Darboux Curves in Three dimensional Walker Manifold

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Abstract

In this paper, we investigate the characterization for darboux curves in a strictWalker 3-manifold and a classication for ruled surface, which is generated by darboux curve in Walker 3-manifold is obtained. An example was given at the end.

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ABSTRACT: In this paper, we investigate the characterization for darboux curves in a strict Walker 3-manifold and a classification for ruled surface, which is generated by darboux curve in Walker 3-manifold is obtained. An example was given at the end.

Key Words: Curves, Darboux curve, ruled surfaces Walker manifolds.

Contents

1. Introduction

The study of submanifolds of a given ambiant space is a naturel interesting problem which enriches our knowledge and understanding of the geometry of the space itself. Here the ambiant space we will consider is a Lorentzian three-manifold admitting a parallel null vector field called strict Walker manifold. It is known that Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. For details see [1,2].
Darboux curves in the Euclidean space were studied by Saban [9] and were generalised by Ergin [4]. For a curve α on a surface in the Euclidean 3-space the function $\mathbb{D} = \langle \alpha''', U\rangle = \kappa_n' - \kappa_g\tau_g$ is called Darboux function of α where U is normal vector field of surface, κ_n, κ_g and τ_g are normal curvature, geodesic curvature and geodesic torsion. Darboux function is equal to zero for Darboux curves. For more informations about Darboux curves see [6,8].
In Minkowski 3-space timelike Darboux curves on a timelike surface were studied by Ergin [5]. In [7], Suroǧlu et al. study and classify modifieded translation surfaces in $Heis3$ and investigate conditions of being minimal surface. Also, they obtain the characterizations of points on this surface.
The study of ruled surfaces of a given ambiant space is a naturel and interesting problem. A surface S in a manifold M is said to be ruled if every point of S is on (a open geodesic segment) in M that lies in S (see [3]). Locally a ruled surface is made by a one parameter family of geodesic segments.
In this paper we study the darboux curves in a three dimensional Walker manifold. We give the characterization for darboux curves in a strict Walker 3-manifold and a classification for ruled surface, which is generated by darboux curve in Walker 3-manifold is obtained.
The paper is organised as follow: in section 2 we give some preliminaries results about Walker 3-manifold and basic notions about curves lyung on a surfaces. In section 3 we give the main results of this paper with an example for illustrate the obtained results.

2. Preliminaries

A Walker n-manifold is a pseudo-Riemannian manifold, which admits a field of null parallel r-planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker ([1]). Walker has derived adapted coordinates to a parallel plan field. Hence, the metric of a three-dimensional Walker manifold (M, g_f^ϵ) with coordinates (x, y, z) is expressed as

$$g_f^\epsilon = dx \circ dz + \epsilon dy^2 + f(x, y, z)dz^2 \tag{2.1}$$

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and its matrix form as

$$g_f^\epsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & f \end{pmatrix} \quad \text{with inverse} \quad (g_f^\epsilon)^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some function $f(x, y, z)$, where $\epsilon = \pm 1$ and thus $D = \mathrm{Span}\partial_x$ as the parallel degenerate line field. Notice that when $\epsilon = 1$ and $\epsilon = -1$ the Walker manifold has signature $(2, 1)$ and $(1, 2)$ respectively, and therefore is Lorentzian in both cases.

It follows after a straightforward calculation that the Levi-Civita connection of any metric (2.1) is given by:

$$\begin{aligned} \nabla_{\partial_x}\partial z &= \frac{1}{2}f_x\partial_x, \quad \nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \\ \nabla_{\partial_z}\partial z &= \frac{1}{2}(ff_x + f_z)\partial_x + \frac{1}{2}f_y\partial_y - \frac{1}{2}f_x\partial_z \end{aligned} \tag{2.2}$$

where ∂_x, ∂_y and ∂_z are the coordinate vector fields $\frac{\partial}{\partial_x}$, $\frac{\partial}{\partial_y}$ and $\frac{\partial}{\partial_z}$, respectively. Hence, if (M, g_f^ϵ) is a strict Walker manifolds i.e., $f(x, y, z) = f(y, z)$, then the associated Levi-Civita connection satisfies

$$\nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \quad \nabla_{\partial_z}\partial z = \frac{1}{2}f_z\partial_x - \frac{\epsilon}{2}f_y\partial_y. \tag{2.3}$$

Note that the existence of a null parallel vector field (i.e $f = f(y, z)$) simplifies the non-zero components of the Christoffel symbols and the curvature tensor of the metric g_f^ϵ as follows:

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2}f_y, \; \Gamma_{33}^1 = \frac{1}{2}f_z, \; \Gamma_{33}^2 = -\frac{\epsilon}{2}f_y \tag{2.4}$$

Starting from local coordinates (x, y, z) for which (2.1) holds, it is easy to check that

$$e_1 = \partial_y, \;\; e_2 = \frac{2-f}{2\sqrt{2}}\partial_x + \frac{1}{\sqrt{2}}\partial_z, \;\; e_3 = \frac{2+f}{2\sqrt{2}}\partial_x - \frac{1}{\sqrt{2}}\partial_z$$

are local pseudo-orthonormal frame fields on (M, g_f^ϵ), with $g_f^\epsilon(e_1, e_1) = 1$, $g_f^\epsilon(e_2, e_2) = \epsilon$ and $g_f^\epsilon(e_3, e_3) = 1$. Thus the signature of the metric g_f^ϵ is $(1, \epsilon, -1)$.
Let now u and v be two vectors in M. Denoted by $(\vec{i}, \vec{j}, \vec{k})$ the canonical frame in $\mathbb{R}^3$.
The vector product of u and v in (M, g_f^ϵ) with respect to the metric g_f^ϵ is the vector denoted by $u \times v$ in M defined by

$$g_f^\epsilon(u \times v, w) = \det(u, v, w) \tag{2.5}$$

for all vector w in M, where $\det(u, v, w)$ is the determinant function associated to the canonical basis of $\mathbb{R}^3$. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then by using (2.5), we have:

$$u \times v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) \vec{i} - \epsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \vec{k} \tag{2.6}$$

Let $\alpha : I \subset \mathbb{R} \longrightarrow (M, g_f^\epsilon)$ be a curve parametrized by its arc-length s.
The Frenet frame of α is the vectors T, N and B along α where T is the tangent, N the principal normal and B the binormal vector. They satisfied the Frenet formulas

$$\left\{ \begin{array}{lcc} \nabla_T T &=& \epsilon_2 \kappa N \\ \nabla_T N &=& -\epsilon_1 \kappa T - \epsilon_3 \tau B \\ \nabla_T B &=& \epsilon_2 \tau N \end{array} \right. \tag{2.7}$$

where κ and τ are respectively the curvature and the torsion of the curve α, with $\epsilon_1 = g_f(T; T); \epsilon_2 = g_f(N; N)$ and $\epsilon_3 = g_f(B, B)$.

Now let $\alpha : I \subset \mathbb{R} \longrightarrow (M, g_f^\epsilon)$ be a curve lying in a surface S (spacelike or timelike) in M. Let U be the unit normal of S, the Darboux frame is given by $\{T, Y, U\}$, where T is the tangent vector of the curve $\alpha(s)$ and $Y = U \times T$.
The usual transformations between the Walker Frenet frame and the Darboux takes the form

$$Y = \cos\theta N - \sin\theta B \tag{2.8}$$
$$U = \sin\theta N + \cos\theta B, \tag{2.9}$$

where θ is an angle between the surface normal vector N and the binormal vector B of α.
Derivating Y along the curve alpha we get

$$\nabla_T Y = \cos\theta\nabla_T N - \theta' \sin\theta N - \sin\theta\nabla_T B - \theta' \cos\theta B.$$

Using the Frenet equation in (2.7) we have

$$\nabla_T Y = \cos\theta(-\epsilon_1\kappa T - \epsilon_3\tau B) - \theta' \sin\theta N - \sin\theta(\epsilon_2\tau N) - \theta' \cos\theta B.$$

Now we suppose that the principal normal and the binormal have the same sign. then we get

$$\nabla_T Y = -\epsilon_1\kappa \cos\theta T - (\theta + \epsilon_2\tau)U \tag{2.10}$$

The same calculus gives

$$\nabla_T U = -\epsilon_1\kappa \sin\theta T + (\theta + \epsilon_2\tau)Y. \tag{2.11}$$

Then the Walker Darboux equation is expressed as

$$\begin{cases} \nabla_T T = -\epsilon_2\kappa_g Y - \epsilon_2\kappa_n U \\ \nabla_T Y = -\epsilon_1\kappa_g T - \tau_g U \\ \nabla_T U = -\epsilon_1\kappa_n T + \tau_g Y, \end{cases} \tag{2.12}$$

where κ_g, κ_g and τ_g are the geodesic curvature, normal curvature and geodesic torsion of $\alpha(s)$ on S, respectively. Also, (2.12) implies

$$\begin{gathered} \kappa_g = -\kappa\cos\theta, \qquad \kappa_n = -\kappa\sin\theta \\ \kappa^2 = \kappa_g^2 + \kappa_n^2, \text{ and } \tau = \tau_g - \frac{\kappa_g\kappa_n' - \kappa_n\kappa_g'}{\kappa^2}. \end{gathered} \tag{2.13}$$

3. Main results

Let S be a spacelike regular surface and $\alpha : I \subset \mathbb{R} \longrightarrow M_f$ be an unit speed curve lying on the surface S.

Theorem 3.1 *Let $\alpha : I \subset \mathbb{R} \longrightarrow M_f$ be an unit speed curve in a Walker 3-manifold. If α is a Darboux curve on the surface S, then*

$$-\epsilon_1\epsilon_2\kappa^2\kappa_n + 2\kappa_g'\tau_g + \tau_g^2\kappa_n + \kappa_g\tau_g' - \kappa_n'' = 0. \tag{3.1}$$

Proof: The function $\mathbb{D}$ of a Darboux curve on the surface S is given by

$$\mathbb{D} = g^\epsilon(\nabla_T^3 T, U), \tag{3.2}$$

where g_f^ϵ is the Walker metric of M_f.
Now let us compute $\nabla_T^3 T$.
Using the equation in (2.12), we get

$$\nabla_T(\nabla_T T) = \epsilon_1\epsilon_2\kappa^2 T - \epsilon_2(\kappa_g' + \tau_g\kappa_n)Y + \epsilon_2(\tau_g\kappa_g - \kappa_n')U. \tag{3.3}$$

Differentiating (3.3) with respect to s and using (2.12), we get

$$\begin{aligned}\nabla^3_T T &= \epsilon_1\epsilon_2(\kappa\kappa' + \kappa_g'\kappa_g + \kappa_n'\kappa_n)T \\ &+(-\epsilon_1\kappa^2\kappa_g - \epsilon_2\kappa_g'' - \epsilon_2\tau_g'\kappa_n - 2\tau_g\kappa_n' + \epsilon_2\kappa_g\tau_g^2)Y \\ &+(-\epsilon_1\kappa^2\kappa_n + 2\epsilon_2\kappa_g'\tau_g + \epsilon_2\tau_g^2\kappa_n + \epsilon_2\kappa_g\tau_g' - \epsilon_2\kappa_n'')U. \end{aligned} \tag{3.4}$$

Finally combine the equations (3.2) and (3.4) and we get

$$\mathbb{D} = -\epsilon_1\kappa^2\kappa_n + 2\epsilon_2\kappa_g'\tau_g + \epsilon_2\tau_g^2\kappa_n + \epsilon_2\kappa_g\tau_g' - \epsilon_2\kappa_n''$$

and that end the proof. □

Proposition 3.1 *Let α be an unit speed curve and S be a ruled surface in (M_f, g_f^ϵ) which is parametrized as*

$$S(s,u) = \alpha(s) + uT(s). \tag{3.5}$$

If α is a darboux curve, then

$$\epsilon_1\epsilon_2\kappa^2\kappa_g + \kappa_g'' + \tau_g'\kappa_n + 2\epsilon_2\tau_g\kappa_n' - \kappa_g\tau_g^2 = 0 \tag{3.6}$$

Proof: We recall that the normal unit Y of the surface S is given by the formula

$$U = \frac{S_s \times S_u}{\|S_s \times S_u\|}, \tag{3.7}$$

where $\times$ is the vector product defined in (2.6).
From equation (3.5) we have

$$\begin{aligned} S_s(s,u) &= T(s) + u\nabla_T T \\ &= (1 - \epsilon_2\kappa_g u)T - \epsilon_2\kappa_n U \end{aligned}$$

and

$$S_u(s,u) = T(s).$$

By the vector product defined in (2.6), we have

$$S_s \times S_u = -\epsilon\epsilon_2\kappa_n Y \tag{3.8}$$

and

$$\|S_s \times S_u\| = \epsilon\kappa_n. \tag{3.9}$$

From equations (3.8)-(3.9) and the formula (3.7), the unit normal vector field of the surface S is

$$U = -\epsilon_2 Y. \tag{3.10}$$

If α is a darboux curve, from (3.2)-(3.3) and (3.10), we have (3.6). □

Example 3.1 *We will consider that the function $f = f(s,u)$ wich defines the geometry of the strict Walker manifold is given by*

$$f(s) = -2ae^{-2s}, \quad a \in \mathbb{R}, \quad 0 < a < 1. \tag{3.11}$$

We consider the curve α given by

$$\alpha(s) = (-ae^{-s}, s, e^s), \quad s \in \mathbb{R}. \tag{3.12}$$

So we have

$$T(s) = (ae^{-s}, 1, e^{s}), \ \ s \in \mathbb{R}. \tag{3.13}$$

An easy computation show that $g_f^{\epsilon}(\alpha'(s), \alpha'(s)) = \epsilon = \epsilon_1$.
From (3.12) and (3.13), the ruled surface which is parametrized in (3.5) is

$$S(s,u) = \left(ae^{-s}(u-1), s+u, e^{s}(u+1)\right). \tag{3.14}$$

The unit normal vector field of the surface S is

$$U = \epsilon\epsilon_u \left(-\frac{3}{2}e^{-s}, \epsilon, -\frac{1}{2a}e^{s}\right). \tag{3.15}$$

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