On the dynamics of the singularly perturbed Logistic difference equation with two different continuous arguments

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Abstract

Here, we study the dynamics of the singularly perturbed logistic difference equation with two different continuous arguments. First of all, local stability of the fixed points is investigated by analyzing the corresponding characteristic equations of the linearized equations. Secondly, we illustrate that the considered system exhibits Hopf bifurcation. A discretized analogue of the original system is obtained using the method of steps. Local stability and bifurcation analysis of the discretized system are investigated. Explicit conditions for the occurrence of a variety of complex dynamics such as fold and Neimark-Sacker bifurcations are reached. We compare the results with those of the associated difference equation with continuous argument when the perturbation parameter $\epsilon \longrightarrow 0$ and with those of the logistic delay differential equation with two different delays when $\epsilon \longrightarrow 1$. Finally, numerical simulations including Lyapunov exponent, bifurcation diagrams and phase portraits are carried out to confirm the theoretical analysis obtained and to illustrate more complex dynamics of the system.

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Abstract

Here, we study the dynamics of the singularly perturbed logistic difference equation with two different continuous arguments. First of all, local stability of the fixed points is investigated by analyzing the corresponding characteristic equations of the linearized equations. Secondly, we illustrate that the considered system exhibits Hopf bifurcation. A discretized analogue of the original system is obtained using the method of steps. Local stability and bifurcation analysis of the discretized system are investigated. Explicit conditions for the occurrence of a variety of complex dynamics such as fold and Neimark-Sacker bifurcations are reached. We compare the results with those of the associated difference equation with continuous argument when the perturbation parameter $\epsilon \longrightarrow 0$ and with those of the logistic delay differential equation with two different delays when $\epsilon \longrightarrow 1$. Finally, numerical simulations including Lyapunov exponent, bifurcation diagrams and phase portraits are carried out to confirm the theoretical analysis obtained and to illustrate more complex dynamics of the system.

Keywords: Logistic equation, Singular perturbation, Local stability, Bifurcation, Chaos.

1 Introduction

In many problems it is meaningless not to have dependence on the past. Time delays occur so often that to ignore them is to ignore reality [19]. A singularly perturbed equation is a differential equation involving at least one delay term and the highest derivative is multiplied by a small parameter [3, 14, 27, 30]. It arises in applications where delays and perturbations play a role [2, 6, 23, 26].
One of the principal mathematical instruments of modern nonlinear dynamics is

the difference equation with continuous argument given in the form

$$x(t) = f(x(t-1)), \qquad\qquad t \in [0, T]. \tag{1.1}$$

Let $\epsilon \in (0, 1]$, the equation

$$\epsilon \frac{dx}{dt} + x(t) = f(x(t-1)), \qquad\qquad t \in [0, T],$$

is considered as a singular perturbation of the difference equation with continuous argument(1.1) [16, 17]. We can consider the left hand side of (1.2) as an approximation to the function $x(t + \epsilon)$:

$$x(t + \epsilon) = x(t) + \epsilon \frac{dx}{dt} + ... \quad .$$

So, it is reasonable to expect that singular perturbed equations to behave as its associated difference equation with continuous argument when the perturbation parameter $\epsilon \longrightarrow 0$ [18].
Models which have only one delay are often used when the other delays are small and insignificant to dynamical behaviors [22]. However, this assumption may not be applicable in many cases. Furthermore, there are systems that single delay can not stabilize, however, adding a second delay can stabilize the same system [25]. Therefore, models with multiple delays are of great interest. These equations have significant physical and biological background and exhibit rich dynamics [4, 5, 11–13, 15, 21, 28, 29].

Consider the Logistic difference equation with two different continuous arguments

$$x(t) = \rho x(t-1)(1 - x(t-2)), \qquad\qquad t \in [0, T] \tag{1.2}$$

and its singularly perturbed equation given in the form

$$\epsilon \frac{dx}{dt} + x(t) = \rho x(t-1)(1 - x(t-2)) \qquad\qquad t \in [0, T] \tag{1.3}$$

with the initial condition

$$x = x_0, \qquad\qquad t \leq 0.$$

In this paper, local stability of the fixed points of (1.3) is investigated by analyzing the corresponding characteristic equations of the linearized equations. Secondly, we illustrate that the considered equation (1.3) exhibits Hopf bifurcation. A discretized analogue of (1.3) is obtained using the method of steps. Local stability and bifurcation analysis of the discretized system are investigated. Also, we compare the results of (1.3) when the perturbation parameter $\epsilon \longrightarrow 1$ with the results that introduced in [10] concerning with the logistic differential difference equation with two different delays

$$\frac{dx}{dt} = -x(t) + \rho x(t-1)(1 - x(t-2)), \qquad t \in [0, T],$$

$$x(t) = x_0, \qquad t \leq 0.$$

In section 3, numerical simulations are carried out to confirm the theoretical analysis obtained and to illustrate more complex dynamics of the system.

2 Main results

2.1 Existence and uniqueness

Theorem 1. *Problem (1.3) has a unique solution $x \in C[0, T]$, $0 \leq x(t) \leq 1$.*

Proof. Define the operator F from $C[0, T]$ into $C[0, T]$ by

$$Fx(t) = x_0 e^{\frac{-t}{\epsilon}} + \frac{\rho}{\epsilon} \int_0^t e^{\frac{s-t}{\epsilon}} x(s-1)(1 - x(s-2)) ds.$$

Now we want to show that F is contraction :

$$|Fx - Fy| = \frac{\rho}{\epsilon} \int_1^t e^{\frac{s-t}{\epsilon}} |x(s-1)(1 - x(s-2)) - y(s-1)(1 - y(s-2))| ds$$

$$\leq \frac{\rho}{\epsilon} \int_1^t e^{\frac{s-t}{\epsilon}} [|x(s-1) - y(s-1)| + |x(s-2) - y(s-2)|] ds$$

$$\leq \frac{\rho}{\epsilon} [\max_{[1,T]} |x(s-1) - y(s-1)| \int_1^t e^{\frac{s-t}{\epsilon}} ds + \max_{[1,T]} |x(s-2) - y(s-2)| \int_1^t e^{\frac{s-t}{\epsilon}} ds].$$

Hence,

$$\begin{aligned} \|Fx - Fy\|_{[0,T]} &\leq \rho \|x - y\|_{[0,T]} (2 - e^{\frac{-(T-1)}{\epsilon}} - e^{\frac{-(T-1)}{\epsilon}}) \\ &\leq 2\rho \|x - y\|_{[0,T]}. \end{aligned}$$

If $\rho < \frac{1}{2}$, then F is contraction map and the solution of (1.2) exists uniquely. $\square$

2.2 Local stability and Hopf bifurcation

There are two fixed points of (1.3) namely $(x_1)_{fix} = 0$ and $(x_2)_{fix} = 1 - \frac{1}{\rho}$.

It is easy to obtain the conditions for local asymptotic stability of the fixed point $(x_1)_{fix} = 0$ by checking the eigenvalues of the linearized system [24].

The linearized equation at the neighborhood of $(x_1)_{fix} = 0$ is

$$\epsilon \frac{dx}{dt} = -x(t) + \rho x(t-1). \tag{2.1}$$

Assuming a trial solution $x(t) = e^{\lambda t}$. Then the characteristic equation reads

$$\epsilon\lambda + 1 - \rho e^{-\lambda} = 0. \tag{2.2}$$

Lemma 1. *[14] All roots of the characteristic equation $\lambda + c + be^{-\lambda} = 0$, where c and b are real, have negative real parts if and only if*

$$c > -1, \quad c + b > 0, \quad b < \sqrt{c^2 + \xi^2},$$

where $\xi = -c\ tan\xi$, $0 < \xi < \pi$ if $c \neq 0$ and $\xi = \frac{\pi}{2}$ if $c = 0$.

Applying lemma 1 to to equation(2.2) with $c = \frac{1}{\epsilon}$, $b = \frac{-\rho}{\epsilon}$, we get the next theorem.

Theorem 2. *The fixed point $(x_1)_{fix} = 0$ of (1.3) is unstable if $\rho < \rho_0$ or $\rho > 1$ where $\rho_0 = -\sqrt{1 + (\epsilon\xi)^2}$, $\xi = \frac{-tan(\xi)}{\epsilon}$, $0 < \xi < \pi$, and is stable if $\rho_0 < \rho < 1$.*

Now we discuss the Hopf bifurcation.

Theorem 3. *When the parameter ρ passes through the critical value $\rho_1 = \epsilon\rho_0 = -\epsilon\sqrt{\frac{1}{(\epsilon)^2} + \xi^2}$, $\xi = \frac{-1}{\epsilon} tan\xi$, $0 < \xi < \pi$, there is a Hopf bifurcation.*

Proof. Assume that $\lambda = i\omega_0$, $\omega_0 \in R^+$ is a pure imaginary solution of (2.2) for some parameter value $\rho = \rho_*$. This leads to the following equations

$$i\omega_0 + \frac{1}{\epsilon} - \frac{\rho_*}{\epsilon} e^{-i\omega_0} = 0,$$

$$\frac{1}{\epsilon} - \frac{\rho_*}{\epsilon} cos(\omega_0) = 0,$$

$$\omega_0 + \frac{\rho_*}{\epsilon} sin(\omega_0) = 0,$$

$$\frac{1}{\epsilon} = \frac{\rho_*}{\epsilon} cos(\omega_0),$$

$$\omega_0 = \frac{\rho_*}{\epsilon} sin(\omega_0),$$

$$\omega_0^2 + \frac{1}{\epsilon^2} = \frac{\rho_*}{\epsilon^2}[cos(\omega_0)^2 + sin(\omega_0)^2] = \frac{\rho_*}{\epsilon^2},$$

$$\rho_* = \pm\epsilon\sqrt{\frac{1}{(\epsilon)^2} + \omega_0^2},$$

$$\omega_0 = \frac{-1}{\epsilon} tan\omega_0.$$

By Theorem 2, $\rho_* = -\epsilon\sqrt{\frac{1}{\epsilon^2} + \omega_0^2}$ is the critical value of ρ, where ω_0 is the root of $\omega_0 = \frac{-1}{\epsilon} tan\omega_0$, $0 < \omega_0 < \pi$.

The condition $\frac{d(Re(\lambda))}{d\rho} \mid_{\rho=\rho_*} \neq 0$ is the last condition for occurrence of a Hopf bifurcation.
To show that this condition is satisfied, let $\lambda = k(\rho) + i\omega(\rho)$ and using (2.2), we get

$$k + i\omega + \frac{1}{\epsilon} - \frac{\rho}{\epsilon} e^{-k-i\omega} = 0,$$

$$k + \frac{1}{\epsilon} - \frac{\rho}{\epsilon} e^{-k} cos\omega = 0, \tag{2.3}$$

$$\omega + \frac{\rho}{\epsilon} e^{-k} sin\omega = 0, \tag{2.4}$$

differentiate (2.3) and (2.4) with respect to ρ, we obtain

$$\epsilon\frac{dk}{d\rho} - e^{-k}cos(\omega) + \rho e^{-k}cos(\omega)\frac{dk}{d\rho} + \rho e^{-k}sin(\omega)\frac{d\omega}{d\rho} = 0,$$

$$\epsilon\frac{d\omega}{d\rho} + e^{-k}sin(\omega) + \rho e^{-k}cos(\omega)\frac{d\omega}{d\rho} - \rho e^{-k}sin(\omega)\frac{dk}{d\rho} = 0,$$

Solving for $\frac{dk}{d\rho}$, we obtain

$$\frac{d(Re(\lambda))}{d\rho} \mid_{\rho=\rho_*} = \frac{d(Re(\lambda))}{d\rho} \mid_{k=0,\omega=\omega_0,\rho=\rho_*},$$

$$= \frac{\epsilon cos(\omega_0) + \rho_*}{(\epsilon + \rho_* cos(\omega_0))^2 + (\rho_* sin(\omega_0))^2},$$

$$= \frac{\epsilon\rho_* cos(\omega_0) + \rho_*^2}{\rho_*[(\epsilon + \rho_* cos(\omega_0))^2 + (\rho_* sin(\omega_0))^2]},$$

$$= \frac{\epsilon + \rho_*^2}{\rho_*[(\epsilon + \rho_* cos(\omega_0))^2 + (\rho_* sin(\omega_0))^2]} \neq 0.$$

This completes the proof. □

The linearized equation at the neighborhood of $(x_2)_{fix} = 1 - \frac{1}{\rho}$ is

$$\epsilon\frac{dy}{dt} = -y(t) + y(t-1) - (\rho - 1)y(t-2), \tag{2.5}$$

where $y(t) = x(t) - (1 - \frac{1}{\rho})$.
The characteristic equation is of the form

$$\epsilon\lambda + 1 - e^{-\lambda} + (\rho - 1)e^{-2\lambda} = 0. \tag{2.6}$$

Theorem 4. *When the parameter ρ passes through the critical value $\rho = \rho_* = 1 - \sqrt{(cos(\omega_0) - 1)^2 + (\epsilon\omega_0 + sin(\omega_0))^2}$, $\omega_0 = \frac{1}{\epsilon}[(cos(\omega_0) - 1)tan(2\omega_0) - sin(\omega_0)]$, there is a Hopf bifurcation.*

Proof. Assume that $\lambda = i\omega_0$, $\omega_0 \in R^+$ is a pure imaginary solution of (2.6) for some parameter value $\rho = \rho_*$. This leads to the following equations

$$i\epsilon\omega_0 + 1 - e^{-i\omega_0} + (\rho_* - 1)e^{-i2\omega_0} = 0,$$

$$1 - cos(\omega_0) + (\rho_* - 1)cos(2\omega_0) = 0,$$

$$\epsilon\omega_0 + sin(\omega_0) - (\rho_* - 1)sin(2\omega_0) = 0,$$

$$(\rho_* - 1)^2 = (cos(\omega_0) - 1)^2 + (\epsilon\omega_0 + sin(\omega_0))^2,$$

$$\rho_* = 1 \pm \sqrt{(cos(\omega_0) - 1)^2 + (\epsilon\omega_0 + sin(\omega_0))^2},$$

$$\frac{\epsilon\omega_0 + sin(\omega_0)}{cos(\omega_0) - 1} = \frac{sin(2\omega_0)}{cos(2\omega_0)},$$

$$\omega_0 = \frac{1}{\epsilon}[(cos(\omega_0) - 1)tan(2\omega_0) - sin(\omega_0)].$$

The condition $\frac{d(Re(\lambda))}{d\rho} |_\rho \neq 0$ is the last condition for occurrence of a Hopf bifurcation.
To show that this condition is satisfied, let $\lambda = k(\rho) + i\omega(\rho)$ and using (2.6), we get

$$\epsilon[k+i\omega]+1-e^{-k-i\omega}+(\rho-1)e^{-2(k-i\omega)}=0,$$

$$\epsilon k+1-e^{-k}cos\omega+(\rho-1)e^{-2k}cos(2\omega)=0, \tag{2.7}$$

$$\epsilon\omega+e^{-k}sin\omega-(\rho-1)e^{-2k}sin\omega=0, \tag{2.8}$$

differentiate (2.7) and (2.8) with respect to ρ, we obtain

$$\epsilon\frac{dk}{d\rho}+e^{-k}cos(\omega)\frac{dk}{d\rho}+e^{-k}sin(\omega)\frac{d\omega}{d\rho}-2(\rho-1)e^{-2k}cos(2\omega)\frac{dk}{d\rho}$$

$$+e^{-2k}cos(\omega)-2(\rho-1)e^{-2k}sin(2\omega)\frac{d\omega}{d\rho}=0,$$

$$\begin{aligned}\epsilon\frac{d\omega}{d\rho} &- e^{-k}sin(\omega)\frac{dk}{d\rho}+e^{-k}cos(\omega)\frac{d\omega}{d\rho}-e^{-2k}sin(2\omega)\\ &+ 2(\rho-1)e^{-2k}sin(2\omega)\frac{dk}{d\rho}-2(\rho-1)e^{-2k}cos(2\omega)\frac{d\omega}{d\rho}=0,\end{aligned}$$

Solving for $\frac{dk}{d\rho}$, we obtain

$$\frac{d(Re(\lambda))}{d\rho}\mid_{\rho=\rho_*}=\frac{dk}{d\rho}\mid_{k=0,\omega=\omega_0,\rho=\rho_*},$$

$$=\frac{2(\rho_*-1)-\epsilon cos(2\omega_0)-sin(2\omega_0)sin(\omega_0)-cos(2\omega_0)cos(\omega_0)}{[\epsilon+cos(\omega_0)-2(\rho_*-1)cos(2\omega_0)]^2+[sin(\omega_0)-2(\rho_*-1)sin(2\omega_0)]^2]}$$

$$=\frac{2(\rho_*-1)-\epsilon cos(2\omega_0)-cos(\omega_0)}{[\epsilon+cos(\omega_0)-2(\rho_*-1)cos(2\omega_0)]^2+[sin(\omega_0)-2(\rho_*-1)sin(2\omega_0)]^2]}.$$

Using (2.7) at $k=0$, $\rho=\rho_*$, $\omega=\omega_0$, we get

$$=\frac{2\rho_*-3-\epsilon cos(2\omega_0)-(\rho_*-1)cos(2\omega_0)}{[\epsilon+cos(\omega_0)-2(\rho_*-1)cos(2\omega_0)]^2+[sin(\omega_0)-2(\rho_*-1)sin(2\omega_0)]^2}.$$

It is clear that for $0<\rho_*<1$ and $0<\epsilon<1$, $\frac{d(Re(\lambda))}{d\rho}\mid_{\rho=\rho_*}\neq 0$.
Hence, at $\rho=\rho_*=1-\sqrt{(cos(\omega_0)-1)^2+(\epsilon\omega_0+sin(\omega_0))^2}$, the condition $\frac{d(Re(\lambda))}{d\rho}\mid_{\rho=\rho_*}\neq 0$ is satisfied.

□

We can see that as $\epsilon\to 1$, we get the same results obtained in [10].

2.3 The discretized system

The I.V.P (1.3) can be writen as

$$\epsilon \frac{dx}{dt} = -x(t) + \rho x(t-1)(1 - y(t-1)), \tag{2.9}$$

$$y(t) = x(t-1), \tag{2.10}$$

$$x(t) = y(t) = x_0, \qquad t \leq 0.$$

The method of steps is used to get a discretized analogue of the system (2.9)-(2.10) as follows [14]:

Let $t \in (0, 1]$, then

$$y_1(t) = x_0,$$

$$\begin{aligned} x_1(t) &= x_0 e^{\frac{-t}{\epsilon}} + \frac{\rho}{\epsilon} \int_0^t e^{\frac{s-t}{\epsilon}} x(s-1)(1 - (y(s-1)))ds \\ &= x_0 e^{\frac{-t}{\epsilon}} + \rho x_0 (1 - y_0)(1 - e^{\frac{-t}{\epsilon}}). \end{aligned}$$

Let $t \longrightarrow 1$, then

$$y_1(1) = x_0,$$

$$x_1(1) = x_0 e^{\frac{-1}{\epsilon}} + \rho x_0 (1 - y_1)(1 - e^{\frac{-1}{\epsilon}}).$$

For $t \in (1, 2]$, when $t \leq 1$, take $x(t) = x_1 = x_1(1)$, $y_1(t) = y_1(1) = y_1$, then

$$y_2(t) = x_1(t),$$

$$\begin{aligned} x_2(t) &= x_0 e^{\frac{-(t-1)}{\epsilon}} + \frac{\rho}{\epsilon} \int_1^t e^{\frac{s-t}{\epsilon}} x_1 (1 - (y_1))ds \\ &= x_1 e^{\frac{-(t-1)}{\epsilon}} + \rho x_1 (1 - y_1)(1 - e^{\frac{-(t-1)}{\epsilon}}). \end{aligned}$$

Let $t \longrightarrow 2$, then

$$y_2(1) = x_1,$$

$$x_2(2) = x_1(1) e^{\frac{-1}{\epsilon}} + \rho x_1(1)(1 - y_1(1))(1 - e^{\frac{-1}{\epsilon}}).$$

For $t \in (2, 3]$, when $t \leq 2$, take $x(t) = x_2 = x_2(2)$, $y_2(t) = y_2(2) = y_2$, then

$$y_3(t) = x_2(t),$$

$$x_3(t) = x_2 e^{\frac{-(t-2)}{\epsilon}} + \frac{\rho}{\epsilon} \int_2^t e^{\frac{s-t}{\epsilon}} x_2 (1 - (y_2))ds$$

$$= x_2 e^{\frac{-(t-2)}{\epsilon}} + \rho x_1(1-y_2)(1-e^{\frac{-(t-2)}{\epsilon}}).$$

Let $t \longrightarrow 3$, then

$$y_3(3) = x_2,$$

$$x_3(3) = x_2 e^{\frac{-1}{\epsilon}} + \rho x_2(1-y_2(1))(1-e^{\frac{-1}{\epsilon}}).$$

Repeating the process we deduce that the solution of (1.3) is given by

$$y_{n+1}(t) = x_n(t),$$

$$x_{n+1}(t) = x_n e^{\frac{-(t-n)}{\epsilon}} + \rho x_n(1-y_n)(1-e^{\frac{-(t-n)}{\epsilon}}).$$

Let $t \longrightarrow n+1$, then

$$y_{n+1} = x_n,$$

$$x_{n+1} = x_n e^{\frac{-1}{\epsilon}} + \rho x_n(1-y_n)(1-e^{\frac{-1}{\epsilon}}). \tag{2.11}$$

2.4 Local stability and bifurcation analysis of the discretized system

The system (2.11) has two fixed points $(x_1^*, y_1^*) = (0,0)$ and $(x_2^*, y_2^*) = (1-\frac{1}{\rho}, 1-\frac{1}{\rho})$.
(1) At $(x_1^*, y_1^*) = (0,0)$:

The Jacobian matrix calculated $(x_1^*, y_1^*) = (0,0)$ reads

$$J(0,0) = \begin{pmatrix} e^{\frac{-1}{\epsilon}} + \rho(1-e^{\frac{-1}{\epsilon}}) & 0 \\ 1 & 0 \end{pmatrix}.$$

The characteristic equation

$$\lambda^2 - \lambda(e^{\frac{-1}{\epsilon}} + \rho(1-e^{\frac{-1}{\epsilon}})) = 0,$$

has two roots $\lambda_1 = 0$ and $\lambda_2 = e^{\frac{-1}{\epsilon}} + \rho(1-e^{\frac{-1}{\epsilon}})$.

We can see that $\lambda_2 = 1$ at $\rho = 1$, $\lambda_2 > 1$ for $\rho > 1$ and $\lambda_2 < 1$ when $\rho < 1$, then we have

Proposition 1. *The fixed point $fix_1 = (0,0)$ is*
1. a sink if $\rho < 1$,
2. a saddle if $\rho > 1$,
3. a non-hyperbolic if $\rho = 1$.

The bifurcation associated with the appearance of an eigenvalue $\lambda = 1$ is called a fold bifurcation and its condition implies that

$$det(J(0,0,\rho^*) - I_2) = 0,$$

where I_2 is is the unit 2x2 matrix [20].

Lemma 2. *If $\rho = 1$, then system (2.11) admits a fold bifurcation at $(x_1^*, y_1^*) = (0,0)$.*

Proof. The condition of the fold bifurcation gives

$$det\begin{pmatrix} e^{\frac{-1}{\epsilon}} + \rho^*(1-e^{\frac{-1}{\epsilon}}) - 1 & 0 \\ 1 & -1 \end{pmatrix} = 0,$$

$$1 - e^{\frac{-1}{\epsilon}} - \rho^*(1-e^{\frac{-1}{\epsilon}}) = 0,$$

$$\rho^*(1-e^{\frac{-1}{\epsilon}}) = 1 - e^{\frac{-1}{\epsilon}},$$

then $\rho^* = 1$. □

(2) At $(x_2^*, y_2^*) = (1-\frac{1}{\rho}, 1-\frac{1}{\rho})$:

The Jacobian matrix calculated $(x_2^*, y_2^*) = (1-\frac{1}{\rho}, 1-\frac{1}{\rho})$ reads

$$J(1-\frac{1}{\rho}, 1-\frac{1}{\rho}) = \begin{pmatrix} 1 & -(\rho-1)(1-e^{\frac{-1}{\epsilon}}) \\ 1 & 0 \end{pmatrix}.$$

The characteristic equation reads

$$\lambda^2 - \lambda + (\rho-1)(1-e^{\frac{-1}{\epsilon}}) = 0.$$

Lemma 3. *[1] Let $F(\lambda) = \lambda^2 + P\lambda + Q$. Suppose that $F(1) > 0$, and $F(\lambda) = 0$ has two roots λ_1 and λ_2. Then*

1. *$F(-1) > 0$ and $Q < 1$ if and only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$;*
2. *$F(-1) < 0$ if and only if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$);*
3. *$F(-1) > 0$ and $Q > 1$ if and only if $|\lambda_1| > 1$ and $|\lambda_2| > 1$;*
4. *$F(-1) = 0$ and $P \neq 0, 2$ if and only if $\lambda_1 = -1$ and $|\lambda_2| \neq 1$;*
5. *$P^2 - 4Q < 0$ and $Q = 1$ if and only if λ_1 and λ_2 are complex and $|\lambda_{1,2}| = 1$.*

Proposition 2. *The fixed point* $(x_2^*, y_2^*) = (1 - \frac{1}{\rho}, 1 - \frac{1}{\rho})$ *is*
1. a sink if $1 < \rho < \frac{2}{1-e^{\frac{-1}{\epsilon}}}$,
2. a source if $\rho > \frac{2}{1-e^{\frac{-1}{\epsilon}}}$.

Definition 1. *The bifurcation correspondence to the existence of* $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$ *is called a Neimark-Sacker bifurcation [20].*

Lemma 4. *If* $\rho = 1 + \frac{1}{1-e^{\frac{-1}{\epsilon}}}$, *then system (2.11) admits a Neimark-Sacker bifurcation at* $(x_1^*, y_1^*) = (0, 0)$.

Proof. The characteristic equation

$$\lambda^2 - \lambda + (\rho - 1)(1 - e^{\frac{-1}{\epsilon}}) = 0,$$

has two roots

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4(\rho - 1)(1 - e^{\frac{-1}{\epsilon}})}}{2}. \tag{2.12}$$

We can see that when

$$1 - 4(\rho - 1)(1 - e^{\frac{-1}{\epsilon}}) < 0,$$

the two roots are complex. Then for $\rho > 1 + \frac{1}{4(1-e^{\frac{-1}{\epsilon}})}$, we can write

$$\lambda_{1,2} = \frac{1 \pm i\sqrt{4(\rho - 1)(1 - e^{\frac{-1}{\epsilon}}) - 1}}{2}.$$

Suppose that $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$ for some parameter value $\rho = \rho^* > 1 + \frac{1}{4(1-e^{\frac{-1}{\epsilon}})}$, then

$$\lambda_1\lambda_2 = \frac{1 - 1 + 4(\rho^* - 1)(1 - e^{\frac{-1}{\epsilon}})}{4} = 1,$$

$$(\rho^* - 1)(1 - e^{\frac{-1}{\epsilon}}) = 1,$$

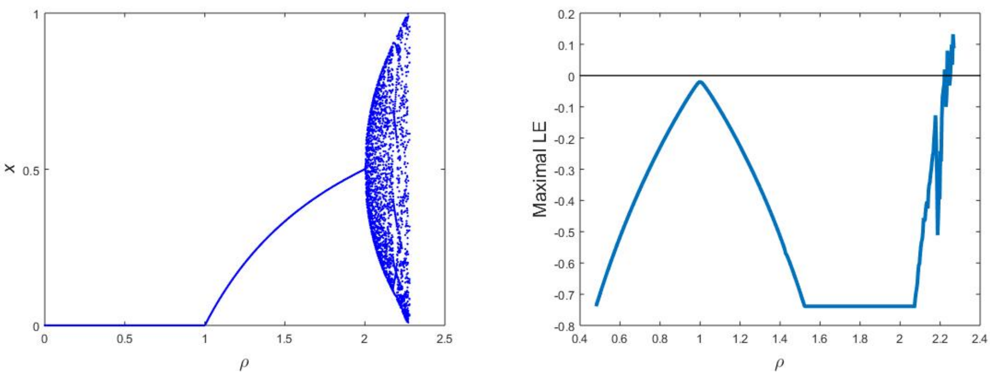
$$\rho^* = 1 + \frac{1}{1 - e^{\frac{-1}{\epsilon}}}. \tag{2.13}$$

Thus at $\rho = \rho^* = 1 + \frac{1}{1-e^{\frac{-1}{\epsilon}}}$, we have $\lambda_{1,2} = e^{\pm\frac{i\pi}{3}}$ and the system admits a Neimark-Sacker bifurcation. □

We can see that as $\epsilon \to 1$, we get the same results obtained in [10].

3 Numerical simulations

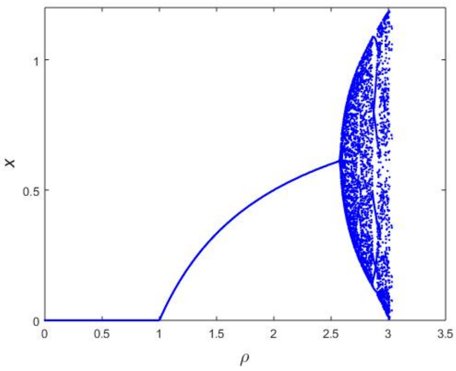
In this section, we perform numerical simulations to confirm theoretical analysis obtained.
In Figure 1, we can see that as $\epsilon \to 0$ in (2.11), we get the same results obtained in [9].



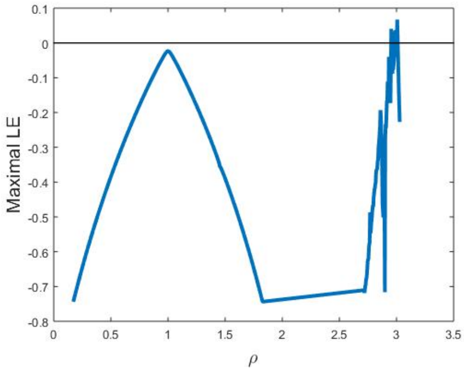
(a) Bifurcation diagram of (2.11) as $\epsilon \to 0$. (b) Liapunov exponent of (2.11) as $\epsilon \to 0$.

Figure 1: Dynamics of the system(2.1) as $\epsilon \to 0$.

In Figure 3 and Figure 4, we can see that as $\epsilon \to 1$ in (2.11), we get the same results obtained in [10].

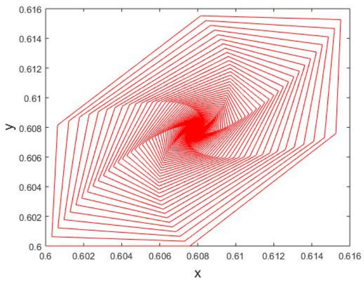


(a) Bifurcation diagram of (2.11) as $\epsilon \to 1$.

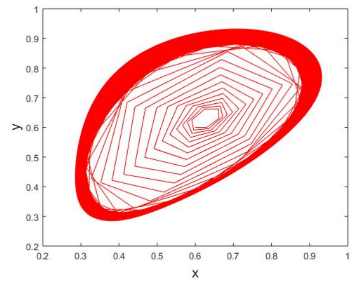


(b) Liapunov exponent of (2.11) as $\epsilon \to 1$.

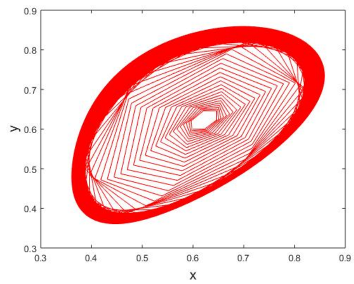
Figure 2: Dynamics of the system(2.11) as $\epsilon \to 1$.



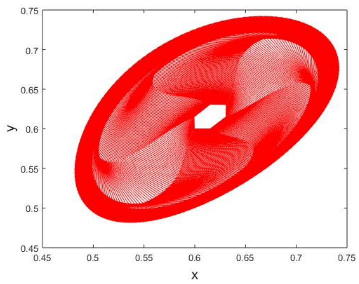
(a) $\rho = 2.55$



(b) $\rho = 2.6$



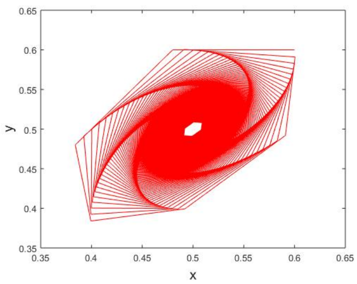
(c) $\rho = 2.65$



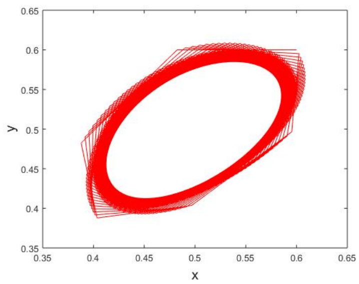
(d) $\rho = 2.7$

Figure 3: Phase portraits of (2.11) for different values of ρ as $\epsilon \to 1$.

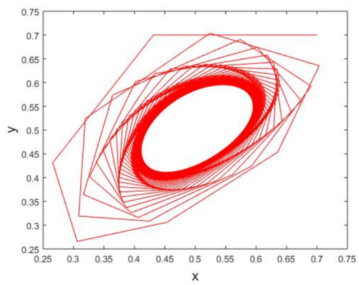
Figure 5 illustrate more complex dynamics of the system by giving phase portraits of the system (2.11) for different values of ρ at which the map is chaotic and different values of ϵ.



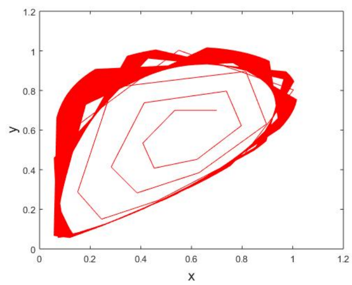
(a) $\rho = 2$ and $\epsilon = 0.001$



(b) $\rho = 2.01$ and $\epsilon = 0.01$



(c) $\rho = 2.03$ and $\epsilon = 0.25$



(d) $\rho = 2.42$ and $\epsilon = 0.5$

Figure 4: Phase portraits of the system (2.11) for different values of ρ and ϵ.

4 Conclusion

In this work, we studied the dynamics of the singularly perturbed logistic difference equation with two different continuous arguments. First of all, we obtained fixed points and discussed their local stability by analyzing the corresponding characteristic equations of the linearized equations. secondly, we show that the equation exhibits Hopf bifurcation and we have reached explicit conditions for its occurrence. Then, the method of steps is applied to obtain a discrete analogue of the considered system. We investigated local stability conditions of the fixed points of the discretized system. Explicit conditions for the occurrence of a variety of complex dynamics such as fold and Neimark-Sacker bifurcations are reached. By letting the perturbation parameter tends to one, it is illustrated that the singularly perturbed logistic difference equation with two different continuous arguments behaves as the logistic delay differential equation with two delays. Finally, numerical simulations including Lyapunov exponent, bifurcation diagram and phase portraits carried out to confirm the theoretical analysis obtained and to illustrate more complex dynamics of the system.

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