Some Properties and Integral representation of Extended Generalized Bessel Matrix Functions

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Some Properties and Integral representation of Extended Generalized Bessel Matrix Functions

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Abstract

In this paper, we discuss the Extended Genarlized Bessel Matrix functions and its Integral representations. Some properties of Extended Generalized Bessel Matrix function including the connection with Laguerre Matrix polynomial, Jacobi Matrix Polynomial have also been obtained.

Keywords: Generalized Wright Hypergeometric Matrix function, Extended Generalized Bessel Matrix function, Integral representation, Laguerre Matrix Polynomial, Jacobi Matrix Polynomial.

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1 Introduction, Preliminaries and Motivation

The Bessel functions, also well-known as the circular cylinder function, is the most frequently used special function in the field of Mathematical Physics. No other special functions have received such detailed treatment in willingly available treaties [23] as have the Bessel functions. In fact a Bessel function is generally defined as a particular solution of a linear differential equation of the second order known as Bessel's equation [23].

We are motivated the works of Jódar et al., who established and deliberate the Bessel Matrix function of first kind and Hypergeometric Matrix functions in [7, 8, 9, 10, 12, 21]. Shehata et al. [22] defined and studied the Extension of Bessel matrix functions. In sequel to the study, we introduce Extended Generalized Bessel matrix function.

For our aim, we begin by recalling some well-known results and facts.
Let $\wp$ is a matrix in $C^{N\times N}$, and its $\sigma(\wp)$ denotes the set of all eigenvalues of $\wp$. The 2-norm of $\wp$ is denoted by $\|\wp\|$, and described as

$$\|\wp\| = \sup_{y\neq 0} \frac{\|\wp y\|_2}{\|y\|_2},$$

where for any vector y in C^N is the Euclidean norm of y.

If $f(z)$ and $g(z)$ are holomorphic functions defined in open set Ω in complex plane, R and S are matrices in $C^{N\times N}$ such that $\sigma(R) \subset \Omega$, $\sigma(S) \subset \Omega$ and $RS = SR$, then [16, P. 558],

$$f(\wp)g(S) = g(S)f(\wp) \tag{1.1}$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$, which is an entire function in complex plane. The image of R under the action of Γ^{-1}, denoted by $\Gamma^{-1}(\wp)$, is a well-defined matrix.
If $\wp + nI$ is an invertable matrix for every non-negative integer n, then $\Gamma(R)$ is an invertible matrix [16],

$$(\wp)(\wp + I)\ldots(\wp + (n-1)I)\Gamma^{-1}(\wp + nI) = \Gamma^{-1}(\wp) \tag{1.2}$$

is well defined. For any matrix $\wp$ in $C^{N\times N}$ the Pochhammer symbol is defined [16],

$$(\wp)_n = (\wp)(R + I)\ldots(\wp + (n-1)I) = \Gamma(\wp + nI)\Gamma^{-1}(\wp), n \geq 1 : (\wp)_0 = I \tag{1.3}$$

Jódar and Cortés [7, P. 91, Theorem 1] have proved that for a positive stable matrix R in $C^{N\times N}$ and an integer $n \geq 1$,

$$\Gamma(\wp) = \lim_{n\to\infty}(n-1)![(\wp)_n]^{-1}n^{\wp} \tag{1.4}$$

If $\wp$ and S are positive stable matrices in $C^{N\times N}$, then the Gamma matrix function [7, P. 91] and the Beta matrix function [7, P. 92] are defined as,

$$\Gamma(\wp) = \int_0^\infty e^{-t}t^{\wp-1}dt; t^{\wp-1} = \exp((\wp - I)\ln t) \tag{1.5}$$

and

$$\beta(\wp, S) = \int_0^1 t^{\wp-I}(1-t)^{S-I}dt \tag{1.6}$$

respectively. Subsequently if $\wp$ and S are commuting positive stable matrices then

$$\beta(\wp, S) = \beta(S, \wp), \tag{1.7}$$

commutatively is a necessary condition for the symmetry of the Beta matrix function [7, P. 93]. Also, we have

$$\beta(\wp, S) = \Gamma(\wp)\Gamma(S)\Gamma^{-1}(\wp + S), \tag{1.8}$$

and

$$\int_0^\infty t^{\wp-I}(1+t)^{-(\wp+S)}dt = \int_0^\infty t^{\wp-I}(1-t)^{S-I}dt \tag{1.9}$$

where $\wp$ and S are commuting matrices in $C^{N\times N}$ such that $\wp$, S and $\wp + S$ are positive stable matrices.

Jódar et al. [8] introduced Bessel matrix functions of the first kind of order M as,

$$J_M(z) = \sum_{k=0}^{\infty}\frac{\Gamma^{-1}(M + k + I)}{k!}\left(\frac{-z^2}{4}\right)^{k+M}, \tag{1.10}$$

where $|z| < \infty$, $|\arg z| < \pi$.

Jódar and Cortés [10, P. 210 section 3] defined the hypergeometric matrix function

$$F\left(J, M; L; z\right) = \sum_{k=0}^{\infty} \frac{(J)_k (M)_k [(L)_k]^{-1}}{k!} z^k, \tag{1.11}$$

where J, K, L be matrices in $C^{N\times N}$, $C + nI$ is invertable for all integers $n \geq 0$.

In 1994, Jódar et al. [12, P. 57] introduced the nth Laguerre matrix polynomial as,

$$L_n^{(l,\xi)}\left(x\right) = \sum_{k=0}^{n} \frac{(-1)^k \xi^k}{k!\ (n-k)!} (l+I)_n [(l+I)_k]^{-1} x^k. \tag{1.12}$$

where l be matrices in $C^{N\times N}$, ξ be a complex number with $\Re\left(\xi\right) > 0$.

The Jacobi matrix polynomials $P_n{}^{(y,z)}\left(x\right)$, for parameter matrices A and B whose eigenvalues, λ, all satisfy $\Re\left(\lambda\right) > 0$ and $yz = zy$. For $n > 0$, the n-th Jacobi matrix polynomial is defined [2, P. 793],

$$p_n{}^{(y,z)}\left(x\right) = \frac{(y+I)_n}{n!} {}_2F_1 \left[\begin{matrix} -nI,\ y+z+(n+1)I \\ A+I \end{matrix} \middle| \begin{matrix} \frac{1-y}{2} \end{matrix}\right] \tag{1.13}$$

In 2018, Maged G. Bin-Saad and Nabel Saleh [15, P. 8] defined the following matrix version of Kampé de Fériet of double hypergeometric function as follows

$$F_{l:m,n}^{p:q,k} \left[\begin{matrix} (A_p) : (B_q) ; (C_k) \\ (D_1) : (E_m) ; (F_n) \end{matrix} \middle| x, y \right] = \sum_{r,s=0}^{\infty} \frac{\prod\limits_{j=1}^{p} (A_j)_{r+s} \prod\limits_{j=1}^{q} (B_j)_r \prod\limits_{j=1}^{k} (C_j)_s}{r!s!}$$
$$\left[\prod_{j=1}^{l} (D_j)_{r+s}\right]^{-1} \left[\prod_{j=1}^{m} (E_j)_r\right]^{-1} \left[\prod_{j=1}^{n} (F_j)_s\right]^{-1} x^r y^s, \tag{1.14}$$

for all matrices in (1.15) are in $C^{N\times N}$, such that $\prod\limits_{j=1}^{l} (D_j + rI + sI)$, $\prod\limits_{j=1}^{m} (E_j + rI)$, $\prod\limits_{j=1}^{n} (F_j + sI)$ are all invertable for integers $(r, s) \geqslant 0$.

They also defined the following matrix version of Srivastava and Daoust[14] multi-variable hypergeometric function:

$$F_{4:0;1}^{2:0;0} \left[\begin{matrix} (P_1 : 3, 2), (P_2 : 1, 2) : --; --; \\ (Q_1 : 2, 3), (Q_2 : 1, 1), (Q_3 : 1, 1), (R : 1, 1) : --; (S_1 : 1) \end{matrix} \middle| x, y \right]$$
$$= \sum_{r,s=0}^{\infty} \frac{(P_1)_{2r+3s} (P_2)_{1r+2s}}{r!s!} \left[(Q_1)_{2r+3s}\right]^{-1} \left[(Q_2)_{r+s}\right] \left[(Q_3)_{r+s}\right] \left[(I)_{r+s}\right] \left[(S_1)_s\right] x^r y^s \tag{1.15}$$

for all matrices in (1.15) are in $C^{N\times N}$, such that $(D_j + rI + sI)$ $where\, j = 1, 2, 3,$ $(R + rI + sI)$, $(S_1 + sI)$ are all invertable for integers $(r, s) \geqslant 0$.

Fox-Wright function [4] defined as following form,

$$ {}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{matrix} (\alpha_i, Y_i)_{1,p} \\ (\beta_j, Z_j)_{1,q} \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 k) \cdots \Gamma(\alpha_p + A_p k)}{\Gamma(\beta_1 + B_1 k) \cdots \Gamma(\beta_q + B_q k)} \frac{z^k}{k!}, \tag{1.16} $$

where $i = 1, 2, \cdots, p;$ $\;j = 1, 2, \cdots, q$ and $z, \alpha_i, \beta_j \in \mathbf{C}$, and the coefficients $A_1, ..., A_p \in \mathbf{R}^+$ and $B_1, ..., B_q \in \mathbf{R}^+$ satisfying the following condition

$$ \sum_{j=1}^{q} Z_j - \sum_{i=1}^{p} Y_i > -1. \tag{1.17} $$

In 2013, Salehbhai et al. [18, P. 2] introduced Extended Generalized Bessel Function in following manner,

$$ {}_h^m J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1 + m\nu + hk)\; k!} \left(\frac{z}{2}\right)^{2k+v}, \tag{1.18} $$

where, $h \in N$, $|z| < \infty$, $|\arg z| < \pi$, $m\; and\, v \in C$.

We need to use the following facts in our study.
For $\wp$ is a matrix in $C^{N\times N}$, we have [10],

$$ (\wp)_{m+n} = (\wp)_M (\wp + m)_n \tag{1.19} $$

The duplication formula for the Gamma function [10, P. 276],

$$ (2\eta)! = 2^{2\eta}(1)_\eta \left(\frac{1}{2}\right)_\eta, (2\eta + 1)! = 2^{2\eta}(1)_\eta \left(\frac{3}{2}\right)_\eta, \tag{1.20} $$

where, $\eta > 0$.

We also need the following successive integral formula [17, P. 22, Entry 2.47]

$$ \int_0^\infty y^{\mu-1} \left(y + a + \sqrt{y^2 + 2ay}\right)^{-\lambda} dy = 2\lambda a^{-\lambda} \left(\frac{a}{2}\right)^\mu \frac{\Gamma(2\mu)\,\Gamma(\lambda - \mu)}{\Gamma(\lambda + \mu + 1)}. \tag{1.21} $$

Main Results:

2 Generalized Wright Hypergeometric Matrix Function

In this section, we establish Genralized Wright Hypergeometric Matrix function and its convergence condition.

The Generalized Wright Hypergeometric matrix function ${}_p\psi_q\left(Y_1, Y_2, ..., Y_P;\ Z_1, Z_2, ...Z_q;\ z\right)$ defined by

$$
{}_p\psi_q\left(z\right) = {}_p\psi_q\left(\begin{matrix}(A_i, \alpha_i)_1^p & ; z \\ (B_j, \beta_j)_1^q & \end{matrix}\right)
$$

$$
= \sum_{n=0}^{\infty} \frac{\Gamma\left(A_1+\alpha_1 n\right)\Gamma\left(A_2+\alpha_2 n\right).......\Gamma\left(A_p+\alpha_p n\right)\Gamma^{-1}\left(B_1+\beta_1 n\right)\Gamma^{-1}\left(B_2+\beta_2 n\right)....\Gamma^{-1}\left(B_q+\beta_q n\right)z^n}{n!} \tag{2.1}
$$

is bounded for matrices $Y_i,\ Z_j \in C^{N\times N}$, $1 \leq i \leq p$, $1 \leq j \leq q$ such that $\beta_j + nI;\ 1 \leq j \leq q$ are invertable for all integers $n \geq 0$.

We are interested to study of the Generalized Wright Hypergeometric matrix function and Extended Generalized Bessel matrix function. We also establish an important property of the Generalized Wright Hypergeometric matrix function by proving the following result.

Theorem 2.1. *Let $Y_1, Y_2, ..., Y_P,\ Z_1, Z_2, ..., Z_q$ be matrices in $C^{N\times N}$ such that*

$$
\gamma\left(Z_1\right)+\gamma\left(Z_2\right)+...+\gamma\left(Z_q\right) > \delta\left(Y_1\right)+\delta\left(Y_2\right)+...+\delta\left(Y_p\right) \tag{2.2}
$$

and $p \leq 1+q$. Then the series (2.1) is absolutely convergent for $|z| = 1$.

Proof. From (2.2), there exists a positive number η such that

$$
\gamma\left(Z_1\right)+\gamma\left(Z_2\right)+...+\gamma\left(Z_q\right)-\left(\delta\left(Y_1\right)+\delta\left(Y_2\right)+...+\delta\left(Y_p\right)\right) = 2\eta \tag{2.3}
$$

Now,

$$
n^{1+\eta}\left[\frac{\Gamma\left(Y_1+\alpha_1 n\right)\Gamma\left(Y_2+\alpha_2 n\right)...\Gamma\left(Y_p+\alpha_p n\right)\Gamma^{-1}\left(Z_1+\beta_1 n\right)\Gamma^{-1}\left(Z_2+\beta_2 n\right)...\Gamma^{-1}\left(Z_q+\beta_q n\right)}{n!}\right]
$$

$$
\begin{aligned}
&= \frac{n^{1+\eta}}{n(n-1)!}\left[\left(\frac{n^{-A_1}\Gamma\left(Y_1+\alpha_1 n\right)}{(n-1)!}\right)n^{Y_1}...\left(\frac{n^{-Y_p}\Gamma\left(Y_p+\alpha_p n\right)}{(n-1)!}\right)n^{Y_p}...\right. \\
&\quad \left.\times\left((n-1)!\Gamma^{-1}\left(Z_1+\beta_1 n\right)n^{Z_1}.n^{-Z_1}\right)...\left((n-1)!\Gamma^{-1}\left(Z_q+\beta_q n\right)n^{Z_q}.n^{-Z_q}\right)(n-1)!^{p-q-1}\right]
\end{aligned} \tag{2.4}
$$

On taking $\delta\left(-Z_i\right) = -\gamma\left(Z_i\right)$,
next, employing the *Schür* decomposition of matrix A [12]

$$
\left\|e^{Yt}\right\| \leq e^{t\alpha(Y)}\sum_{s=0}^{r-1}\frac{\left(\left\|Y\right\|r^{\frac{1}{2}}t\right)^s}{s!}\ \left(t \geq 0\right), \tag{2.5}
$$

$$
\left\|t^{Y}\right\| \leq \left\|e^{Y\ln t}\right\| \leq t^{\alpha(Y)}\sum_{s=0}^{r-1}\frac{\left(\left\|Y\right\|r^{\frac{1}{2}}\ln t\right)^s}{s!}\ \left(t \geq 1\right). \tag{2.6}
$$

On using equation (2.5) and (2.6), this yields

$$\left\|n^{Y_1}\right\| ... \left\|n^{Yp}\right\| \left\|n^{Z_1}\right\| ... \left\|n^{Z_q}\right\|$$

$$\leqslant n^{\delta(Y_1)+\delta(Y_2)+...+\delta(Y_p)-\gamma(Z_1)-\gamma(Z_2)...-\gamma(Z_q)}$$

$$\times\left\{\sum_{s=0}^{r-1}\frac{\left(\|Y_1\| r^{\frac{1}{2}}\ln n\right)^s}{s!}\right\}\left\{\sum_{s=0}^{r-1}\frac{\left(\|Y_2\| r^{\frac{1}{2}}\ln n\right)^s}{s!}\right\}...\left\{\sum_{s=0}^{r-1}\frac{\left(\|Y_p\| r^{\frac{1}{2}}\ln n\right)^s}{s!}\right\}... \tag{2.7}$$

$$\times\left\{\sum_{s=0}^{r-1}\frac{\left(\|Z_1\| r^{\frac{1}{2}}\ln n\right)^s}{s!}\right\}\left\{\sum_{s=0}^{r-1}\frac{\left(\|Z\|_2 r^{\frac{1}{2}}\ln n\right)^s}{s!}\right\}...\left\{\lim_{x\to\infty}\sum_{s=0}^{r-1}\frac{\left(\|Z_q\| r^{\frac{1}{2}}\ln n\right)^s}{s!}\right\}$$

$$\leq n^{-2\eta}\left\{\sum_{s=0}^{r-1}\frac{\left(\max\left\{\|A_1\|...\|A_p\|, \|B_1\|...\|B_q\|\right\} r^{\frac{1}{2}}\ln n\right)^s}{s!}\right\}^{p+q} \tag{2.8}$$

From (2.4) and (2.8), we get

$$\lim_{n\to\infty} n^{1+\eta}\left\|\frac{\Gamma(Y_1+\alpha_1 n)\Gamma(Y_2+\alpha_2 n)...\Gamma(Y_p+\alpha_p n)\Gamma^{-1}(Z_1+\beta_1 n)\Gamma^{-1}(Z_2+\beta_2 n)....\Gamma^{-1}(Z_q+\beta_q n)}{n!}\right\|$$

$$\leq \lim_{n\to\infty} n^{-\eta}S\left\|\frac{n^{-Y_1}\Gamma(Y_1+\alpha_1 n)}{(n-1)!}\right\|...\left\|\frac{n^{-Y_p}\Gamma(Y_p+\alpha_p n)}{(n-1)!}\right\|...\left\|\left((n-1)!\Gamma^{-1}\left(Z_1+\beta_1 n\right)n^{-Z_1}\right)\right\| \tag{2.9}$$

$$\times...\left\|\left((n-1)!\Gamma^{-1}\left(Z_q+\beta_q n\right)n^{-Zq}\right)\right\|(n-1)!^{p-q-1},$$

where,

$$S=\left\{\sum_{s=0}^{r-1}\frac{\left(\max\left\{\|Y_1\|...\|Y_p\|, \|Z_1\|...\|Z_q\|\right\} r^{\frac{1}{2}}\ln n\right)^s}{s!}\right\}^{p+q} \tag{2.10}$$

Since,

$$\lim_{n\to\infty} n^{-\eta}S=0, \tag{2.11}$$

$$\lim_{n\to\infty} n^{1+\eta}\left\|\frac{\Gamma(Y_1+\alpha_1 n)\Gamma(Y_2+\alpha_2 n)...\Gamma(Y_p+\alpha_p n)\Gamma^{-1}(Z_1+\beta_1 n)\Gamma^{-1}(Z_2+\beta_2 n)...\Gamma^{-1}(Z_q+\beta_q n)}{n!}\right\|, \tag{2.12}$$

$$\to 0$$

for $\|z\|=1$ and $p\leq q+1$.

Further using comparision theorem of numerical series of positive numbers, leads to (2.1). □

3 Extended Generalized Bessel Matrix Function

In this section, we introduce Extended Generalized Bessel Matrix Function in terms of Generalized Wright Hypergeometric Matrix Function. We also obtain integral representation of

${}_{A}^{B}J_{M}$.

The Extended Generalized Bessel Matrix function is defined as follows.

$$ {}_{A}^{B}J_{M}\left(z\right)=\left(\frac{z}{2}\right)^{M}{}_{0}\psi_{1}\left[\begin{array}{c} - \\ (BM+I;A) \end{array}\middle|\left(\frac{-z^{2}}{4}\right)\right] \tag{3.1} $$

where, A, B and M are matrices in $C^{N\times N}$ satisfied the conditions $\Re\left(a\right)>0$ for all eigenvalues $a\in\sigma\left(A\right)$, $\Re\left(m\right)>-1$, $\Re\left(b\right)>-1$ for all eigenvalues $m\in\sigma\left(M\right)$, $b\in\sigma\left(B\right)$ respectively. Also $\left(Ak+BM+I\right)$ is matrix in $C^{N\times N}$ such that $\left(Ak+BM+I\right)$ is an invertable matrix for every integer $k\geq 0$.

In light of (2.1), we can write,

$$ {}_{0}\psi_{1}\left[\begin{array}{c} - \\ (BM+I;A) \end{array}\middle|\left(\frac{-z^{2}}{4}\right)\right]=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma^{-1}\left(BM+Ak+I\right)}{k!}\left(\frac{-z^{2}}{4}\right)^{k}. \tag{3.2} $$

Arrive at conclusion that Extended Generalized Bessel matrix function ${}_{A}^{B}J_{M}$ [?] is entire function.

Theorem 3.1. *For $|z|<\infty$; $|\arg z|<\pi$ and $n\in N$, the Extended Generalized Bessel Matrix function defined by (3.1) satisfy the following integral representation*

$$ {}_{A}^{B}J_{M}\left(z\right)=\frac{\left(\frac{z}{2}\right)^{M}}{2\pi i}\int\limits_{c}\exp\left(sI+\frac{z^{2}s^{-A}}{4}\right)s^{-(BM+1)I}ds \tag{3.3} $$

Proof. The contour integral representation for the reciprocal gamma function is given as [13],

$$ \frac{1}{\Gamma\left(z\right)}=\frac{1}{2\pi i}\int\limits_{c}e^{t}t^{-z}dt \tag{3.4} $$

On using (3.4), we can say that

$$ \Gamma^{-1}\left(BM+Ak+I\right)=\frac{1}{2\pi i}\int\limits_{c}e^{s}s^{-(BM+Ak+I)}ds \tag{3.5} $$

From (3.5) and (3.1), we get

$$ {}_{A}^{B}J_{M}\left(z\right)=\sum_{k=0}^{\infty}\frac{z^{2k+M}}{2\pi i\ 2^{2k+M}\ k!}\int\limits_{c}e^{s}s^{-(BM+Ak+I)}ds \tag{3.6} $$

On interchanging the order of the integration and summation in the R.H.S. of (3.6), we find that

$$ {}_{A}^{B}J_{M}\left(z\right)=\frac{Z^{M}\,2^{-M}}{2\pi i}\int\limits_{c}e^{s}s^{-(BM+I)}\sum_{k=0}^{\infty}\frac{z^{2k}\left(S^{-A}\right)^{k}}{k!\,2^{2k}}ds \tag{3.7} $$

After further simplification leads to,

$$ {}_{A}^{B}J_{M}(z) = \frac{\left(\frac{z}{2}\right)^{M}}{2\pi i} \int\limits_{c} \exp\left(sI + \frac{z^{2}s^{-A}}{4}\right) s^{-(BM+1)I} ds. \tag{3.8} $$

□

Theorem 3.2. *For δ, M, $B \in C$, $\Re(\delta) > 0$, $\Re(M) > 0$, $\Re(A) > 0$, a, $\lambda \in R^{+}$, $0 < \Re(\delta) < \lambda$ the Mellin-transform type integral formula involving the Extended Generalized Bessel Matrix function is as follows.*

$$ \int\limits_{0}^{\infty} x^{\delta-1}\left(x + a + \sqrt{x^{2}+2ax}\right)^{-\lambda} {}_{A}^{B}J_{M}\left(\frac{y}{x + a + \sqrt{x^{2}+2ax}}\right) dx $$
$$ = \Gamma(2\delta)\ 2^{1-\delta-M} a^{\delta-\lambda-M} \left(\frac{y}{2}\right)^{M} {}_{2}\psi_{3}\left[\begin{matrix}(\lambda+M+1,2),(\lambda+M-\delta,2)\\ (BM+I;A),\ (\lambda+M-\delta,2),(\lambda+M,2)\end{matrix}\middle| \left(\frac{-y^{2}}{4a^{2}}\right)\right] \tag{3.9} $$

Proof. From equation (3.1), we have

$$ I = \sum_{n=0}^{\infty} \frac{(-1)^{n}\Gamma^{-1}(BM+Ak+I)\, y^{2n+M}}{2^{2n+M}} \left(\frac{1}{x + a + \sqrt{x^{2}+2ax}}\right)^{2n+M} \times \int\limits_{0}^{\infty} x^{\delta-1}\left(x + a + \sqrt{x^{2}+2ax}\right)^{-\lambda} dx. \tag{3.10} $$

On simplification, we get,

$$ I = \sum_{n=0}^{\infty} \frac{(-1)^{n}\Gamma^{-1}(BM+Ak+I)\, y^{2n+M}}{2^{2n+M}} \int\limits_{0}^{\infty} x^{\delta-1}\left(x + a + \sqrt{x^{2}+2ax}\right)^{-\lambda-M-2n} dx. \tag{3.11} $$

Use of (1.21), the equation (3.11) becomes,

$$ I = \sum_{n=0}^{\infty} 2(-1)^{n}(\lambda+M+2n)\, a^{-\lambda-M-2n}\left(\frac{a}{2}\right)^{\delta} \times \frac{\Gamma(2\delta)\ \Gamma(\lambda+M+2n-\delta)\ \Gamma^{-1}(BM+Ak+I)\, y^{2n+M}}{\Gamma(\lambda+v+2n+\delta+1)\,\Gamma(\lambda+M+2n\quad+\delta+I)\, 2^{2n+M}}. \tag{3.12} $$

Finally, we arrive at

$$ I = \frac{\Gamma(2\delta)\ 2^{1-\delta-M} a^{\delta-\lambda-M} y^{M}}{2^{M}} \times \sum_{n=0}^{\infty} a^{-2n} \frac{(\lambda+M+2n)\,\Gamma(\lambda+M+2n)\ \Gamma^{-1}(BM+Ak+I)\,\Gamma^{-1}(\lambda+M+2n\ +\delta+I)}{\Gamma(\lambda+M+2n)\ n!}\left(\frac{-y^{2}}{4}\right)^{n}. \tag{3.13} $$

□

4 Integral involving Laguerre matrix polynomial with ${}_{A}^{B}J_{M}$

In this section, we discuss other integral formula associated with Laguerre matrix polynomial $L_n^{(l,\xi)}(x)$ and Extended Generalized Bessel Matrix function ${}_{A}^{B}J_{M}$.

Theorem 4.1. *Let α be a matrix in $C^{N\times N}$ and β be a complex number with $\Re(\beta)>0$, $\Re(\sigma\mp i\mu)>0$, $\Re(\nu+\gamma+1)>0$, $n\in N_0$ and ${}_{A}^{B}J_{M}$ is defined as (3.1). Then the integral is of the general form*

$$\int_0^\infty x^\gamma e^{-\sigma x} L_n^{(\alpha,\beta)}(x)\, {}_{A}^{B}J_{M}(\mu x)\,dx$$
$$=\sum_{k=0}^{n}\frac{(-\beta)^k\Gamma(n+\alpha+1)(-1)^k\mu^\nu\Gamma(BM+1)\Gamma(\nu+\gamma+k+1)}{k!\,\Gamma(n-k+1)\Gamma(\alpha+k+1)\Gamma(BM+Ak+1)2^\nu\Gamma(BM+1)\sigma^{\nu+\gamma+k+1}} \tag{4.1}$$
$$\times {}_2F_A\left[\left.\begin{matrix}\frac{\nu+k+\gamma+1}{2},\frac{\nu+k+\gamma+2}{2}\\ \frac{BM+1}{A},\frac{BM+2}{A},\frac{BM+3}{A},...,\frac{BM+A}{A}\end{matrix}\right|\frac{-\mu^2}{\sigma^2}\right]$$

Proof. Let L.H.S. of (4.1) is I and applying definition (3.1), we get

$$I=\sum_{k=0}^{n}\frac{(-\beta)^k\Gamma(n+\alpha+1)(-1)^k\mu^{2k+\nu}}{k!\,\Gamma(n-k+1)\Gamma(\alpha+k+1)\Gamma(BM+Ak+1)}\int_0^\infty x^{2k+\nu}x^k x^\gamma e^{-\sigma x}dx, \tag{4.2}$$

We can write equation (4.2),

$$I=\sum_{k=0}^{n}\frac{(-\beta)^k\Gamma(n+\alpha+1)(-1)^k\mu^{2k+\nu}}{k!\,\Gamma(n-k+1)\Gamma(\alpha+k+1)\Gamma(BM+Ak+1)}\int_0^\infty e^{-\sigma x}x^{3k+\gamma+\nu}dx. \tag{4.3}$$

On using the Gamma matrix function [7], and evaluating the inner integral, this gives

$$\int_0^\infty e^{-\sigma x}x^{3k+\gamma+\nu}dx=\frac{\Gamma(3k+\nu+\gamma+1)}{\sigma^{\nu+3k+\gamma+1}}. \tag{4.4}$$

From equation (4.3) and (4.4),

$$I=\sum_{k=0}^{n}\frac{(-\beta)^k\Gamma(n+\alpha+1)(-1)^k\mu^{2k+\nu}\Gamma(3k+\nu+\gamma+1)}{k!\,\Gamma(n-k+1)\Gamma(\alpha+k+1)\Gamma(BM+Ak+1)\sigma^{\nu+3k+\gamma+1}}. \tag{4.5}$$

Subsequently, using Legendre duplication formula [10, p. 276], this leads to

$$\Gamma(3k+\nu+\gamma+1)=\frac{2^{3k+\nu+\gamma+1}}{\sqrt{\pi}}\Gamma\left(\frac{\nu+3k+\gamma+1}{2}\right)\Gamma\left(\frac{\nu+3k+\gamma+2}{2}\right), \tag{4.6}$$

where,

$$\Gamma\left(\frac{\nu+2k+k+\gamma+1}{2}\right)=\left(\frac{\nu+k+\gamma+1}{2}\right)_k\left(\frac{\nu+k+\gamma+1}{2}\right), \tag{4.7}$$

and

$$\Gamma\left(\frac{\nu+2k+k+\gamma+2}{2}\right)=\left(\frac{\nu+k+\gamma+2}{2}\right)_k\left(\frac{\nu+k+\gamma+2}{2}\right). \tag{4.8}$$

On using (4.6) to (4.8), one can obtain

$$\frac{2^{3k+\nu+\gamma+1}\left(\frac{\nu+k+\gamma+1}{2}\right)\left(\frac{\nu+k+\gamma+2}{2}\right)}{\sqrt{\pi}}=\Gamma\left(\nu+\gamma+k+1\right). \tag{4.9}$$

From (4.6) to (4.9) and using Legendre Duplication formula [10, P. 276] , we obtain

$$I=\sum_{k=0}^{n}\frac{(-\beta)^k\Gamma\left(n+\alpha+1\right)(-1)^k\mu^{2k+\nu}\Gamma\left(BM+1\right)\Gamma\left(\nu+\gamma+k+1\right)2^{2k}}{k!\,\Gamma\left(n-k+1\right)\Gamma\left(\alpha+k+1\right)\Gamma\left(BM+Ak+1\right)2^{2k+\nu}\Gamma\left(BM+1\right)\sigma^{\nu+3k+\gamma+1}}\times\left(\frac{\nu+k+\gamma+1}{2}\right)_k\left(\frac{\nu+k+\gamma+2}{2}\right)_k, \tag{4.10}$$

The use of (1.11), lead to desired result (4.1). □

5 Integral involving Jacobi matrix polynomial with ${}^{B}_{A}J_M$

In this section, we derive two stimulating integrals involving the product of ${}^{B}_{A}J_M$ with Jacobi matrix polynomial $P_n^{(A,B)}(x)$ which are expressed in terms of Kampé de Fériet and Srivastava and Daoust matrix functions.

Theorem 5.1. *For $\Re(\nu)>-1$, $\nu\in\sigma(A)$, $0<\Re(\mu)$, $\Re(\mu)\in\sigma(P)$, the following integral formula in terms of Kampé de Fériet holds true:*

$$\begin{aligned}
&\int_0^\infty t^{K-I}(1+t)^{-(K+L)}\,{}^{B}_{A}J_M\left(\frac{x}{1+t}\right)P_n^{(C,D)}\left(1-\frac{zx}{1+t}\right)dt\\
&=\Theta\left\{F^{2:0:4}_{2:0:5}\left[\begin{array}{c}\Delta\left(2;M+L\right):--;\Delta\left(2;-nI\right),\Delta\left(2;C+D+(n+1)I\right)\\ \Delta\left(2;M+K+L\right):\Delta\left(2A:BM+I\right),\Delta\left(2;C+I\right),\frac{1}{2}I\end{array}\middle|\frac{-x^2}{4},\frac{z^2x^2}{4}\right]\right.\\
&+\frac{nxz}{2}\left(M+L\right)\left(C+D+(n+1)I\right)\left(M+K+L\right)^{-1}(C+I)^{-1}(BM+I)_1\\
&\left.\times F^{2:0:4}_{2:0:5}\left[\begin{array}{c}\Delta\left(2;M+K+L\right):--;\Delta\left(2;-nI+I\right),\Delta\left(2;C+D+(n+2)I\right)\\ \Delta\left(2;M+K+L+2I\right):\Delta\left(2A:BM+2I\right),\Delta\left(2;C+2I\right),\frac{3}{2}I\end{array}\middle|\frac{-x^2}{4},\frac{z^2x^2}{4}\right]\right\},
\end{aligned} \tag{5.1}$$

where $\Theta=\frac{1}{n!}(C+I)_n\Gamma^{-1}\left(BM+I\right)\Gamma^{-1}\left(M+K+L\right)\Gamma\left(M+L\right)\Gamma\left(P\right)\left(\frac{x}{2}\right)^M$ *and*

$$\Delta\left(m;S\right)=\frac{S}{m},\ \frac{S+I}{m},...,\frac{S+mI+I}{m};\ m\geqslant 1.$$

Proof. We denote the first part of postulate (5.1) by I, further employing (3.1) and (1.13), we get,

$$
\begin{aligned}
I = & \int_0^\infty t^{K-I}(1+t)^{-(K+L)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma^{-1}\left(BM + Ak + I\right) \left(\frac{x}{2\,(1+t)}\right)^{2mI+M} \\
& \times \sum_{k=0}^{n} \frac{(C+I)_n (C+D+(n+1)\,I)_k (-nI)_k [(C+I)_k]^{-1}}{n!\,k!} \left(\frac{1-\left(1-\frac{zx}{1+t}\right)}{2}\right)^k .
\end{aligned} \tag{5.2}
$$

On computing, we find that

$$
\begin{aligned}
I = & \sum_{m=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^m \Gamma^{-1}\left(BM + Ak + I\right) (\beta + I)_n (C+D+(n+1)\,I)_k (-nI)_k [(C+I)_k]^{-1}}{n!\,m!\,k!} \\
& \times \left(\frac{x}{2}\right)^{2mI+M} \left(\frac{zx}{2}\right)^k \int_0^\infty t^{P-I}(1+t)^{-(K+L+M+kI+2mI)} dt
\end{aligned} \tag{5.3}
$$

From (1.9) and (5.3), we arrive at,

$$
\begin{aligned}
I = & \sum_{m=0}^{\infty} \sum_{k=0}^{n} \frac{(C+I)_n (C+D+(n+1)\,I)_k (-nI)_k [(C+I)_k]^{-1}}{n!\,m!\,k!} \left(\frac{-x^2}{4}\right)^m \left(\frac{zx}{2}\right)^k \\
& \times \left[(M+K+L)_{2m+k}\right]^{-1} (M+L)_{2m+k} \Gamma\left(K\right) \Gamma\left(M+L\right) \Gamma^{-1}\left(M+K+L\right) \left(BM+I\right)_{Ak}^{-1} \Gamma^{-1}\left(BM+I\right).
\end{aligned} \tag{5.4}
$$

Now, on splitting the k- series into in even and odd terms [19, section 20, P. 200], we get,

$$
\begin{aligned}
I = \Theta \Bigg\{ & \sum_{m=0}^{\infty} \sum_{k=0}^{n} \frac{\left[(M+K+L)_{2m+2k}\right]^{-1} (M+L)_{2m+2k} \left(BM+I\right)_{2Ak}^{-1}}{m!\ (2k)!} \\
& \times (C+D+(n+1)\,I)_{2k} (-nI)_{2k} [(C+I)_{2k}]^{-1} \left(\frac{-x^2}{4}\right)^m \left(\frac{-zx}{2}\right)^{2k} \\
& + \sum_{m=0}^{\infty} \sum_{k=0}^{n} \frac{\left[(M+K+L)_{2m+2k+I}\right]^{-1} (M+L)_{2m+2k+I} \left(BM+I\right)_{2Ak+2A}^{-1}}{m!\ (2k+1)!} \\
& \times (C+D+(n+1)\,I)_{2k+1} (-nI)_{2k+1} \left[(C+I)_{2k+1}\right]^{-1} \left(\frac{-x^2}{4}\right)^m \left(\frac{-zx}{2}\right)^{2k+1} \Bigg\} .
\end{aligned} \tag{5.5}
$$

From (1.19), (1.20) and (5.5), yields,

$$I = \Theta\left\{\sum_{m=0}^{\infty}\sum_{k=0}^{n}\frac{\left[(M+K+L)_{2m+2k}\right]^{-1}(-nI)_{2k}(C+D+(n+1)\,I)_{2k}[(C+I)_{2k}]^{-1}\left(\frac{I}{2}\right)_k^{-1}}{m!}\right.$$

$$\times(M+L)_{2m+2k}\,(BM+I)_{2Ak}^{-1}+\frac{nxz}{2}\,(M+L)\,(C+D+(n+1)\,I)\,(M+K+L)^{-1}(C+I)^{-1}$$

$$(BM+I)_1+\sum_{m=0}^{\infty}\sum_{k=0}^{n}\frac{\left[(M+K+L)_{2m+2k}\right]^{-1}(-nI+I)_{2k}(C+D+(n+2)\,I)_{2k}[(M+K+L+2I)_{2k}]^{-1}}{m!\,(2k+1)!}$$

$$\left.\times\,(BM+I)_{2Ak+1}^{-1}\,[(C+I)_{2k}]^{-1}\left(\frac{3I}{2}\right)_k^{-1}\left(\frac{-x^2}{4}\right)^{2k}\left(\frac{zx}{2}\right)^{2k}\right\},\tag{5.6}$$

afterwards applying (1.14), leads to (5.1). □

Theorem 5.2. *For $\Re(\nu)>-1$, $\nu\in\sigma(A)$, $0<\Re(\mu)$, $\Re(\mu)\in\sigma(P)$, the following integral formula in terms of Srivastava and Daoust function holds true.*

$$\int_0^{\infty} t^{P-I}(1+t)^{-(P+Q)}\,{}_A^BJ_M\left(\frac{x}{1+t}\right)P_n^{(B,C)}\left(1-\frac{zx}{1+t}\right)dt$$
$$=\left(\frac{x}{2}\right)^M\Gamma(P)\,\Gamma^{-1}(M+Q)\,\Gamma^{-1}(BM+I)\,\Gamma^{-1}(M+P+Q)$$
$$\times F_{4:0:1}^{2:0:0}\left[\begin{matrix}(M+Q:2,3)\,,(B+C+I:1,2)\,;\,\text{\_\_\_\_};\,\text{\_\_\_}\\(M+P+Q:2,3)\,,\Delta\,(BM+I:A,A)\,,(B+C+I:1,1)\,,(1:1,1):\,\text{\_\_};\,(B+I:1)\end{matrix}\middle|\,\frac{-x^2}{4},\frac{zx^3}{8}\right]\tag{5.7}$$

Proof. We consider the L.H.S. of assertion (5.7) as I and applying (3.1) and (1.13), afterwards using following lemma [19]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A\,(n,k)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A\,(n+k,k),\tag{5.8}$$

and further interchanging the sequence of integration and summation, we get

$$I=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+k+k}\Gamma^{-1}\,(BM+A\,(n+k)+I)\,(B+I)_{n+k}(B+C+(n+k+1)\,I)_k(-n-k)_k}{n!k!\,(n+k)!}$$

$$[(B+I)_k]^{-1}\left(\frac{x}{2}\right)^{M+2nI+2kI}\left(\frac{zx}{2}\right)^k\int_0^{\infty}t^{P-I}(1+t)^{-(M+P+Q+2nI+3kI)}dt.\tag{5.9}$$

From (1.9) and (5.9), we get

$$I=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+2k}\Gamma^{-1}\left(BM+A\left(n+k\right)+I\right)\Gamma\left(BM+I\right)\left(B+I\right)_{n+k}\left(B+C+\left(n+k+1\right)I\right)_{k}}{\Gamma\left(BM+I\right)n!k!\left(n+k-1\right)!}$$
$$\left[(B+I)_{k}\right]^{-1}\left(\frac{x}{2}\right)^{M+2nI+2kI}\left(\frac{zx}{2}\right)^{k}\int_{0}^{1}t^{P-I}(1-t)^{M+Q+3k+2n-1}dt. \tag{5.10}$$

On employing the definition of Beta matrix function (1.8), we obtain

$$I=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+2k}\left[(BM+I)_{A(n+k)}\right]^{-1}\Gamma^{-1}\left(BM+I\right)\left(B+C+I\right)_{2k+n}\left[\left((B+C+I)_{k+n}\right)\right]^{-1}}{n!k!(1)_{k+n}}$$
$$\left[(B+I)_{k}\right]^{-1}\left(\frac{x}{2}\right)^{M+2nI+2kI}\left(\frac{zx}{2}\right)^{k}\frac{\Gamma\left(P\right)\Gamma\left(M+Q+3kI+2nI\right)}{\Gamma\left(P+M+Q+3kI+2nI\right)}. \tag{5.11}$$

After simplification leads to,

$$I=\left(\frac{x}{2}\right)^{M}\Gamma\left(P\right)\Gamma^{-1}\left(M+Q\right)\Gamma^{-1}\left(BM+I\right)\Gamma^{-1}\left(M+P+Q\right)$$
$$\times\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M+Q)_{2nI+3kI}(B+C+I)_{2k+n}}{n!k!\left(1\right)_{k+n}}\left[(M+P+Q)_{2nI+3kI}\right]^{-1}\left[(BM+I)_{A(n+k)}\right]^{-1}$$
$$\times\left[\left((B+C+I)_{k+n}\right)\right]^{-1}\left[(B+I)_{k}\right]^{-1}\left(\frac{-x^{2}}{4}\right)^{n}\left(\frac{zx^{3}}{8}\right)^{k}, \tag{5.12}$$

on account of (1.15), this leads to assertion (5.7). □

6 Concluding remark

We obtained an Extended Generalized Wright Hypergeometric function and Extended Generalized Bessel matrix function. Some properties of these extended functions such as convergence condition , integral formula, and integral representations have been obtained. Obtained results can play important role in Mathematical Physics and classical analysis.

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