatrix}}\upsilon+\tfrac{\mathbf{W}}{\rho}.\tag{84}$$

From (65) we give

$$\nabla^{\begin{pmatrix}D_1,D_2,D_3\\ \lambda_1,\lambda_2,\lambda_3\end{pmatrix}}\cdot\upsilon=0.\tag{85}$$

Thus, the scaling-law Navier-Stokes-type equations of the scaling-law flows can be written as follows:

$$\begin{cases}\left(\lambda_0 D_0 t^{D_0-1}\right)\partial_t^{(1)}(\rho\upsilon)+\upsilon\cdot\nabla^{\begin{pmatrix}D_1,D_2,D_3\\ \lambda_1,\lambda_2,\lambda_3\end{pmatrix}}(\rho\upsilon)\\ =-\nabla^{\begin{pmatrix}D_1,D_2,D_3\\ \lambda_1,\lambda_2,\lambda_3\end{pmatrix}}p+\mu\nabla^{\begin{pmatrix}2D_1,2D_2,2D_3\\ \lambda_1,\lambda_2,\lambda_3\end{pmatrix}}\upsilon+\mathbf{W},\\ \nabla^{\begin{pmatrix}D_1,D_2,D_3\\ \lambda_1,\lambda_2,\lambda_3\end{pmatrix}}\cdot\upsilon=0,\end{cases}\tag{86}$$

or

$$\begin{cases} \left(\lambda_0 D_0 t^{D_0-1}\right) \partial_t^{(1)} \upsilon + \upsilon \cdot \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \upsilon \\ = -\frac{1}{\rho} \nabla^{(\alpha,\beta,\gamma)} p + \frac{\mu}{\rho} \nabla^{\begin{pmatrix} 2D_1, 2D_2, 2D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \upsilon + \frac{\mathbf{w}}{\rho}, \\ \nabla^{\begin{pmatrix} D_1, D_2, D_3 \\ \lambda_1, \lambda_2, \lambda_3 \end{pmatrix}} \cdot \upsilon = 0. \end{cases} \tag{87}$$

On putting $D_0 = D_1 = D_2 = D_3 = 1$ and $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1$ in (86) and (87), we obtain the Navier-Stokes equations [33,38].

5 Conclusion

In the present study, the scaling-law vector calculus, which is connected between the vector calculus and fractal geometry, was proposed due to the calculus with respect to monotone functions. The works of Gauss-Ostrogradsky, Stokes and Green were extended based on the scaling-law vector calculus. Making use of the material scaling-law derivative and transport theorem of the scaling-law flows, the conservations of the mass and momentums for the scaling-law flow were considered, and the scaling-law Navier-Stokes-type equations of the scaling-law flows were discussed in detail. The obtained formulas are efficient and accurate for solving the challenge for the scaling-law flows.

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