Existence and concentration of positive solutions for a fractional Schr\”odinger logarithmic equation

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Abstract

In this paper, we study the existence and concentration of positive solutions for the following fractional Schr\”odinger logarithmic equation: \begin{equation*} \left\{ \begin{aligned} & \varepsilon^{2s} (-\Delta)^{s} u+V(x)u =u\log u^2,\ x\in \mathbb{R}^N,\\ &u\in H^s(\mathbb{R}^N), \end{aligned} \right. \end{equation*} where $\varepsilon > 0$ is a small parameter, $N>2s,$ $s \in (0 ,1), (-\Delta)^{s}$ is the fractional Laplacian, the potential V is a continuous function having a global minimum. Using variational method to modify the nonlinearity with the sum of a $C^1 $ functional and a convex lower semicontinuous functional, we prove the existence of positive solutions and concentration around of a minimum point of V when ε tends to zero.

Existence and concentration of positive solutions for a fractional Schrödinger logarithmic equation *

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Abstract

In this paper, we study the existence and concentration of positive solutions for the following fractional Schrödinger logarithmic equation:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = u \log u^2, \ x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $N > 2s$, $s \in (0,1)$, $(-\Delta)^s$ is the fractional Laplacian, the potential V is a continuous function having a global minimum. Using variational method to modify the nonlinearity with the sum of a C^1 functional and a convex lower semicontinuous functional, we prove the existence of positive solutions and concentration around of a minimum point of V when ε tends to zero.

Keywords: Fractional Schrödinger logarithmic problem, Variational methods, Positive solutions.

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1 Introduction and main results

In this paper, we investigate the existence and concentration of a positive solution for the following fractional Schrödinger logarithmic equation:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = u \log u^2, \ x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $N > 2s$, $s \in (0,1)$, $(-\Delta)^s$ is the fractional Laplacian which may be defined for any $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough by setting

$$(-\Delta)^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \mathrm{d}y, \ x \in \mathbb{R}^N \tag{1.2}$$

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along functions $u \in C_0^\infty(\mathbb{R}^N)$, $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$. The fractional Laplace operator can be viewed as the infinitesimal generators of a Lévy stable diffusion processes (see [7]). This operator arises in the description of various phenomena in the applied sciences, such as phase transitions, materials science, conservation laws, minimal surfaces, water waves, optimization, plasma physics and so on. Please see [7, 14, 20] and the references therein for a more detailed introduction.

The fractional logarithmic Schrödinger equation (1.1) is a generalization of the classical nonlinear Schrödinger equation with logarithmic nonlinearity [12]. When $s = 1$, (1.1) stems from the classical logarithmic Schrödinger equation

$$iu_t + \Delta u + u \log |u|^2 = 0. \tag{1.3}$$

The classical logarithmic Schrödinger equation has been ruled out as a fundamental quantum wave equation by very accurate experiments on neutron diffraction, it has been extensively studied in the mathematical and physical literature (see [9], [12], [13], [18] and the references therein). Thus, it is natural for us to consider the logarithmic Schrödinger equation with fractional Laplacian.

Recently, by means of nonsmooth critical point theory, d'Avenia et al. [15] studied the existence of multiple standing waves solutions to the following fractional logarithmic Schrödinger equations of the type

$$\begin{cases} (-\Delta)^s u + \omega u = u \log u^2, \ x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N) \end{cases} \tag{1.4}$$

for $s \in (0, 1)$ and $N > 2s$. They also investigated the Hölder regularity of the weak solutions. Zhang and Hu in [29] used the fractional logarithmic Sobolev inequality and Galerkin method constructing and estimating the norm of the approximate solutions, they gave some properties of the family of potential wells and obtained existence of global solution for a initial boundary value problem for a class of fractional logarithmic Schrödinger equation. The analysis of concentration phenomenon of solutions for the nonlinear fractional Schrödinger equation

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u), \ x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N) \end{cases} \tag{1.5}$$

has attracted the attention from many researchers. Dávila et al. in [16] proved that (1.5) has multi-peak solutions via Liapunov-Schmidt reduction method when $f(u) = u^p$ with $p \in (1, 2_s^* - 1)$ and the potential V verifies the following conditions:

$$V \in C^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ and } \inf_{x \in \mathbb{R}^N} V(x) > 0.$$

By means of the Lyusternik-Shnirelmann and Morse theories, Figueiredo and Siciliano in [17] proved a multiplicity result for (1.5) with $f \in C^1$ and satisfying some additional hypotheses. He and Zou ([19]) investigated the existence and the concentration of positive solutions for a class of fractional Schrödinger equations involving the critical Sobolev exponent. Shen et al. in [23] investigated the existence of ground state solutions for a fractional Choquard equation involving a general nonlinearity. With the penalization method and the Ljusternik-Schnirelmann theory, Ambrosio in [6] got the multiplicity of positive solutions of (1.5) under the some assumptions f and the potential V is a positive continuous potential with local minimum. In [5] the author used penalization technique and Ljusternik-Schnirelmann theory to study the multiplicity and concentration of positive solutions to (1.5) when the potential V has a local minimum. It is quite natural to ask: what's going to happen with logarithmic nonlinear terms in (1.5) ?

When $s = 1$, (1.1) reduces to be

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = u \log u^2, \ x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{1.6}$$

Alves and deMorais Filho in [1] established the existence and concentration of positive solutions to problem (1.6) when $V(x)$ is a continuous function verifying the following condition

(V_1) $\quad V_\infty := \liminf_{|x|\to+\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) = V_0.$

Later, Alves and Ji in [3] considered the multiple positive solutions to problem (1.6) under the same assumption (V_1). In rapid sequence, they in [2] got the existence and concentration of positive solutions for (1.6) via penalization method when $V(x)$ satisfies a local assumption:

(V_2) $\quad V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > -1$;

(V_3) There exists an open and bounded set $\Lambda \subset \mathbb{R}^N$ satisfying

$$-1 < V_0 = \inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

Motivated by studies found in the above mentioned papers, in the present paper we intend to study the existence and concentration of positive solution for the problem (1.1), where the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function verifying the condition (V_1). Here, we will consider only the case $V_\infty < \infty$, since the case $V_\infty = \infty$ is simpler, due to compact embeddings in the Lebesgue spaces $L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*)$. Moreover, using the same reasoning of [27], we can assume without loss of generality that $V_0 > -1$. To the best of our knowledge, this is a new attempt to study concentration of positive solutions for a fractional Schrödinger equation involving logarithmic nonlinearity.

By a change of variable, (1.1) is equivalent to the easier handle equation

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u = u \log u^2, \ x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N). \end{cases} \tag{1.7}$$

The weak solutions to (1.7) can be found as critical points of the Euler-Lagrange functional I_ε defined by

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) + 1)|u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x.$$

Unfortunately, due to the singularity of the logarithm at the origin, the functional fails to be finite as well of class C^1 on $H^s(\mathbb{R}^N)$. Due to this loss of smoothness, it is convenient to work in a suitable Banach space endowed with a Luxemburg type norm in order to make functional I_ε well defined and C^1 smooth. This space allows to control the singularity of the logarithmic nonlinearity at infinity and at the origin. Aiming this approach, we consider the reflexive the Banach space

$$H^s_\varepsilon(\mathbb{R}^N) = \Big\{ u \in H^s(\mathbb{R}^N); \int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 \mathrm{d}x < +\infty \Big\}$$

with the norm

$$\|u\|_\varepsilon = \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + (V(\varepsilon x) + 1)|u|^2) \mathrm{d}x \Big)^{\frac{1}{2}},$$

then the energy functional I_ε is well-defined and of class C^1 on H^s_ε.

We say that $u \in H^s_\varepsilon$ is a (weak) solution of problem (1.7) if $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ (i.e., $I_\varepsilon(u) < \infty$) and for any $v \in H^s_\varepsilon$,

$$\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(\varepsilon x) uv) \mathrm{d}x = \int_{\mathbb{R}^N} uv \log u^2 \mathrm{d}x. \tag{1.8}$$

Now we state the main result of this paper:

Theorem 1.1. *Suppose that the potential V satisfies (V_1). Then there is an $\varepsilon_0 > 0$ such that problem (1.1) has a positive solution $u_\varepsilon \in H^s_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, if u_ε denotes one of these solutions and x_ε is a global maximum point of u_ε, then we have $V(x_\varepsilon) \to V_0$ as $\varepsilon \to 0$.*

Remark 1.1. *When the potential V satisfies $(V_2) - (V_3)$, how about the existence and concentration of positive solutions for problem (1.1)? That is something we are going to consider later.*

Let us point out some comments on the approach we chose to prove Theorem 1.1 and the difficulties we faced. The existence of the logarithmic nonlinearity leads the energy functional associated is not continuous. There are functions $u \in H^s(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u \log u^2 \mathrm{d}x = +\infty$, then the energy functional I_ε fails to be finite and C^1 smooth on $H^s_\varepsilon(\mathbb{R}^N)$, some estimates for this problem are very delicate and different from those used in the Schrödinger equation (1.5) . In the present paper, we shall modify the nonlinearity in a special way and then to work with a modified problem. In order to prove that the solutions obtained for the modified problem are solutions of the original problem we shall make some estimates when $\varepsilon > 0$ is sufficient small. For our study of the logarithmic Schrödinger equations, the equality of the type $I_\varepsilon(u) - \frac{1}{2}\langle I'_\varepsilon(u), u\rangle = \frac{1}{2}\int_{\mathbb{R}^N} u^2 \mathrm{d}x$ is very important.

The outline of the present paper is as follows. In Section 2, we give some notations and recall some useful lemmas for the fractional Sobolev spaces. In Section 3, some preliminaries of problem (1.1) are given. In section 4, we study the autonomous problem of (1.1). Section 5 is devoted to prove the existence of solution for (1.7) when ε is small enough. In the last section, we provide a existence result for (1.1).

2 Variational framework for problem (1.1)

In this section, we outline the variational framework for problem (1.1) and give some preliminary Lemmas. Now we give some notations about $H^s(\mathbb{R}^N)$ first.

For $s \in (0,1)$, we define the homogeneous fractional Sobolev space $D^{s,2}(\mathbb{R}^N, \mathbb{R})$ by

$$D^{s,2}(\mathbb{R}^N) = \Big\{ u \in L^{2_s^*}(\mathbb{R}^N) : \ |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^N) \Big\},$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the seminorm

$$[u]_s = \Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \Big)^{\frac{1}{2}} = \Big(\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \mathrm{d}\xi \Big)^{\frac{1}{2}}.$$

It is well known that $H^s(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$ for $2 \le p \le 2_s^*$, and for any $s \in (0,1)$, there exists a best constant $S_s > 0$ such that

$$S_s = \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x}{\Big(\int_{\mathbb{R}^N} |u|^{2_s^*} \mathrm{d}x \Big)^{2/2_s^*}}. \tag{2.1}$$

The fractional Sobolev space $H^s(\mathbb{R}^N)$ can be described by means of the Fouier transform as follows

$$H^s(\mathbb{R}^N) = \Big\{ u \in L^2(\mathbb{R}^N) : \ \int_{\mathbb{R}^N} (|\xi|^{2s} + 1) |\hat{u}|^2 \mathrm{d}\xi < \infty \Big\},$$

which is endowed with the standard scalar product and norm

$$(u,v) = \int_{\mathbb{R}^N} (|\xi|^{2s} + 1) \hat{u} \bar{\hat{v}} \mathrm{d}\xi, \quad \|u\|_{H^s}^2 = \int_{\mathbb{R}^N} (|\xi|^{2s} + 1) |\hat{u}|^2 \mathrm{d}\xi.$$

In view of Plancherel's theorem, we have

$$(u,v) = \int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv) \mathrm{d}x, \quad \|u\|_{H^s}^2 = \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2) \mathrm{d}x.$$

In order to overcome the lack of smoothness of I_ε, we decompose it into a sum of a C^1 functional plus a convex lower semicontinuous functional by following the approach explored in [21] and [25]. For $\delta > 0$, let us define the following functions:

$$F_1(\tau) = \begin{cases} 0, & \tau = 0, \\ -\frac{1}{2}\tau^2 \log \tau^2, & 0 < |\tau| < \delta, \\ -\frac{1}{2}\tau^2 (\log \delta^2 + 3) + 2\delta|\tau| - \frac{1}{2}\delta^2, & |\tau| \ge \delta, \end{cases}$$

and

$$F_2(\tau) = \begin{cases} 0, & 0 \le |\tau| < \delta, \\ \frac{1}{2}\tau^2 \Big(\log \frac{\tau^2}{\delta^2} - 3 \Big) + 2\delta|\tau| - \frac{1}{2}\delta^2, & |\tau| \ge \delta. \end{cases}$$

Therefore,

$$F_2(\tau) - F_1(\tau) = \frac{1}{2}\tau^2 \log \tau^2, \quad \forall \tau \in \mathbb{R}, \tag{2.2}$$

and the functional I_ε can be rewritten as

$$I_\varepsilon(u) = \Phi_\varepsilon(u) + \Psi(u), u \in H^s_\varepsilon, \tag{2.3}$$

where

$$\Phi_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} [|(-\Delta)^{\frac{s}{2}} u|^2 + (V(\varepsilon x)+1)|u|^2]\mathrm{d}x - \int_{\mathbb{R}^N} F_2(u)\mathrm{d}x, \tag{2.4}$$

and

$$\Psi(u) = \int_{\mathbb{R}^N} F_1(u)\mathrm{d}x. \tag{2.5}$$

Remark 2.1. *From [21] and [25], we get*

$$F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R}). \tag{2.6}$$

If $\delta > 0$ is small enough,

$$F_1 \textit{ is convex}, \quad F_1 \textit{ is even and } F_1'(\tau)\tau \geq 0, \ \tau \in \mathbb{R}, \tag{2.7}$$

and

$$\textit{the function } F_1(\tau) \geq 0 \textit{ and then } \Psi(\tau) \geq 0, \ \forall \tau \in H^s_\varepsilon. \tag{2.8}$$

Remark 2.2. *By a simple observation, it is easy to see that*

$$\begin{aligned} &\frac{F_2'(\tau)}{\tau} \textit{ is nondecreasing for } \tau > 0, \\ &\frac{F_2'(\tau)}{\tau} \textit{ is strictly increasing for } \tau > \delta, \\ &\lim_{\tau \to +\infty} \frac{F_2'(\tau)}{\tau} = +\infty, \end{aligned} \tag{2.9}$$

and

$$F_2'(\tau) \geq 0 \textit{ for } \tau > 0 \textit{ and } F_2'(\tau) > 0 \textit{ for } \tau > \delta.$$

Hereafter, $\delta > 0$ is fixed in such a way that the above properties hold. For each fixed $p \in (2, 2^*_s)$, there is $C > 0$ such that

$$|F_2'(\tau)| \leq C|\tau|^{p-1}, \ \forall \ \tau \in \mathbb{R}. \tag{2.10}$$

Using above Remarks, it follows that $\Phi_\varepsilon \in C^1(H^s_\varepsilon, \mathbb{R})$, and Ψ is convex and lower semicontinuous, but Ψ is not a C^1 functional since we are working on $\mathbb{R}^N$. Due to this fact, we recall a kind of critical point theorem (Theorem 2.4). Now we need some definitions that may be firstly appeared in [26].

Definition 2.1. *Suppose that E be a Banach space, E' be the dual space of E and $\langle \cdot, \cdot \rangle$ be the duality paring between E' and E. Let $J(u) = \Phi(u) + \Psi(u), \forall u \in E$, where $\Phi \in C^1(E, \mathbb{R})$ and Ψ is convex and lower semicontinuous.*

(i) *The sub-differential $\partial J(u)$ of the functional J at a point $u \in E$ is the following set*

$$\{w \in E' : \langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \geq \langle w, v-u\rangle,\ \forall v \in E\}. \tag{2.11}$$

(ii) *A critical point of J is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.,*

$$\langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \geq 0,\ \forall v \in E. \tag{2.12}$$

(iii) *A Palais-Smale sequence at level c for J is a sequence $\{u_n\} \subset E$ such that $J(u_n) \to c$ and there is a numerical sequence $\tau_n \to 0^+$ with*

$$\langle \Phi'(u_n), v-u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n\|v-u_n\|,\ \forall v \in E. \tag{2.13}$$

(iv) *The functional J satisfies the Palais-Smale condition at level c ($(PS)_c$ condition for short) if all Palais-Smale sequences at level c has a convergent subsequence.*

(v) *The effective domain of J is the set $D(J) = \{u \in E : J(u) < +\infty\}$.*

In the sequel, we list some properties on I_ε which canbe found in [21, 25, 27].

Lemma 2.1. *Let I_ε satisfy (2.3), then:*

(i) *If $u \in D(I_\varepsilon)$ is a critical point of I_ε, then*

$$\langle I'_\varepsilon(u), v-u\rangle = \langle \Phi'_\varepsilon(u), v-u\rangle + \Psi(v) - \Psi(u) \geq 0,\ \forall v \in H^s_\varepsilon, \tag{2.14}$$

that is

$$\begin{aligned}\int_{\mathbb{R}^N} [(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}(v-u) + (V(\varepsilon x)+1)u(v-u)]\mathrm{d}x - \int_{\mathbb{R}^N} F_1(v)\mathrm{d}x - \int_{\mathbb{R}^N} F_1(u)\mathrm{d}x \\ \geq \int_{\mathbb{R}^N} F_2'(u)(v-u)\mathrm{d}x, \quad \forall\ v \in H^s_\varepsilon.\end{aligned}$$

(ii) ([*25, 27*]) *For each $u \in D(I_\varepsilon)$ such that $\|I_\varepsilon(u)\| < +\infty$, we have $\partial I_\varepsilon(u) \neq \emptyset$, that is, there is $w \in (H^s_\varepsilon)'$, which is denoted by $w = I'_\varepsilon(u)$, such that*

$$\langle \Phi'_\varepsilon(u), v-u\rangle + \int_{\mathbb{R}^N} F_1(v)\mathrm{d}x - \int_{\mathbb{R}^N} F_1(u)\mathrm{d}x \geq \langle w, v-u\rangle, \forall\ v \in H^s_\varepsilon.$$

(iii) ([*21*] *Lemma 2.4* (i)) *If a function $u \in D(I_\varepsilon)$ is a critical point of I_ε, then u is a solution of (1.7).*

(iv) ([*21*] *Lemma 2.4* (ii)) *If $(u_n) \subset H^s_\varepsilon$ is a Palais-Smale sequence, then*

$$\langle I'_\varepsilon(u_n), z\rangle = o_n(1)\|z\|_\varepsilon,\ \forall\ z \in H^s_\varepsilon(\mathbb{R}^N). \tag{2.15}$$

(v) ([*25*] *Lemma 2.2*) *If Ω is a bounded domain with regular boundary then Ψ (and hence I_ε) is of class C^1 in $H^s(\Omega)$.*

We are going to give a very useful conclusion which will be used later. Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \le \phi(x) \le 1, x \in \mathbb{R}^N$;

$$\phi(x) = \begin{cases} 1, & \text{for } |x| \le 1, \\ 0, & \text{for } |x| \ge 2. \end{cases}$$

For a given $R > 0$ and $u \in D(I_\varepsilon)$, let us define $\phi_R(x) = \phi(\frac{x}{R})$ and $u_R(x) = \phi_R(x)u(x)$. Then we have the following preliminary result.

Lemma 2.2. *For any $\varepsilon > 0, u_R \to u$ in H_ε^s as $R \to +\infty$.*

Proof. It is readily seen that

$$\begin{aligned}
[u_R - u]_s^2 &= \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{[(u(x) - u(y))(\phi_R(x) - 1) + u(y)(\phi_R(x) - \phi_R(y))]^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\le 2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2 (\phi_R(x) - 1)^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\quad + 2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(y)|^2 (\phi_R(x) - \phi_R(y))^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{2.16}$$

Note that $u \in H_\varepsilon^s, |\phi_R(y) - 1| \le 2$ and $\phi_R(y) - 1 \to 0$ a.e. as $R \to \infty$. Then, the Dominated Convergence Theorem yields

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2 (\phi_R(x) - 1)^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \to 0. \tag{2.17}$$

In the following, we will prove that

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(y)|^2 (\phi_R(x) - \phi_R(y))^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \to 0. \tag{2.18}$$

We present the proof process here for completeness even if the proof is similar to Lemma 2.1 in [4].

Note that

$$\begin{aligned}
\mathbb{R}^{2N} &= ((\mathbb{R}^N - B_{2R}) \times (\mathbb{R}^N - B_{2R})) \cup (\mathbb{R}^N \times B_{2R}) \cup (B_{2R} \times (\mathbb{R}^N - B_{2R})) \\
&=: X_R^1 \cup X_R^2 \cup X_R^3.
\end{aligned}$$

Then

$$\begin{aligned}
&\iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(y)|^2 \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&= \iint_{X_R^1} |u(y)|^2 \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \iint_{X_R^2} |u(y)|^2 \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&\quad + \iint_{X_R^3} |u(y)|^2 \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{2.19}$$

In what follows, we estimate each integral in (2.19).

(i) For $(x,y)\in X_R^1:=(\mathbb{R}^N-B_{2R})\times(\mathbb{R}^N-B_{2R})$. Since $\phi_R(x)=\phi_R(y)=0$ in $\mathbb{R}^N\setminus B_{2R}$, we have

$$\iint_{X_R^1}|u(y)|^2\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y=0. \tag{2.20}$$

(ii) For $(x,y)\in X_R^2:=\mathbb{R}^N\times B_{2R}$,

$$\begin{aligned}
&\iint_{X_R^2}|u(y)|^2\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&=\int_{B_{2R}}|u(y)|^2dy\int_{\{x\in\mathbb{R}^N:|x-y|\le 2R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&\quad+\int_{B_{2R}}|u(y)|^2dy\int_{\{x\in\mathbb{R}^N:|x-y|\ge 2R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&:=A_{2R}+A_{2R}^c.
\end{aligned}$$

By the definition of ϕ, using $0\le\phi\le 1$, there is $\xi=\frac{y}{R}+\tau\frac{x-y}{R}, \tau\in(0,1)$ such that

$$\begin{aligned}
A_{2R}&=\int_{B_{2R}}|u(y)|^2dy\int_{\{x\in\mathbb{R}^N:|x-y|\le 2R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&=\int_{B_{2R}}|u(y)|^2dy\int_{\{x\in\mathbb{R}^N:|x-y|\le 2R\}}\frac{|\nabla\phi(\xi)|^2|\frac{x-y}{R}|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&=R^{-2}|\nabla\phi|^2_{L^\infty(\mathbb{R}^N)}\int_{B_{2R}}|u(y)|^2\mathrm{d}y\int_{\{x\in\mathbb{R}^N:|x-y|\le 2R\}}\frac{1}{|x-y|^{N+2s-2}}\mathrm{d}x\\
&\le CR^{-2}\int_{B_{2R}}|u(y)|^2\mathrm{d}y\int_0^{2R}\frac{1}{r^{N+2s-2}}r^{N-1}\mathrm{d}r\\
&\le CR^{-2s}\int_{B_{2R}}|u(y)|^2\mathrm{d}y.
\end{aligned} \tag{2.21}$$

Since $0\le\phi(x)\le 1$, we get $|\phi_R(x)-\phi_R(y)|^2<4$, then

$$\begin{aligned}
A_{2R}^c&=\int_{B_{2R}}|u(y)|^2dy\int_{\{x\in R^N:|x-y|\ge 2R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&\le 4\int_{B_{2R}}|u(y)|^2\mathrm{d}y\int_{\{x\in\mathbb{R}^N:|x-y|>2R\}}\frac{1}{|x-y|^{N+2s}}\mathrm{d}x\\
&\le C\int_{B_{2R}}|u(y)|^2\mathrm{d}y\int_{2R}^{+\infty}\frac{1}{r^{N+2s}}r^{N-1}\mathrm{d}r\\
&\le CR^{-2s}\int_{B_{2R}}|u(y)|^2\mathrm{d}y.
\end{aligned}$$

From above two inequalities, we have

$$\iint_{X_R^2}\frac{|u(y)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\le CR^{-2s}\int_{B_{2R}}u^2(y)\mathrm{d}y\le CR^{-2s} \tag{2.22}$$

since u is bounded in H^s_ε.

(iii) For $(x,y)\in B_{2R}\times(\mathbb{R}^N-B_{2R})$,

$$\begin{aligned}
&\iint_{X_R^3}|u(y)|^2\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{R}^N\setminus B_{2R}}|u(y)|^2\mathrm{d}y\int_{\{x\in B_{2R}:|x-y|\le R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N\setminus B_{2R}}|u(y)|^2\mathrm{d}y\int_{\{x\in B_{2R}:|x-y|>R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&=:A_{R,\varepsilon}+B_{R,\varepsilon}.
\end{aligned}\tag{2.23}$$

If $|x-y|\le R$, then $|y|\le|x-y|+|x|\le 3R$. Hence, similar with (2.21), we get

$$\begin{aligned}
A_{R,\varepsilon}&=\int_{\mathbb{R}^N\setminus B_{2R}}|u(y)|^2\mathrm{d}y\int_{\{x\in B_{2R}:|x-y|\le R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&\le C\int_{B_{3R}}|u(y)|^2\mathrm{d}y\int_{\{x\in\mathbb{R}^N:|x-y|\le R\}}\frac{1}{|x-y|^{N+2s-2}}\mathrm{d}x\\
&\le CR^{-2}\int_{B_{3R}}|u(y)|^2\mathrm{d}y\int_0^R\frac{1}{r^{N+2s-2}}r^{N-1}\mathrm{d}r\\
&\le CR^{-2s}\int_{B_{3R}}|u(y)|^2\mathrm{d}y.
\end{aligned}\tag{2.24}$$

On the other hand, if $|x-y|\ge R$, we know that for any $K>4$, it gets $B_{2R}\times(\mathbb{R}^N-B_{2R})\subset(B_{2R}\times B_{KR})\cup(B_{2R}\times(\mathbb{R}^N-B_{KR}))$, thus we get

$$\begin{aligned}
B_{R,\varepsilon}&=\int_{B_{KR}}|u_n(y)|^2\mathrm{d}y\int_{\{x\in B_{2R}:|x-y|\ge R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N-B_{KR}}|u_n(y)|^2\mathrm{d}y\int_{\{x\in B_{2R}:|x-y|\ge R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&=:B^1_{R,\varepsilon}+B^2_{R,\varepsilon}.
\end{aligned}\tag{2.25}$$

It follows from $|\phi_R(x)-\phi_R(y)|^2<4$ that

$$\begin{aligned}
B^1_{R,\varepsilon}&=\int_{B_{KR}}|u_n(y)|^2\mathrm{d}y\int_{\{x\in B_{2R}:|x-y|\ge R\}}\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\\
&\le 4\int_{B_{KR}}|u(y)|^2\mathrm{d}y\int_{\{x\in B_{2R}:|x-y|\ge R\}}\frac{1}{|x-y|^{N+2s}}\mathrm{d}x\\
&\le C\int_{B_{KR}}|u(y)|^2\mathrm{d}y\int_R^{+\infty}\frac{1}{r^{N+2s}}r^{N-1}\mathrm{d}r\\
&\le CR^{-2s}\int_{B_{KR}}|u(y)|^2\mathrm{d}y.
\end{aligned}\tag{2.26}$$

For $(x,y)\in B_{2R}\times(\mathbb{R}^N-B_{KR})$, we have $|x-y|>|y|-|x|=\frac{|y|}{2}+\frac{|y|}{2}-|x|\ge\frac{|y|}{2}+\frac{KR}{2}-2R>$

$\frac{|y|}{2}$ since $K > 4$. Then

$$
\begin{aligned}
B^2_{R,\varepsilon} &= \int_{\mathbb{R}^N - B_{KR}} \mathrm{d}y \int_{\{x\in B_{2R}:|x-y|\geq R\}} |u(y)|^2 \frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x \\
&\leq C\int_{\mathbb{R}^N - B_{KR}} |u(y)|^2\mathrm{d}y \int_{\{x\in B_{2R}:|x-y|\geq R\}} \frac{1}{|x-y|^{N+2s}}\mathrm{d}x \\
&\leq CR^N \int_{\mathbb{R}^N - B_{KR}} \frac{|u(y)|^2}{|y|^{N+2s}}\mathrm{d}y \\
&\leq CR^N \Big(\int_{\mathbb{R}^N - B_{KR}} u^{2^*_s}(y)\mathrm{d}y\Big)^{\frac{2}{2^*_s}} \Big(\int_{\mathbb{R}^N - B_{KR}} |y|^{-(N+2s)\frac{2^*_s}{2^*_s-2}}\mathrm{d}y\Big)^{\frac{2}{2^*_s}} \\
&\leq CR^N \Big(\int_{\mathbb{R}^N - B_{KR}} u^{2^*_s}(y)\mathrm{d}y\Big)^{\frac{2}{2^*_s}} \Big(\int_{KR}^{+\infty} |r|^{-(N+2s)\frac{2^*_s}{2^*_s-2}} r^{N-1}\mathrm{d}r\Big)^{\frac{2}{2^*_s}} \\
&\leq CK^{-N} \Big(\int_{\mathbb{R}^N - B_{KR}} u^{2^*_s}(y)\mathrm{d}y\Big)^{\frac{2}{2^*_s}}.
\end{aligned}
\tag{2.27}
$$

Hence from (2.23) to (2.27), it follows that

$$
\iint_{X^3_R} |u(y)|^2 \frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \leq CR^{-2s} + CK^{-N} \tag{2.28}
$$

since u is bounded in H^s_ε. By combining (2.20), (2.22) and (2.28), we get

$$
\iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(y)|^2 \frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \leq CR^{-2s} + K^{-N} \tag{2.29}
$$

for some $C > 0$ independent of R and ε. Then, we can infer that

$$
\begin{aligned}
&\limsup_{R\to\infty} \iint_{\mathbb{R}^N\times\mathbb{R}^N} |u_n(y)|^2 \frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&= \limsup_{K\to\infty}\limsup_{R\to\infty} \iint_{\mathbb{R}^N\times\mathbb{R}^N} |u_n(y)|^2 \frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = 0.
\end{aligned}
$$

Hence, (2.18) is proved.

Note that, it follows from the Dominated Convergence Theorem that

$$
\int_{\mathbb{R}^N} V(\varepsilon x+1)|u_R(x)-u(x)|^2\mathrm{d}x = \int_{\mathbb{R}^N} V(\varepsilon x+1)(\phi_R(x)-1)^2u^2(x)\mathrm{d}x \to 0 \text{ as } R\to\infty.
$$

Thus $\|u_R - u\|_\varepsilon \to 0$ as $R\to\infty$. Thus the proof of the theorem is completed. □

As a consequence of the above properties, we have the following result:

Lemma 2.3. *If $u \in D(I)$ and $\|I'(u)\| < +\infty$, then $F_1'(u)u \in L^1(\mathbb{R}^N)$.*

Proof. Let $\xi \in C^\infty_0(\mathbb{R}^N)$ be such that $0 \leq \xi(x) \leq 1, x\in\mathbb{R}^N; \xi(x) = 1$ for $|x| \leq 1$, and $\xi(x) = 0$ for $|x| \geq 2$. For a given $R > 0$ and $u \in D(I)$, let us define $\xi_R(x) = \xi(\frac{x}{R})$ and $u_R(x) = \xi_R(x)u(x)$.

By (ii) in Lemma 2.1 ,

$$\langle \Phi_\varepsilon'(u), u_R\rangle + \int_{\mathbb{R}^N} F_1'(u)u_R dx = \langle w, u_R\rangle \tag{2.30}$$

for some $w \in H_\varepsilon^s$. Hence, since $u_R \to u$ in H_ε^s as $R \to +\infty$ from Lemma 2.2, by (2.30) and Lemma 2.1 (v), we get that $\int_{\mathbb{R}^N} F_1'(u)u_R dx \le C$ for $R > 0$ large enough.

Combing the convergence $u_R(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $R \to +\infty$ and (2.7), it follows from Fatou's Lemma that

$$0 \le \int_{\mathbb{R}^N} F_1'(u)u\mathrm{d}x \le \liminf_{R\to+\infty} \int_{\mathbb{R}^N} F_1'(u)u_R\mathrm{d}x \le C.$$

This inequality implies that $F_1'(u)u \in L^1(\mathbb{R}^N)$. □

An immediate consequence of the Lemma 2.3 is the following

Corollary 2.1. *For each $u \in D(I_\varepsilon) \setminus \{0\}$ with $\|I_\varepsilon(u)\| < +\infty$, we have*

$$\langle I_\varepsilon'(u), u\rangle = \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}u|^2 + V(\varepsilon x)|u|^2)\mathrm{d}x - \int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x,$$

and

$$I_\varepsilon(u) - \frac{1}{2}\langle I_\varepsilon'(u), u\rangle = \frac{1}{2}\int_{\mathbb{R}^N} u^2\mathrm{d}x. \tag{2.31}$$

Remark 2.3. *(2.31) is very important for our study, we will use fractional logarithmic Sobolev inequality to verify the boundedness of (PS) sequence(see Lemma 4.2).*

Corollary 2.2. *If $\{u_n\} \subset H_\varepsilon^s$ is a (PS) sequence for I_ε, then $\langle I_\varepsilon'(u_n), u_n\rangle = o_n(1)\|u_n\|_\varepsilon$. If $\{u_n\}$ is bounded, we have*

$$I_\varepsilon(u_n) = I_\varepsilon(u_n) - \frac{1}{2}\langle I_\varepsilon'(u_n), u_n\rangle + o_n(1)\|u_n\|_\varepsilon = \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 dx + o_n(1)\|u_n\|_\varepsilon.$$

Corollary 2.3. *If $u \in H_\varepsilon^s$ is a critical point of I_ε and $v \in H_\varepsilon^s$ verifies $F_1'(u)v \in L^1(\mathbb{R}^N)$, then $\langle I_\varepsilon'(u), v\rangle = 0$.*

The following lemma is a variant of the Brézis-Lieb lemma from [10], the proof follows along the same lines as Lemma 3.1 in [24]. We omit the details here.

Lemma 2.4. *Let $\{u_n\}$ be a bounded sequence in $H^s(\mathbb{R}^N)$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$. Then $u \in H^s(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \Big(|u_n|^2 \log |u_n|^2 - |u_n - u|^2 \log |u_n - u|^2\Big)\mathrm{d}x = \int_{\mathbb{R}^N} |u|^2 \log |u|^2 \mathrm{d}x.$$

In order to get the boundedness of (PS) sequence, we recall the fractional logarithmic Sobolev inequality. For a proof we refer to [11].

Lemma 2.5. *Let f be any function in $H^s(\mathbb{R}^N)$ and $\alpha > 0$. Then*

$$\int_{\mathbb{R}^N} |f(x)|^2 \log\Big(\frac{|f(x)|^2}{\|f\|_{L^2}^2}\Big)\mathrm{d}x + \Big(N + \frac{N}{s}\log\alpha + \log\frac{s\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2s})}\Big)\|f\|_{L^2}^2 \leq \frac{\alpha^2}{\pi^s}\|(-\Delta)^{\frac{s}{2}} f\|_{L^2}^2. \quad (2.32)$$

At last of this section, we give a mountain pass theorem without (PS) condition which is a consequence of the Mountain Pass Theorem with (PS) condition due to Szulkin [26].

Theorem 2.4. *(Mountain Pass Theorem without (PS) condition) Let E be a real Banach space and $J : E \to (-\infty, +\infty]$ be a functional such that:*

(i) $J(u) = \Psi_0(u) + \Psi_1(u), u \in E$, with $\Psi_0 \in C^1(E, \mathbb{R})$ and $\Psi_1 : E \to (-\infty, +\infty]$ is convex, $\Psi_1 \not\equiv +\infty$ and is lower semicontinuous;

(ii) there exist constant $\rho, \alpha > 0$ such that $J(0) = 0$ and $J|_{\partial B_\rho} \geq \alpha$;

(iii) there exists some $e \in B_\rho(0)$ such that $J(e) \leq 0$.

If

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)),\ \Gamma = \{\gamma \in C([0, 1], E); \gamma(0) = 0, J(\gamma(1)) < 0\}, \quad (2.33)$$

then, for a given $\varepsilon > 0$ there is $u_\varepsilon \in E$ such that, for $\forall v \in E$,

$$\langle \Psi_0'(u_\varepsilon), v - u_\varepsilon\rangle + \Psi_1(v) - \Psi_1(u_\varepsilon) \geq -3\varepsilon\|v - u_\varepsilon\|, \quad (2.34)$$

and

$$J(u_\varepsilon) \in [c - \varepsilon, c + \varepsilon]. \quad (2.35)$$

From Theorem 2.4, let $\varepsilon = 1/n$, we can get following corollary.

Corollary 2.5. *Under the conditions of Theorem 2.4 , there is a $(PS)_c$ sequence $\{u_n\} \subset E$ for J, that is, $J(u_n) \to c$ and*

$$\langle \Psi_0'(u_n), v - u_n\rangle + \Psi_1(v) - \Psi_1(u_n) \geq -\tau_n\|v - u_n\|,\ \forall\ v \in E$$

with $\tau_n \to 0^+$.

3 The limiting problem

In this section, we consider the limiting problem associated with problem (1.7). Here, without loss of generality, we assume that $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > -1$.

Consider the problem

$$\begin{cases} (-\Delta)^s u + V_0 u = u\log u^2, \text{ in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N). \end{cases} \quad (3.1)$$

The corresponding energy functional associated to (3.1) is denoted by $I_0 : E_0 := E_0(\mathbb{R}^N) \to (-\infty, +\infty]$ and defined as

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} (V_0+1)|u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 \mathrm{d}x,$$

where the Banach space

$$E_0 = \Big\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_0|u|^2 \mathrm{d}x < +\infty\Big\},$$

with the norm

$$\|u\|_0 = \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}u|^2 + (V_0+1)|u|^2)\mathrm{d}x\Big)^{\frac{1}{2}},$$

In the sequel, we are going to find a solution for (3.1). The point is to find a solution which is the limit of a (PS) sequence. Let us start with the following lemmas about Mountain Pass theorem .

Lemma 3.1. *The functional I_0, defined with Φ_0 and Ψ in (2.4) and (2.5) respectively, satisfies the following Mountain Pass geometry:*
(i) there exist $\alpha, \rho > 0$ such that $I_0(u) \geq \alpha$ for any $u \in H^s_0(\mathbb{R}^N)$ with $\|u\|_0 = \rho$;
(ii) there exists $e \in H^s_0(\mathbb{R}^N)$ with $\|e\|_0 > \rho$ such that $I_0(e) < 0$.

Proof. (i): Since $F_1 \geq 0$, one has

$$\begin{aligned} I_0(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}u|^2 + (V_0+1)|u|^2)\mathrm{d}x + \int_{\mathbb{R}^N} F_1(u)\mathrm{d}x - \int_{\mathbb{R}^N} F_2(u)\mathrm{d}x, \\ &\geq \frac{1}{2}\|u\|_0^2 - \int_{\mathbb{R}^N} F_2(u)\mathrm{d}x. \end{aligned}$$

By (2.10), for some $\alpha > 0$ and $\|u\|_0 = \rho > 0$ small enough, we get

$$I_0(u) \geq \frac{1}{2}\|u\|_0^2 - C\|u\|_0^p \geq \alpha > 0 \ \text{ since } p \in (2, 2^*_s).$$

(ii): Let us fix $u \in D(I_0) \setminus \{0\}$ and $t > 0$. Using (2.2), we get

$$\begin{aligned} I_0(tu) &= \frac{t^2}{2}\|u\|_0^2 - \frac{1}{2}\int_{\mathbb{R}^N} t^2u^2 \log(|tu|^2)\mathrm{d}x \\ &= t^2\Big(\frac{1}{2}\|u\|_0^2 - \frac{1}{2}\int_{\mathbb{R}^N} [u^2\log(t^2) + u^2\log(u^2)]\mathrm{d}x\Big) \\ &= t^2\Big[I_0(u) - \log t\int_{\mathbb{R}^N} u^2\mathrm{d}x\Big] \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$. So let $tu = e$, we can get the conclusion. □

From Lemma 3.1, we can define following minimax level:

$$\bar{c}_0 = \inf_{\gamma_0\in\Gamma_0}\sup_{t\in[0,1]} I_0(\gamma_0(t)), \text{ where } \Gamma_0 = \{\gamma_0 \in C([0,1], E_0) : \gamma_0(0) = 0, I_0(\gamma_0(1)) < 0\}.$$

Using Theorem 2.4, there exists a Palais-Smale sequence $\{u_n\}$ at the level c_ε, that is, $I_\varepsilon(u_n) \to c_\varepsilon$ and

$$\int_{\mathbb{R}^N}((-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(v-u_n) + (V(\varepsilon x)+1)\ u_n(v-u_n))\mathrm{d}x - \int_{\mathbb{R}^N} F_2'(u_n)(v-u_n)$$
$$+\int_{\mathbb{R}^N} F_1(v)\mathrm{d}x - \int_{\mathbb{R}^N} F_1(u_n)\mathrm{d}x \geq -\tau_n\|v-u_n\|_\varepsilon,\ \forall v\in H^s_\varepsilon.$$

Next lemma we will show the boundedness of (PS) sequence of I_0 in which we will used the fractional logarithmic Sobolev inequality (See Lemma 2.5).

Lemma 3.2. *All $(PS)_{\bar{c}_0}$-sequences are bounded in H^s_0.*

Proof. Let $\{u_n\} \subset H^s_0$ be a $(PS)_{\bar{c}_0}$ sequence. By Corollary 2.2,

$$\int_{\mathbb{R}^N}|u_n|^2\mathrm{d}x = 2I_0(u_n) - \langle I_0'(u_n), u_n\rangle = 2\bar{c}_0 + o_n(1) + o_n(1)\|u_n\|_0 \leq C + o_n(1)\|u_n\|_0$$

for some $C > 0$. Consequently

$$\|u_n\|_2^2 \leq C + o_n(1)\|u_n\|_0. \tag{3.2}$$

Using Lemma 2.5, we have that

$$\begin{aligned}\|u_n\|_0^2 &= 2I_0(u_n) + \int_{\mathbb{R}^N} u_n^2\log(u_n^2)\mathrm{d}x\\ &\leq C + \|u_n\|_2^2\log\|u_n\|_2^2 - \Big(N + \frac{N}{s}\log\alpha + \log\frac{s\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2s})}\Big)\|u_n\|_2^2 + \frac{\alpha^2}{\pi^s}\|(-\Delta)^{\frac{s}{2}}u_n\|_{L^2}^2.\end{aligned} \tag{3.3}$$

Thus, for $\alpha > 0$ and $\delta > 0$ small and by (3.2), we have

$$\|u_n\|_0^2 \leq C + o_n(1)\|u_n\|_0^{1+\delta} + o_n(1)\|u_n\|_0,$$

and so $\{u_n\}$ is bounded in H^s_0. □

The following lemma is important for the proof of Lemma 3.5.

Lemma 3.3. *Assume that hypothesis (V_1) is satisfied. For each $u \in E_0$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) := I_0(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in (t_u, ∞), i.e. the function $g_u(t)$ achieves a positive maximum at the unique critical point $t_u > 0$, characterized as*

$$I_0(u) = \frac{2\log t_u + 1}{2}\int_{\mathbb{R}^N}|u|^2dx.$$

Proof. Since

$$\begin{aligned} g_u(t) : &= I_0(tu) = \frac{t^2}{2}\|u\|_0^2 + \int_{\mathbb{R}^N} F_1(tu)\mathrm{d}x - \int_{\mathbb{R}^N} F_2(tu)\mathrm{d}x \\ &= \frac{t^2}{2}\|u\|_0^2 - \frac{1}{2}\int_{\mathbb{R}^N} t^2u^2\log(|tu|^2)\mathrm{d}x \\ &= t^2\Big[I_0(u) - \log t\int_{\mathbb{R}^N} u^2\mathrm{d}x\Big], \end{aligned}$$

we have $g_u(0) = 0$, $g_u(t) > 0$ for $t > 0$ small and $g_u(t) < 0$ for $t > 0$ large. Therefore, $\max_{t\geq 0} g_u(t)$ is achieved at a global maximum point $t = t_u > 0$ verifying $g_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_0$.

Now we claim that $t_u > 0$ is unique. Indeed, suppose that there exist $t_2 > t_1 > 0$ such that $g_u'(t_1) = g_u'(t_2) = 0$. Then, for $i = 1, 2$,

$$t_i\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + t_i\int_{\mathbb{R}^N}(V_0+1)|u|^2\mathrm{d}x dx - \int_{\mathbb{R}^N} F_2'(t_iu)udx + \int_{\mathbb{R}^N} F_1'(t_iu)udx = 0.$$

Hence,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \int_{\mathbb{R}^N}(V_0+1)|u|^2\mathrm{d}x - \int_{\mathbb{R}^N}\frac{F_2'(t_iu)u}{t_i}dx + \int_{\mathbb{R}^N}\frac{F_1'(t_iu)u}{t_i}dx = 0,$$

which implies that

$$\int_{\mathbb{R}^N}\Big(\frac{F_2'(t_2u)u}{t_2} - \frac{F_2'(t_1u)u}{t_1}\Big)dx = \int_{\mathbb{R}^N}\Big(\frac{F_1'(t_2u)u}{t_2} - \frac{F_1'(t_1u)u}{t_1}\Big)dx.$$

From (2.9), we get the left side of above equality is positive. For the right side of above equality, we have

$$\begin{aligned} &\int_{\mathbb{R}^N}\Big(\frac{F_1'(t_2u)u}{t_2} - \frac{F_1'(t_1u)u}{t_1}\Big)dx \\ &= \int_{\{x:|u|<\frac{\delta}{t_2}\}}\Big(\frac{F_1'(t_2u)u}{t_2} - \frac{F_1'(t_1u)u}{t_1}\Big)dx \\ &\quad + \int_{\{x:\frac{\delta}{t_2}<|u|<\frac{\delta}{t_1}\}}\Big(\frac{F_1'(t_2u)u}{t_2} - \frac{F_1'(t_1u)u}{t_1}\Big)dx \\ &\quad + \int_{\{x:|u|>\frac{\delta}{t_1}\}}\Big(\frac{F_1'(t_2u)u}{t_2} - \frac{F_1'(t_1u)u}{t_1}\Big)dx \\ &= \int_{\{x:|u|<\frac{\delta}{t_2}\}} u^2\log(\frac{t_1}{t_2})^2dx + \int_{\{x:|u|>\frac{\delta}{t_1}\}}\Big(\frac{1}{t_2} - \frac{1}{t_1}\Big)2\delta udx \\ &\quad + \int_{\{x:\frac{\delta}{t_2}<|u|<\frac{\delta}{t_1}\}}\Big(u^2\log\frac{(t_1u)^2}{\delta^2} + 2u(\frac{\delta}{t_2} - u)\Big)dx. \end{aligned}$$

A direct computation shows that the right side of the last last equality is negative, which is a contradiction. Hence $t_u > 0$ is unique. $\square$

Remark 3.1. *From above Lemma, any $u \in D(I_0) \setminus \{0\}$ and every ray $\{tu; t > 0\}$ intersects the set*

$$\mathcal{N}_0 = \Big\{ u \in D(I_0) \setminus \{0\}; I_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \Big\}$$

at exactly the unique point $\tilde{t}u$. So in this way, we get $\tilde{t} = 1$ if and only if $u \in \mathcal{N}_0$.

The following vanishing Lemma is a version of the concentration-compactness principle proved by P. L. Lions ([28]).

Lemma 3.4. *Let $\{u_n\}$ be a bounded sequence in $E_0 \setminus \{0\}$ and satisfies*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2 \mathrm{d}x = 0,$$

where $R > 0$. Then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for every $2 < t < 2_s^*$.

Define

$$c_0 := \inf_{u\in\mathcal{N}_0} I_0(u). \tag{3.4}$$

Replacing V_0 by V_∞, we can define the energy level $c_\infty = \inf_{\mathcal{N}_\infty} I_\infty$ corresponding to problem (3.1). Using the definition of c_0 and c_∞, it follows that $c_0 < c_\infty$.

The next lemma shows that the mountain pass level $\bar{c}_0$ in (4.1) is the ground state energy for the functional I_0, it also establishes an important relation between $\bar{c}_0$ and c_0.

Lemma 3.5. (*a*) $\bar{c}_0 > 0$;

(*b*) $\bar{c}_0 = c_0 := \inf_{u\in\mathcal{N}_0} I_0(u)$.

Proof. (a): Similar to the proof in Lemma 4.1 (i).

(b): Let $u \in \mathcal{N}_0$ and let us consider $I_0(t^*u) < 0$ for some $t^* > 0$. If $\gamma_0 : [0,1] \to E_0$ is the continuous path $\gamma_0(t) = t \cdot t^*u$, then

$$\bar{c}_0 = \inf_{\gamma_0\in\Gamma_0} \sup_{t\in[0,1]} I_0(\gamma_0(t)) \le \sup_{t\in[0,1]} I_0(\gamma_0(t)) \le \sup_{t\ge0} I_0(tu) = I_0(u) \tag{3.5}$$

and consequently $\bar{c}_0 \le \inf_{u\in\mathcal{N}_0} I_0(u)$.

To prove the reverse inequality, by Lemma 3.1, there exists a $(PS)_{c_0}$ sequence $\{u_n\} \subset E_0$ for I_0. By Lemma 3.2, the sequence $\{u_n\}$ is bounded in E_0. Next we will prove that

$$\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x \not\to 0. \tag{3.6}$$

Indeed, on the contrary, by Lemma 3.4, we would have that $u_n \to 0$ in $L^p(\mathbb{R}^N), \forall p \in (2, 2_s^*)$. Then, by (2.10) we would get

$$\int_{\mathbb{R}^N} F_2'(u_n)u_n \mathrm{d}x \to 0. \tag{3.7}$$

On the other hand, by the second part of (2.7) and (3.7) we obtain

$$\begin{aligned}\|u_n\|_0^2 + \int_{\mathbb{R}^N} F_1'(u_n)u_n \mathrm{d}x =& \langle I_0'(u_n), u_n\rangle + \int_{\mathbb{R}^N} F_2'(u_n)u_n \mathrm{d}x \\ =& o_n(1)\|u_n\|_0 + \int_{\mathbb{R}^N} F_2'(u_n)u_n \mathrm{d}x = o_n(1),\end{aligned} \tag{3.8}$$

then it follows from $\int_{\mathbb{R}^N} F_1'(u_n)u_n \mathrm{d}x \geq 0$ that $u_n \to 0$ in E_0 and $F_1'(u_n)u_n \to 0$ in $L^1(\mathbb{R}^N)$. Since F_1 is convex, even and $F_1(t) \geq F_1(0) = 0$ for all $t \in \mathbb{R}$, we derive that $0 \leq F_1(t) \leq F_1'(t)t$ for all $t \in \mathbb{R}$. Hence, $F_1(u_n) \to 0$ in $L^1(\mathbb{R}^N)$ and so $I_0(u_n) \to 0$, but this contradicts the fact that $\bar{c}_0 > 0$ (part (a) above) and (3.6) is proved.

Hence there are constants a and b such that

$$0 < a \leq \int_{\mathbb{R}^N} u_n^2 \mathrm{d}x \leq b, \ \forall n \in \mathbb{N}. \tag{3.9}$$

For each u_n, let $t_n > 0$ be such that $t_n u_n \in \mathcal{N}_0$. Recalling that

$$I_0(t_n u_n) = \frac{1}{2} \int_{\mathbb{R}^N} |t_n u_n|^2 \mathrm{d}x, \tag{3.10}$$

or equivalently

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} (V_0 + 1)|u_n|^2 \mathrm{d}x - \int_{\mathbb{R}^N} u_n^2 \log |t_n u_n|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x,$$

and

$$\langle I_0'(u_n), u_n\rangle = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V_0 |u_n|^2 \mathrm{d}x - \int_{\mathbb{R}^N} u_n^2 \log |u_n|^2 \mathrm{d}x = o_n(1),$$

we have that

$$o_n(1) = 2 \log t_n \int_{\mathbb{R}^N} u_n^2 \mathrm{d}x.$$

This equality combines with (3.9) to result $t_n \to 1$. On the other hand, by (3.10) and Corollary 2.2,

$$\inf_{u \in \mathcal{N}_0} I_0(u) \leq I_0(t_n u_n) = \frac{t_n^2}{2} \int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x = t_n^2 (I_0(u_n) + o_n(1)\|u_n\|_0) = t_n^2 I_0(u_n) + o_n(1).$$

Passing to the limit in this inequality, the reverse inequality $\inf_{u \in \mathcal{N}_0} I_0(u) \leq \bar{c}_0$ holds. □

The next result and Remark imply that the weak limit of a $(PS)_{c_0}$ sequence is non-trivial.

Lemma 3.6. *Let $\{u_n\} \subset E_0$ be a $(PS)_{c_0}$-sequence for I_0. Then, only one of the alternatives below holds:*

(i) $u_n \to 0$ in E_0;

(ii) there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(y_n)} |u_n|^2 \mathrm{d}x \geq \beta > 0.$$

Proof. Assume that (ii) does not occur, it means that for all $R > 0$,

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \mathrm{d}x = 0.$$

Since $\{u_n\} \subset E_0$ be a $(PS)_{c_0}$-sequence for I_0, arguing as the same in the proof of Lemma 4.2 we can see that $\{u_n\}$ is bounded in E_0. Then we can use Lemma 3.4 to get that

$$u_n \to 0 \text{ in } L^t(\mathbb{R}^N), \quad \forall\, 2 < t < 2_s^*. \tag{3.11}$$

It follows from $\langle I_0'(u_n), u_n\rangle = o_n(1)$ that

$$\begin{aligned} o_n(1) = \langle I_0'(u_n), u_n\rangle &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V_0 |u_n|^2 \mathrm{d}x - \int_{\mathbb{R}^N} u_n^2 \log u_n^2 \mathrm{d}x \\ &\geq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V_0 |u_n|^2 \mathrm{d}x - C \int_{\{u_n^2 \geq \frac{1}{e}\}} u_n^p \mathrm{d}x, \end{aligned}$$

then from (3.11), we get $u_n \to 0$ in E_0. $\square$

Remark 3.2. *By the above Lemma, if u is the weak limit of a $(PS)_{c_0}$-sequence $\{u_n\}$ of the functional I_0, then we may assume that $u \neq 0$. Otherwise we have that $u_n \rightharpoonup 0$ and $u_n \not\to 0$, then by Lemma 3.4, there exists $\{y_n\} \subset \mathbb{R}^N$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n(x)|^2 \mathrm{d}x \geq \beta > 0.$$

Set $v_n(x) := u_n(x + y_n)$, obviously, $\{v_n\}$ is also a $(PS)_{c_0}$-sequence of I_0, and there exists $v \in E_0$ such that $v_n \rightharpoonup v$ in H_0^s with $v \not\to 0$.

Now, we prove the following result for the autonomous problem (3.1).

Theorem 3.1. *Problem (3.1) has a positive ground state solution.*

Proof. Similar to the proof of Lemma 4.1 and Lemma 4.2, we can get that I_0 possesses a bounded $(PS)_{c_0}$-sequence $\{u_n\} \subset E_0$ such that, as $n \to \infty$,

$$I_0(u_n) \to c_0 \text{ and } I_0'(u_n) \to 0,$$

then we may assume that $u_n \rightharpoonup u$ in E_0.

Moreover, since $\langle I_0'(u_n)\varphi\rangle = o_n(1)$, for all $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$, by Foutou's lemma we obtain that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle I_0'(u_n), \varphi\rangle \\ &= \lim_{n\to\infty} \Big[\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi + V_0 u_n \varphi \mathrm{d}x - \int_{\mathbb{R}^N} u_n \varphi \log u_n^2 \mathrm{d}x \Big] \\ &\geq \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + V_0 u \varphi \mathrm{d}x - \int_{\mathbb{R}^N} u \varphi \log u^2 \mathrm{d}x \\ &= \langle I_0'(u), \varphi\rangle. \end{aligned} \tag{3.12}$$

In particular, if $\langle I_0'(u), u\rangle < 0$, for $t \geq 0$, let

$$\xi(t) := \langle I_0'(tu), tu\rangle.$$

Then $\xi(1) = \langle I_0'(u), u\rangle < 0$. Since

$$\int_{\mathbb{R}^N} u_n^2 \log u_n^2 \mathrm{d}x \leq C \int_{\{u_n^2 \geq \frac{1}{e}\}} u_n^p \mathrm{d}x, \ p > 2,$$

we have

$$\begin{aligned}
\xi(t) &= \langle I_0'(tu), tu\rangle \\
&= t^2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + V_0|u|^2 \mathrm{d}x\Big) - \int_{\mathbb{R}^N} (tu)^2 \log(tu)^2 \mathrm{d}x \\
&\geq t^2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + V_0|u|^2 \mathrm{d}x\Big) - Ct^p \int_{\mathbb{R}^N} u^p \mathrm{d}x \\
&> 0
\end{aligned}$$

for $t > 0$ small. Since $\xi(t)$ is continuous, there exists a $t_0 \in (0,1)$ such that $\xi(t_0) = \langle I_0'(t_0 u), t_0 u\rangle = 0$, that is $t_0 u \in \mathcal{N}_0$, and then

$$\begin{aligned}
c_0 &\leq I_0(t_0 u) = I_0(t_0 u) - \frac{1}{2}\langle I_0'(t_0 u), t_0 u\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} (t_0 u)^2 \mathrm{d}x \\
&< \frac{1}{2}\int_{\mathbb{R}^N} u^2 \mathrm{d}x \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2 \mathrm{d}x \\
&= \liminf_{n\to\infty} [I_0(u_n) - \frac{1}{2}\langle I_0'(u_n), u_n\rangle] \\
&= c_0,
\end{aligned} \tag{3.13}$$

which is a contradiction, i.e., $\langle I_0'(u), u\rangle = 0$ holds true. Thus, $u \in \mathcal{N}_0$. Using the fact that $\langle I_0'(u), u^-\rangle = 0$ we can see that $u \geq 0$ in $\mathbb{R}^N$. Moreover $u > 0$ by the maximum principle.

Next we will prove that $I_0(u) = c_0$. In fact, by $u \in \mathcal{N}_0$ and Fatou's Lemma we have

$$\begin{aligned}
c_0 &\leq I_0(u) = I_0(u) - \frac{1}{2}\langle I_0'(u), u\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} u^2 \mathrm{d}x \\
&\leq \frac{1}{2}\liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2 \mathrm{d}x \\
&= \liminf_{n\to\infty} [I_0(u_n) - \frac{1}{2}\langle I_0'(u_n), u_n\rangle] \\
&= c_0,
\end{aligned} \tag{3.14}$$

Combing with Lemma 3.5, we get that problem (3.1) has a positive ground state solution. □

4 Existence of a solution for (1.7)

The main goal of this section is proving the existence of solution for (1.7) when ε is small enough. Similar to Lemma 3.1 and Lemma 3.2, we get following lemmas:

Lemma 4.1. *For all $\varepsilon > 0$, the functional I_ε, defined with Φ_ε and Ψ in (2.4) and (2.5), respectively, satisfies the following Mountain Pass geometry.*
(i) there exist $\alpha, \rho > 0$ such that $I_\varepsilon(u) \geq \alpha$ for any $u \in H^s_\varepsilon(\mathbb{R}^N)$ with $\|u\|_\varepsilon = \rho$;
(ii) there exists $e \in H^s_\varepsilon(\mathbb{R}^N)$ with $\|e\|_\varepsilon > \rho$ such that $I_\varepsilon(e) < 0$.

Then we can define the minimax level

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \sup_{t\in[0,1]} I_\varepsilon(\gamma(t)),\ \Gamma_\varepsilon = \{\gamma \in C([0,1], H^s_\varepsilon); \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}. \tag{4.1}$$

Using Theorem 2.4, there exists a Palais-Smale sequence $\{u_n\}$ at the level c_ε, that is, $I_\varepsilon(u_n) \to c_\varepsilon$ and

$$\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u_n(-\Delta)^{\frac{s}{2}}(v-u_n) + (V(\varepsilon x)+1)\ u_n(v-u_n))\mathrm{d}x - \int_{\mathbb{R}^N} F_2'(u_n)(v-u_n)$$
$$+ \int_{\mathbb{R}^N} F_1(v)\mathrm{d}x - \int_{\mathbb{R}^N} F_1(u_n)\mathrm{d}x \geq -\tau_n\|v-u_n\|_\varepsilon,\ \forall v \in H^s_\varepsilon.$$

Lemma 4.2. *All $(PS)_{c_\varepsilon}$-sequences are bounded in H^s_ε.*

Lemma 4.1 and Lemma 4.2 guarantee the existence of a $(PS)_{c_\varepsilon}$-sequence $\{u_n\}$ for the functional I_ε. So we are going to prove that this sequence converges to a weak solution which is a critical point of I_ε and therefore is a solution of (1.7). In this paper our approach gives additional informations on the convergences of (PS)-sequences, which are used in order to get concentration as $\varepsilon \to 0$.

Proposition 4.1. *For a fixed $\varepsilon > 0$, let $\{u_n\} \subset H^s_\varepsilon$ be a $(PS)_{c_\varepsilon}$-sequences for the functional I_ε. Then, for sufficiently small $\varepsilon > 0$, there exists $u_\varepsilon \in H^s_\varepsilon$ such that $u_n \rightharpoonup u_\varepsilon$ in H^s_ε and also*

$$u_n \to u_\varepsilon \text{ in } L^p(\mathbb{R}^N),\ \forall p \in [2, 2^*_s). \tag{4.2}$$

Proof. Let $\varepsilon > 0$ be fixed for a while. From the proof of Lemma 4.1 and Lemma 4.2, we can get that I_ε possesses a bounded $(PS)_{c_\varepsilon}$-sequence $\{u_n\} \subset H^s_\varepsilon$ such that, as $n \to \infty$,

$$I_\varepsilon(u_n) \to c_\varepsilon \text{ and } I'_\varepsilon(u_n) \to 0,$$

then there exists $u_\varepsilon \in H^s_\varepsilon$ such that

$$u_n \rightharpoonup u_\varepsilon \quad \text{weakly in } \ H^s_\varepsilon, \tag{4.3}$$
$$u_n \to u_\varepsilon \quad \text{strongly in } \ L^t_{loc}(\mathbb{R}^N), \tag{4.4}$$
$$u_n \to u_\varepsilon \quad \text{a.e. in } \ \mathbb{R}^N. \tag{4.5}$$

We may assume that $u_n \rightharpoonup u_\varepsilon$ for some $u_\varepsilon \in H^s_\varepsilon$. We are to prove (4.2).

Firstly, let us prove the following claim which will be useful soon.

Claim 5.1. $F_1'(u_\varepsilon)u_\varepsilon \in L^1(\mathbb{R}^N)$ and $\langle I_\varepsilon'(u_\varepsilon), u_\varepsilon\rangle \le 0$.

Let $\phi \in C_0^\infty(\mathbb{R}^N), 0 \le \phi \le 1, \phi \equiv 1$ in $B_1(0)$ and $\phi \equiv 0$ in $B_2^c(0)$, then define $\phi_R(\cdot) := \phi(\cdot/R)$, it results by (2.15) with $z = \phi_R u_n$ that

$$\begin{aligned}&\int_{\mathbb{R}^N}((-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(\phi_R u_n) + (V(\varepsilon x)+1)\phi_R|u_n|^2)\mathrm{d}x + \int_{\mathbb{R}^N} F_1'(u_n)u_n\phi_R dx\\ &= \int_{\mathbb{R}^N} F_2'(u_n)u_n\phi_R dx + o_n(1).\end{aligned}$$

Fixing R and passing to the limit $n \to \infty$ in the above equality, then we get

$$\begin{aligned}&\int_{\mathbb{R}^N}((-\Delta)^{\frac{s}{2}}u_\varepsilon(-\Delta)^{\frac{s}{2}}(\phi_R u_\varepsilon) + (V(\varepsilon x)+1)\phi_R|u_\varepsilon|^2)\mathrm{d}x + \int_{\mathbb{R}^N} F_1'(u_\varepsilon)u_\varepsilon\phi_R dx\\ &= \int_{\mathbb{R}^N} F_2'(u_\varepsilon)u_\varepsilon\phi_R dx + o_n(1).\end{aligned} \tag{4.6}$$

Observe that

$$\begin{aligned}&\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_\varepsilon(-\Delta)^{\frac{s}{2}}(\phi_R u_\varepsilon)\mathrm{d}x\\ &= \frac{c(\varepsilon,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u_\varepsilon(x)-u_\varepsilon(y))(u_\varepsilon\phi_R(x)-u_\varepsilon\phi_R(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ &= \frac{c(\varepsilon,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\phi_R(x)|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ &\quad + \frac{c(\varepsilon,s)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{u_\varepsilon(y)(u_\varepsilon(x)-u_\varepsilon(y))(\phi_R(x)-\phi_R(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{4.7}$$

From the Hölder inequality and the boundedness of $\{u_\varepsilon\}$ in H_ε^s, it follows that

$$\begin{aligned}&\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{u_\varepsilon(y)(u_\varepsilon(x)-u_\varepsilon(y))(\phi_R(x)-\phi_R(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ &\le\Big(\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}}\Big(\iint_{\mathbb{R}^N\times\mathbb{R}^N}|u_\varepsilon(y)|^2\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}}\\ &\le C\Big(\iint_{\mathbb{R}^N\times\mathbb{R}^N}|u_\varepsilon(y)|^2\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.8}$$

From (2.18) in Lemma 2.2, we see that

$$\lim_{R\to\infty}\iint_{\mathbb{R}^N\times\mathbb{R}^N}|u_\varepsilon(y)|^2\frac{|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = 0. \tag{4.9}$$

Then, putting inequlities (4.6)-(4.9) together, using that $F_1'(t)t \ge 0$ for all $t \in R$ and applying Fatou's lemma in (4.6), as $R \to \infty$, we can get $\langle I_\varepsilon'(u_\varepsilon), u_\varepsilon\rangle \le 0$.

To proceed further, we need to use the Concentration Compactness Principle, due to Lions [22], employed to the following sequence

$$\rho_n(x) := \frac{|u_n(x)|^2}{|u_n|_2^2}, \forall\, x \in \mathbb{R}^N.$$

This principle assures that only one of the following statements holds for a subsequence of ρ_n, still denoted by ρ_n :
(**Vanishing**)

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n \mathrm{d}x = 0, \quad \forall\ R>0. \tag{4.10}$$

(**Compactness**) There exists a sequence of points $\{y_n\} \subset \mathbb{R}^N$ such that for all $\eta > 0$ there exists $R > 0$ such that

$$\int_{B_{R(y_n)}} \rho_n \mathrm{d}x \geq 1-\eta, \quad \forall\ n \in \mathbb{N}. \tag{4.11}$$

(**Dichotomy**) There exist $\{y_n\} \subset \mathbb{R}^N, \alpha \in (0,1), R_1 > 0, R_n \to \infty$ such that the functions $\rho_{1,n}(x) := \chi_{B_{R_1}(y_n)}(x)\rho_n(x)$ and $\rho_{2,n}(x) := \chi_{B^c_{R_n}(y_n)}(x)\rho_n(x)$ satisfy

$$\int_{\mathbb{R}^N} \rho_{1,n} \mathrm{d}x \to \alpha \quad \text{and} \quad \int_{\mathbb{R}^N} \rho_{2,n} \mathrm{d}x \to 1-\alpha. \tag{4.12}$$

In order to get that $\{\rho_n\}$ verifies the Compactness condition, we must exclude the others two possibilities.

Firstly, the vanishing case (4.10) cannot occur, otherwise we conclude that $|u_n|_p \to 0$, and so $F_2'(u_n)u_n \to 0$ in $L^1(\mathbb{R}^N)$. Then argue as lemma 3.5, we can get that $u_n \to 0$ in H^s_ε, which is a contradiction with (3.6).

Now we show that Dichotomy also does not hold. Suppose that Dichotomy is the case. Under this assumption, as far as the sequence $\{y_n\}$ is concerned, there are two possible situations to be considered.

• $\{y_n\}$ is bounded:

In this case, for some $\tau > 0$ and for sufficiently large n, it follows from the first convergence in (4.12) and assertion (3.9) that

$$\int_{B_{R_1}(y_n)} |u_n|_2^2 \mathrm{d}x = |u_n|_2^2 \int_{\mathbb{R}^N} \rho_{1,n} \mathrm{d}x \geq \tau.$$

Then for all n sufficiently large, choosing $R_0 > 0$ such that $B_R(y_n) \subset B_{R_0}(0)$, it follows that

$$\int_{B_{R_0}(0)} |u_n|^2 \mathrm{d}x \geq \tau.$$

Since $u_n \rightharpoonup u$ in H^s_ε, it follows from the Compact Sobolev imbedding that

$$\int_{B_{R_0}(0)} |u_\varepsilon|^2 \mathrm{d}x \geq \tau > 0 \quad \text{and then} \quad u_\varepsilon \not\equiv 0.$$

By Claim 5.1, $\langle I_\varepsilon'(u_\varepsilon), u_\varepsilon\rangle \leq 0$. Then by the Remark **??**, there is a unique $t_\varepsilon \in (0,1]$ such that $t_\varepsilon u_\varepsilon \in \mathcal{N}_\varepsilon$.

Using Corollary 2.2,

$$I_\varepsilon(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x + o_n(1),$$

and then

$$c_\varepsilon = \lim_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x. \tag{4.13}$$

Since $t_\varepsilon u_\varepsilon \in \mathcal{N}_\varepsilon$, by Fatou's Lemma and similar to Lemma 3.5 part (b), we have

$$\begin{aligned} c_\varepsilon = I_\varepsilon(t_\varepsilon u_\varepsilon) = \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} |u_\varepsilon|^2 \mathrm{d}x \leq \frac{1}{2} \int_{\mathbb{R}^N} |u_\varepsilon|^2 \mathrm{d}x \\ \leq \liminf_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x = c_\varepsilon. \end{aligned} \tag{4.14}$$

Thus the convergence $|u_n|_2 \to |u_\varepsilon|_2$ holds. This together with the weak convergence implies that $u_n \to u_\varepsilon$ in $L^2(\mathbb{R}^N)$.

As $u_n \to u_\varepsilon$ in $L^2(\mathbb{R}^N)$ and $\{y_n\}$ is bounded, the convergence

$$\int_{B^c_{R_n}(y_n)} |u_n|^2 \mathrm{d}x \to 0 \tag{4.15}$$

holds.

On the other hand, by (3.9) and by the second convergence in (4.12) there are $\tau_1 > 0$ and $n_0 > 0$ such that

$$\int_{B^c_{R_n}(y_n)} |u_n|^2 \mathrm{d}x \geq \tau_1 > 0, \quad \forall\ n > n_0. \tag{4.16}$$

But this fact contradicts (4.15).

• $\{y_n\}$ is unbounded:

In this case, we proceed analogously as in the bounded case. Aiming this, we define the sequence

$$v_n(x) := u_n(x + y_n),\ x \in \mathbb{R}^N. \tag{4.17}$$

Hence $\{v_n\} \subset H^s_\varepsilon(\mathbb{R}^N)$ is bounded, then, up to subsequence, we may assume that $v_n \rightharpoonup v_\varepsilon$ and by the first part of (4.12) we have $v_\varepsilon \not\equiv 0$.

Analogously to the previous case we need the following claim

Claim 5.2. $F_1'(v_\varepsilon)v_\varepsilon \in L^1(\mathbb{R}^N)$ and $\langle I'_\infty(v_\varepsilon), v_\varepsilon\rangle \leq 0$.

In the proof of this claim, we use the equality

$$\langle I'_\varepsilon(u_n), \phi_R(\cdot - y_n)u_n\rangle = o_n(1). \tag{4.18}$$

After a change of variables, this equality is transformed into

$$\begin{aligned} &\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}}(\phi_R v_n) + (V(\varepsilon(x+y_n)) + 1)\phi_R |v_n|^2)\mathrm{d}x + \int_{\mathbb{R}^N} F_1'(v_n)v_n\phi_R dx \\ &= \int_{\mathbb{R}^N} F_2'(v_n)v_n\phi_R dx + o_n(1). \end{aligned} \tag{4.19}$$

Therefore, similarly to the proof of Claim 5.1, by (V_1) and the fact that $|y_n| \to \infty$, as $n \to \infty$, Claim 5.2 holds.

By Theorem 3.1, there exists the infimum in (3.4) such that $c_0 = I_0(u_0)$ for some positive function $u_0 \in \mathcal{N}_0$. Note that, if $\varphi \in C_0^\infty(\mathbb{R}^N), 0 \le \varphi \le 1, \varphi \equiv 1$ in $B_1(0)$ and $\varphi \equiv 0$ in $B_2^c(0)$, defining $\varphi_R(\cdot) := \varphi(\cdot/R)$ and $u_R(x) = \varphi_R(x)u_0(x)$, we have that

$$u_R \to u_0 \text{ in } H_\varepsilon^s \text{ as } R \to +\infty.$$

Fixing $R > 0$ and arguing as in the proof of (3.5), for a fixed $\varepsilon > 0$ we find

$$c_\varepsilon \le \max_{t\in[0,+\infty)} I_\varepsilon(tu_R) = I_\varepsilon(t_\varepsilon u_R),$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_R|^2 + (V(\varepsilon(x)) + 1)|u_R|^2)\mathrm{d}x + \int_{\mathbb{R}^N} \frac{F_1'(t_\varepsilon u_R)u_R}{t_\varepsilon} dx \\ &= \int_{\mathbb{R}^N} \frac{F_2'(t_\varepsilon u_R)u_R}{t_\varepsilon} dx.\end{aligned} \tag{4.20}$$

Since $V(\varepsilon x) \to V_0$ as $\varepsilon \to 0$, by the Lebesgue Dominated Convergence theorem, we have from the left side of the above equality that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_R|^2 + (V(\varepsilon x) + 1)|u_R|^2)\mathrm{d}x = \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_R|^2 + (V_0 + 1)|u_R|^2)\mathrm{d}x.$$

Assuming $t_\varepsilon \to +\infty$ as $\varepsilon \to 0$, it is easy to verify that the right side of the equality (4.24) goes to $+\infty$ as $\varepsilon \to 0$, which is a contradiction. Thus, $\{t_\varepsilon\}$ is bounded in $\mathbb{R}$ for ε small enough. Moreover, since

$$\begin{aligned}I_\varepsilon(t_\varepsilon u_R) &= I_0(t_\varepsilon u_R) + \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} (V(\varepsilon x) - V_0)|u_R|^2)\mathrm{d}x \\ &\le I_0(t_R u_R) + \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} (V(\varepsilon x) - V_0)|u_R|^2)\mathrm{d}x,\end{aligned}$$

where $t_R > 0$ satisfies

$$I_0(t_R u_R) = \max_{t\in[0,+\infty)} I_0(tu_R).$$

Using $\sup_{x\in B_R(0)} |V(\varepsilon x) - V_0| \to 0$ as $\varepsilon \to 0$, we get

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le \limsup_{\varepsilon\to 0} I_\varepsilon(t_\varepsilon u_R) \le I_0(t_R u_R). \tag{4.21}$$

Now, we use the fact that $\{t_R\}$ is also bounded for R large enough, $u_R \le u_0$ and F_1 is increasing for $t \ge 0$ to deduce that

$$F_1(t_{R_n} u_{R_n}) \to F_1(ku_0) \text{ in } L^1(\mathbb{R}^N)$$

for some $k > 0$. Since $u_0 \in \mathcal{N}_0$, we can ensure that $F_1(ku_0) \in L^1(\mathbb{R}^N)$ for all $k \ge 0$. Thus, if $R_n \to +\infty$ and $t_{R_n} \to t_*$, the Lebesgue Dominated Convergence theorem yields

$$F_1(t_{R_n} u_{R_n}) \to F_1(t_* u_0) \text{ in } L^1(\mathbb{R}^N),$$

and

$$F_1'(t_{R_n}u_{R_n})t_{R_n}u_{R_n} \to F_1'(t_*u_0)t_*u_0 \text{ in } L^1(\mathbb{R}^N).$$

As an immediate consequence, $t_R \to 1$ as $R \to +\infty$ and

$$I_0(t_R u_R) \to I_0(u_0) \text{ as } R \to +\infty.$$

This combined with (4.21) gives

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le I_0(u_0) = c_0.$$

Because for $\forall \varepsilon > 0, u \in D(I_\varepsilon)$, it has that $I_\varepsilon(u) \ge I_0(u)$, and then by part (b) in Lemma 3.5 with $\varepsilon = 0$, the reverse inequality holds:

$$\liminf_{\varepsilon\to 0} c_\varepsilon \ge c_0.$$

Therefore,

$$\lim_{\varepsilon\to 0} c_\varepsilon = c_0. \tag{4.22}$$

As before, replacing t_ε by t_∞, we conclude that

$$\begin{aligned} c_\infty \le I_\infty(t_\infty v_\varepsilon) &= \frac{t_\infty^2}{2}\int_{\mathbb{R}^N} |v_\varepsilon|^2 \mathrm{d}x \le \frac{1}{2}\int_{\mathbb{R}^N} |v_\varepsilon|^2 \mathrm{d}x \\ &\le \liminf_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x \le \lim_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x \\ &= \lim_{n\to\infty} I_\varepsilon(u_n) = c_\varepsilon. \end{aligned} \tag{4.23}$$

But using the definition of c_0 and c_∞, it follows that $c_0 < c_\infty$, from (4.22) that is

$$\lim_{\varepsilon\to 0} c_\varepsilon = c_0 < c_\infty. \tag{4.24}$$

So for small ε, (4.23) is in contradiction with assertion (4.24). Thus Dichotomy does not occur in any case, and then Compactness must hold.

To reach our goal, let us state the last claim:

Claim 5.3 The sequence of points $\{y_n\} \subset \mathbb{R}^N$ in (4.11) is bounded.

We argue by contradiction. If the sequence of points $\{y_n\}$ is unbounded, that is, up to subsequence, $|y_n| \to +\infty$, then similar to the case of Dichotomy, where $\{y_n\}$ were unbounded, we can get that $c_\varepsilon \ge c_\infty$, which is a contradiction for small ε.

In view of Claim 5.3, for a given $\eta > 0$, there exists $R > 0$ such that, by (4.11),

$$\int_{B_R^c(0)} \rho_n \mathrm{d}x < \eta, \quad \forall\ n \in \mathbb{N},$$

it is equivalent to

$$\int_{B_R^c(0)} |u_n|^2 \mathrm{d}x \le \eta |u_n|_2^2 < b\eta, \quad \forall\ n \in \mathbb{N}, \tag{4.25}$$

where $b = \sup_{n\in\mathbb{N}} |u_n|_2^2$. Since $u_\varepsilon \in L^2(\mathbb{R}^N)$, there exists $R_0 > 0$ such that

$$\int_{B^c_{R_0}(0)} |u_\varepsilon|^2 \mathrm{d}x \le \eta. \tag{4.26}$$

Then, for $R_1 \ge \max\{R, R_0\}$ due to the convergence $u_n \to u_\varepsilon$ in $L^2(B_{R_1}(0))$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_{R_1}(0)} |u_n - u_\varepsilon|^2 \mathrm{d}x < \eta, \ \forall\ n \ge n_0. \tag{4.27}$$

Then, by (4.25), (4.26) and (4.27), it follows that if $n \ge n_0$,

$$\int_{\mathbb{R}^N} |u_n - u_\varepsilon|^2 \mathrm{d}x \le \eta + \int_{B^c_{R_1}(0)} |u_n - u_\varepsilon|^2 \mathrm{d}x \le \eta + \int_{B^c_{R_1}(0)} |u_n|^2 \mathrm{d}x + \int_{B^c_{R_1}(0)} |u_\varepsilon|^2 \mathrm{d}x \le C\eta$$

for some C that does not depend on η. As η is arbitrary, we can conclude that $u_n \to u_\varepsilon$ in $L^2(\mathbb{R}^N)$.

Since $\{u_n\}$ is bounded in $L^{2^*}(\mathbb{R}^N)$ by interpolation on the Lebesgue spaces, it follows that

$$u_n \to u_\varepsilon \text{ in } L^p(\mathbb{R}^N), \quad \forall\ 2 \le p < 2^*.$$

□

Corollary 4.1. *For the sequence $\{u_n\} \subset H^s_\varepsilon$ in Proposition 4.1 and for small $\varepsilon > 0$, we have the convergence*

$$\int_{\mathbb{R}^N} F_2'(u_n) u_n \mathrm{d}x \to \int_{\mathbb{R}^N} F_2'(u_\varepsilon) u_\varepsilon \mathrm{d}x, \quad \text{as} \quad n \to \infty. \tag{4.28}$$

Proof. It follows from (2.10) and Proposition 4.1. □

Proof of the existence of positive solution of (1.7) for small ε. Let $\{u_n\} \subset H^s_\varepsilon$ be the $(PS)_{c_\varepsilon}$-sequence for $I_\varepsilon, v \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$ be sufficiently small. From (2.13), we have

$$\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} (v - u_n) + (V(\varepsilon x) + 1) u_n (v - u_n)) \mathrm{d}x + \int_{\mathbb{R}^N} F_2'(u_n)(v - u_n) \mathrm{d}x$$
$$+ \int_{\mathbb{R}^N} F_1(v) \mathrm{d}x - \int_{\mathbb{R}^N} F_1(u_n) \mathrm{d}x \ge -\tau_n \|v - u_n\|_{H^s_\varepsilon}, \ \tau_n \to 0^+,$$

then, passing to the limit as $n \to \infty$, using that F_1 is lower semicontinuous, Proposition 4.1 and Corollary 4.1, we obtain

$$\int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u_\varepsilon (-\Delta)^{\frac{s}{2}} (v - u_\varepsilon) + (V(\varepsilon x) + 1) u_n (v - u_\varepsilon)) \mathrm{d}x + \int_{\mathbb{R}^N} F_2'(u_\varepsilon)(v - u_\varepsilon) \mathrm{d}x$$
$$+ \int_{\mathbb{R}^N} F_1(v) \mathrm{d}x - \int_{\mathbb{R}^N} F_1(u_\varepsilon) \mathrm{d}x \ge 0,$$

that is

$$\langle \Phi_\varepsilon'(u_\varepsilon), v - u_\varepsilon \rangle + \Psi(v) - \Psi(u_\varepsilon) \ge 0, \ \forall v \in H^s_\varepsilon,$$

i.e., u_ε is a critical point of I_ε for small $\varepsilon > 0$. By (iii) in Lemma 2.1, u_ε is a solution of (1.7). Since $I_\varepsilon(u_\varepsilon) = c_\varepsilon$, we can use the same arguments explored in [15] to conclude that $u_\varepsilon \in C^2(\mathbb{R}^N)$ with

$$u_\varepsilon(x) > 0, \ \forall\, x \in \mathbb{R}^N \quad \text{or} \quad u_\varepsilon(x) < 0, \ \forall\, x \in \mathbb{R}^N.$$

Since

$$f(t) = \begin{cases} t \log t^2, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0 \end{cases}$$

is an odd function, without loss of generality we can assume that u_ε is a positive function. □

Before concluding this section, we would like to point out that the function

$$v_\varepsilon(x) = u_\varepsilon(x/\varepsilon), \ \forall\, x \in \mathbb{R}^N$$

is a positive solution of (1.1), that is,

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s v_\varepsilon + V(x) v_\varepsilon = v_\varepsilon \log v_\varepsilon^2, \ \text{in} \ \ \mathbb{R}^N, \\ v_\varepsilon \in H^s_\varepsilon(\mathbb{R}^N). \end{cases}$$

5 The concentration of solutions

Lemma 5.1. *Let $\varepsilon_n \to 0$ and set : $u_n = u_{\varepsilon_n}$. Then, for some subsequence of $\{u_n\}$, there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that*

$$\int_{B_R(y_n)} |u_n|^2 \mathrm{d}x \geq \beta > 0, \ \forall\, n \in \mathbb{N}. \tag{5.1}$$

Proof. This proof follows the similarity to that in Lemma 3.6, if $\int_{B_R(y_n)} |u_n|^2 \mathrm{d}x \to 0$, then we get $u_n \to 0$ in H^s_ε. On the other hand, by Lemma 3.1 there is $\rho > 0$ such that

$$0 < \rho \leq c_{\varepsilon_n} = I_{\varepsilon_n}(u_n) = I_{\varepsilon_n}(u_n) - \frac{1}{2}\langle I'_{\varepsilon_n}(u_n), u_n\rangle = \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x,$$

which is a contradiction. □

Lemma 5.2. *Let $\varepsilon_n \to 0$, we have that $\varepsilon_n y_n \to y_0$ for some $y_0 \in \mathbb{R}$ with*

$$V(y_0) = V_0 = \inf_{x \in \mathbb{R}^N} V(x). \tag{5.2}$$

Let $u_n \in \mathcal{N}_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to c_0$. Then there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n)$ has a convergent subsequence in H^s_ε.

Proof. Since $u_n \in \mathcal{N}_{\varepsilon_n}$ and $\lim\limits_{n\to\infty} I_{\varepsilon_n}(u_n) = c_0$, it is easy to get that $\{u_n\}$ is bounded in H^s_ε and $\|u_n\|_{\varepsilon_n} \nrightarrow 0$. Let $\tilde{u}_n(x) = u_n(x + y_n)$, so $\{\tilde{u}_n\}$ is bounded in H^s_ε, then up to a subsequence, we have $\tilde{u}_n \rightharpoonup \tilde{u} \neq 0$ in H^s_ε and $\tilde{u}_n(x) \to \tilde{u}(x)$ *a.e.* in $\mathbb{R}^N$. Fix $t_n > 0$ such that $t_n\tilde{u}_n \in \mathcal{N}_0$ and set $\tilde{y}_n = \varepsilon_n y_n$, then we can see that

$$c_0 \leqslant I_{V_0}(t_n\tilde{u}_n) \leqslant I_{\varepsilon_n}(t_n u_n) \leqslant I_{\varepsilon_n}(u_n) = c_0 + o_n(1),$$

which gives that $\lim\limits_{n\to\infty} I_0(t_n\tilde{u}_n) = c_0 > 0$. In particular, $v_n \not\to 0$ in E_0. Set $v_n := t_n\tilde{u}_n$, combining $I_0(v_n) \to c_0$ and $v_n \in \mathcal{N}_0$, we know that $\{v_n\}$ is a bounded sequence. Applying Lemma 5.1,

$$\beta_0 = \liminf_{n\to\infty} \int_{B(y_n,r)} |u_n|^2 \mathrm{d}x = \liminf_{n\to\infty} \int_{B(0,r)} |\tilde{u}_n|^2 \mathrm{d}x \leqslant C \liminf_{n\to\infty} \|\tilde{u}_n\|_0^2.$$

For large n, we have $0 < \frac{\beta_0}{2C} < \|\tilde{u}_n\|_0^2$, then

$$0 \leqslant \frac{\beta_0}{2C} t_n^2 < \|t_n\tilde{u}_n\|_0^2 = \|v_n\|_0^2 \leqslant C.$$

Hence $\{t_n\}$ is bounded.

Now, we may assume that $t_n \to t^* > 0$. If $t^* = 0$, by using the boundedness of $\{\tilde{u}_n\}$ we have $t_n\tilde{u}_n =: v_n \to 0$ in E_0. This is $\lim\limits_{n\to\infty} I_0(t_n\tilde{u}_n) = 0$, which contradicts $c_0 > 0$. Thus, up to a subsequence, we may assume that

$$v_n \to v_0 = t^*\tilde{u} \not\equiv 0 \text{ in } E_0, \qquad \tilde{u}_n \to \frac{1}{t^*} v_0 = \tilde{u} \text{ in } E_0.$$

In order to complete the proof of the lemma, we show that $\{\tilde{y}_n\}$ is bounded in $\mathbb{R}^N$. We argue by contradiction, up to a subsequence, we assume that there is a sequence such that $|\tilde{y}_n| \to \infty$ as $n \to \infty$, and since a similar equality to (4.18) holds, i.e.,

$$\langle I'_{\varepsilon_n}(u_n), (\varphi_R(\cdot - y_n)u_n)\rangle = 0,$$

a similar inequality to (4.21) also holds. Thus, passing to the limit as $R \to +\infty$ we obtain that $\langle I'_\infty(u), u\rangle \leq 0$. Because $u = 0$, there is $t \in (0, 1]$ such that $tw \in \mathcal{N}_\infty$.

Therefore, by (4.24),

$$\begin{aligned} c_\infty \leq I_\infty(t\tilde{u}) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\tilde{u}(x)|^2 \mathrm{d}x \leq \liminf_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} |\tilde{u}_n|^2 \mathrm{d}x \leq \liminf_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x \\ &\leq \limsup_{n\to\infty}\Big(I_{\varepsilon_n}(u_n) - \frac{1}{2}\langle I'_{\varepsilon_n}(u_n)u_n\rangle\Big) = \limsup_{n\to\infty}\langle I'_{\varepsilon_n}(u_n), u_n\rangle = \limsup_{n\to\infty} c_{\varepsilon_n} = c_0 < c_\infty, \end{aligned} \tag{5.3}$$

which is a contradiction. Hence, we may assume that $\epsilon_n y_n \to y_0$ for some $y_0 \in \mathbb{R}^N$. Arguing as (5.3), we may achieve that

$$c_0 = \lim_{n\to\infty} c_{\varepsilon_n} \geq c_{V(y_0)}$$

and consequently assertion (5.2) holds, because $V(y_0) > V_0$ yields $c_0 < c_{V(y_0)}$.

To assure the second part of the theorem, with the same notations of (4.18), replacing φ_R by a function φ with compact support, we have that $\langle I'_0(\tilde{u}), \varphi\rangle = 0$ and then $\langle I'_0(\tilde{u}), \tilde{u}\rangle = 0$.

Applying the same ideas employed in Proposition 4.1, we conclude that $\tilde{u}_n \to \tilde{u}$ in $L^p(\mathbb{R}^N)$ for all $p \in [2, 2_s^*)$, and therefore $\int_{\mathbb{R}^N} F_2'(\tilde{u}_n)\tilde{u}_n \mathrm{d}x \to \int_{\mathbb{R}^N} F_2'(\tilde{u})\tilde{u}\mathrm{d}x$.

Finally, using the equalities $\langle I_0'(\tilde{u}), \tilde{u}\rangle = 0$ and

$$\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\tilde{u}_n|^2 + V(\varepsilon_n y_n + \varepsilon_n x)|\tilde{u}_n|^2)\mathrm{d}x + \int_{\mathbb{R}^N} F_1'(\tilde{u}_n)\tilde{u}_n\phi_R \mathrm{d}x$$
$$= \int_{\mathbb{R}^N} F_2'(\tilde{u}_n)\tilde{u}_n\phi_R \mathrm{d}x + o_n(1),$$

we get

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\tilde{u}_n|^2 \mathrm{d}x \to \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\tilde{u}|^2 \mathrm{d}x.$$

and the proof finishes. □

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