An Stochastic Epidemic Model with Two Types of Infectious Diseases and Vertical Transmission

Wang xunyang[1], Qihong Shi[1], Huang canyun[1], and Hao yixin[1]

[1]Lanzhou University of Technology

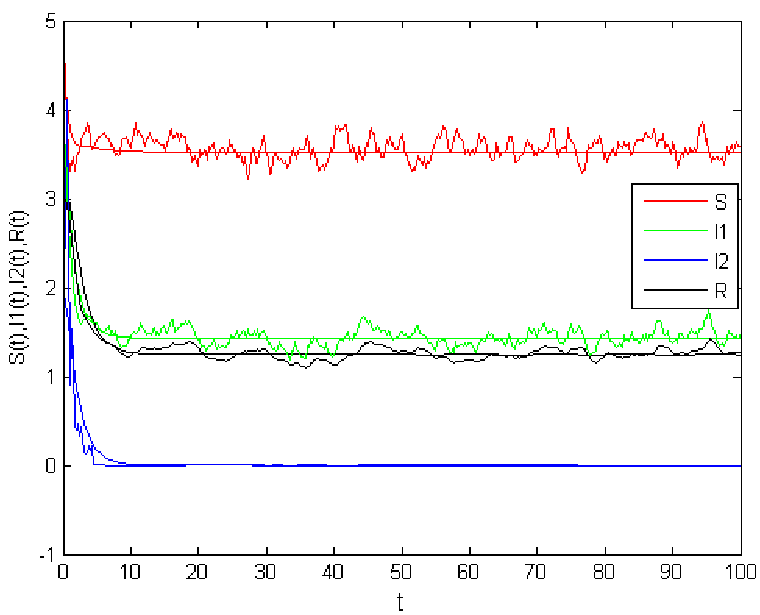
April 05, 2024

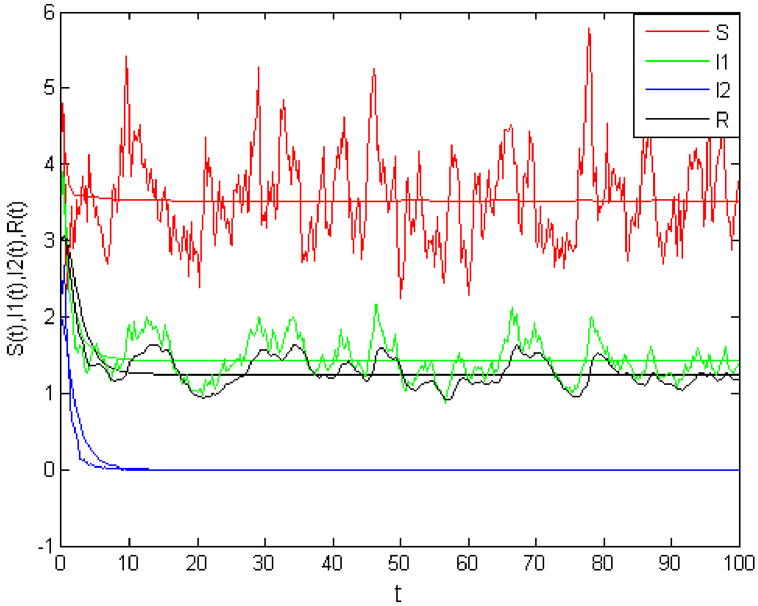
Abstract

In this paper, a stochastic epidemic model with two different types of infectious diseases that spread through both horizontal and vertical transmission is investigaed. By constructing suitable Lyapunov functions and applying It$^{o's formulaaswellasChebyshev's inequality,thesufficien}$

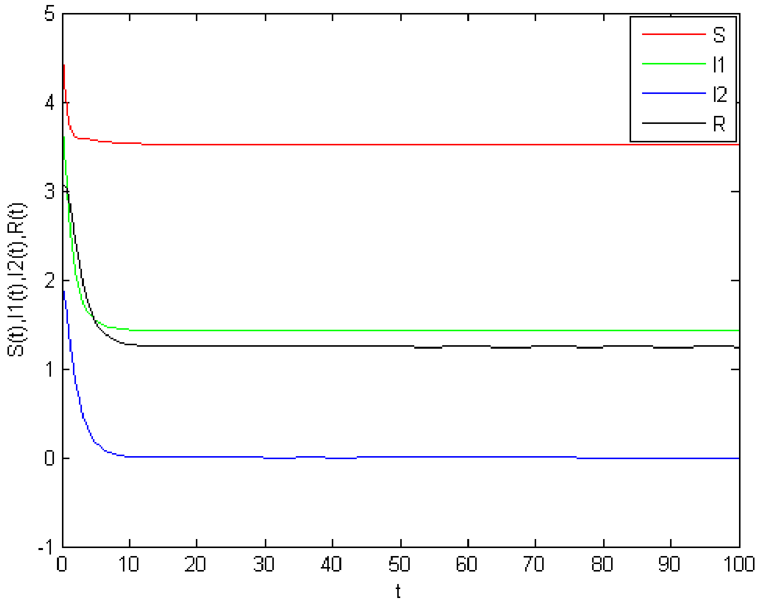
Hosted file

`submission1.tex` available at https://authorea.com/users/735047/articles/711651-an-stochastic-epidemic-model-with-two-types-of-infectious-diseases-and-vertical-transmission

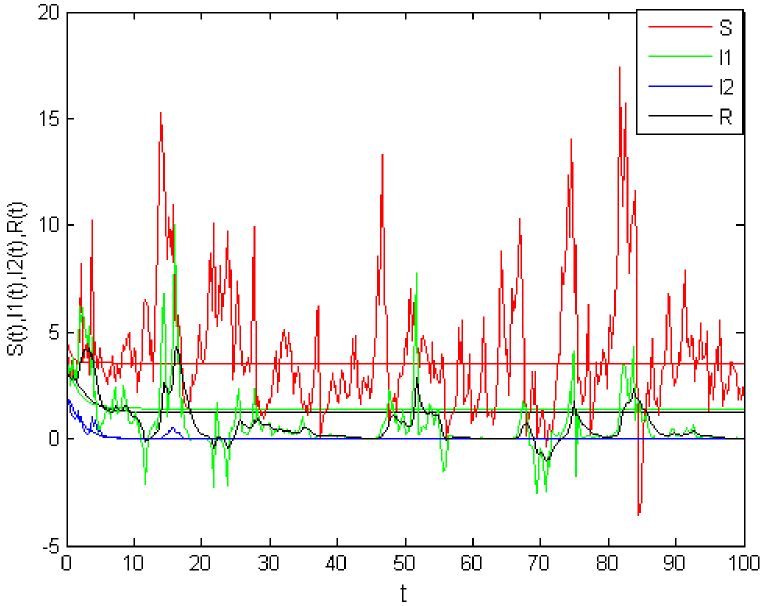




S
I1
I2
R
S(t),I1(t),I2(t),R(t)
t



S
I1
I2
R
S(t),I1(t),I2(t),R(t)
t


S
I1
I2
R
S(t),I1(t),I2(t),R(t)
t
20
15
10
5
0
-5
0
10
20
30
40
50
60
70
80
90
100

A Stochastic Epidemic Model with Two Types of Infectious Diseases and Vertical Transmission*

Xunyang Wang[1,2]†, Canyun Huang[1], Yixin Hao[1], Qi Hong Shi[1]

1.Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu, China, 730050
2.Postdoctoral Research Station of State Grid Gansu Electric Power Research Institute, No. 249, Wanxin North Road, Lanzhou, Gansu, China, 730000
E-mail: 12198114@163.com

Abstract

In this paper, a stochastic epidemic model with two different types of infectious diseases that spread through both horizontal and vertical transmission is investigated. To indicate our model is well-posed, the existence and uniqueness of positive solution is proved at the beginning. By constructing suitable *Lyapunov* functions and applying *Itô*'s formula as well as *Chebyshev*'s inequality, the sufficient conditions for stochastic ultimate boundedness is also established, furthermore, when some main parameters and all the stochastically perturbed intensity satisfy a certain relationship, the stochastic permanence is finally proved. The reliability of theoretical results are further illustrated by numerical simulations.

Key words: Vertical transmission; *Itô*'s formula; Stochastic ultimate boundedness; Stochastic permanence

1 Introduction

In all periods of the development of human society, there are arduous struggles against various infectious diseases[1]. To make matters worse, multiple infectious diseases often exist on human individuals at the same time, and the coordinated and cross-infection between infectious diseases makes the course of the disease more complicated and difficult to deal with. We often refer to this situation as the parallel development of multiple infectious diseases[2]. The probability of several infectious diseases in a patient at the same time is related to the environment, susceptible population and human immunity. If the sanitary environment is poor, and there are sewage, excreta, animal and plant residues, mosquitoes and mice everywhere, the probability of N kinds of infectious diseases will be very high. If people with bad living habits, sanitation workers, medical staff in the infection department and other personnel are frequently exposed to a large number of pathogenic microorganisms, the probability of suffering from N kinds of infection rate is also very

*Supported by the Natural Science Foundation of Gansu Province,China(No.20JR5RA460)
†Corresponding author

high. In addition, people with immune system defects or injuries have a high rate of N infectious diseases[3]. For example, patients with advanced AIDS have a 100% chance of contracting N infectious diseases[4,5].

Recently, the research on modelling two types of infectious diseases draw many applied mathematicians' attention[6-11]. In [6], the authors discussed an epidemic model with double hypothesis combined two different transmission mechanisms. In [7], a stochastically perturbed SIRS epidemic model with two viruses is formulated to investigate the effect of intensities of white noise on each population, and further discussed the dynamics of threshold around disease-free equilibrium as well as endemic equilibrium. In [8], they proposed new mathematical models with nonlinear incidence rate and double epidemic hypothesis. Among them, Ackleh and Allen, Naji and Hussien also studied the epidemic model with multi-disease and vertical transmission[10,11].

As we all know, when modelling, the incidence is the key factor in epidemiological model, and it is defined as the rate at which susceptible becomes infectious. It is usually assumed to be constant or deterministic function in the previous deterministic models. In fact, it is inevitably affected by environmental white noise, which has attracted the attention of scholars. In recent years, a number of scholars have established a series of stochastic epidemic models and achieved great results by introducing random perturbation into deterministic model[12-15]. The reference[11] have discussed the dynamic behavior of the solution of the system under some different parameters and initial values, on this basis, we study a stochastic epidemic model with two types of infectious diseases and vertical transmission. By constructing suitable *Lyapunov* function and using *Itô*'s formula and *Chebyshev*'s inequality, the dynamic behaviors of the model are analyzed. Finally, numerical simulations are used to confirm our obtained theoretical results.

2 Model formulation

The reference [11] researched the following differential equations:

$$\begin{cases} \frac{dS}{dt} = \Lambda - (\frac{\beta_1 I_1}{1+I_1} + \beta_2 I_2)S + (\gamma - p_2)I_2 - \mu S - p_1 I_1 + \eta R, \\ \frac{dI_1}{dt} = \frac{\beta_1 S I_1}{1+I_1} - (\mu + \alpha_1 + \delta - p_1)I_1, \\ \frac{dI_2}{dt} = \beta_2 S I_2 - (\mu + \alpha_2 + \gamma - p_2)I_2, \\ \frac{dR}{dt} = \delta I_1 - (\eta + \mu)R, \end{cases} \tag{2.1}$$

where $S(t)$ represents the number of susceptible individuals at time t; $I_1(t)$ and $I_2(t)$ that represents the number of infected individuals at time t and $R(t)$ that represents the number of recovered individuals at time t, the initial condition $S(0) > 0, I_1(0) > 0, I_2(0) > 0, R(0) > 0$, the total population $N(t) = S(t) + I_1(t) + I_2(t) + R(t)$. There is a constant number of populations entering to the deterministic system with recruitment rate $\Lambda > 0$. There is a vertical transmission of both of the diseases; that is, the infectious individual gives birth to a new infected individual of rates $0 \le p_1 \le 1$ and $0 \le p_2 \le 1$ for the disease I_1 and I_2, respectively. Consequently, $p_1 I_1$ and $p_2 I_2$ individuals enter into infected compartments I_1 and I_2, respectively, and the same quantities are disappearing from recruitment in the susceptible compartment. The diseases are transmitted by contact, between the individuals in the S compartment and those in $I_i (i = 1, 2)$ compartments with nonlinear incidence rate for I_1 that is give by $\frac{\beta_1 S I_1}{1+I_1}$, in which $\beta_1 > 0$ represents the infection force rate, and linear incidence rate for I_2 that is given by $\beta_2 SI_2$, where $\beta_2 > 0$ represents the infection rate. The individuals in the I_1 compartment are facing death due to the disease with infection death rate $\alpha_1 \geq 0$. They recover from disease and get immunity with a recover rate $\delta > 0$. The individuals in the I_2 compartment are facing death due to the disease with infection death rate $\alpha_2 \geq 0$. They also recover from the disease but return back to be susceptible with recovery rate $\gamma > 0$. The individuals in the R compartment are losing immunity rate $0 \leq \eta < 1$. There is a natural death rate $\mu > 0$ for the individuals in the population. Finally, it is assumed that both the diseases cannot be transmitted to the same individual simultaneously. Moreover, to insure that the recruitment Λ in the susceptible compartment is always positive, the following hypotheses are assumed to be hold always:

$$\delta \geq p_1, \gamma \geq p_2.$$

In this paper, taking into account the effect of randomly fluctuating environment, we assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the parameter β_1, β_2,

$$\beta_1 \to \beta_1 + \sigma_1 \dot{B_1}(t), \quad \beta_2 \to \beta_2 + \sigma_2 \dot{B_2}(t). \tag{2.2}$$

where $B_i(t)(i = 1, 2)$ is standard Brownian motions with $B(0) = 0$, and with intensity of white noise $\sigma_i^2 > 0(i = 1, 2)$. The stochastic version corresponding to the deterministic system (2.1) takes the following form:

$$\begin{cases} dS = [\Lambda - (\frac{\beta_1 I_1}{1+I_1} + \beta_2 I_2)S + (\gamma - p_2)I_2 - \mu S - p_1 I_1 + \eta R]dt \\ -\frac{\sigma_1 SI_1}{1+I_1} dB_1(t) - \sigma_2 SI_2 dB_2(t), \\ dI_1 = [\frac{\beta_1 SI_1}{1+I_1} - (\mu + \alpha_1 + \delta - p_1)I_1]dt + \frac{\sigma_1 SI_1}{1+I_1} dB_1(t), \\ dI_2 = [\beta_2 SI_2 - (\mu + \alpha_2 + \gamma - p_2)I_2]dt + \sigma_2 SI_2 dB_2(t), \\ dR = [\delta I_1 - (\eta + \mu)R]dt, \end{cases} \tag{2.3}$$

Next, we give some basic theory in stochastic differential equation (see[16]).

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions(i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbf{P} - null$ sets). $B(t)$ be an n-dimensional standard Brownian motion defined on the space.

In general, consider the n-dimensional stochastic differential equation of $It\hat{o}$ type

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad t \geq t_0, \tag{2.4}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ : x_i > 0,\ 1 \leq i \leq n\}$. Define the differential operator $\mathbf{L}$ associated with system (2.4) by

$$\mathbf{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} [g^\tau(x,t)g(x,t)]_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If $\mathbf{L}$ acts on a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, then

$$\mathbf{L}V(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace[g^\tau(x,t)V_{xx}(x,t)g(x,t)],$$

where $V_t = \frac{\partial V}{\partial t}, V_x = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}), V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{n\times n}$.

Lemma 1 ($It\hat{0}'s\ formula$)([16]) Let $x(t)$ be an $It\hat{o}$ process on $t \geq 0$ of system (2.4), $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+;\mathbb{R}_+)$, then the function $V(x(t),t)$ is again an $It\hat{o}$ process with the stochastic differential given by

$$dV(x(t),t) = \mathbf{L}V(x(t),t)dt + V_x(x(t),t)g(x(t),t)dB(t).$$

Lemma 2 ($Chebyshev's\ inequality$)([16])

$$\mathbf{E}(|X| \geq c) \leq c^{-p}\mathbf{E}|X|^p, \quad p > 1.$$

3 Existence and uniqueness of positive solution

In this section, in order to show our model is well-posed, we prove that there is a unique global positive solution of system (2.3) by using the Lpapunov analysis method and $It\hat{o}'s$ formula.

Theorem 3.1 There is a unique solution $(S(t), I_1(t), I_2(t), R(t))$ of system (2.3) on $t \geq 0$ for any initial value $(S(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}^4_+$, and the solution will remain in $\mathbb{R}^4_+$ with probability 1, namely, $(S(t), I_1(t), I_2(t), R(t)) \in \mathbb{R}^4_+$ for all $t \geq 0$ almost surely.

proof. Since the coefficients of the equation are locally Lipschitz continuous for any given initial value $(S(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}^4_+$, there is a unique local solution $(S(t), I_1(t), I_2(t), R(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time[15]. To show this solution is global, we need to show that $\tau_e = \infty$ a.s.. Let $k_0 \geq 0$ be sufficiently large so that $S(0), I_1(0), I_2(0)$ and $R(0)$ all lie within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0,\tau_e) : S(t) \notin (\frac{1}{k}, k)\ or\ I_1(t) \notin (\frac{1}{k}, k)\ or\ I_2(t) \notin (\frac{1}{k}, k)\ or\ R(t) \notin (\frac{1}{k}, k)\},$$

where we set $\inf \emptyset = \infty$(as usual $\emptyset$ denotes the empty set). Obviously, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim\limits_{k\to\infty} \tau_k$, hence $\tau_\infty \leq \tau_e$ a.s.. To complete the proof, all we need to show that $\tau_\infty = \infty$ a.s.. If this statement is false, then there exist a pair of constants $T > 0$ and $\varepsilon \in (0,1)$ such that

$$\mathbf{P}\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there is an integer $k_1 \geq k_0$ such that for all $k \geq k_1$

$$\mathbf{P}\{\tau_k \leq T\} \geq \varepsilon. \tag{3.1}$$

Define a C^2-function V: $\mathbb{R}^4_+ \to \mathbb{R}_+$ by

$$V(S, I_1, I_2, R) = (S - 1 - \ln S) + (I_1 - 1 - \ln I_1) + (I_2 - 1 - \ln I_2) + (R - 1 - \ln R).$$

The nonnegativity of this function can be seen from $u - 1 - \ln u \geq 0,\ u > 0$.

Let $k \geq k_0$ and $T > 0$ be arbitrary. Applying the $It\hat{o}'s$ formula, we obtain

$$dV(S, I_1, I_2, R) = \mathbf{L}V dt + \frac{\sigma_1}{1+I_1}(I_1 - S)dB_1(t) + \sigma_2(I_2 - S)dB_2(t), \tag{3.2}$$

where

$$\mathbf{L}V = [\Lambda - \mu(S + I_1 + I_2 + R) - \alpha_1 I_1 - \alpha_2 I_2] - \frac{\Lambda + (\gamma - p_2)I_2 - p_1 I_1 + \eta R}{S}$$

$$-\frac{\beta_1(S-I_1)}{1+I_1}-\beta_2(S-I_2)-\frac{\delta I_1}{R}+(4\mu+\alpha_1+\alpha_2+\delta+\gamma+\eta-p_1-p_2)$$
$$+\frac{\sigma_1^2(S^2+I_1^2)}{2(1+I_1)^2}+\frac{1}{2}\sigma_2^2(S^2+I_2^2)$$
$$\leq \Lambda+4\mu+\alpha_1+\alpha_2+\delta+\gamma+\eta+\frac{\sigma_1^2(S^2+I_1^2)}{2(1+I_1)^2}+\frac{1}{2}\sigma_2^2(S^2+I_2^2)\triangleq H,$$

where H is a positive constant. Hence,

$$dV(S,I_1,I_2,R)\leq Hdt+\frac{\sigma_1(I_2-S)}{1+I_1}dB_1(t)+\sigma_2(I_2-S)dB_2(t).$$

Integrating both sides of the above inequality from 0 to $\tau_k\wedge T$, we get

$$\int_0^{\tau_k\wedge T}dV(S(s),I_1(s),I_2(s),R(s))\leq\int_0^{\tau_k\wedge T}Hds$$
$$+\int_0^{\tau_k\wedge T}[\frac{\sigma_1(I_1(s)-S(s))}{1+I_1(s)}dB_1(s)+\sigma_2(I_2(s)-S(s))dB_2(s)],$$

where $\tau_k\wedge T=\min\{\tau_k,T\}$. Then taking the expectations leads to

$$\mathbf{E}V(S(\tau_k\wedge T),I_1(\tau_k\wedge T),I_2(\tau_k\wedge T),R(\tau_k\wedge T))\leq V(S(0),I_1(0),I_2(0),R(0))+HT.$$

Set $\Omega_k=\{\tau_k\leq T\}$ for $k\geq k_1$ and from (3.1), we have $\mathbf{P}(\Omega_k)\geq\varepsilon$. For every $\nu\in\Omega_k$, $S(\tau_k,\nu)$, $I_1(\tau_k,\nu)$, $I_2(\tau_k,\nu)$, $R(\tau_k,\nu)$ equals either k or $\frac{1}{k}$; hence $V(S(\tau_k,\nu),I_1(\tau_k,\nu),I_2(\tau_k,\nu),R(\tau_k,\nu))$ is no less than $\min\{k-1-\ln k,\frac{1}{k}-1-\ln\frac{1}{k}\}$.

Then we obtain

$$V(S(0),I_1(0),I_2(0),R(0))+HT\geq\mathbf{E}[1_{\Omega_k(\nu)}V(S(\tau_k),I_1(\tau_k),I_2(\tau_k),R(\tau_k))]$$
$$\geq\varepsilon\min\{k-1-\ln k,\frac{1}{k}-1-\ln\frac{1}{k}\},$$

where $1_{\Omega_k(\nu)}$ is the indicator function of Ω_k.

Letting $n\to\infty$ leads to the contradiction $\infty=V(S(0),I_1(0),I_2(0),R(0))+HT<\infty$. This completes the proof.

4 Stochastic ultimate boundedness

Definition 4.1 The solution $X(t)=(S(t),I_1(t),I_2(t),R(t))$ of system (2.3) are said to be stochastically ultimately bounded, if for any $\varepsilon\in(0,1)$, there is a positive constant $\chi=\chi(\varepsilon)$, such that for any initial value $(S(0),I_1(0),I_2(0),R(0))\in\mathbb{R}_+^4$, the solution $X(t)$ has the property that

$$\limsup_{t\to\infty}\mathbf{P}\{|X(t)|>\chi\}<\varepsilon.$$

Theorem 4.1 The solution of system (2.3) are stochastically ultimately bounded for any initial value $(S(0),I_1(0),I_2(0),R(0))\in\mathbb{R}_+^4$.

proof. From Theorem 3.1, the solution will remain in $\mathbb{R}^4_+$ for any $t \geq 0$ almost surely. Define a function

$$V(S, I_1, I_2, R) = S^\theta + I_1^\theta + I_2^\theta + R^\theta,$$

for $(S, I_1, I_2, R) \in \mathbb{R}^4_+$ and $\theta \in (0, 1)$. By $It\hat{o}$'s formula, we obtain

$$dV(S, I_1, I_2, R) = [\theta S^{\theta-1}(\Lambda + (\gamma - p_2)I_2 + \eta R) + \theta(\frac{\beta_1 S I_1^\theta}{1 + I_1} + \beta_2 S I_2^\theta + \delta R^{\theta-1} I_1)]dt$$

$$-\theta[\frac{\beta_1 S^\theta I_1}{1 + I_1} + \beta_2 S^\theta I_2 + \mu S^\theta + p_1 S^{\theta-1} I_1 + I_1^\theta(\mu + \alpha_1 + \delta - p_1) + I_2^\theta(\mu + \alpha_2 + \gamma - p_2) + R^\theta(\eta + \mu)]dt$$

$$+\frac{1}{2}\theta(\theta - 1)[\frac{\sigma_1^2}{(1 + I_1)^2}(S^\theta I_1^2 + S^2 I_1^\theta) + \sigma_2^2(S^\theta I_2^2 + S^2 I_2^\theta)]dt$$

$$+\frac{\sigma_1\theta}{1 + I_1}(S I_1^\theta - S^\theta I_1)dB_1(t) + \sigma_2\theta(S I_2^\theta - S^\theta I_2)dB_2(t)$$

$$\leq \theta S^{\theta-1}(\Lambda + (\gamma - p_2)I_2 + \eta R)dt + \frac{1}{2}\theta(\theta - 1)[\frac{\sigma_1^2}{(1 + I_1)^2}(S^\theta I_1^2 + S^2 I_1^\theta) + \sigma_2^2(S^\theta I_2^2 + S^2 I_2^\theta)]dt$$

$$+[\theta(\frac{\beta_1 S I_1^\theta}{1 + I_1} + \beta_2 S I_2^\theta + \delta R^{\theta-1} I_1) + (S^\theta + I_1^\theta + I_2^\theta + R^\theta) - V(S, I_1, I_2, R)]dt$$

$$+\frac{\sigma_1\theta}{1 + I_1}(S I_1^\theta - S^\theta I_1)dB_1(t) + \sigma_2\theta(S I_2^\theta - S^\theta I_2)dB_2(t)$$

$$\leq [C - V(S, I_1, I_2, R)]dt + \frac{\sigma_1\theta}{1 + I_1}(S I_1^\theta - S^\theta I_1)dB_1(t) + \sigma_2\theta(S I_2^\theta - S^\theta I_2)dB_2(t),$$

where $C > 0$ is a suitable constant.

Based on Theorem 3.1 and the above inequality, we have

$$d(e^t V(S, I_1, I_2, R)) = e^t V(S, I_1, I_2, R)dt + e^t dV(S, I_1, I_2, R)$$

$$\leq Ce^t dt + \frac{\sigma_1\theta}{1 + I_1}(S I_1^\theta - S^\theta I_1)dB_1(t) + \sigma_2\theta(S I_2^\theta - S^\theta I_2)dB_2(t).$$

Integrating both sides of the above inequality from 0 to t, we get

$$e^t V(S(t), I_1(t), I_2(t), R(t)) \leq V(S(0), I_1(0), I_2(0), R(0)) + C(e^t - 1)$$

$$+\int_0^t [\frac{\sigma_1\theta}{1 + I_1(s)}(S(s)I_1^\theta(s) - S^\theta(s)I_1(s))dB_1(s) + \sigma_2\theta(S(s)I_2^\theta(s) - S^\theta(s)I_2(s))dB_2(s)].$$

Then taking the expectations leads to

$$e^t \mathbf{E}V(S(t), I_1(t), I_2(t), R(t)) \leq V(S(0), I_1(0), I_2(0), R(0)) + C(e^t - 1)$$

$$\Rightarrow \mathbf{E}V(S(t), I_1(t), I_2(t), R(t)) \leq e^{-t}V(S(0), I_1(0), I_2(0), R(0)) + C(1 - e^{-t})$$

$$\Rightarrow \limsup_{t\to\infty} \mathbf{E}V(S(t), I_1(t), I_2(t), R(t)) \leq C.$$

Note that

$$|X(t)|^\theta = (S^2(t) + I_1^2(t) + I_2^2(t) + R^2(t))^{\frac{\theta}{2}}$$

$$\leq 4^{\frac{\theta}{2}} \max\{S^\theta(t), I_1^\theta(t), I_2^\theta(t), R^\theta(t)\} \leq 4^{\frac{\theta}{2}} V(S, I_1, I_2, R),$$

then we get

$$\limsup_{t\to\infty} \mathbf{E}|X(t)|^\theta \leq 4^{\frac{\theta}{2}} \limsup_{t\to\infty} \mathbf{E}V(S, I_1, I_2, R) \leq 4^{\frac{\theta}{2}} C < \infty.$$

Therefore, there exists a positive constant δ_1 such that

$$\limsup_{t\to\infty} \mathbf{E}|\sqrt{X(t)}| \leq \delta_1.$$

For any $\varepsilon > 0$, set $\chi = \frac{\delta_1^2}{\varepsilon^2}$, then by *Chebyshev*'s inequality

$$\mathbf{P}\{|X(t)| > \chi\} \leq \frac{\mathbf{E}|\sqrt{X(t)}|}{\sqrt{\chi}}.$$

Thus, we obtain

$$\limsup_{t\to\infty} \mathbf{P}\{|X(t)| > \chi\} \leq \frac{\delta_1}{\sqrt{\chi}} = \varepsilon,$$

which yields the required assertion.

Theorem 3.1 together with Theorem 4.1 indicates that the solution to our model is non-explosive in condition that all the parameters and intensities remain positive.

5 Stochastic permanence

Definition 5.1 The solution $X(t) = (S(t), I_1(t), I_2(t), R(t))$ of system (2.3) are said to be stochastically permanent, if for any $\varepsilon \in (0, 1)$, there exists a pair of positive constants $\lambda = \lambda(\varepsilon)$ and $\chi = \chi(\varepsilon)$ such that for any initial value $(S(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}_+^4$, the solution $X(t)$ has the properties

$$\liminf_{t\to\infty} \mathbf{P}\{|X(t)| \leq \chi\} \geq 1 - \varepsilon, \qquad \liminf_{t\to\infty} \mathbf{P}\{|X(t)| \geq \lambda\} \geq 1 - \varepsilon.$$

Theorem 5.1 For any initial value $(S(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}_+^4$, the solution $X(t) = (S(t), I_1(t), I_2(t), R(t))$ satisfies

$$\limsup_{t\to\infty} \mathbf{E}(|X(t)|^{-\vartheta}) \leq Q, \tag{5.1}$$

where ϑ is an arbitrary positive constant satisfying

$$\frac{\vartheta + 1}{2}(\mu + \max\{1, \alpha_1, \alpha_2\} + 2\max\{\sigma_1^2, \sigma_2^2\}) < \Lambda, \tag{5.2}$$

$$Q = \frac{4\vartheta(4k\vartheta\Lambda + C^2)}{4k\vartheta\Lambda} \max\{1, (\frac{2\vartheta\Lambda + C + \sqrt{4k\vartheta\Lambda + C^2}}{2\vartheta\Lambda})^{\vartheta-1}\}, \tag{5.3}$$

in which

$$0 < k < \vartheta[2\Lambda - (\mu + \max\{1, \alpha_1, \alpha_2\} + 2\max\{\sigma_1^2, \sigma_2^2\})], \tag{5.4}$$

$$C = k + \vartheta(\mu + \max\{1, \alpha_1, \alpha_2\} + 2\max\{\sigma_1^2, \sigma_2^2\}). \tag{5.5}$$

proof. Define a function $V(S,I_1,I_2,R)=\frac{1}{S+I_1+I_2+R}$, $(S(t),I_1(t),I_2(t),R(t))\in\mathbb{R}^4_+$; using $It\hat{o}'s$ formula, we get

$$dV(S,I_1,I_2,R)=[\mu V+V^2(\alpha_1I_1+\alpha_2I_2-\Lambda)+2V^3(\frac{\sigma_1^2S^2I_1^2}{(1+I_1)^2}+\sigma_2^2S^2I_2^2)]dt.$$

Choosing a positive constant ϑ that satisfies (5.2) and applying $It\hat{o}'s$ formula, we obtain

$$\mathbf{L}(1+V)^{\vartheta}=\vartheta(1+V)^{\vartheta-1}[\mu V+V^2(\alpha_1I_1+\alpha_2I_2-\Lambda)+2V^3(\frac{\sigma_1^2S^2I_1^2}{(1+I_1)^2}+\sigma_2^2S^2I_2^2)]=\vartheta(1+V)^{\vartheta-1}G,$$

where $G=\mu V+V^2(\alpha_1I_1+\alpha_2I_2-\Lambda)+2V^3(\frac{\sigma_1^2S^2I_1^2}{(1+I_1)^2}+\sigma_2^2S^2I_2^2)$.

Since

$$V^2(\alpha_1I_1+\alpha_2I_2)<V^2(S+\alpha_1I_1+\alpha_2I_2+R)$$
$$<V^2\max\{1,\alpha_1,\alpha_2\}(S+I_1+I_2+R)=V\max\{1,\alpha_1,\alpha_2\},$$
$$2V^3(\frac{\sigma_1^2S^2I_1^2}{(1+I_1)^2}+\sigma_2^2S^2I_2^2)<2V^3\max\{\sigma_1^2,\sigma_2^2\}(S^2+S^2I_2^2)<2V\max\{\sigma_1^2,\sigma_2^2\},$$

Therefore,

$$G<(\mu+\max\{1,\alpha_1,\alpha_2\}+2\max\{\sigma_1^2,\sigma_2^2\})V-\Lambda V^2.$$

Let $k>0$ be sufficiently small such that it satisfies (5.4), by $It\hat{o}$'s formula

$$\mathbf{L}(e^{kt}(1+V)^{\vartheta})=ke^{kt}(1+V)^{\vartheta}+e^{kt}\mathbf{L}(1+V)^{\vartheta}$$
$$<e^{kt}(1+V)^{\vartheta-1}[k(1+V)+\vartheta(\mu+\max\{1,\alpha_1,\alpha_2\}+2\max\{\sigma_1^2,\sigma_2^2\})V-\vartheta\Lambda V^2]$$
$$=e^{kt}(1+V)^{\vartheta-1}[k+CV-\vartheta\Lambda V^2]\le e^{kt}W,$$

where

$$W=\frac{4k\vartheta\Lambda+C^2}{4k\vartheta\Lambda}\max\{1,(\frac{2\vartheta\Lambda+C+\sqrt{4k\vartheta\Lambda+C^2}}{2\vartheta\Lambda})^{\vartheta-1}\},$$

and C have been defined in the statement of the theorem.

Thus,

$$d(e^{kt}(1+V)^{\vartheta})\le We^{kt}dt.$$

Integrating both sides of the above inequality from 0 to t, we get

$$e^{kt}(1+V)^{\vartheta}\le(1+V(0))^{\vartheta}+\frac{W}{k}(e^{kt}-1)\le(1+V(0))^{\vartheta}+\frac{W}{k}e^{kt}.$$

$$\Rightarrow(1+V)^{\vartheta}\le\frac{W}{k}+(1+V(0))^{\vartheta}e^{-kt}$$

$$\Rightarrow\mathbf{E}(1+V)^{\vartheta}\le\frac{W}{k}+(1+V(0))^{\vartheta}e^{-kt}.$$

Therefore, we obtain

$$\limsup_{t\to\infty}\mathbf{E}(V(t))^{\vartheta}\le\limsup_{t\to\infty}\mathbf{E}(1+V)^{\vartheta}\le\frac{W}{k}.$$

For $(S, I_1, I_2, R) \in \mathbb{R}^4_+$, we know that

$$(S + I_1 + I_2 + R)^{\vartheta} \leq 4^{\vartheta}(S^2 + I_1^2 + I_2^2 + R^2)^{\frac{\vartheta}{2}} \leq 4^{\vartheta}|X(t)|^{\vartheta};$$

consequently,

$$\limsup_{t\to\infty} \mathbf{E}(\frac{1}{|X(t)|^{\vartheta}}) \leq 4^{\vartheta} \limsup_{t\to\infty} \mathbf{E}(V(t))^{\vartheta} \leq \frac{4^{\vartheta}W}{k} = Q,$$

which completes the proof.

Theorem 5.2 Assume $\max\{\sigma_1^2, \sigma_2^2\} < \Lambda - \frac{\mu+\max\{1,\alpha_1,\alpha_2\}}{2}$, then the solutions of system (2.3) are stochastically permanent.

proof. From Theorem 4.1, we have $\mathbf{P}\{|X(t)| > \chi\} < \varepsilon$ which implies $\mathbf{P}\{|X(t)| \leq \chi\} \leq 1 - \varepsilon$. This follows that $\liminf\limits_{t\to\infty} \mathbf{P}\{|X(t)| \leq \chi\} \geq 1 - \varepsilon$.

By Theorem 5.1, we get $\limsup\limits_{t\to\infty} \mathbf{E}(\frac{1}{|X(t)|^{\vartheta}}) \leq Q$.

For any $\varepsilon > 0$, let $\lambda = \frac{\varepsilon^{\frac{1}{\vartheta}}}{Q^{\frac{1}{\vartheta}}}$; then by *Chebyshev*'s inequality,

$$\mathbf{P}\{|X(t)| < \lambda\} = \mathbf{P}\{\frac{1}{|X(t)|} > \frac{1}{\lambda}\} \leq \lambda^{\vartheta}\mathbf{E}(|X(t)|^{-\vartheta}).$$

Hence,

$$\limsup_{t\to\infty} \mathbf{P}\{|X(t)| < \lambda\} \leq \frac{\varepsilon}{Q} \cdot Q = \varepsilon,$$

which follows that

$$\liminf_{t\to\infty} \mathbf{P}\{|X(t)| \geq \lambda\} \geq 1 - \varepsilon.$$

The proof is completed.

Compared with the existence, uniqueness and boundedness of positive solutions, the stochastic persistence condition of the model solution is much more stringent. Specifically, some main parameters, i.e., the recrument rate Λ, disease-related mortality α_1, α_2, natural mortality μ and all the perturbed intensities σ_1, σ_2 should satisfy $\max\{\sigma_1^2, \sigma_2^2\} < \Lambda - \frac{\mu+\max\{1,\alpha_1,\alpha_2\}}{2}$.

6 Numerical simulations and Conclusions

In this section, for the system (2.3), we will use the Milstein method mentioned in Higham [17] to illustrate our main results.

Consider the following discretization equations:

$$\begin{cases} S_{k+1} = S_k + [\Lambda - (\frac{\beta_1 I_{1k}}{1+I_{1k}} + \beta_2 I_{2k})S_k + (\gamma - p_2)I_{2k} - \mu S_k - p_1 I_{1k} + \eta R_k]\Delta t \\ -\frac{\sigma_1 S_k I_{1k}}{1+I_{1k}}\sqrt{\Delta t}\xi_k - \sigma_2 S_k I_{2k}\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2}(\frac{S_k I_{1k}}{1+I_{1k}})^2(\xi_k^2 - 1)\Delta t + \frac{\sigma_2^2}{2}S_k^2 I_{2k}^2(\xi_k^2 - 1)\Delta t, \\ I_{1k+1} = I_{1k} + [\frac{\beta_1 S_k I_{1k}}{1+I_{1k}} - (\mu + \alpha_1 + \delta - p_1)I_{1k}]\Delta t + \frac{\sigma_1 S_k I_{1k}}{1+I_{1k}}\sqrt{\Delta t}\xi_k \\ +\frac{\sigma_1^2}{2}(\frac{S_k I_{1k}}{1+I_{1k}})^2(\xi_k^2 - 1)\Delta t, \\ I_{2k+1} = I_{2k} + [\beta_2 S_k I_{2k} - (\mu + \alpha_2 + \gamma - p_2)I_{2k}]\Delta t + \sigma_2 S_k I_{2k}\sqrt{\Delta t}\xi_k \\ +\frac{\sigma_2^2}{2}S^2 I_{2k}^2(\xi_k^2 - 1)\Delta t, \\ R_{k+1} = R_k + [\delta I_{1k} - (\eta + \mu)R_k]\Delta t, \end{cases} \tag{6.1}$$

where $\xi_k(k = 1, \cdots, n)$ is the Guassian random variables which follow $N(0, 1)$.

We choose the parameters by $\Lambda = 2, \beta_1 = 0.75, \beta_2 = 0.3, \gamma = 0.75, p_1 = 0.01, p_2 = 0.05, \mu = 0.3, \eta = 0.5, \delta = 0.7, \alpha_1 = 0.1, \alpha_2 = 0.6$, and initial value $(S(0), I_1(0), I_2(0), R(0))$ =(5, 4, 2, 3). Then the corresponding pathwise estimation of the solutions of system (2.3) are shown in Fig.1. Let $\sigma_1 = 0.04, \sigma_2 = 0.06$, the solutions of system (2.3) are stochastically permanent(Fig.1(a)). Let $\sigma_1 = 0.2, \sigma_2 = 0.1$ and the condition of Theorem (5.2) is satisfied, we can see that the larger intensity of the white noise will weaken the stability of the system(Fig.1(b)).

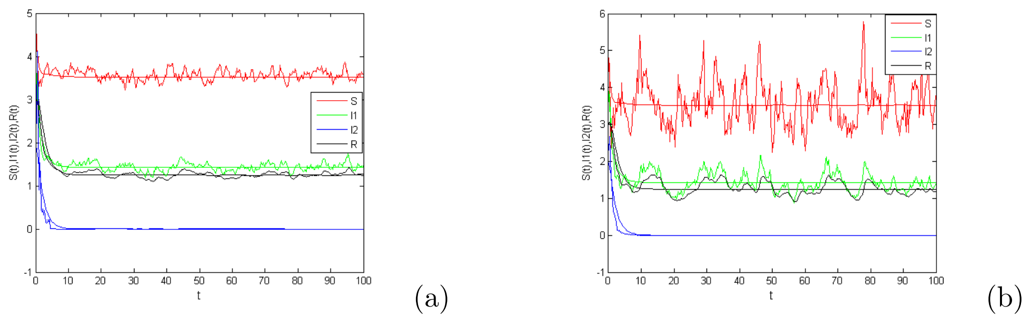


Figure 1: Solutions of system (2.3) with different noise. Other parameters and initial condition are given in text. (a): $\sigma_1 = 0.04, \sigma_2 = 0.06$; (b): $\sigma_1 = 0.2, \sigma_2 = 0.1$.

When $\sigma_1 = \sigma_2 = 0.0$, system (2.3) will be deterministic and its time-series plots shown by Fig.2(a). We choose $\sigma_1 = 1.0, \sigma_2 = 1.3$ which does not satisfy the condition of Theorem (5.2), then the noise can force the population to become largely fluctuating. In this case, the solutions of system (2.3) are not stochastically permanent(Fig.2(b)).

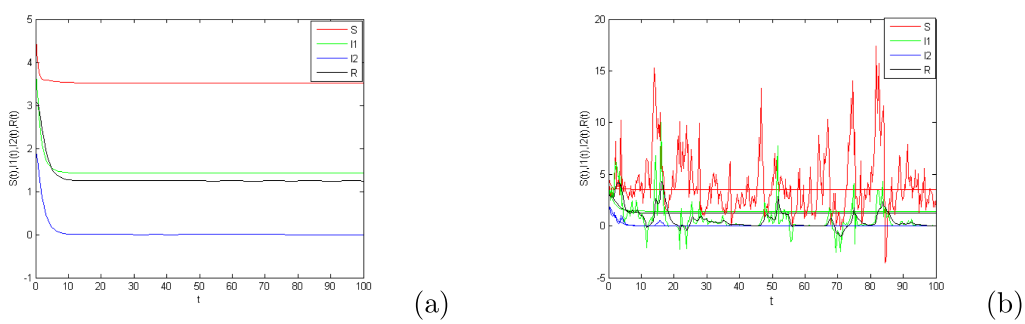


Figure 2: Solutions of system (2.3) with different noise. Other parameters and initial condition are given in text. (a): $\sigma_1 = 0.0, \sigma_2 = 0.0$; (b): $\sigma_1 = 1.0, \sigma_2 = 1.3$.

References

[1] Andrewes C. H., Natural History of Infectious Disease[J]. American Journal of Public Health & the Nations Health, 1953, 43(12):1592.

[2] Massinissa S. M., Farah A., Piers M., Bernadette M., The WHO R and D Blueprint: 2018 review of emerging infectious diseases requiring urgent research and development efforts[J]. Antiviral Research, 2018, 159: 63-67.

[3] Konstantin P. L., Mewa S., Roberto C., Maria L. G., A multi-antigen print immunoassay for the development of serological diagnosis of infectious diseases[J]. Journal of Immunological Methods, 2000, 242(1C2):91-100.

[4] Bhunu C.P., Garira W., Mukandavire Z., Modeling HIV/AIDS and Tuberculosis Coinfection[J]. Bulletin of Mathematical Biology, 2009, 71: 1745C1780.

[5] Eduardo M., Marcelo N. Burattini, Francisco A. B. C., Hyung M. Y., etc, Modeling the interaction between aids and tuberculosis[J]. Mathematical & Computer Modelling, 1993, 17(9):7-21.

[6] Brahim B., Mohamed E. F., Aziz L., Regragui T., Analysis of a deterministic and a stochastic epidemic model with two distinct epidemics hypothesis[J], Physica A: Statistical Mechanics and its Applications, 2019,534,122321.

[7] Rajasekar S.P., Pitchaimani M., Qualitative analysis of stochastically perturbed SIRS epidemic model with two viruses[J]. Chaos, Solitons & Fractals, 2019, 118: 207-221.

[8] Meng X. Z. , Zhao S.N.,Feng T., Zhang T.H., Dynamics of a novel nonlinear stochastic SIS epidemic model with double epidemic hypothesis[J]. Journal of Mathematical Analysis and Applications, 2016, 433(1): 227-242.

[9] Martcheva M., A non-autonomous multi-strain SIS epidemic model[J]. Journal of Biological Dynamics, 2009, 3(2-3): 235-251.

[10] Ackleh A.S., Allen L.J., Competitive exclusion and coexistence for pathogens in an epidemic model with variable population size[J] Journal of Mathematical Biology, 2003, 47(2): 153-168.

[11] Naji R.K. , Hussien R.M., The dynamics of epidemic model with two types of indectious diseases and vertical transmission[J]. Journal of Applied Mathematics, 2016, 2016(5): 1-16.

[12] Dalal N., Greenhalgh D., Mao X.R., A stochastic model of AIDS and condom use[J]. Journal of Mathematical Analysis and Applications, 2007, 325(1): 36-57.

[13] Ji C.Y., Jiang D.Q., Shi N.Z., The behavior of an SIR epidemic model with stochastic perturbation[J]. Stochastic Analysis and Application, 2012, 30(5): 755-773.

[14] Ji C.Y., Jiang D.Q., Shi N.Z., Multigroup SIR epidemic model with stochastic perturbation[J]. Physica A Statistical Mechanics and Its Applications, 2011, 390(10): 1747-1762.

[15] Zhao Y.N., Jiang D.Q., The threshold of a stochastic SIS epidemic model with vaccination[J]. Applied Mathematics and Computation, 2014, 243: 718-727.

[16] X. Mao, Stochastic differential equations and their applications[M], Chichester Horwood Publishing, 1997.

[17] Higham D. J., An algorithmic introduction to numerical simulation of stochastic differential equations[J]. Siam Review, 2001, 43(3): 525-546.