Residual-based a posteriori error estimates for the h-p version of the finite element discretization of the elliptic Robin boundary control problem.

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Abstract

In this paper, we analyzed a priori and a posteriori error estimates for the h - p version of the finite element discretization of the elliptic Robin boundary control problem. The conforming \$h-p\$ finite element method is used. First, we established the optimality conditions for the continuous and discrete optimal control problems, respectively. Then, a priori error estimates of the \$h-p\$ finite element discretization for the optimal control problem are derived rigorously. Moreover, residual-based a posteriori error estimates are established for the coupled state and control approximations. Such estimators can be used to construct reliable adaptive methods for optimal control problems.

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Abstract

In this paper, we analyzed a priori and a posteriori error estimates for the $h-p$ version of the finite element discretization of the elliptic Robin boundary control problem. The conforming $h-p$ finite element method is used. First, we established the optimality conditions for the continuous and discrete optimal control problems, respectively. Then, a priori error estimates of the $h-p$ finite element discretization for the optimal control problem are derived rigorously. Moreover, residual-based a posteriori error estimates are established for the coupled state and control approximations. Such estimators can be used to construct reliable adaptive methods for optimal control problems.

Key words : Optimal control problems, $h-p$ element method, elliptic Robin boundary Control problem, a priori error estimates, a posteriori error estimates of residual type.

1 Introduction

Optimal control problems have become a very active and successful research area and it can be used in many sciences and engineering. They have various application backgrounds in the operation of physical, social, and economic processes. Concerning the analysis of optimal control problems with PDEs constraints, we must mention the pioneer works in this field, such as [6, 8, 16, 19, 23, 31], for an overview of optimal control problems for more details.

The finite element method have been widely used in PDEs-constraints optimal control problems. There have been many works in literature on the finite element approximation of optimal control problems [5, 20, 21, 24], and so on.

The $h-p$ version of the finite element which is the general version of finite element method has been applied to many practical problems. We mention the pioneering works [3, 4, 25]. It seems suitable to apply the $h-p$ version methods to approximate optimal control problems. Here, we only mention the following works in [9–12]. Yanping Chen and Yijie Lin, in [10], presented a posteriori error analysis for the $h-p$ finite element approximation of convex optimal control problems. Before obtaining the $h-p$ a posteriori error estimates for the coupled control and state approximations, they derived a new quasi-interpolation operator of Clément type and a new quasi-interpolation operator of the Scott-Zhang type that preserves homogeneous boundary conditions.

A priori error estimate consists in increasing the discretization error in a given norm (or semi-norm) by a quantity which depends on the exact solution (in general not known explicitly). It is generally used to prove the convergence of the method under certain assumptions of regularity on the exact solution. But these assumptions in practice are not often verified.

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Moreover, the estimator is in term of the exact solution of the model, this shows that the a priori estimates cannot be used for an algorithm of mesh adaptation (which is very important in the finite element method because with this method, the discrete solution strongly depends on the mesh used). Hence, the notion of a posteriori error analysis which now makes it possible to set up (thanks to an algorithm) an adapted mesh. For optimal control problems, there are many works on a priori error estimates (see, e.g., [**?**, 1, 17, 18, 27–29]) for the standard finite element method.

That is why, after the a priori error analysis, we focused on the adaptability of the mesh, i.e. to build a tool that allows, after a first resolution of the problem on a coarse decomposition, a decomposition choice that is the best suited to the problem (the flexibility of the method with joints that greatly facilitate the construction of the new decomposition), which is a posteriori error estimator. A posteriori error estimates are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. For optimal control problems, there many works on a posteriori error estimates for $h-p$ finite element method, [9–12, 14, 21, 22]. They are essential to design adaptive mesh refinement algorithms which equidistribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuška and Rheinboldt [2], adaptive finite element methods based on a posteriori error estimates have been extensively investigated. What we propose to do here is to extend in the case of spectral elements a part of the results obtained by Verfiirth [32], concerning indicators constructed from the residue of the equation. The extension is not evident because an important step in the demonstration in finite elements are based on inverse inequalities [26], which are known to be bad in spectral methods, so it is a matter of changing of the indicators appropriately.

However, as part of the $h-p$ version, two choices of "refinement" are possible in the areas where the indicator reveals poor convergence; either divide the domain to reduce the h_k setting, or increase the maximum p_k degree. The idea to determine the best strategy is to use a spectral decomposition of the indicator in an appropriate base of the Polynomials. Examination of the behaviour of the coefficients allow then to find the best refinement: it is indeed preferable to increase the p_k when all coefficients are of the same size and to decrease the h_k when the high-order coefficients are small.

The Robin boundary condition is a general form of the isolation boundary condition for convection-diffusion equations and can be viewed as the trade-off Dirichlet and Neumann boundary conditions. They are applied to electromagnetic problems or boundary conditions of convection, for example in heat transmission problems [15] and in modeling the convection between the conducting bodies (see [7, 13]). So, in this paper, the elliptic Robin boundary control problem is approximated by $h-p$ version of finite element method. We mainly consider the following optimal control problem:

$$\min_{u\in U_{ad}} J(u,y) := \frac{1}{2}\int_{\Omega}(y-y_d)^2 + \frac{\varepsilon}{2}\int_{\partial\Omega} u^2 \tag{1.1}$$

Subject to the state equation

$$-\,div(a\nabla y) = f \quad \text{in} \quad \Omega \tag{1.2}$$

With the boundary control conditions(**Robin**)

$$(a\nabla y).n = \frac{\beta}{\kappa}(u-y) \quad \text{on} \quad \partial\Omega \tag{1.3}$$

In our paper, Ω will be an open bounded subset of $\mathbb{R}^2$, with Lipschitz continuous boundary $\partial\Omega$. We also assume that Ω is polygonal. The nonempty, closed and convex constraint $U_{ad} = \{u \in L^2(\partial\Omega) : ||u||^2_{0,\partial\Omega} \leq \zeta^2\}$, n is outwards unit normal vector from Ω, ε and κ are strictly positive constants. $\beta : \partial\Omega \to (0,\infty)$ is a positive function, $\beta \in L^\infty(\partial\Omega)$ with $\int_{\partial\Omega}\beta^2 > 0$, $y_d \in L^2(\Omega)$ is the observation, u is the control variable, y is the state variable and $f \in L^2(\Omega)$. $a(\cdot) = \left(a_{ij}(\cdot)\right)_{2\times 2} \in \left(W^{1,\infty}(\Omega)\right)^{2\times 2}$ such that there is a constant δ satisfying, for all vector $x \in \mathbb{R}^2$, $x^t a x \geq \delta ||x||^2_{\mathbb{R}^2}$, where x^t denote the transpose of x. Also for simplicity, we suppose in this paper that a is constant matrix.

For $\Omega \subset \mathbb{R}^2$ a polygonal convex domain, we set $L^2(\Omega) = H^0(\Omega), H^k(\Omega), H^k_0(\Omega),\ \ k \geq 0$ integer, denote the usual Sobolev spaces. For $u \in H^k(\Omega)$ we denote by $||u||_{k,\Omega}$ and $|\cdot|_{k,\Omega}$ the usual norm and semi-norm respectively. If I is an interval or a segment, then we define $H^k(I)$, $||\cdot||_{k,I}$ analogously.

Introduce the function spaces $U = L^2(\partial\Omega) \supset U_{ad} \ni u$ as the control space, $V = H^1(\Omega)$ as the state space. To avoid the excessive use of constants, generic constants in the estimations will be denoted by C.

The outline of this paper is as follows: In section 2, $h-p$ version of the finite element methods for the optimal control problem are constructed, then optimality conditions for both exact and discrete system are given. A priori error estimates for the optimal control problems are obtained in section 3. In section 4, a posteriori error estimates for the optimal control problems are obtained.

2 Optimality of the optimal control problem and its finite element approximation

In this section, we shall find out the optimality conditions of the optimal problem (1.2)-(1.3) and describe its $h-p$ finite element approximation.

2.1 Optimality of the problem

Let's assume that y is enough regular, so for any regular test function $v \in C^1(\overline{\Omega})$, integration by parts yields as

$$\int_\Omega (a\nabla y)\nabla v - \int_{\partial\Omega}(\alpha u).v + \int_{\partial\Omega}(\alpha y).v \tag{2.1}$$

The weak formulation for the boundary control value problem (1.1)-(1.3) using Green's formula is of the form : find $y \in V$ such that

$$A(y,v) = l(v),\ \forall\ v \in V. \tag{2.2}$$

Where

$$A(y,v) := \int_\Omega (a\nabla y)\nabla v + \int_{\partial\Omega}\alpha y v, \quad l(v) := \int_\Omega f v + \int_{\partial\Omega}\alpha u v \quad \text{and} \quad \alpha = \frac{\beta}{\kappa}.$$

The extension to $v \in V$ is possible since for $y \in V$ both sides are continuous with respect to $v \in V$ and since $C^1(\overline{\Omega})$ is dense in V.

It follows from the above assumptions on a that there exists constants $c_1, c_2, c_3 > 0$ such that $\forall\ v, y \in V$

$$A(v,v) \geq c_1\,||v||^2_{1,\Omega}, \quad A(y,v) \leq c_2\,||y||_{1,\Omega}||v||_{1,\Omega} \text{ and } |l(v)| \leq c_3\,||v||_{1,\Omega}$$

Therefore, following the work done in [30, 31] one can prove the following optimality conditions:

Theorem 2.1. *The pair $(u,y) \in U_{ad} \times V$ is the optimal solution of the control problem (1.1)-(1.2) if only if there exists a unique pair $(p,\lambda) \in V \times \mathbb{R}^+$ such that (u,y,p,λ) satisfies the following optimality conditions :*

$$(OCP-OPT) \quad \begin{cases} (i)\ A(y,v) = (\alpha u, v)_{0,\partial\Omega} + (f,v)_{0,\Omega} \quad \forall\, v \in V \\ (ii)\ A(q,p) = (y - y_d, q)_{0,\Omega} \quad \forall\, q \in V \\ (iii)\ \ (\alpha p + \varepsilon u, w - u)_{0,\partial\Omega} \geq 0 \quad \forall\, w \in U_{ad}, \\ (iv)\ p + (\varepsilon + \lambda) u = 0 \end{cases} \tag{2.3}$$

where λ satisfies

$$\lambda = \begin{cases} constant \geq 0, & \|u\|^2_{0,\partial\Omega} = \zeta^2 \\ \quad 0 & \|u\|^2_{0,\partial\Omega} \leq \zeta^2 \end{cases} \tag{2.4}$$

Where λ is a Lagrange multiplier and p is the adjoint state.

2.2 $H-p$ finite element method

In this subsection, we construct the hp finite element approximation of the optimal control problems where we assume that Ω is polygonal. We divide the domain Ω into N_τ non-overlapping sub-domains (elements) $\tau_i, 1 \leq i \leq N_\tau$:

$$\overline{\Omega} = \bigcup_{i=1}^{N_\tau} \overline{\tau}_i, \quad \tau_i \bigcap \tau_j = \emptyset, \quad i \neq j, \quad 1 \leq i,j \leq N_\tau \tag{2.5}$$

Let $\mathcal{T}$ be a local quasi-uniform partitioning of $\overline{\Omega}$ into non-overlapping regular elements τ , and $\hat{\tau} = (-1,1)^2$ be the reference element.

Let $\mathcal{T}_U$ the a partitioning of $\partial\Omega$, such that

$$\partial\Omega = \bigcup_{j=1}^{N_\tau} \overline{\gamma}_j \quad \gamma_i \bigcap \gamma_j = \emptyset, \quad i \neq j, \quad 1 \leq i,j \leq N_\tau \tag{2.6}$$

where $\gamma_j \ \ j = 1,2,\cdots N_\tau$ are the open sides of the boundary $\partial\Omega$. We further require that $P_i \in \partial\Omega$ where $P_i (i = 1,\cdots,I)$ is the set associated with the triangulation $\mathcal{T}$.

We let $\mu(\mathcal{T})$ denote all edges, and let $\mu_0(\mathcal{T})$ denote all edges which lie on the boundary $\partial\Omega$.

Each element τ can be the image of the reference element $\hat{\tau}$ under an affine map $F_\tau : \hat{\tau} \to \tau$. We write $h_\tau(h_\gamma) := diam\tau(\gamma)$, and assume that the triangulation is χ-shape regular, i.e.,

$$h_\tau{}^{-1}\|F'_\tau\| + h_\tau\|(F'_\tau)^{-1}\| \leq \chi \tag{2.7}$$

For χ-shape regular meshes $\mathcal{T}$ on the domain Ω, we associate with each element $\tau \in \mathcal{T}$ a polynomial degree $p \in \mathbb{N}_0$, these polynomial degrees p are collected into the polynomial degree vector $\mathbf{p} = \{p\}$.

For χ-shape regular meshes γ on the boundary $\partial\Omega$, we associate with each element $\gamma \in \mathcal{T}_U$ a polynomial degree $q \in \mathbb{N}_0$.

Then we can define the spaces of hp finite element approximation $U^{\mathbf{p}}(\mathcal{T}_U, \partial\Omega), S^{\mathbf{p}}(\mathcal{T},\Omega)$ as follows:

$$\begin{aligned} U^{\mathbf{p}}(\mathcal{T}_U, \partial\Omega) &:= \{u \in L^2(\partial\Omega) : u|_{e\in\mu_0(\mathcal{T})} \circ F_\tau \in \mathcal{Q}_p\} \\ S^{\mathbf{p}}(\mathcal{T},\Omega) &:= \{v \in H^1(\Omega) : v|_{e\in\mu(\mathcal{T})} \circ F_\tau \in \mathcal{Q}_p\} \end{aligned}$$

where $\mathcal{Q}_p$ denotes the spaces of polynomials in $\hat{\tau}$ of degree $\leq p,q$ in each variable, respectively.

As to polynomial degree distribution $\mathbf{p}$, similarly to (2.7), we assume that the polynomial degrees of neighboring elements are comparable, i.e., there is a constant $\chi > 0$ such that

$$\chi^{-1}(p_\tau) \le p_{\tau'} + 1 \le \chi(p_\tau + 1) \;\; \forall \;\; \tau, \tau' \in \mathcal{T}, \;\; \tau \cap \tau' \neq \emptyset \tag{2.8}$$

Convergence is obtained either by increasing the degree of the polynomials or increasing the number of elements N_τ.

Let $U_{h,p} := U_{ad} \bigcap U^{\mathbf{p}}(\mathcal{T}_U, \partial\Omega)$ be the space of approximation of the control, and let $S^{\mathbf{p}}(\mathcal{T}, \Omega)$ be the space of approximation of the state and co-state. Then the $h-p$ spectral element approximation of optimal control problem reads as follows:

$$(OCP)^{hp} \left\{ \begin{array}{l} \min\limits_{u \in U_{hp}} J(u_{hp}, y_{hp}) = \dfrac{1}{2}\displaystyle\int_\Omega (y_{hp} - y_d)^2 + \dfrac{\varepsilon}{2}\int_{\partial\Omega} u_{hp}^2 \\ A(y_{hp}, v_{hp}) = (\alpha u_{hp}, v_{hp})_{0,\partial\Omega} + (f, v_{hp})_{0,\Omega}, \forall\, v_{hp} \in S^{\mathbf{p}}(\mathcal{T}, \Omega). \end{array} \right. \tag{2.9}$$

The following theorem shows the existence and uniqueness of the solution of the above system.

Theorem 2.2. *Assume that the initial assumptions hold. There exists a unique solution* (y_{hp}, u_{hp}) *for the minimization problem* $(OCP)^{hp}$ *such that* $y_{hp} \in S^{\mathbf{p}}(\mathcal{T}, \Omega)$*,* $u \in U_{hp}$*.*

Proof. Let $\{(y_{hp}^n, u_{hp}^n,)\}_{n=1}^\infty$ be a minimization sequence for the system $(OCP)^{hp}$, then it is clear that $\{u_{hp}^n\}_{n=1}^\infty$ are bounded in U_{hp}. Thus there is a subsequence of $\{u_{hp}^n\}_{n=1}^\infty$(still denoted by $\{u_{hp}^n\}_{n=1}^\infty$) such that u_{hp}^n converges weakly to u_{hp} in U_{hp}. For the subsequence $\{u_{hp}^n\}_{n=1}^\infty$, we have

$$A(y_{hp}^n, v_{hp}) = (\alpha u_{hp}^n, v_{hp})_{0,\partial\Omega} + (f, v_{hp})_{0,\Omega}, \quad \forall\, v_{hp} \in S^{\mathbf{p}}(\mathcal{T}, \Omega). \tag{2.10}$$

Let $v_{hp} = y_{hp}^n$ in (4.10) to give

$$A(y_{hp}^n, y_{hp}^n) = (\alpha u_{hp}^n, y_{hp}^n)_{0,\partial\Omega} + (f, y_{hp}^n)_{0,\Omega}, \quad \forall\, y_{hp}^n \in S^{\mathbf{p}}(\mathcal{T}, \Omega). \tag{2.11}$$

For each $\{y_{hp}\}_{n=1}^\infty$, we have y_{hp}^n is a solution of (2.11) and $||y||_{1,\Omega} \le C\big(||f||_{0,\Omega} + ||u||_{0,\partial\Omega}\big)$. Thus we have that $\{y_{hp}\}_{n=1}^\infty$ is bounded set in $S^{\mathbf{p}}(\mathcal{T}, \Omega)$. Thus

$$\left\{ \begin{array}{l} u_{hp}^n \to u_{hp} \text{ weakly in } U_{hp} \\ y_{hp}^n \to y_{hp} \text{ weakly in } S^{\mathbf{p}}(\mathcal{T}, \Omega) \end{array} \right. \tag{2.12}$$

So, we have

$$A(y_{hp}, v_{hp}) = (\alpha u_{hp}, v_{hp})_{0,\partial\Omega} + (f, v_{hp})_{0,\Omega}, \quad \forall\, v_{hp} \in S^{\mathbf{p}}(\mathcal{T}, \Omega). \tag{2.13}$$

Since F is a convex functional on $L^2(\Omega)$ and Q is a strictly convex functional on $L^2(\partial\Omega)$, we have $F(y_{hp}) + Q(u_{hp}) \le \underline{\lim}\,\big(F(y_{hp}^n) + Q(u_{hp}^n)\big)$. Then (y_{hp}, u_{hp}) is the unique solution of $(OCP)^{hp}$. This complete the proof of the theorem. □

Furthermore, the following first order optimality conditions are satisfied.

Lemma 2.3. *The pair* $(u_{hp}, y_{hp}) \in U_{hp} \times S^{\mathbf{p}}(\mathcal{T}, \Omega)$ *is the optimal solution of the control problem (2.9) if only if there exists a pair* $(p_{hp}, \lambda_{hp}) \in (S^{\mathbf{p}}(\mathcal{T}, \Omega), \mathbb{R}^+)$ *such that* $(u_{hp}, y_{hp}, p_{hp}, \lambda_{hp})$ *satisfies the following optimality conditions* $(OCP - OPT)^{hp}$

$$\left\{ \begin{array}{l} (i)\;\; A(y_{hp}, v_{hp}) = (\alpha u_{hp}, v_{hp})_{0,\partial\Omega} + (f, v_{hp})_{0,\Omega}, \;\; \forall\, v_{hp} \in S^{\mathbf{p}}(\mathcal{T}, \Omega) \\ (ii)\;\; A(q_{hp}, p_{hp}) = (y_{hp} - y_d, q_{hp})_{0,\Omega}, \;\; \forall\, q_{hp} \in S^{\mathbf{p}}(\mathcal{T}, \Omega) \\ (iii)\;\; (\alpha p_{hp} + \varepsilon u_{hp}, w_{hp} - u_{hp})_{0,\partial\Omega} \ge 0 \quad \forall\, w_{hp} \in U_{hp}, \\ (iv)\;\; p_{hp} + (\varepsilon + \lambda_{hp}) u_{hp} = 0. \end{array} \right. \tag{2.14}$$

where λ_{hp} *satisfies*

$$\lambda_{hp} = \begin{cases} constant \geq 0, & ||u_{hp}||^2_{0,\partial\Omega} = \zeta^2 \\ 0 & ||u_{hp}||^2_{0,\partial\Omega} < \zeta^2 \end{cases} \tag{2.15}$$

3 A priori error estimates

In this section, we study a priori error estimates of the hp finite element approximations. Here we note that Ω is a convex open domain with Lipschitz boundary $\partial\Omega$, and in the light of the optimality conditions, we have $y \in V$.

To derive a priori estimates, we need to prove some important results that will be used later.

Lemma 3.1. *Let* $(u_{hp}, hp, p_{hp}, \lambda_{hp})$ *be the solution of the optimality conditions* $(OCP - OPT)^{hp}$ *then there exists a constant* C *(independent of* h *and* p*) such that*

$$max\{||u_{hp}||_{0,\partial\Omega}, ||y_{hp}||_{1,\Omega}, ||p_{hp}||_{1,\Omega}, |\lambda_{hp}|\} \leq C \tag{3.1}$$

Proof. Introduce the solution $(w_{hp}, y_{hp}(w_{hp})) \in U_{hp} \times S^{\mathbf{p}}(\mathcal{T}, \Omega)$ of the optimal control problem. Since the pair (u_{hp}, y_{hp}) is the optimal solution of the optimal control problem, we have

$$\begin{aligned} J(u_{hp}, y_{hp}) &= \frac{1}{2}\int_\Omega (y_{hp} - y_d)^2 + \frac{\varepsilon}{2}\int_{\partial\Omega} u^2_{hp} \\ &= \frac{1}{2}\sum_{\tau\in\mathcal{T}}\int_{\mathcal{T}} (y_{hp} - y_d)^2 + \frac{\varepsilon}{2}\sum_{e\in\mu_0}\int_e u^2_{hp} \\ &\leq \frac{1}{2}\sum_{\tau\in\mathcal{T}}\int_{\mathcal{T}} (y_{hp}(v_{hp}) - y_d)^2 + \frac{\varepsilon}{2}\sum_{e\in\mu_0}\int_e v^2_{hp} \leq C(e, \tau, y_d) \end{aligned}$$

Which implies that

$$\frac{\varepsilon}{2}||u_{hp}||2_{0,\partial\Omega} \leq C(e, \tau, y_d)$$

and can be rewritten

$$||u_{hp}||^2_{0,\partial\Omega} \leq \frac{2}{\varepsilon}C(e, \tau, y_d)$$

finally we have

$$||u_{hp}||_{0,\partial\Omega} \leq C \text{ and } ||y_{hp} - y_d||_{0,\Omega} \leq C$$

It is well known that

$$\text{for all } y \in V, u \in U_{ad}, ||y||_{1,\Omega} \leq C(||f||_{0,\Omega} + ||u||_{0,\partial\Omega})$$

Since $U_{hp} \subset U_{ad}$ and $S^{\mathbf{p}}(\mathcal{T}, \Omega) \subset V$, we have $||y_{hp}||_{1,\Omega} \leq C(||f||_{0,\Omega} + ||u_{hp}||_{0,\partial\Omega})$
Therefore

$$||y_{hp}||_{1,\Omega} \leq C$$

According to (2.14) (ii) and (iv), we obtain

$$||p_{hp}||_{1,\Omega} \leq c||y_{hp} - y_d||_{0,\Omega} \leq C \quad \text{and} \quad |\lambda_{hp}| \leq C$$

□

We introduce the auxiliary system to obtain a priori error estimates for hp finite element method : finding $(y_{hp}(u), p_{hp}(u)) \in S^{\mathbf{p}}(\mathcal{T},\Omega) \times S^{\mathbf{p}}(\mathcal{T},\Omega)$ such that

$$\begin{cases} (i)\ A(y_{hp}(u), v_{hp}) = (\alpha u, v_{hp})_{0,\partial\Omega} + (f, v_{hp})_{0,\Omega},\ \forall\, v_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega) \\ (ii)\ A(q_{hp}, p_{hp}(u)) = (y - y_d, q_{hp})_{0,\Omega},\ \forall\, q_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega) \end{cases}$$

This auxiliary system will allow me to derive a priori error estimates for the optimal control problem.

The following error bounded for the interpolation operator I_{hp} can be derived by using a result for the standard interpolation operator based on the reference domain and the technique employed in the $h-p$ finite element method. See [4] lemma 4.5.

Lemma 3.2. *Let $h = max\{h_{\tau_i}, 1 \le i \le N_t\}$, the for all $v \in H^s(\Omega), s \le 1$ it holds*

$$||v - I_{hp}v||_{t,\Omega} \le C\frac{h^{\mu-t}}{p^{s-t}}||v||_{s,v} \quad \forall\ \ v \in V \cap H^s(\Omega) \tag{3.2}$$

Where $\mu = min\{p+1, s\}$ and $t = 0, 1$.

We define the projection operator π_p^h as follows : $\forall\ v \in V$, find $\pi_p^h v \in S^{\mathbf{p}}(\mathcal{T},\Omega)$ such that

$$A(\pi_p^h v - v, v_{hp}) = 0, \ \ \forall\ \ v_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega)$$

where A is a V-elliptic, continuous bilinear form.

Lemma 3.3. *Let $\pi_p^h : V \cap H^s(\Omega) \longrightarrow S^{\mathbf{p}}(\mathcal{T},\Omega)$ such that for any $0 \le t \le s$*

$$||u - \pi_p^h u||_{t,\Omega} \le C\frac{h^{\mu-t}}{p^{s-t}}||u||_{s,\Omega}\ \ \forall\ \ u \in V \cap H^s(\Omega),\ where\ \mu = min\{p+1, s\} \tag{3.3}$$

Lemma 3.4. *Let (u, y, p, λ) be the optimal solution of optimality conditions $(OCP - OPT)$, $(y_{hp}(u), p_{hp}(u))$ be the solution of the auxiliary system, there holds*

$$||y - y_{hp}(u)||_{1,\Omega} + ||p - p_{hp}(u)||_{1,\Omega} \le C\frac{h^{\mu-1}}{p^{m-1}}\left(||y||_{m,\Omega} + ||p||_{m,\Omega}\right) \tag{3.4}$$

Where $\mu = min\{p+1, m\}$

Proof. In the light of the auxiliary system and optimality conditions $(OCP - OPT)$, we obtain

$$A(y - y_{hp}(u), v_{hp}) = 0 \tag{3.5}$$

$$A(q_{hp}, p - p_{hp}(u)) = 0 \tag{3.6}$$

Using (3.5) with Poincaré inequality and Céa lemma, $\ \forall\ \ w_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega)$

$$\begin{aligned} c||y - y_{hp}(u)||^2_{1,\Omega} &\le A(y - y_{hp}(u), y - y_{hp}(u)) \\ &\le A(y - y_{hp}(u), y - w_{hp}) + A(y - y_{hp}(u), w_{hp} - y_{hp}) \\ &\le A(y - y_{hp}(u), y - w_{hp}) \\ &\le M||y - y_{hp}(u)||_{1,\Omega}||y - w_{hp}||_{1,\Omega} \\ &\le M||y - y_{hp}(u)||_{1,\Omega} \inf_{\forall w_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega)} ||y - w_{hp}||_{1,\Omega} \end{aligned}$$

Thus we have,

$$||y - y_{hp}(u)||_{1,\Omega} \le C \inf_{\forall w_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega)} ||y - w_{hp}||_{1,\Omega}$$

From (3.4), we have

$$\begin{aligned} ||y - y_{hp}(u)||_{1,\Omega} &\leq C \inf_{\forall w_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega)} ||y - w_{hp}||_{1,\Omega} \\ &\leq C||y - \pi_p^h y||_{1,\Omega} \\ &\leq C \frac{h^{\mu-1}}{p^{m-1}} ||y||_{m,\Omega} \end{aligned}$$

Similarly, using (3.6) with Poincaré inequality and Céa lemma, we have $\forall \; q_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega)$

$$\begin{aligned} c||p - p_{hp}(u)||^2_{1,\Omega} &\leq A(p - p_{hp}(u), p - p_{hp}(u)) \\ &\leq A(p - q_{hp}, p - p_{hp}(u)) + A(q_{hp} - p_{hp}, p - p_{hp}(u)) \\ &\leq A(p - q_{hp}, p - p_{hp}(u)) \\ &\leq M||p - p_{hp}(u)||_{1,\Omega}||p - q_{hp}||_{1,\Omega} \\ &\leq M||p - p_{hp}(u)||_{1,\Omega} \inf_{\forall q_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega)} ||p - q_{hp}||_{1,\Omega} \end{aligned}$$

Thus we have,

$$||p - p_{hp}(u)||_{1,\Omega} \leq C \inf_{\forall q_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega)} ||p - q_{hp}||_{1,\Omega}$$

From (3.3), we have

$$\begin{aligned} ||p - p_{hp}(u)||_{1,\Omega} &\leq C \inf_{\forall q_{hp} \in S^{\mathbf{p}}(\mathcal{T},\Omega)} ||p - q_{hp}||_{1,\Omega} \\ &\leq C||p - \pi_p^h p||_{1,\Omega} \\ &\leq C \frac{h^{\mu-1}}{p^{m-1}} ||p||_{m,\Omega} \end{aligned}$$

□

Lemma 3.5. *Let* $(u_{hp}, y_{hp}, p_{hp}, \lambda_{hp})$ *be the optimal solution of the optimality conditions* $(OCP{-}OPT)^{hp}$*,* $(y_{hp}, p_{hp}(u))$ *be the solution of the auxiliary system, there holds the following estimates*

$$||p_{hp} - p_{hp}(u)||_{1,\Omega} + ||y_{hp} - y_{hp}(u)||_{1,\Omega} \leq C\{||u - u_{hp}||_{0,\partial\Omega} + ||y - y_{hp}(u)||_{0,\Omega}\} \tag{3.7}$$

Proof. Combining the auxiliary and the discrete systems, we obtain

$$A(y_{hp} - y_{hp}(u), v_{hp}) = (\alpha(u_{hp} - u), v_{hp})_{0,\partial\Omega} \tag{3.8}$$

$$A(q_{hp}, p_{hp} - p_{hp}(u)) = (y_{hp} - y, q_{hp})_{0,\Omega} \tag{3.9}$$

Letting $v_{hp} = y_{hp} - y_{hp}(u)$, (3.8) becomes

$$A(y_{hp} - y_{hp}(u), y_{hp} - y_{hp}(u)) = (\alpha(u_{hp} - u), y_{hp} - y_{hp}(u))_{0,\partial\Omega}$$

by cœrcivity, we have :

$$\begin{aligned} c||y_{hp} - y_{hp}(u)||^2_{1,\Omega} &\leq A(y_{hp} - y_{hp}(u), y_{hp} - y_{hp}(u)) \\ &= (\alpha(u_{hp} - u), y_{hp} - y_{hp}(u))_{0,\partial\Omega} \\ &\leq ||u_{hp} - u||_{0,\partial\Omega}||y_{hp} - y_{hp}(u)||_{0,\partial\Omega} \end{aligned}$$

which implies

$$||y_{hp} - y_{hp}(u)||_{1,\Omega} \leq C||u_{hp} - u||_{0,\partial\Omega} \tag{3.10}$$

Letting $q_{hp} = p_{hp} - p_{hp}(u)$, (4.25) becomes

$$A(p_{hp} - p_{hp}(u), p_{hp} - p_{hp}(u)) = (y_{hp} - y, p_{hp} - p_{hp}(u))_{0,\Omega}$$

by coercivity, we obtain

$$\begin{aligned} c||p_{hp} - p_{hp}(u)||^2_{1,\Omega} &\leq A(p_{hp} - p_{hp}(u), p_{hp} - p_{hp}(u)) \\ &= (y_{hp} - y, p_{hp} - p_{hp}(u))_{0,\Omega} \\ &\leq ||y_{hp} - y||_{0,\Omega}||p_{hp} - p_{hp}(u)||_{0,\Omega} \end{aligned}$$

Then,

$$\begin{aligned} ||p_{hp} - p_{hp}(u)||_{1,\Omega} &\leq C||y_{hp} - y||_{0,\Omega} \\ &\leq C(||y_{hp} - y_{hp}(u)||_{0,\Omega} + ||y - y_{hp}(u)||_{0,\Omega}) \\ &\leq C(||y_{hp} - y_{hp}(u)||_{1,\Omega} + ||y - y_{hp}(u)||_{0,\Omega}) \\ &\leq C(||u_{hp} - u||_{0,\partial\Omega} + ||y - y_{hp}(u)||_{0,\Omega}) \end{aligned}$$

This completes the proof. □

Lemma 3.6. *Let (u, y, p, λ) and $(u_{hp}, y_{hp}, p_{hp}, \lambda_{hp})$ the optimal solution of optimality conditions $(OCP-OPT)$ and $(OCP - OPT)^{hp}$ respectively. There holds*

$$||u - u_{hp}||_{0,\partial\Omega} \leq C\frac{h^{\mu-1}}{p^{m-1}}(||y||_{m,\Omega} + ||p||_{m,\Omega}) \tag{3.11}$$

Proof.

$$\begin{aligned} (p - p_{hp}, q_{hp})_{0,\partial\Omega} &= (-(\lambda + \varepsilon)u + (\lambda_{hp} + \varepsilon)u_{hp}, q_{hp})_{0,\partial\Omega} \\ &= (-(\lambda + \varepsilon)u + (\lambda + \varepsilon)u_{hp} - (\lambda + \varepsilon)u_{hp} + (\lambda_{hp} + \varepsilon)u_{hp}, q_{hp})_{0,\partial\Omega} \\ &= (\lambda_{hp} - \lambda)(u_{hp}, q_{hp})_{0,\partial\Omega} + (\lambda + \varepsilon)(u_{hp} - u, q_{hp})_{0,\partial\Omega} \end{aligned} \tag{3.12}$$

Let

$$q_{hp} = \zeta^2\alpha(u - u_{hp}) - \mathcal{R}u_{hp} = \zeta^2\alpha(u - u_{hp}) - (\alpha(u - u_{hp}), u_{hp})_{0,\partial\Omega}.u_{hp}$$

such that $(u_hp, q_{hp}) = 0$

$$\begin{aligned} (u_{hp}, q_{hp})_{0,\partial\Omega} &= (u_{hp}, \zeta^2\alpha(u - u_{hp}) - (\alpha(u - u_{hp}), u_{hp})u_{hp})_{0,\partial\Omega} \\ &= \zeta^2(u_{hp}, \alpha(u - u_{hp}))_{0,\partial\Omega} - (u_{hp}, (\alpha(u - u_{hp}), u_{hp})u_{hp})_{0,\partial\Omega} \\ &= \zeta^2(u_{hp}, \alpha(u - u_{hp}))_{0,\partial\Omega} - (u_{hp}, u_{hp})(\alpha(u - u_{hp}), u_{hp})_{0,\partial\Omega} \\ &= 0 \end{aligned}$$

Where $||u_{hp}||^2_{0,\partial\Omega} = \zeta^2$

$$\begin{aligned}
\mathcal{R} &= (\alpha(u-u_{hp}), u_{hp})_{0,\partial\Omega} = \frac{-1}{\lambda_{hp}+\varepsilon}(\alpha(u-u_{hp}), -(\lambda_{hp}+\varepsilon)u_{hp})_{0,\partial\Omega} \\
&= \frac{-1}{\lambda_{hp}+\varepsilon}(\alpha(u-u_{hp}), p_{hp})_{0,\partial\Omega}, \text{ using (3.8)} \\
&= \frac{1}{\lambda_{hp}+\varepsilon}\left[(a\nabla(y_{hp}-y_{hp}(u)), \nabla p_{hp})_{0,\Omega} + (\alpha(y_{hp}-y_{hp}(u)), p_{hp})_{0,\partial\Omega}\right]
\end{aligned}$$

$$\begin{aligned}
|\mathcal{R}| &= \frac{1}{\lambda_{hp}+\varepsilon}\left|(a\nabla(y_{hp}-u_{hp}(u)), \nabla p_{hp})_{0,\Omega} + (\alpha(y_{hp}-y_{hp}(u)), p_{hp})_{0,\partial\Omega}\right| \\
&\leq \frac{1}{\lambda_{hp}+\varepsilon}\left[|(a\nabla(y_{hp}-u_{hp}(u)), \nabla p_{hp})_{0,\Omega}| + |(\alpha(y_{hp}-y_{hp}(u)), p_{hp})_{0,\partial\Omega}|\right] \\
&\leq C||y_{hp}-y_{hp}(u)||_{0,\Omega}
\end{aligned}$$

$$\begin{aligned}
\lambda_{hp}((u-u_{hp}), q_{hp})_{0,\partial\Omega} &= \lambda_{hp}((u-u_{hp}), \zeta^2\alpha(u-u_{hp}) - (\alpha(u-u_{hp}), u_{hp})_{0,\partial\Omega}.u_{hp})_{0,\partial\Omega} \\
&= \lambda_{hp}\zeta^2((u-u_{hp}), \alpha(u-u_{hp}))_{0,\partial\Omega} - \lambda_{hp}((u-u_{hp}), (\alpha(u-u_{hp}), \\
&\quad u_{hp})_{0,\partial\Omega}.u_{hp})_{0,\partial\Omega}
\end{aligned}$$

Then, using (4.25), we have :

$$\begin{aligned}
\lambda_{hp}\zeta^2((u-u_{hp}), \alpha(u-u_{hp}))_{0,\partial\Omega} &= \lambda_{hp}((u-u_{hp}), q_{hp})_{0,\partial\Omega} + \lambda_{hp}((u-u_{hp}), (\alpha(u-u_{hp}), \\
&\quad u_{hp})_{0,\partial\Omega}.u_{hp})_{0,\partial\Omega} \\
&= \frac{\lambda_{hp}}{\lambda+\varepsilon}(p_{hp}-p, q_{hp})_{0,\partial\Omega} + \lambda_{hp}((u-u_{hp}), \mathcal{R}u_{hp})_{0,\partial\Omega}
\end{aligned}$$

Assuming that there exists δ such that $\alpha(x) \geq \delta$ a.e on $\partial\Omega$. Thus

$$((u-u_{hp}), \alpha(u-u_{hp}))_{0,\partial\Omega} \geq \delta||u-u_{hp}||^2$$

Then

$$\lambda_{hp}\zeta^2\delta||u-u_{hp}||^2_{0,\partial\Omega} \leq \frac{\lambda_{hp}}{\lambda+\varepsilon}|(p_{hp}-p, q_{hp})_{0,\partial\Omega} + \lambda_{hp}((u-u_{hp}), \mathcal{R}u_{hp})_{0,\partial\Omega}|$$

which can be rewritten as

$$||u-u_{hp}||_{0,\partial\Omega} \leq c(\iota)(||p-p_{hp}||_{0,\Omega} + |\mathcal{R}|) + \iota||u-u_{hp}||_{0,\partial\Omega}$$

therefore,

$$||u-u_{hp}||_{0,\partial\Omega} \leq C(||p-p_{hp}||_{0,\Omega} + ||y_{hp}-y_{hp}(u)||_{0,\Omega})$$

Finally, we can arrive at

$$||u-u_{hp}||_{0,\partial\Omega} \leq C\frac{h^{\mu-1}}{p^{m-1}}(||y||_{m,\Omega} + ||p||_{m,\Omega}) \tag{3.13}$$

□

Theorem 3.7 (Convergence). *Let (u, y, p) and (u_{hp}, y_{hp}, p_{hp}) be the solutions of $(OCP-OPT)$ and $(OCP-OPT)^{hp}$, respectively. Assuming that $(y, p) \in H^m(\Omega) \times H^m(\Omega)(m \geq 1)$, h and p lager* *enough, we obtain the following a priori error estimates*

$$||u-u_{hp}||_{0,\partial\Omega}+||y-y_{hp}||_{1,\Omega}+||p-p_{hp}||_{1,\Omega}\leq C\frac{h^{\mu-1}}{p^{m-1}}(||y||_{m,\Omega}+||p||_{m,\Omega}) \tag{3.14}$$

where $\mu=min\{p+1,m\}$

Proof.

$$\begin{aligned}
||y-y_{hp}||_{1,\Omega} &\leq ||y-y_{hp}(u)||_{1,\Omega}+||y_{hp}(u)-y_{hp}||_{1,\Omega}\\
||p-p_{hp}||_{1,\Omega} &\leq ||p-p_{hp}(u)||_{1,\Omega}+||p_{hp}(u)-p_{hp}||_{1,\Omega}
\end{aligned}$$

Then using (3.4) and (3.7), we have

$$||y-y_{hp}||_{1,\Omega}+||p-p_{hp}||_{1,\Omega} \leq C_1\frac{h^{\mu-1}}{p^{m-1}}(||y||_{m,\Omega}+||p||_{m,\Omega})+C_2\{||u-u_{hp}||_{0,\partial\Omega}+||y-y_{hp}(u)||_{0,\Omega}\}$$

which can be rewritten using (3.13) as

$$\begin{aligned}
||u-u_{hp}||_{0,\partial\Omega}+||y-y_{hp}||_{1,\Omega}+||p-p_{hp}||_{1,\Omega} &\leq C_1\frac{h^{\mu-1}}{p^{m-1}}||p||_{m,\Omega}+C_3\frac{h^{\mu-1}}{p^{m-1}}||y||_{m,\Omega}\\
&+ C_4\frac{h^{\mu-1}}{p^{m-1}}(||y||_{m,\Omega}+||p||_{m,\Omega})\\
&\leq C\frac{h^{\mu-1}}{p^{m-1}}(||y||_{m,\Omega}+||p||_{m,\Omega})
\end{aligned}$$

□

4 A posteriori error estimates

In this section, a posteriori error estimates for the hp finite element approximation for the optimal control problems. we introduce two lemmas which generalize the well-known Clément-type interpolation operators and also two Lemmas which generalize the polynomial inverse inequalities.

Lemma 4.1 ([25]). *Let $\mathcal{T}$ be a χ-shape regular triangulation of a domain $\Omega\in\mathbb{R}$, and let $\mathbf{p}$ be a polynomial degree distribution which is comparable. Then there exists a bounded linear operator $I^{hp}: L^2(\Omega)\to \mathbf{S}^{\mathbf{p}}(\mathcal{T},\Omega)$ and there exists a constant $C>0$, which depends only on χ, such that for every $u\in H^1(\Omega)$ and all elements $\tau\in\mathcal{T}$ and all edges $e\in\varepsilon(\mathcal{T})$*

$$||u-I^{hp}u||_{0,\tau}+\frac{h_\tau}{p_\tau}||\nabla(u-I^{hp}u)||_{0,\tau}\leq C\frac{h_\tau}{p_\tau}||\nabla u||_{0,w_\tau}, \tag{4.1}$$

$$||u-I^{hp}u||_{L^2(e)}\leq C\sqrt{\left(\frac{h_e}{p_e}\right)}||\nabla u||_{0,w_e} \tag{4.2}$$

where h_e is the length of the edge e and $p_e=max\{p_\tau,p_{\tau'}\}$, where τ, τ' are element sharing the edge e, and w_τ , w_e are patches covering τ and e with a few layers, respectively.

Scott-Zhang-type approximation

Let a set $\mathcal{B} \subset \varepsilon(\mathcal{T})$ of boundary edges of the triangulation $\mathcal{T}$ be given, i.e,

$$\mathcal{B} \subset \varepsilon(\mathcal{T}) \quad \text{and} \quad b \in \partial\Omega, \quad \forall\, b \in \mathcal{B} \tag{4.3}$$

Next, we define for $q \in (1, \infty)$ the space

$$W^{1,q}_{\mathcal{B},p} := \{u \in W^{1,q}(\Omega) : u|_b \circ F_b \in \mathcal{P}_{p_b} \text{ for all } b \in \mathcal{B} \text{ and (1.1) holds}\} \tag{4.4}$$

where the continuity condition (1.1) is

$$\text{for alla } b, b' \text{ and } V \in \Lambda(b) \cap \Lambda(b') \text{ there holds } \lim_{x \to V x \in b} u(x) = \lim_{x \to V x \in b'} u(x). \tag{4.5}$$

Lemma 4.2 (Scott-Zhang-type quasi interpolation [25]). *. Let $\mathcal{T}$ be a γ-shape regular triangulation of a domain $\Omega \subset \mathbb{R}^2$ and let $\mathbf{p}$ be a polynomial degree distribution which is comparable. Then there exists a linear operator $\pi^{hp} : H^1(\Omega) \to S^p(\mathcal{T}, \Omega)$ such that*

$$\left(\pi^{hp} u\right) |b = u|b \quad \forall \;\; b \in \mathcal{B}$$

Furthermore, there exists a constant $C > 0$ depending only on γ and q such that for all elements $\tau \in \mathcal{T}$ and all edges $e \in \varepsilon(\mathcal{T})$

$$||u - \pi^{hp} u||_{0,\tau} + \frac{h_\tau}{p_\tau} ||\nabla(u - \pi^{hp} u)||_{0,\tau} \leq C \frac{h_\tau}{p_\tau} ||\nabla u||_{0,w^4_\tau}, \tag{4.6}$$

$$||u - \pi^{hp} u||_{0,e} \leq C \sqrt{\left(\frac{h_\tau}{p_\tau}\right)} ||\nabla u||_{0,w^4_e} \tag{4.7}$$

Analysis of the error indicators requires polynomial inverse estimates in weighted Sobolev spaces in multi-dimensions. Under this consideration, the weight functions: $\Phi_{\hat{\tau}}(x) := dist(x, \partial\hat{\tau})$ on the reference element $\hat{\tau}$ should be introduced (see [26] for more details). For an arbitrary element $\tau \in \mathcal{T}$, set $\Phi_\tau = c_\tau \Phi_{\hat{\tau}} \circ F_\tau^{-1}$, where is c_τ a scaling factor which is chosen such that $\int_\tau \Phi_\tau dxdy = meas(\tau)$.).

We have following lemmas reads as follows:

Lemma 4.3. *Let $\hat{\tau}$ be the reference square as defined previously , let the weight function $\Phi_{\hat{\tau}}$ be defined above. Let $\gamma, \beta \in \mathbb{R}$, satisfying $-1 < \gamma < \beta$ and $\delta \in [0, 1]$. Then for all polynomial $\psi_p \in \mathcal{Q}_p$*

$$\int_{\hat{\tau}} \Phi_{\hat{\tau}} |\nabla \psi_p|^2 dxdy \leq C_1 p^2 \int_{\hat{\tau}} |\psi_p|^2 dxdy, \tag{4.8}$$

$$\int_{\hat{\tau}} (\Phi_{\hat{\tau}})^\gamma \psi_p^2 dxdy \leq C_2 p^{2(\beta-\gamma)} \int_{\hat{\tau}} (\Phi_{\hat{\tau}})^\beta \psi_p^2 dxdy, \tag{4.9}$$

$$\int_{\hat{\tau}} (\Phi_{\hat{\tau}})^{2\delta} |\nabla \psi_p|^2 dxdy \leq C_3 p^{2(2-\delta)} \int_{\hat{\tau}} (\Phi_{\hat{\tau}})^\delta \psi_p^2 dxdy, \tag{4.10}$$

where $C_i, i = 1, 2, 3$, are constants, C_2 is dependent on β and γ, and C_3 is dependent on δ

Lemma 4.4. *Let $\hat{\tau}$ be the reference square defined as previously , $\theta \in (1/2, 1]$. Set $\hat{e} = (0, 1) \times \{0\}$. Then there exists a constant $C_\theta > 0$, which is dependent on θ, such that the followings hold. For every univariate polynomial $\psi \in \mathcal{Q}_p$ and every $\epsilon \in (0, 1]$ there exists an extension $v_{\hat{e}} \in H^1(\hat{\tau})$ such that*

(i) $v_{\hat{e}}|_{\hat{e}} = \psi.\Phi^\theta_{\hat{e}}$ *and* $v_{\hat{e}}|_{\partial\hat{\tau} - \hat{e}} = 0$ *;*

(ii) $||v_{\hat{e}}||^2_{0,\hat{\tau}} \leq C_\theta \epsilon ||\psi.\Phi_{\hat{e}}^{\theta/2}||^2_{0,\hat{e}}$ *;*

(iii) $||\nabla v_{\hat{e}}||^2_{0,\hat{\tau}} \leq C_\theta(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||\psi.\Phi_{\hat{e}}^{\theta/2}||^2_{0,\hat{e}}$

where $\Phi_{\hat{e}}$ is the weight function defined above, and $\hat{\tau}$ is the reference element such that $\hat{e} \subset \partial\hat{\tau}$.

4.1 A posteriori upper error estimates

In this subsection, we will derive upper a posteriori error estimates of residual type.

Definition 4.5. *We first define the following notations:*

$$\eta^2 := \sum_{\tau\in\mathcal{T}} \sum_{i=1}^{6} \eta^2_{i,\tau} \tag{4.11}$$

$$\begin{aligned}
\eta^2_{1,\tau} &:= \frac{h^2_\tau}{p^2_\tau}||f + div(a\nabla y_{hp})||^2_{0,\tau} && (4.12)\\
\eta^2_{2,\tau} &:= \sum_{e\in\mu(\tau)-\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||[(a\nabla y_{hp}).n_e]||^2_{0,e} && (4.13)\\
\eta^2_{3,\tau} &:= \sum_{e\in\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||\alpha(u_{hp} - y_{hp}) - [(a\nabla y_{hp}).n_e]||^2_{0,e} && (4.14)\\
\eta^2_{4,\tau} &:= \frac{h^2_\tau}{p^2_\tau}||y_{hp} - y_d + div(a^*\nabla p_{hp})||^2_{0,\tau} && (4.15)\\
\eta^2_{5,\tau} &:= \sum_{e\in\mu(\tau)-\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||[(a^*\nabla p_{hp}).n_e]||^2_{0,e} && (4.16)\\
\eta^2_{6,\tau} &:= \sum_{e\in\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||\alpha p_{hp} + [(a^*\nabla p_{hp}).n_e]||^2_{0,e} && (4.17)
\end{aligned}$$

where we denote the jump of v across the edges by $[v]$, and n_e is the unit outer normal on e.

To derive a posteriori upper error estimates for the optimal control problem, we need to introduce the auxiliary system : find $(y(u_{hp}), p(u_{hp})))$ such that :

$$\left\{ \begin{array}{l} A(y(u_{hp}), v) = (\alpha u_{hp}, v)_{0,\partial\Omega} + (f, v)_{0,\Omega}, \ \forall\ v \in V \\ A(q, p(u_{hp})) = (y(u_{hp}) - y_d, q)_{0,\Omega}, \ \forall\ q \in V \end{array} \right. \tag{4.18}$$

Using the above auxiliary system, we now prove the following lemmas to obtain a posteriori error estimates of the optimal control problem.

Lemma 4.6. *Let (y, u, p, λ) and $(y_{hp}, u_{hp}, p_{hp}, \lambda_{hp})$ be the solution of optimality conditions $(OCP - OPT)$ and $(OCP - OPT)^{hp}$, respectively. Let $(y(u_{hp}), p(u_{hp}))$ be the solution of auxiliary system. Then we have q*

$$||y(u_{hp}) - y||_{1,\Omega} \leq C||u_{hp} - u||_{0,\partial\Omega} \tag{4.19}$$

and

$$||p - p(u_{hp})||_{1,\Omega} \leq C||y_{hp} - y(u_{hp})||_{1,\Omega} \tag{4.20}$$

Proof. It follows from the continuous optimality conditions and the auxiliary system that

$$A(y(u_{hp}) - y, v) = (\alpha(u_{hp} - u)), v)_{0,\partial\Omega} \tag{4.21}$$

$$A(q, p(u_{hp}) - y) = (y(u_{hp}) - y, q)_{0,\Omega} \tag{4.22}$$

Letting $v = y(u_{hp}) - y$ in (4.21), we have

$$A(y(u_{hp}) - y, y(u_{hp}) - y) = (\alpha(u_{hp} - u)), y(u_{hp}) - y)_{0,\partial\Omega}$$

The above expression implies that

$$\begin{aligned} c||y(u_{hp}) - y||^2_{1,\Omega} &\leq A(y(u_{hp}) - y, y(u_{hp}) - y) \\ &= (\alpha(u_{hp} - u)), y(u_{hp}) - y)_{0,\partial\Omega} \\ &\leq \delta||u_{hp} - u||_{0,\partial\Omega}||y(u_{hp}) - y||_{0,\partial\Omega} \\ &\leq \delta||u_{hp} - u||_{0,\partial\Omega}||y(u_{hp}) - y||_{0,\Omega} \end{aligned}$$

which implies

$$||y(u_{hp}) - y||_{1,\Omega} \leq C||u_{hp} - u||_{0,\partial\Omega} \tag{4.23}$$

Letting $q = p(u_{hp}) - p$ in (4.22), we have

$$A(p(u_{hp}) - p, p(u_{hp}) - p) = (y(u_{hp}) - y, p(u_{hp}) - p)_{0,\Omega}$$

The above expression implies that:

$$\begin{aligned} c||p(u_{hp}) - p||^2_{1,\Omega} &\leq A(p(u_{hp}) - p, p(u_{hp}) - p) \\ &= (y(u_{hp}) - y, p(u_{hp}) - p)_{0,\Omega} \\ &\leq C||y(u_{hp}) - y||_{0,\Omega}||p(u_{hp}) - p||_{0,\Omega} \end{aligned}$$

Which implies that :

$$||p(u_{hp}) - p||_{1,\Omega} \leq C||y(u_{hp}) - y||_{1,\Omega} \tag{4.24}$$

□

Lemma 4.7. *Let* (u, y, p, λ) *and* $(u_{hp}, y_{hp}, p_{hp}, \lambda_{hp})$ *be the solution of optimality conditions* $(OCP - OPT)$ *and* $(OCP - OPT)^{hp}$ *respectively. Let* $(y(u_{hp}), p(u_{hp}))$ *be the solution of auxiliary system. Then we have :*

$$||u - u_{hp}||_{0,\partial\Omega} \leq C||y(u_{hp}) - y_{hp}||_{0,\Omega}$$

Proof.

$$\begin{aligned} (p - p_{hp}, q)_{0,\partial\Omega} &= (-(\lambda + \varepsilon)u + (\lambda_{hp} + \varepsilon)u_{hp}, q)_{0,\partial\Omega} \\ &= (-(\lambda + \varepsilon)u + (\lambda_{hp} + \varepsilon)u - (\lambda_{hp} + \varepsilon)u + (\lambda_{hp} + \varepsilon)u_{hp}, q)_{0,\partial\Omega} \\ &= (\lambda_{hp} - \lambda)(u, q)_{0,\partial\Omega} + (\lambda_{hp} + \varepsilon)(u_{hp} - u, q)_{0,\partial\Omega} \end{aligned} \tag{4.25}$$

Let

$$q = \zeta^2\alpha(u - u_{hp}) - \mathcal{C}u = \zeta^2\alpha(u - u_{hp}) - (\alpha(u - u_{hp}), u)_{0,\partial\Omega}.u$$

such that $(u, q) = 0$

$$\begin{aligned}(u, q)_{0,\partial\Omega} &= (u, \zeta^2\alpha(u - u_{hp}) - (\alpha(u - u_{hp}), u)u)_{0,\partial\Omega}\\ &= \zeta^2(u, \alpha(u - u_{hp}))_{0,\partial\Omega} - (u, (\alpha(u - u_{hp}), u)u)_{0,\partial\Omega}\\ &= \zeta^2(u, \alpha(u - u_{hp}))_{0,\partial\Omega} - (u, u)(\alpha(u - u_{hp}), u)_{0,\partial\Omega}\\ &= 0\end{aligned}$$

Where $||u||^2_{0,\partial\Omega} = \zeta^2$

$$\begin{aligned}\mathcal{C} &= (\alpha(u - u_{hp}), u)_{0,\partial\Omega} = \frac{-1}{\lambda + \varepsilon}(\alpha(u - u_{hp}), -(\lambda + \varepsilon)u)_{0,\partial\Omega}\\ &= \frac{-1}{\lambda + \varepsilon}(\alpha(u - u_{hp}), p)_{0,\partial\Omega}, \text{ using } (4.21)\\ &= \frac{1}{\lambda + \varepsilon}[(a\nabla(y(u_{hp}) - y), \nabla p)_{0,\Omega} + (\alpha(y(u_{hp}) - y)), p)_{0,\partial\Omega}]\end{aligned}$$

$$\begin{aligned}|\mathcal{C}| &= \frac{1}{\lambda + \varepsilon}|(a\nabla(y(u_{hp}) - y), \nabla p)_{0,\Omega} + (\alpha(y(u_{hp}) - y)), p)_{0,\partial\Omega}|\\ &\leq \frac{1}{\lambda + \varepsilon}[||(a\nabla(y(u_{hp}) - y), \nabla p)_{0,\Omega}| + |(\alpha(y(u_{hp}) - y)), p)_{0,\partial\Omega}|]\\ &\leq C||y(u_{hp}) - y||_{0,\Omega}\end{aligned}$$

$$\begin{aligned}\lambda_{hp}((u - u_{hp}), q)_{0,\partial\Omega} &= \lambda_{hp}((u - u_{hp}), \zeta^2\alpha(u - u_{hp}) - (\alpha(u - u_{hp}), u)_{0,\partial\Omega}.u)_{0,\partial\Omega}\\ &= \lambda_{hp}\zeta^2((u - u_{hp}), \alpha(u - u_{hp}))_{0,\partial\Omega} - \lambda_{hp}((u - u_{hp}), (\alpha(u - u_{hp}),\\ &\quad u)_{0,\partial\Omega}.u)_{0,\partial\Omega}\end{aligned}$$

Then, using (4.25), we have :

$$\begin{aligned}\lambda_{hp}\zeta^2((u - u_{hp}), \alpha(u - u_{hp}))_{0,\partial\Omega} &= \lambda_{hp}((u - u_{hp}), q)_{0,\partial\Omega} + \lambda_{hp}((u - u_{hp}), (\alpha(u - u_{hp}),\\ &\quad u)_{0,\partial\Omega}.u)_{0,\partial\Omega}\\ &= \frac{\lambda_{hp}}{\lambda_{hp} + \varepsilon}(p_{hp} - p, q)_{0,\partial\Omega} + \lambda_{hp}((u - u_{hp}), \mathcal{C}u)_{0,\partial\Omega}\end{aligned}$$

Assuming that there exists δ such that $\alpha(x) \geq \delta$ a.e on $\partial\Omega$. Thus

$$((u - u_{hp}), \alpha(u - u_{hp}))_{0,\partial\Omega} \geq \delta||u - u_{hp}||^2$$

Then

$$\lambda_{hp}\zeta^2\delta||u - u_{hp}||^2_{0,\partial\Omega} \leq \frac{\lambda_{hp}}{\lambda_{hp} + \varepsilon}|(p_{hp} - p, q)_{0,\partial\Omega} + \lambda_{hp}((u - u_{hp}), \mathcal{C}u)_{0,\partial\Omega}|$$

which can be rewritten as

$$||u - u_{hp}||_{0,\partial\Omega} \leq c(\iota)(||p - p_{hp}||_{0,\Omega} + |\mathcal{C}|) + \iota||u - u_{hp}||_{0,\partial\Omega}$$

Finally, applying the above expressions, we can get

$$||u - u_{hp}||_{0,\partial\Omega} \leq C||y(u_{hp}) - y_{hp}||_{0,\Omega}$$

$\square$

Lemma 4.8. *Let* $(u_{hp}, y_{hp}, p_{hp}, \lambda_{hp})$ *and* $(y(u_{hp}), p(u_{hp})$ *be the solution of the optimality conditions* $(OCP - OPT)^{hp}$ *and the auxiliary system. Then,*

$$||y(u_{hp}) - y_{hp}||^2_{1,\Omega} + ||p(u_{hp}) - p_{hp}||^2_{1,\Omega} \leq C\eta^2 \tag{4.26}$$

Proof. It follows from the discrete optimality conditions $(OCP - OPT)^{hp}$ and the auxiliary system, we have :

$$\begin{aligned} A(y(u_{hp}) - y_{hp}, v_{hp}) &= 0 \ \forall v_{hp} \in S^p(\mathcal{T}, \Omega) && (4.27) \\ A(q_{hp}, p(u_{hp}) - p_{hp}) &= (y(u_{hp}) - y_{hp}, q_{hp}) \ \forall q_{hp} \in S^p(\mathcal{T}, \Omega) && (4.28) \end{aligned}$$

Let $e^p = p(u_{hp}) - p_{hp}$ and let $e^p_I = \pi^{hp} e^p$, where π^{hp} is the Scott-Zhang type interpolator defined in lemma(4.9). Applying the standard residual techniques (see, e.g., [32]). Then it follows from the projection equation, Green's formula, and Holder's inequality that

$$\begin{aligned} c||p(u_{hp}) - p_{hp}||^2_{1,\Omega} &\leq A(e^p, e^p) \\ &= A(p(u_{hp}), e^p) - A(p_{hp}, e^p) \\ &= A(p(u_{hp}), e^p) - A(p_{hp}, e^p - e^p_I) - A(p_{hp}, e^p_I) \\ &= (y(u_{hp}) - y_d, e^p)_{0,\Omega} - A(p_{hp}, e^p - e^p_I) - (y_{hp} - y_d, e^p_I)_{0,\Omega} \\ &= (y(u_{hp}) - y_{hp}, e^p)_{0,\Omega} - A(p_{hp}, e^p - e^p_I) + (y_{hp} - y_d, e^p - e^p_I)_{0,\Omega} \\ &= (y(u_{hp}) - y_{hp}, e^p)_{0,\Omega} + (y_{hp} - y_d, e^p - e^p_I)_{0,\Omega} - \int_\Omega (a^* \nabla p_{hp}).\nabla(e^p - e^p_I) \\ &- \int_{\partial\Omega} \alpha p_{hp}(e^p - e^p_I) \\ &= \sum_{\tau\in\mathcal{T}} \left\{ \int_\tau div(a^* \nabla p_{hp})(e^p - e^p_I) - \int_{\partial\tau} (a^* \nabla p_{hp}).n_\tau (e^p - e^p_I) \right\} - \int_{\partial\Omega} \alpha p_{hp}(e^p - e^p_I) \\ &+ (y(u_{hp}) - y_{hp}, e^p)_{0,\Omega} + (y_{hp} - y_d, e^p - e^p_I)_{0,\Omega} \\ &= \sum_{\tau\in\mathcal{T}} \left\{ \int_\tau (y_{hp} - y_d + div(a^* \nabla p_{hp}))\,(e^p - e^p_I) - \int_{\partial\tau} (a^* \nabla p_{hp}).n_\tau (e^p - e^p_I) \right\} \\ &- \sum_{e\in\mu_0(\tau)} \int_e \alpha p_{hp}(e^p - e^p_I) + (y(u_{hp}) - y_{hp}, e^p)_{0,\Omega} \\ &= \sum_{\tau\in\mathcal{T}} \int_\tau (y_{hp} - y_d + div(a^* \nabla p_{hp})(e^p - e^p_I) - \sum_{e\in\mu(\tau)-\mu_0(\tau)} \int_e [(a^* \nabla p_{hp}).n_e](e^p - e^p_I) \\ &- \sum_{e\in\mu_0(\tau)} \int_e (\alpha p_{hp} + [a^* \nabla p_{hp}.n_e])(e^p - e^p_I) + (y(u_{hp}) - y_{hp}, e^p)_{0,\Omega} \end{aligned}$$

where we used (4.38) and (4.28), we have :

$$
\begin{aligned}
c||p(u_{hp})-p_{hp}||_{1,\Omega}^2 &\leq \sum_{\tau\in\mathcal{T}}||(y_{hp}-y_d+div(a^*\nabla p_{hp})||_{0,\tau}||(e^p-e_I^p)||_{0,\tau}\\
&+ \sum_{e\in\mu(\tau)-\mu_0(\tau)}||[(a^*\nabla p_{hp}).n_e]||_{0,e}||(e^p-e_I^p)||_{0,e}\\
&+ \sum_{e\in\mu_0(\tau)}||(\alpha p_{hp}+[a^*\nabla p_{hp}.n_e]||_{0,e}||(e^p-e_I^p)||_{0,e}\\
&+ ||y(u_{hp})-y_{hp}||_{0,\Omega}||e^p||_{0,\Omega}
\end{aligned}
\tag{4.29}
$$

And using the Theorem(4.9), we have :

$$
\begin{aligned}
c||p(u_{hp})-p_{hp}||_{1,\Omega}^2 &\leq C\sum_{\tau\in\mathcal{T}}\frac{h_\tau}{p_\tau}||(y_{hp}-y_d+div(a^*\nabla p_{hp})||_{0,\tau}||\nabla e^p||_{0,w_\tau^4}\\
&+ C\sum_{e\in\mu(\tau)-\mu_0(\tau)}\sqrt{\frac{h_\tau}{p_\tau}}||[(a^*\nabla p_{hp}).n_e]||_{0,e}||\nabla e^p||_{0,w_\tau^4}\\
&+ C\sum_{e\in\mu_0(\tau)}\sqrt{\frac{h_\tau}{p_\tau}}||(\alpha p_{hp}+[a^*\nabla p_{hp}.n_e]||_{0,e}||\nabla e^p||_{0,w_\tau^4}\\
&+ C||y(u_{hp})-y_{hp}||_{0,\Omega}||e^p||_{0,\Omega}\\
&\leq C(\sigma)\left\{\sum_{\tau\in\mathcal{T}}\left(\frac{h_\tau}{p_\tau}\right)^2||(y_{hp}-y_d+div(a^*\nabla p_{hp})||_{0,\tau}^2\right.\\
&+ \sum_{e\in\mu(\tau)-\mu_0(\tau)}\frac{h_\tau}{p_\tau}||[(a^*\nabla p_{hp}).n_e]||_{0,e}^2\\
&+ \sum_{e\in\mu_0(\tau)}\frac{h_\tau}{p_\tau}||(\alpha p_{hp}+[a^*\nabla p_{hp}.n_e]||_{0,e}^2\\
&+ \left.||y(u_{hp})-y_{hp}||_{0,\Omega}^2\right\}+\sigma||e^p||_{1,\Omega}^2
\end{aligned}
\tag{4.30}
$$

Setting $\sigma=\frac{c}{2}$, we can obtain :

$$
\begin{aligned}
||p(u_{hp})-p_{hp}||_{1,\Omega}^2 &\leq C\sum_{\tau\in\mathcal{T}}\frac{h_\tau^2}{p_\tau^2}||y_{hp}-y_d+div(a^*\nabla p_{hp})||_{0,\tau}^2\\
&+ C\sum_{\tau\in\mathcal{T}}\sum_{e\in\mu(\tau)-\mu_0(\tau)}\frac{h_\tau}{2p_\tau}||[(a^*\nabla p_{hp}).n_e]||_{0,e}^2\\
&+ C\sum_{\tau\in\mathcal{T}}\sum_{e\in\mu_0(\tau)}\frac{h_\tau}{2p_\tau}||\alpha p_{hp}+[(a^*\nabla p_{hp}).n_e]||_{0,e}^2\\
&+ C||y(u_{hp})-y_{hp}||_{0,\Omega}^2
\end{aligned}
$$

Thus, we have :

$$
||p(u_{hp})-p_{hp}||_{1,\Omega}^2\leq C\sum_{\tau\in\mathcal{T}}(\eta_{4,\tau}^2+\eta_{5,\tau}^2+\eta_{6,\tau}^2)+C||y(u_{hp})-y_{hp}||_{0,\Omega}^2 \tag{4.31}
$$

Similarly, let $e^y = y(u_{hp}) - y_{hp}$, and let $e_I^y = \pi^{hp} e^y$, where π^{hp} is the Scott-Zhang type interpolator defined in lemma(4.9). Applying the standard residual techniques (see, e.g. [32]). Then it follows from the projection equation, Green's formula, and Holder's inequality that :

$$\begin{aligned} c||y(u_{hp}) - y_{hp}||^2_{1,\Omega} &\leq A(e^y, e^y) = A(e^y, e^y - e_I^y) + A(e^y, e^y) \\ &= A(y(u_{hp}) - y_{hp}, e^y - e_I^y) \\ &= A(y(u_{hp}), e^y - e_I^y) - A(y_{hp}, e^y - e_I^y \end{aligned}$$

$$\begin{aligned} c||y(u_{hp}) - y_{hp}||^2_{1,\Omega} &\leq \sum_{\tau \in \mathcal{T}} \left\{ \int_\tau (f + div(a \nabla y_{hp}))(e^y - e_I^y) - \sum_{\tau \in \mathcal{T}} \int_{\partial \tau} [(a \nabla y_{hp}).n_\tau](e^y - e_I^y) \right\} \\ &+ \sum_{e \in \mu_0(\tau)} \int_e (\alpha(u_{hp} - y_{hp}))(e^y - e_I^y) \\ &= \sum_{\tau \in \mathcal{T}} \int_\tau (f + div(a \nabla y_{hp}))(e^y - e_I^y) - \sum_{e \in \mu(\tau) - \mu_0(\tau)} \int_e [(a \nabla y_{hp}).n_e](e^y - e_I^y) \\ &+ \sum_{e \in \mu_0(\tau)} \int_e (\alpha(u_{hp} - y_{hp}) - [a \nabla y.n_e])(e^y - e_I^y) \end{aligned}$$

where we used (4.38) and (4.27), we have :

$$\begin{aligned} c||y(u_{hp}) - y_{hp}||^2_{1,\Omega} &\leq C \sum_{\tau \in \mathcal{T}} ||f + div(a \nabla y_{hp})||_{0,\tau} ||e^y - e_I^y||_{0,\tau} \\ &+ C \sum_{e \in \mu(\tau) - \mu_0(\tau)} ||[(a \nabla y_{hp}).n_e]||_{0,e} ||e^y - e_I^y||_{0,e} \\ &+ C \sum_{e \in \mu_0(\tau)} ||\alpha(u_{hp} - y_{hp}) - [a \nabla y.n_e]||_{0,e} ||e^y - e_I^y||_{0,e} \end{aligned}$$

And using the Theorem(4.9), we have :

$$\begin{aligned} c||y(u_{hp}) - y_{hp}||^2_{1,\Omega} &\leq C \sum_{\tau \in \mathcal{T}} \frac{h_\tau}{p_\tau} ||f + div(a \nabla y_{hp})||_{0,\tau} ||\nabla e^y||_{0,w_\tau^e} \\ &+ C \sum_{e \in \mu(\tau) - \mu_0(\tau)} \sqrt{\frac{h_\tau}{p_\tau}} ||[(a \nabla y_{hp}).n_e]||_{0,e} ||\nabla e^y||_{0,e} \\ &+ C \sum_{e \in \mu_0(\tau)} \sqrt{\frac{h_\tau}{p_\tau}} ||\alpha(u_{hp} - y_{hp}) - [a \nabla y.n_e]||_{0,e} ||\nabla e^y||_{0,e} \\ &\leq C(\sigma) \sum_{\tau \in \mathcal{T}} \frac{h_\tau^2}{p_\tau^2} ||f + div(a \nabla y_{hp})||^2_{0,\tau} \\ &+ C(\sigma) \sum_{e \in \mu(\tau) - \mu_0(\tau)} \frac{h_\tau}{p_\tau} ||[(a \nabla y_{hp}).n_e]||^2_{0,e} \\ &+ C(\sigma) \sum_{e \in \mu_0(\tau)} \frac{h_\tau}{p_\tau} ||\alpha(u_{hp} - y_{hp}) - [a \nabla y.n_e]||^2_{0,e} + \sigma ||e^y||^2_{1,\Omega} \end{aligned}$$

Setting $\sigma = \frac{c}{2}$, we can have :

$$\begin{aligned} ||y(u_{hp}) - y_{hp}||^2_{1,\Omega} &\leq C\sum_{\tau\in\mathcal{T}} \frac{h_\tau^2}{p_\tau^2}||f + div(a\nabla y_{hp})||^2_{0,\tau} \\ &+ C\sum_{\tau\in\mathcal{T}} \sum_{e\in\mu(\tau)-\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||[(a\nabla y_{hp}).n_e]||^2_{0,e} \\ &+ C\sum_{\tau\in\mathcal{T}} \sum_{e\in\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||\alpha(u_{hp} - y_{hp}) - [a\nabla y.n_e]||^2_{0,e} \end{aligned}$$

$$||y(u_{hp}) - y_{hp}||^2_{1,\Omega} \leq C\sum_{\tau\in\mathcal{T}}(\eta^2_{1,\tau} + \eta^2_{2,\tau} + \eta^2_{3,\tau}) \tag{4.32}$$

Then, □

Theorem 4.9. *Let (u, y, p, λ) and $(u_{hp}, y_{hp}, p_{hp}, \lambda_{hp})$ be the solution of optimality conditions $(OCP - OPT)$ and $(OCP - OPT)^{hp}$ respectively. Then we have that*

$$||u - u_{hp}||^2_{0,\partial\Omega} + ||y - y_{hp}||^2_{1,\Omega} + ||p - p_{hp}||^2_{1,\Omega} \leq C\eta^2 \tag{4.33}$$

Proof. Applying the lemmas(4.6)-(4.8), in summary we have the following estimate of $u - u_{hp}$:

$$||u - u_{hp}||_{0,\partial\Omega} \leq ||y(u_{hp}) - y_{hp}||_{0,\Omega} \leq C\eta$$

The next step of the proof is to estimate $y - y_{hp}$:

$$\begin{aligned} ||y - y_{hp}||_{1,\Omega} &\leq ||y - y(u_{hp})||_\Omega + ||y(u_{hp}) - y_{hp}||_\Omega \\ &\leq C||u - u_{hp}||_{0,\partial\Omega} + ||y(u_{hp}) - y_{hp}|| \leq C\eta \end{aligned}$$

The final step of the proof is to estimate error $p - p_{hp}$:

$$\begin{aligned} ||p - p_{hp}||_{1,\Omega} &\leq ||p - p(u_{hp})||_{1,\Omega} + ||p(u_{hp}) - p_{hp}||_{1,\Omega} \\ &\leq C||y - y(u_{hp})||_{0,\partial\Omega} + ||p(u_{hp}) - p_{hp}|| \leq C\eta \end{aligned}$$

□

4.2 A posteriori lower error estimates

In this part, we discussed lower a posteriori bounds, that means the efficiency of the error estimates established in the Theorem 4.9.

Lemma 4.10. *Let (y, p, u, λ) and $(y_{hp}, p_{hp}, u_{hp}, \lambda_{hp})$ be the solution of optimality conditions $(OCP - OPT)$ and $(OCP - OPT)^{hp}$ respectively. Then,*

$$\eta^2_{1,\tau} \leq C\left(p_\tau^2||y - y_{hp}||^2_{1,\tau} + p_\tau^{2\beta}\frac{h_\tau^2}{p_\tau^2}||\pi_\tau f - f||^2_{0,\tau}\right) \tag{4.34}$$

$$\begin{aligned} \eta^2_{4,\tau} &\leq Cp_\tau^{2\beta}\frac{h_\tau^2}{p_\tau^2}(||y_{hp} - y||^2_{0,\tau} + ||\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d||^2_{1,\tau}) \\ &+ Cp_\tau^2||p - p_{hp}||^2_{1,\tau} \end{aligned} \tag{4.35}$$

Where π_τ is the L^2-project operator on the space of polynomials of degree p_τ on the element τ, $1/2 < \beta \le 1$ and the constant C depends on α.

Proof. let Φ_τ be the weight function defined before lemma4.8.

(i) Upper bound of $\eta^2_{4,\tau} = \dfrac{h_\tau^2}{p_\tau^2}||y_{hp} - y_d + div(a^*\nabla p_{hp})||^2_{0,\tau}$.

Define $w_\tau = (\pi_\tau(y_{hp}) - \pi_\tau(y_d) + div(a^*\nabla p_{hp}))\Phi^\alpha_\tau$, $\frac{1}{2} < \alpha \le 1$.

$$\begin{aligned}
||w_\tau\Phi_\tau^{\frac{-\alpha}{2}}||^2_{0,\tau} &= \int_\tau (\pi_\tau(y_{hp}) - \pi_\tau(y_d) + div(a^*\nabla p_{hp}))w_\tau \\
&= \int_\tau (y - y_d + div(a^*\nabla p_{hp}))w_\tau + \int_\tau (y_{hp} - y)w_\tau + \int_\tau (\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d)w_\tau \\
&= A(w_\tau, p) - \left[\int_\tau (a^*\nabla p_{hp})w_\tau + \int_{\partial\tau} \alpha p_{hp} w_\tau\right] + \int_\tau (y_{hp} - y)w_\tau \\
&+ \int_\tau (\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d)w_\tau \\
&= A(w_\tau, p - p_{hp}) + \int_\tau (y_{hp} - y)w_\tau + \int_\tau (\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d)w_\tau \\
&\le C||p - p_{hp}||_{1,\tau}|w|_{1,\tau} + ||(y_{hp} - y)\Phi_\tau^{\frac{\alpha}{2}}||_{0,\tau}||w_\tau\Phi_\tau^{\frac{-\alpha}{2}}||_{0,\tau} \\
&+ ||(\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d)\Phi_\tau^{\frac{\alpha}{2}}||_{0,\tau}||w_\tau\Phi_\tau^{-\frac{\alpha}{2}}||_{0,\tau} \qquad (4.36)
\end{aligned}$$

Then we should estimate w_τ with the H^1 semi-norm. Using the inverse estimates (4.9)-(4.10) with $\beta = \alpha, \gamma = 2(\alpha - 1)$(note that we have $\gamma = 2(\alpha - 1) > -1$ when $\alpha > \frac{1}{2}$), $\delta = \alpha$, and the affine transformation F_τ, we have :

$$\begin{aligned}
|w_\tau|^2_{1,\Omega} &\le 2\int_\tau \Phi_\tau^{2\alpha}|\nabla(\pi_\tau(y_{hp}) - \pi_\tau(y_d) + div(a^*\nabla p_{hp}))|^2 \\
&+ 2\int_\tau (\pi_\tau(y_{hp}) - \pi_\tau(y_d) + div(a^*\nabla p_{hp}))^2|\Phi_\tau^\alpha|^2 \\
&\le C\frac{p_\tau^{2(2-\alpha)}}{h_\tau^2}\int_\tau \Phi_\tau^\alpha(\pi_\tau(y_{hp}) - \pi_\tau(y_d) + div(a^*\nabla p_{hp}))^2 \\
&+ \frac{C}{h_\tau^2}\int_\tau \Phi_\tau^{2(\alpha-1)}(\pi_\tau(y_{hp}) - \pi_\tau(y_d) + div(a^*\nabla p_{hp}))^2 \\
&\le C\frac{p_\tau^{2(2-\alpha)}}{h_\tau^2}\int_\tau \Phi_\tau^\alpha(\pi_\tau(y_{hp}) - \pi_\tau(y_d) + div(a^*\nabla p_{hp}))^2 \\
&= Cp_\tau^{2(1-\alpha)}\frac{p_\tau^2}{h_\tau^2}||w_\tau\Phi_\tau^{-\frac{\alpha}{2}}||^2_{0,\tau} \qquad (4.37)
\end{aligned}$$

Therefore, it follows from (4.36) and (4.37) that :

$$\begin{aligned}
&||w_\tau\Phi_\tau^{\frac{-\alpha}{2}}||_{0,\tau} \\
\le\ & C\big(p_\tau^{1-\alpha}\frac{p_\tau}{h_\tau}||p - p_{hp}||_{1,\tau} + ||(y_{hp} - y)\Phi_\tau^{-\frac{\alpha}{2}}||_{0,\tau} + ||(\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d)\Phi_\tau^{\frac{\alpha}{2}}||_{0,\tau}\big) \\
\le\ & C\big(\frac{p_\tau^{2-\alpha}}{h_\tau}||p - p_{hp}||_{1,\tau} + ||y_{hp} - y||_{0,\tau} + ||\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}\big)
\end{aligned}$$

Furthermore, it follows from (4.38) and (4.9) with $\alpha=\beta$ and $\gamma=0$ that :

$$\begin{aligned}
& ||\pi_\tau(y_{hp})-\pi_\tau(y_d)+div(a^*\nabla p_{hp})||_{0,\tau} \\
\leq\ & Cp_\tau^\beta||\pi_\tau(y_{hp})-\pi_\tau(y_d)+div(a^*\nabla p_{hp})\Phi_\tau^{\frac{\beta}{2}}||_{0,\tau}=Cp_\tau^\beta||w_\tau\Phi_\tau^{-\frac{\beta}{2}}||_{0,\tau} \\
\leq\ & Cp_\tau^\beta(\frac{p_\tau^{2-\beta}}{h_\tau}||p-p_{hp}||_{1,\tau}+||y_{hp}-y||_{0,\tau}+||\pi_\tau(y_{hp})-\pi_\tau(y_d)-y_{hp}+y_d||_{0,\tau})
\end{aligned}$$

Thus,

$$\begin{aligned}
\eta_{4,\tau}^2 &= \frac{h_\tau^2}{p_\tau^2}||y_{hp}-y_d+div(a^*\nabla p_{hp})||_{0,\tau}^2 \\
&\leq C\frac{h_\tau^2}{p_\tau^2}||\pi_\tau(y_{hp})-\pi_\tau(y_d)+div(a^*\nabla p_{hp})||_{0,\tau}^2+C\frac{h_\tau^2}{p_\tau^2}||\pi_\tau(y_{hp})-\pi_\tau(y_d)-y_{hp}+y_d||_{0,\tau}^2 \\
&\leq Cp_\tau^2||p-p_{hp}||_{1,\tau}^2+Cp_\tau^{2\beta}\frac{h_\tau^2}{p_\tau^2}(||y_{hp}-y||_{0,\tau}^2+||\pi_\tau(y_{hp})-\pi_\tau(y_d)-y_{hp}+y_d||_{0,\tau}^2) \\
&\leq Cp_\tau^2||p-p_{hp}||_{1,\tau}^2+Cp_\tau^{2\beta}\frac{h_\tau^2}{p_\tau^2}(||y_{hp}-y||_{0,\tau}^2+||\pi_\tau(y_{hp})-\pi_\tau(y_d)-y_{hp}+y_d||_{0,\tau}^2)
\end{aligned}$$

$$\begin{aligned}
\eta_{4,\tau}^2 &= \frac{h_\tau^2}{p_\tau^2}||y_{hp}-y_d+div(a^*\nabla p_{hp})||_{0,\tau}^2 \\
&\leq Cp_\tau^2||p-p_{hp}||_{1,\tau}^2+Cp_\tau^{2\beta}\frac{h_\tau^2}{p_\tau^2}(||y_{hp}-y||_{0,\tau}^2+||\pi_\tau(y_{hp})-\pi_\tau(y_d)-y_{hp}+y_d||_{1,\tau}^2)
\end{aligned}$$

(ii) Upper bound of $\eta_{1,\tau}^2=\frac{h_\tau^2}{p_\tau^2}||f+div(a\nabla y_{hp})||_{0,\tau}^2$

Similarly, define $v_\tau=(\pi_\tau f+div(a\nabla y_{hp}))\Phi_\tau^\alpha$. Then we have :

$$\begin{aligned}
|v_\tau|_{1,\tau}^2 &\leq Cp_\tau^{2(1-\alpha)}\frac{p_\tau^2}{h_\tau^2}\int_\tau\Phi_\tau^\alpha(\pi_\tau f+div(a\nabla y_{hp}))^2 \\
&= Cp_\tau^{2(1-\alpha)}\frac{p_\tau^2}{h_\tau^2}||v_\tau\Phi_\tau^{-\frac{\alpha}{2}}||_{0,\tau}^2
\end{aligned}$$

Therefore

$$\begin{aligned}
||v_\tau\Phi_\tau^{-\frac{\alpha}{2}}||_{0,\tau}^2 &= \int_\tau(\pi_\tau f+div(a\nabla y_{hp}))v_\tau \\
&= \int_\tau(f+div(a\nabla y_{hp}))v_\tau+\int_\tau(\pi_\tau f-f)v_\tau \\
&= \int_\tau fw_\tau+\int_{\partial\tau}\alpha uv_t-\left[\int_\tau(a\nabla y_{hp})\nabla v_\tau+\int_{\partial\tau}\alpha y_{hp}\right]+\int_\tau(\pi_\tau f-f)v_\tau \\
&= A(y-y_{hp},v_\tau)+\int_\tau(\pi_\tau f-f)v_\tau \\
&\leq ||y-y_{hp}||_{1,\tau}|v_\tau|_{1,\tau}+||(\pi_\tau f-f)\Phi_\tau^{\frac{\alpha}{2}}||_{0,\tau}||v_\tau\Phi_\tau^{-\frac{\alpha}{2}}||_{0,\tau}
\end{aligned}$$

and hence

$$||v_\tau\Phi_\tau^{-\frac{\alpha}{2}}||_{0,\tau}\leq C(\frac{p_\tau^{2-\alpha}}{h_\tau}||y-y_{hp}||_{1,\tau}+||\pi_\tau f-f||_{0,\tau}) \tag{4.38}$$

where, it follows from (4.38) and (4.9) with $\alpha = \beta$ and $\gamma = 0$ that :

$$\begin{aligned} ||\pi_\tau f + div(a\nabla y_{hp})||_{0,\tau} &\leq Cp_\tau^\beta ||(\pi_\tau f + div(a\nabla y_{hp}))\Phi_\tau^{\frac{\beta}{2}}||_{0,\tau} = Cp_\tau^\beta ||w_\tau \Phi_\tau^{-\frac{\beta}{2}}||_{0,\tau} \\ &\leq Cp_\tau^\beta (\frac{p_\tau^{2-\beta}}{h_\tau}||y - y_{hp}||_{1,\tau} + ||\pi_\tau f - f||_{0,\tau}) \end{aligned}$$

using similar techniques

$$\begin{aligned} \frac{h_\tau^2}{p_\tau^2}||f + div(a\nabla y_{hp})||_{0,\tau}^2 &\leq C\frac{h_\tau^2}{p_\tau^2}||\pi_\tau f + div(a\nabla y_{hp})||_{0,\tau}^2 + C\frac{h_\tau^2}{p_\tau^2}||\pi_\tau f + f||_{0,\tau}^2 \\ &\leq Cp_\tau^2||y - y_{hp}||_{1,\tau}^2 + Cp_\tau^{2\beta}\frac{h_\tau^2}{p_\tau^2}||\pi_\tau f - f||_{0,\tau}^2 \end{aligned}$$

Therefore

$$\begin{aligned} \eta_{1,\tau}^2 &= \frac{h_\tau^2}{p_\tau^2}||f + div(a\nabla y_{hp})||_{0,\tau}^2 \\ &\leq C\left(p_\tau^2||y - y_{hp}||_{1,\tau}^2 + p_\tau^{2\beta}\frac{h_\tau^2}{p_\tau^2}||\pi_\tau f - f||_{0,\tau}^2\right) \end{aligned} \tag{4.39}$$

□

In order to obtain a local upper bound for the edge contribution η_2, η_3, η_5 and η_6 we introduce the sets

$$w_\tau = \{\cup\tau' : \tau' \text{ and } \tau \text{ share at least one edge}\} \text{ and } \tilde{w}_\tau := \{\cup\tau : \partial\tau \cap \partial\Omega = e\}$$

We again set $e = \overline{\tau}_1 \cap \overline{\tau}_2$ and $\tau_e = \overline{\tau}_1 \cup \overline{\tau}_2$

Lemma 4.11. *Let (y, p, u, λ) and $(y_{hp}, p_{hp}, u_{hp}, \lambda_{hp})$ be the solution of optimality conditions $(OCP - OPT)$ and $(OCP - OPT)^{hp}$ respectively. Then we have :*

$$\begin{aligned} \eta_{2,\tau}^2 &\leq C\left(p_\tau^{2+2\epsilon}||y - y_{hp}||_{1,w_\tau}^2 + p_\tau^{2+2\epsilon}\frac{h_\tau^2}{p_\tau^2}||\alpha(u - y_{hp})||_{0,w_\tau}^2\right. \\ &+ \left.\frac{h_\tau^2}{p_\tau^{1-4\epsilon}}\sum_{\tau'\subset w_\tau}||\pi_{\tau'}f - f||_{0,\tau'}^2\right); \end{aligned} \tag{4.40}$$

$$\begin{aligned} \eta_5^2 &\leq C\left(p_\tau^{2+2\epsilon}||p - p_{hp}||_{1,w_\tau}^2 + p_\tau^{1+2\epsilon}\frac{h_\tau}{p_\tau}\sum_{e\in\mu(\tau)-\mu_0(\tau)}h_e||\alpha p_{hp}||_{0,e}\right. \\ &+ \left.\frac{h_\tau^2}{p_\tau^{1-4\epsilon}}\sum_{\tau'\subset w_\tau}\left(||y_{hp} - y||_{0,\tau'}^2 + ||\pi_{\tau'}(y_{hp}) - \pi_{\tau'}(y_d) - y_{hp} + y_d||_{0,\tau'}^2\right)\right); \end{aligned} \tag{4.41}$$

$$\begin{aligned} \eta_{3,\tau}^2 &\leq C\left(p_\tau^{2+2\epsilon}||y - y_{hp}||_{1,\tilde{w}_\tau}^2 + p_\tau^{2+2\epsilon}\frac{h_\tau^2}{p_\tau^2}||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2\right. \\ &+ \left.\sum_{\tau\subset\tilde{w}_\tau}\frac{h_\tau^2}{p_\tau^{1-4\epsilon}}||\pi_\tau f - f||_{0,\tau}^2\right); \end{aligned} \tag{4.42}$$

$$\begin{aligned} \eta_{6,\tau}^2 &\leq C\left(p_\tau^{2+2\epsilon}||p - p_{hp}||_{1,\tilde{w}_\tau}^2 + \sum_{\tau\subset\tilde{w}_\tau}\frac{h_\tau^2}{p_\tau^{1-4\epsilon}}(||y_{hp} - y||_{0,\tau}^2\right. \\ &+ \left.||\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}^2)\right). \end{aligned} \tag{4.43}$$

Where h_e is the length of the edge e, π_τ is the L^2-project operator on the space of polynomials of degree p_τ on the element τ, $0 < \epsilon \leq 1/4$ is an arbitrary small positive number and the constant C depends on ϵ.

Proof. To obtain an upper bound for the edge contribution, we will use weight functions associated with the edges and a suitable extension operator. For given element τ with (interior) edge e, we set τ_e to be the union of all the elements sharing the edge e.

(i) Upper bound of $\eta^2_{2,\tau} = \sum\limits_{e\in\mu(\tau)-\mu_0(\tau)} \dfrac{h_\tau}{2p_\tau}||[(a\nabla y_{hp}).n_e]||^2_{0,e}$.

We construct a function $w_e \in H^1(\tau_e)$ with $w_e|_e = [a\nabla y_{hp}.n_e]\Phi_e^\theta$, $\frac{1}{2} < \theta \leq 1$, such that w_e is an affine transformation of v on the reference element and $[a\nabla y_{hp}.n_e]$ will be and ψ in lemma 4.4.

$$\begin{aligned}
||[a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||^2_{0,e} &= ||w_e\Phi_{\tau_e}^{\frac{-\theta}{2}}||^2_{0,e} = \int_e [a\nabla y_{hp}.n_e]w_e \\
&= \int_e [a\nabla(y_{hp}-y).n_e]w_e + \int_e \alpha(u-y)w_e \\
&= A(y_{hp}-y, w_e) + \int_{\tau_e} \big(f + div(a\nabla y_{hp})\big)w_e + \int_e (\alpha u - \alpha y_{hp})w_e \\
&\leq C||y-y_{hp}||_{1,\tau_e}|w_e|_{1,\tau_e} + C||f+div(a\nabla y_{hp})||_{0,\tau_e}||w_e||_{0,\tau_e} \\
&+ C||\alpha(u-y_{hp})||_{0,e}||w_e||_{0,e}
\end{aligned}$$

Using the equivalence affine transformation and lemma 4.4, we obtain the upper bounds for $|w_e|_{1,\tau_e}$ and $||w_e||_{0,\tau_e}$ in term of $||[a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}$.

$$|w_e|^2_{1,\tau_e} \leq C\frac{1}{h_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||[a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||^2_{0,e},$$

$$||w_e||^2_{0,\tau_e} \leq Ch_\tau\epsilon||[a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||^2_{0,e},$$

$$||w_e||^2_{0,e} \leq Ch_e||[a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||^2_{0,e}.$$

where $\epsilon \in (0,1]$ is an arbitrary small positive number. Summing up, we have :

$$\begin{aligned}
||[a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e} &\leq C\left(\Big(\frac{1}{h_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})\Big)^{1/2}||y-y_{hp}||_{1,\tau_e}\right. \\
&+ \left.(h_\tau\epsilon)^{1/2}||f+div(a\nabla y_{hp})||_{0,\tau_e} + (h_e)^{1/2}||\alpha(u-y_{hp})||_{0,e}\right).
\end{aligned}$$

Considering (4.39), we sum up all the edges $e \in \mu(\tau)-\mu_0(\tau)$ and then obtain that :

$$\begin{aligned}
&\sum_{e\in\mu(\tau)-\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||^2_{0,e} \\
\leq\ & C\left(\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||y-y_{hp}||^2_{1,w_\tau} + p_\tau\epsilon\frac{h_\tau^2}{p_\tau^2}||f+div(a\nabla y_{hp})||^2_{0,w_\tau}\right. \\
+\ & \left.p_\tau\frac{h_\tau^2}{p_\tau^2}||\alpha(u-y_{hp})||^2_{0,w_\tau}\right). \\
\leq\ & C\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||y-y_{hp}||^2_{1,w_\tau} + Cp_\tau\frac{h_\tau^2}{p_\tau^2}||\alpha(u-y_{hp})||^2_{0,w_\tau}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad Cp_\tau \epsilon \sum_{\tau' \subset w_\tau} \frac{h_\tau^2}{p_\tau^2} ||f + div(a\nabla y_{hp})||_{0,\tau'}^2 \\
&\leq \quad C\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||y - y_{hp}||_{1,w_\tau}^2 + Cp_\tau \frac{h_\tau^2}{p_\tau^2}||\alpha(u - y_{hp})||_{0,w_\tau}^2 \\
&+ \quad C\epsilon p_\tau^3 \sum_{\tau' \subset w_\tau} ||y - y_{hp}||_{1,\tau'}^2 + C\epsilon p_\tau^{1+2\beta}\frac{h_\tau^2}{p_\tau^2} \sum_{\tau' \subset w_\tau} ||\pi_{\tau'} f - f||_{0,\tau'}^2
\end{aligned}
$$

where ϵ is an arbitrary positive number, and $\frac{1}{2} < \beta \leq 1$ is defined in lemma 4.4. Setting $\epsilon = 1/p_\tau^2$ yields that

$$
\begin{aligned}
&\sum_{e \in \mu(\tau) - \mu_0(\tau)} \frac{h_\tau}{p_\tau} ||[a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
\leq \quad & Cp_\tau ||y - y_{hp}||_{1,w_\tau}^2 + Cp_\tau \frac{h_\tau^2}{p_\tau^2}||\alpha(u - y_{hp})||_{0,w_\tau}^2 + Cp_\tau^{2\beta - 1}\frac{h_\tau^2}{p_\tau^2} \sum_{\tau' \subset w_\tau} ||\pi_{\tau'} f - f||_{0,\tau'}^2
\end{aligned}
$$

Using the inverse estimate in lemma 4.3 and setting $\theta = \beta = 1/2 + \epsilon$ with $0 < \epsilon \leq 1/4$, we have that

$$
\begin{aligned}
\eta_{2,\tau}^2 \quad &= \quad \sum_{e \in \mu(\tau) - \mu_0(\tau)} \frac{h_\tau}{2p_\tau} ||a\nabla y_{hp}.n_e||_{0,e}^2 \\
&\leq \quad Cp_\tau^{2\theta} \sum_{e \in \mu(\tau) - \mu_0(\tau)} \frac{h_\tau}{p_\tau} ||[a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
&\leq \quad Cp_\tau^{1+2\theta}||y - y_{hp}||_{1,w_\tau}^2 + Cp_\tau^{2\theta+1}\frac{h_\tau^2}{p_\tau^2}||\alpha(u - y_{hp})||_{0,w_\tau}^2 \\
&+ \quad Cp_\tau^{2\beta + 2\theta - 1}\frac{h_\tau^2}{p_\tau^2} \sum_{\tau' \subset w_\tau} ||\pi_{\tau'} f - f||_{0,\tau'}^2 \\
&\leq \quad C\left(p_\tau^{2+2\epsilon}||y - y_{hp}||_{1,w_\tau}^2 + p_\tau^{2+2\epsilon}\frac{h_\tau^2}{p_\tau^2}||\alpha(u - y_{hp})||_{0,w_\tau}^2\right. \\
&+ \quad \left.\frac{h_\tau^2}{p_\tau^{1-4\epsilon}} \sum_{\tau' \subset w_\tau} ||\pi_{\tau'} f - f||_{0,\tau'}^2\right)
\end{aligned}
$$

(ii) Upper bound of $\eta_{5,\tau}^2 = \sum_{e \in \mu(\tau) - \mu_0(\tau)} \dfrac{h_\tau}{2p_\tau}||[(a^*\nabla p_{hp}).n_e]||_{0,e}^2$.
Similarly we define $v_e = [a^*\nabla p_{hp}.n_e]\Phi_e^\theta$

$$
\begin{aligned}
||[a^*\nabla p_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \quad &= \quad ||v_e \Phi_{\tau_e}^{\frac{-\theta}{2}}||_{0,e}^2 = \int_e [a^*\nabla p_{hp}.n_e]v_e \\
&= \quad \int_e [a^*\nabla(p_{hp} - p).n_e]v_e - \int_e \alpha p v_e \\
&= \quad A(p_{hp} - p, v_e) + \int_{\tau_e} \left(y - y_d + div(a^*\nabla p_{hp})\right)v_e - \int_e (\alpha p_{hp})v_e \\
&\leq \quad C||p - p_{hp}||_{1,\tau_e}|v_e|_{1,\tau_e} + C||y - y_d + div(a^*\nabla p_{hp})||_{0,\tau_e}||v_e||_{0,\tau_e} \\
&+ \quad C||\alpha p_{hp}||_{0,e}||w_e||_{0,e}
\end{aligned}
$$

Using the equivalence affine transformation and lemma 4.4, we obtain the upper bounds for $|v_e|_{1,\tau_e}$ and $||v_e||_{0,\tau_e}$ in term of $||[a^*\nabla p_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}$.

$$|v_e|_{1,\tau_e}^2 \le C\frac{1}{h_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||[a^*\nabla p_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2,$$

$$||v_e||_{0,\tau_e}^2 \le Ch_\tau\epsilon||[a^*\nabla p_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2,$$

$$||v_e||_{0,e}^2 \le Ch_e||[a^*\nabla p_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2.$$

where $\epsilon \in (0,1]$ is an arbitrary small positive number. Summing up, we have :

$$\begin{aligned}
||[a^*\nabla p_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e} \quad &\le \quad C\left(\left(\frac{1}{h_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})\right)^{1/2}||p - p_{hp}||_{1,\tau_e}\right. \\
&+ \quad \left.(h_\tau\epsilon)^{1/2}||y - y_d + div(a^*\nabla p_{hp})||_{0,\tau_e} + h_e^{1/2}||\alpha p_{hp}||_{0,e}\right).
\end{aligned}$$

Considering (4.39), we sum up all the edges $e \in \mu(\tau) - \mu_0(\tau)$ and then obtain that :

$$\begin{aligned}
&\sum_{e\in\mu(\tau)-\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[a^*\nabla p_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
\le \quad & C\left(\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||p - p_{hp}||_{1,w_\tau}^2 + p_\tau\epsilon\frac{h_\tau^2}{p_\tau^2}||y - y_d + div(a^*\nabla p_{hp})||_{0,w_\tau}^2\right. \\
+ \quad & \left. p_\tau\frac{h_\tau^2}{p_\tau^2}||\alpha p_{hp}||_{0,w_\tau}^2\right). \\
\le \quad & C\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||p - p_{hp}||_{1,w_\tau}^2 + Cp_\tau\frac{h_\tau^2}{p_\tau^2}||\alpha p_{hp}||_{0,w_\tau}^2 \\
+ \quad & Cp_\tau\epsilon\sum_{\tau'\subset w_\tau}\frac{h_\tau^2}{p_\tau^2}||y - y_d + div(a*\nabla p_{hp})||_{0,\tau'}^2 \\
\le \quad & C\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||p - p_{hp}||_{1,w_\tau}^2 + Cp_\tau\frac{h_\tau^2}{p_\tau^2}||\alpha p_{hp}||_{0,w_\tau}^2 + C\epsilon p_\tau^3\sum_{\tau'\subset w_\tau}||p - p_{hp}||_{1,\tau'}^2 \\
+ \quad & C\epsilon p_\tau^{2\beta+1}\frac{h_\tau^2}{p_\tau^2}\sum_{\tau'\subset w_\tau}\left(||y_{hp} - y||_{0,\tau'}^2 + ||\pi_{\tau'}(y_{hp}) - \pi_{\tau'}(y_d) - y_{hp} + y_d||_{0,\tau'}^2\right)
\end{aligned}$$

where ϵ is an arbitrary positive number, and $\frac{1}{2} < \beta \le 1$ is defined in lemma 4.4.
Setting $\epsilon = 1/p_\tau^2$ yields that :

$$\begin{aligned}
&\sum_{e\in\mu(\tau)-\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[a^*\nabla p_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
\le \quad & Cp_\tau||p - p_{hp}||_{1,w_\tau}^2 + Cp_\tau\frac{h_\tau^2}{p_\tau^2}||\alpha p_{hp}||_{0,w_\tau}^2 + Cp_\tau^{2\beta-1}\frac{h_\tau^2}{p_\tau^2}\sum_{\tau'\subset w_\tau}\left(||y_{hp} - y||_{0,\tau'}^2\right. \\
+ \quad & \left.||\pi_{\tau'}(y_{hp}) - \pi_{\tau'}(y_d) - y_{hp} + y_d||_{0,\tau'}^2\right)
\end{aligned}$$

Using the inverse estimate in lemma 4.3 and setting $\theta = \beta = 1/2 + \epsilon$ with $0 < \epsilon \le 1/4$, we have that :

$$
\begin{aligned}
\eta_{5,\tau}^2 &= \sum_{e\in\mu(\tau)-\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||a^*\nabla p_{hp}.n_e||_{0,e}^2 \\
&\leq Cp_\tau^{2\theta} \sum_{e\in\mu(\tau)-\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[a^*\nabla p_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
&\leq Cp_\tau^{1+2\theta}||p-p_{hp}||_{1,w_\tau}^2 + Cp_\tau^{2\theta+1}\frac{h_\tau^2}{p_\tau^2}||\alpha p_{hp}||_{0,w_\tau}^2 + Cp_\tau^{2\beta+2\theta-1} \\
&\quad \frac{h_\tau^2}{p_\tau^2}\sum_{\tau'\subset w_\tau}\left(||y_{hp}-y||_{0,\tau'}^2 + ||\pi_{\tau'}(y_{hp})-\pi_{\tau'}(y_d)-y_{hp}+y_d||_{0,\tau'}^2\right) \\
&\leq C\Bigg(p_\tau^{2+2\epsilon}||p-p_{hp}||_{1,w_\tau}^2 + p_\tau^{2+2\epsilon}\frac{h_\tau^2}{p_\tau^2}||\alpha p_{hp}||_{0,w_\tau}^2 \\
&+ \frac{h_\tau^2}{p_\tau^{1-4\epsilon}}\sum_{\tau'\subset w_\tau}\left(||y_{hp}-y||_{0,\tau'}^2 + ||\pi_{\tau'}(y_{hp})-\pi_{\tau'}(y_d)-y_{hp}+y_d||_{0,\tau'}^2\right)\Bigg)
\end{aligned}
$$

□

To obtain an upper bound for the edge contribution, we will use weight functions associated with the edges and a suitable extension operator. For given element τ with (exterior) edge e.
(iii) Upper bound for $\eta_{3,\tau}^2 = \sum_{e\in\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||\alpha(u_{hp}-y_{hp})-[(a\nabla y_{hp}).n_e]||_{0,e}^2$.
We construct a function $w_e \in H^1(\tau)$ with $w_e|_e = [\alpha(u_{hp}-y_{hp}) - a\nabla y_{hp}.n_e]\Phi_e^\theta$, $\frac{1}{2} < \theta \leq 1$, such that w_e is an affine transformation of v on the reference element and $[\alpha(u_{hp}-y_{hp}) - a\nabla y_{hp}.n_e]$ will be and ψ in lemma 4.4. Let $e \in \mu_0(\tau)$, . Then we have :

$$
\begin{aligned}
||[\alpha(-u_{hp}+y_{hp}) + a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 &= ||w_e\Phi_\tau^{\frac{\theta}{2}}||_{0,e}^2 = \int_e [\alpha(u-y-u_{hp}+y_{hp}) + a\nabla y_{hp}.n_e]w_e \\
&= A(y_{hp}-y, w_e) + \int_\tau \left(f + div(a\nabla y_{hp})\right)w_e + \int_e \alpha(u-u_{hp})w_e \\
&\leq C||y-y_{hp}||_{1,\tau}|w_e|_{1,\tau} + C||f+div(a\nabla y_{hp})||_{0,\tau}||w_e||_{0,\tau} \\
&+ C||\alpha(u-u_{hp})||_{0,e}||w_e||_{0,e}
\end{aligned}
$$

Using the equivalence affine transformation and lemma 4.4, we obtain the upper bounds for $|w_e|_{1,\tau}$ and $||w_e||_{0,\tau}$ in term of $||[\alpha(-u_{hp}+y_{hp}) + a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}$.

$$
|w_e|_{1,\tau}^2 \leq C\frac{1}{h_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||[\alpha(-u_{hp}+y_{hp}) + a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2,
$$

$$
||w_e||_{0,\tau}^2 \leq Ch_\tau\epsilon||[\alpha(-u_{hp}+y_{hp}) + a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2,
$$

$$
||w_e||_{0,e}^2 \leq Ch_e||[\alpha(-u_{hp}+y_{hp}) + a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2.
$$

where $\epsilon \in (0,1]$ is an arbitrary small positive number. Summing up, we have :

$$
||[\alpha(-u_{hp}+y_{hp}) + a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e} \leq C\Bigg(\left(\frac{1}{h_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})\right)^{1/2}||y-y_{hp}||_{1,\tau}
$$

$$+ \quad (h_\tau \epsilon)^{1/2}||f + div(a\nabla y_{hp})||_{0,\tau} + h_e^{1/2}||\alpha(u - u_{hp})||_{0,e}\Bigg).$$

Considering (4.39), we sum up all the edges $e \in \mu_0(\tau)$ and then obtain that :

$$\begin{aligned}
&\sum_{e\in\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[\alpha(-u_{hp} + y_{hp}) + a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
\leq\ & C\left(\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||y - y_{hp}||_{1,\tilde{w}_\tau}^2 + p_\tau \epsilon \frac{h_\tau^2}{p_\tau^2}||f + div(a\nabla y_{hp})||_{0,\tilde{w}_\tau}^2\right. \\
+\ & \left.\right). \\
\leq\ & C\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||y - y_{hp}||_{1,\tilde{w}_\tau}^2 + C\ p_\tau \frac{h_\tau^2}{p_\tau^2}||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2 \\
+\ & C p_\tau \epsilon \frac{h_\tau^2}{p_\tau^2}||f + div(a\nabla y_{hp})||_{0,\tilde{w}_\tau}^2 \\
\leq\ & C\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||y - y_{hp}||_{1,\tilde{w}_\tau}^2 + C\ p_\tau \frac{h_\tau^2}{p_\tau^2}||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2 \\
+\ & C\sum_{\tau\subset\tilde{w}_\tau} \epsilon p_\tau^3||y - y_{hp}||_{1,\tau}^2 + C\sum_{\tau\subset\tilde{w}_\tau} \epsilon p_\tau^{1+2\beta}\frac{h_\tau^2}{p_\tau^2}||\pi_\tau f - f||_{0,\tau}^2
\end{aligned}$$

where ϵ is an arbitrary positive number, and $\frac{1}{2} < \beta \leq 1$ is defined in lemma 4.4.
Setting $\epsilon = 1/p_\tau^2$ yields that :

$$\begin{aligned}
&\sum_{e\in\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[\alpha(-u_{hp} + y_{hp}) + a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
\leq\ & Cp_\tau||y - y_{hp}||_{1,\tilde{w}_\tau}^2 + Cp_\tau\frac{h_\tau^2}{p_\tau^2}||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2 + C\epsilon\sum_{\tau\subset\tilde{w}_\tau} p_\tau^{1+2\beta}\frac{h_\tau^2}{p_\tau^2}||\pi_\tau f - f||_{0,\tau}^2
\end{aligned}$$

Using the inverse estimate in lemma 4.3 and setting $\theta = \beta = 1/2 + \epsilon$ with $0 < \epsilon \leq 1/4$, we have that :

$$\begin{aligned}
\eta_{3,\tau}^2 &= \sum_{e\in\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||[\alpha(-u_{hp} + y_{hp}) + a\nabla y_{hp}.n_e]||_{0,e}^2 \\
&\leq Cp_\tau^{2\theta}\sum_{e\in\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[\alpha(-u_{hp} + y_{hp}) + a\nabla y_{hp}.n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
&\leq Cp_\tau^{1+2\theta}||y - y_{hp}||_{1,\tilde{w}_\tau}^2 + Cp_\tau^{2\theta+1}\frac{h_\tau^2}{p_\tau^2}||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2 \\
&+ C\sum_{\tau\subset\tilde{w}_\tau} p_\tau^{2\beta+2\theta-1}\frac{h_\tau^2}{p_\tau^2}||\pi_\tau f - f||_{0,\tau}^2 \\
&\leq C\left(p_\tau^{2+2\epsilon}||y - y_{hp}||_{1,\tilde{w}_\tau}^2 + p_\tau^{2+2\epsilon}\frac{h_\tau^2}{p_\tau^2}||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2\right. \\
&+ \left.\sum_{\tau\subset\tilde{w}_\tau} \frac{h_\tau^2}{p_\tau^{1-4\epsilon}}||\pi_\tau f - f||_{0,\tau}^2\right)
\end{aligned}$$

(iv) Upper bound for $\eta_{6,\tau}^2 = \sum_{e\in\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||\alpha p_{hp} + [(a^*\nabla p_{hp}).n_e]||_{0,e}^2$.
Similarly we define $[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]\Phi_e^\theta$.

$$\begin{aligned}
||[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 &= ||v_e\Phi_\tau^{\frac{-\theta}{2}}||_{0,e}^2 = \int_e [\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]v_e \\
&= \int_e [a^*\nabla(p_{hp}-p).n_e]v_e + \int_e \alpha(p_{hp}-p)v_e \\
&= A(p_{hp}-p, v_e) + \int_\tau \left(y - y_d + div(a^*\nabla p_{hp})\right)v_e \\
&\leq C||p-p_{hp}||_{1,\tau}|v_e|_{1,\tau} + C||y-y_d+div(a^*\nabla p_{hp})||_{0,\tau}||v_e||_{0,\tau}
\end{aligned}$$

Using the equivalence affine transformation and lemma 4.4, we obtain the upper bounds for $|v_e|_{1,\tau}$ and $||v_e||_{0,\tau}$ in term of $||[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}$.

$$|v_e|_{1,\tau_e}^2 \leq C\frac{1}{h_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2,$$

$$||v_e||_{0,\tau_e}^2 \leq Ch_\tau\epsilon||[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2,$$

where $\epsilon \in (0,1]$ is an arbitrary small positive number. Summing up, we have :

$$\begin{aligned}
||[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e} &\leq C\Bigg(\big(\frac{1}{h_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})\big)^{1/2}||p-p_{hp}||_{1,\tau} \\
&+ (h_\tau\epsilon)^{1/2}\big(||y-y_d+div(a^*\nabla p_{hp})||_{0,\tau}\big)\Bigg).
\end{aligned}$$

Considering (4.39), we sum up all the edges $e\in\mu_0(\tau)$ and then obtain that :

$$\begin{aligned}
&\sum_{e\in\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
&\leq C\left(\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||p-p_{hp}||_{1,\tilde{w}_\tau}^2 + p_\tau\epsilon\frac{h_\tau^2}{p_\tau^2}||y-y_d+div(a^*\nabla p_{hp})||_{0,\tilde{w}_\tau}^2\right) \\
&\leq C\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||p-p_{hp}||_{1,\tilde{w}_\tau}^2 + Cp_\tau\epsilon\sum_{\tau\subset\tilde{w}_\tau}\frac{h_\tau^2}{p_\tau^2}||y-y_d+div(a\nabla p_{hp})||_{0,\tau}^2 \\
&\leq C\frac{1}{p_\tau}(\epsilon p^{2(2-\theta)} + \epsilon^{-1})||p-p_{hp}||_{1,\tilde{w}_\tau}^2 + C\sum_{\tau\subset\tilde{w}_\tau}\epsilon p_\tau^3||p-p_{hp}||_{1,\tau}^2 \\
&+ C\sum_{\tau\subset\tilde{w}_\tau}\epsilon p_\tau^{2\beta+1}\frac{h_\tau^2}{p_\tau^2}\left(||y_{hp}-y||_{0,\tau}^2 + ||\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}^2\right)
\end{aligned}$$

where ϵ is an arbitrary positive number, and $\frac{1}{2} < \beta \leq 1$ is defined in lemma 4.4.
Setting $\epsilon = 1/p_\tau^2$ yields that :

$$\begin{aligned}
&\sum_{e\in\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
&\leq Cp_\tau||p-p_{hp}||_{1,\tilde{w}_\tau}^2 + C\sum_{\tau\subset\tilde{w}_\tau} p_\tau^{2\beta-1}\frac{h_\tau^2}{p_\tau^2}\left(||y_{hp}-y||_{0,\tau}^2 + ||\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}^2\right)
\end{aligned}$$

Using the inverse estimate in lemma 4.3 and setting $\theta = \beta = 1/2 + \epsilon$ with $0 < \epsilon \leq 1/4$, we have that :

$$\begin{aligned}
\eta_{6,\tau}^2 &= \sum_{e\in\mu_0(\tau)} \frac{h_\tau}{2p_\tau}||[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]||_{0,e}^2 \\
&\leq Cp_\tau^{2\theta} \sum_{e\in\mu_0(\tau)} \frac{h_\tau}{p_\tau}||[\alpha p_{hp} + (a^*\nabla p_{hp}).n_e]\Phi_e^{\frac{\theta}{2}}||_{0,e}^2 \\
&\leq Cp_\tau^{1+2\theta}||p - p_{hp}||_{1,\tilde{w}_\tau}^2 + C\sum_{\tau\subset\tilde{w}_\tau} p_\tau^{2\beta+2\theta-1} \\
&\quad \frac{h_\tau^2}{p_\tau^2}(||y_{hp} - y||_{1,\tau}^2 + ||\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}^2) \\
&\leq C\left(p_\tau^{2+2\epsilon}||p - p_{hp}||_{1,\tilde{w}_\tau}^2 + \sum_{\tau\subset\tilde{w}_\tau} \frac{h_\tau^2}{p_\tau^{1-4\epsilon}}(||y_{hp} - y||_{0,\tau}^2\right. \\
&+ \left.||\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}^2)\right)
\end{aligned}$$

□

Theorem 4.12. *Let (y, p, u, λ) and $(y_{hp}, p_{hp}, u_{hp}, \lambda_{hp})$ be the solutions of $(OCP - OPT)$ and $(OCP - OPT)^{hp}$ respectivelyt. Let $\eta_{i,\tau}$, $i = 1, \cdots, 6$ defined by (4.38)-(4.38). Assume that all the conditions in Lemmas 4.10-4.11 are valid. Then we have :*

$$\begin{aligned}
\eta^2 \leq\ & C\sum_{\tau\in\mathcal{T}} p_\tau^{2+2\epsilon}\Big(||p - p_{hp}||_{1,w_\tau}^2 + ||y - y_{hp}||_{1,w_\tau}^2 + ||p - p_{hp}||_{1,\tilde{w}_\tau}^2 + ||y - y_{hp}||_{1,\tilde{w}_\tau}^2 \\
& + \frac{h_\tau^2}{p_\tau^2}(F_1^2 + F_2^2 + E^2)\Big)
\end{aligned}$$

where $0 < \epsilon \leq \frac{1}{4}$ is a small positive number and

$$\begin{aligned}
F_1^2 &= \sum_{\tau'\subset w_\tau} (||\pi_{\tau'}f - f||_{0,\tau'}^2 + ||\pi_\tau(y_{hp}) - \pi_{\tau'}(y_d) - y_{hp} + y_d||_{0,\tau'}^2) \\
F_1^2 &= \sum_{\tau\subset\tilde{w}_\tau} (||\pi_\tau f - f||_{0,\tau}^2 + ||\pi_\tau(y_h p) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}^2) \\
E^2 &= ||\alpha(u - y_{hp})||_{0,w_\tau}^2 + ||\alpha p_{hp}||_{0,w_\tau}^2 + ||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2
\end{aligned}$$

Proof.

$$\begin{aligned}
\eta^2 &= \sum_{\tau\in\mathcal{T}}\sum_{i=1}^{6} \eta_{i,\tau}^2 \\
&\leq C\sum_{\tau\in\mathcal{T}}\left(p_\tau^2||y - y_{hp}||_{1,\tau}^2 + \frac{h_\tau^2}{p_\tau^{1-2\epsilon}}||\pi_\tau f - f||_{0,\tau}^2\right) \\
&\quad +C\sum_{\tau\in\mathcal{T}} \frac{h_\tau^2}{p_\tau^{1-2\epsilon}}(||y_{hp} - y||_{0,\tau}^2 + ||\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d||_{1,\tau}^2) \\
&\quad +C\sum_{\tau\in\mathcal{T}} p_\tau^2||p - p_{hp}||_{1,\tau}^2 + C\sum_{\tau\in\mathcal{T}}\left(p_\tau^{2+2\epsilon}||y - y_{hp}||_{1,w_\tau}^2 + p_\tau^{2+2\epsilon}\frac{h_\tau^2}{p_\tau^2}||\alpha(u - y_{hp})||_{0,w_\tau}^2\right.
\end{aligned}$$

$$
\begin{aligned}
& + \frac{h_\tau^2}{p_\tau^{1-4\epsilon}} \sum_{\tau' \subset w_\tau} ||\pi_{\tau'} f - f||_{0,\tau'}^2 \Bigg) \\
& + \sum_{\tau \in \mathcal{T}} C \Bigg(p_\tau^{2+2\epsilon} ||p - p_{hp}||_{1,w_\tau}^2 + p_\tau^{2+2\epsilon} \frac{h_\tau^2}{p_\tau^2} ||\alpha p_{hp}||_{0,w_\tau}^2 \\
& + \frac{h_\tau^2}{p_\tau^{1-4\epsilon}} \sum_{\tau' \subset w_\tau} \left(||y_{hp} - y||_{0,\tau'}^2 + ||\pi_{\tau'}(y_{hp}) - \pi_{\tau'}(y_d) - y_{hp} + y_d||_{0,\tau'}^2 \right) \Bigg) \\
& + C \sum_{\tau \in \mathcal{T}} \Bigg(p_\tau^{2+2\epsilon} ||y - y_{hp}||_{1,\tilde{w}_\tau}^2 + p_\tau^{2+2\epsilon} \frac{h_\tau^2}{p_\tau^2} ||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2 \\
& + \sum_{\tau \subset \tilde{w}_\tau} \frac{h_\tau^2}{p_\tau^{1-4\epsilon}} ||\pi_\tau f - f||_{0,\tau}^2 \Bigg) \\
& + C \sum_{\tau \in \mathcal{T}} \Bigg(p_\tau^{2+2\epsilon} ||p - p_{hp}||_{1,\tilde{w}_\tau}^2 + \sum_{\tau \subset \tilde{w}_\tau} \frac{h_\tau^2}{p_\tau^{1-4\epsilon}} (||y_{hp} - y||_{0,\tau}^2 \\
& + ||\pi_\tau(y_{hp}) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}^2) \Bigg) \\
\leq \; & C \Bigg(\sum_{\tau \in \mathcal{T}} p_\tau^{2+2\epsilon} (||p - p_{hp}||_{1,w_\tau}^2 + ||y - y_{hp}||_{1,w_\tau}^2 + ||p - p_{hp}||_{1,\tilde{w}_\tau}^2 + ||y - y_{hp}||_{1,\tilde{w}_\tau}^2) \\
& + \sum_{\tau \in \mathcal{T}} p_\tau^{2+2\epsilon} \frac{h_\tau^2}{p^{3-2\epsilon}} (\sum_{\tau' \subset w_\tau} (||\pi_{\tau'} f - f||_{0,\tau'}^2 + ||\pi_\tau (y_{hp} - \pi_{\tau'}(y_d) - y_{hp} + y_d||_{0,\tau'}^2) \\
& + \sum_{\tau \subset \tilde{w}_\tau} (||\pi_\tau f - f||_{0,\tau}^2 + ||\pi_\tau(y_h p) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}^2)) \\
& + \sum_{\tau \in \mathcal{T}} p_\tau^{2+2\epsilon} \frac{h_\tau^2}{p_\tau^2} (||\alpha(u - y_{hp})||_{0,w_\tau}^2 + ||\alpha p_{hp}||_{0,w_\tau}^2 + ||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2) \Bigg)
\end{aligned}
$$

we note that $0 < \epsilon \leq \frac{1}{4}$, we have $3 - 2\epsilon \geq 2$, $3 - 2\epsilon - (2 + 2\epsilon) = 1 - 4\epsilon$.

$$
\begin{aligned}
\eta^2 \; \leq \; & C \sum_{\tau \in \mathcal{T}} p_\tau^{2+2\epsilon} \Bigg((||p - p_{hp}||_{1,w_\tau}^2 + ||y - y_{hp}||_{1,w_\tau}^2 + ||p - p_{hp}||_{1,\tilde{w}_\tau}^2 + ||y - y_{hp}||_{1,\tilde{w}_\tau}^2) \\
& + \frac{h_\tau^2}{p_\tau^2} (\sum_{\tau' \subset w_\tau} (||\pi_{\tau'} f - f||_{0,\tau'}^2 + ||\pi_\tau(y_{hp}) - \pi_{\tau'}(y_d) - y_{hp} + y_d||_{0,\tau'}^2) \\
& \sum_{\tau \subset \tilde{w}_\tau} (||\pi_\tau f - f||_{0,\tau}^2 + ||\pi_\tau(y_h p) - \pi_\tau(y_d) - y_{hp} + y_d||_{0,\tau}^2) \\
& (||\alpha(u - y_{hp})||_{0,w_\tau}^2 + ||\alpha p_{hp}||_{0,w_\tau}^2 + ||\alpha(u - u_{hp})||_{0,\tilde{w}_\tau}^2)) \Bigg)
\end{aligned}
$$

□

Remark 4.13. *It follows from Theorems 4.12 and 4.9 that :*

$$
||u - u_{hp}||_{0,\partial\Omega}^2 + ||y - y_{hp}||_{1,\Omega}^2 + ||p - p_{hp}||_{1,\Omega}^2 \leq C\eta^2 \tag{4.44}
$$

and

$$\eta^2 \leq C \sum_{\tau \in \mathcal{T}} p_\tau^{2+2\epsilon} \Big(||p - p_{hp}||^2_{1,w_\tau} + ||y - y_{hp}||^2_{1,w_\tau} + ||p - p_{hp}||^2_{1,\tilde{w}_\tau} + ||y - y_{hp}||^2_{1,\tilde{w}_\tau} + \frac{h_\tau^2}{p_\tau^2} \big(F_1^2 + F_2^2 + E^2\big) \Big)$$

where F_1^2*,* F_2^2 *and* E_1^2 *are defined in Theorem 4.12 which are all higher order terms under some regularity conditions. Then we obtain the a posteriori error estimates with the upper and lower bounds, although there is a gap of order* p^2 *between the upper and lower bounds.*

5 Discussions

In this paper, we discussed a priori and a posteriori error estimates of the $h-p$-finite element method for boundary convex optimal control problem governed by the elliptic partial differential equations. It is shown that the a posteriori error estimators derived in this paper provide both upper and lower bounds for the approximation errors, although the lower bound is suboptimal in the sense that there is a gap of order p^2 between the upper and lower bounds. In this area there are many important issues that still need to be addressed. For example, studies for more complicated control problems for instance using state constraints instead of control constraints. Furthermore many computational issues have to be addressed. For example, adaptive refinement strategy should be investigated for efficiently implementing adaptive $h-p$-finite element method for optimal control problems.

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