Snap in Motion Along Plane and Space Curves

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Abstract

The rate of change of jerk or the second derivative of acceleration with respect to time has been called snap or jounce. Snap is an important topic that has many applications in mechanics, acoustics and is used to explain the universe's different phenomena. The main purpose of this paper is to study the snap vector in planar and space motion. For the planar motion, the snap vector is resolved into tangential-normal and radial-transverse components. The oscillation of a simple pendulum and central force proportional to distance are chosen as models for the plane motion to show the several geometric properties of the snap vector. Furthermore, we consider a particle moving in the three-dimensional Euclidean space and resolve its snap vector along the tangential direction, the radial direction in the osculating plane, and the other radial direction in the rectifying plane, respectively. The motion of an electron under a constant magnetic field and the motion of a particle along a logarithmic spiral curve are chosen as models for the three-dimensional motion to show the several geometric properties of the snap vector.

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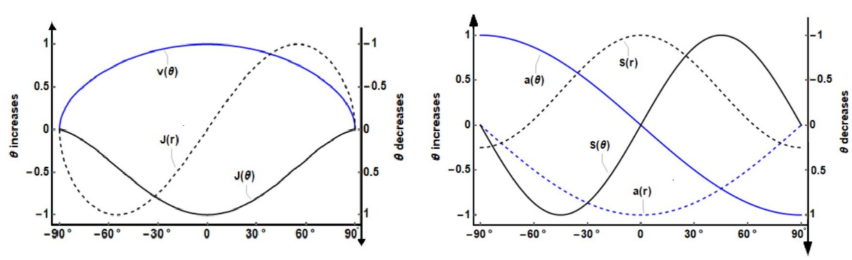

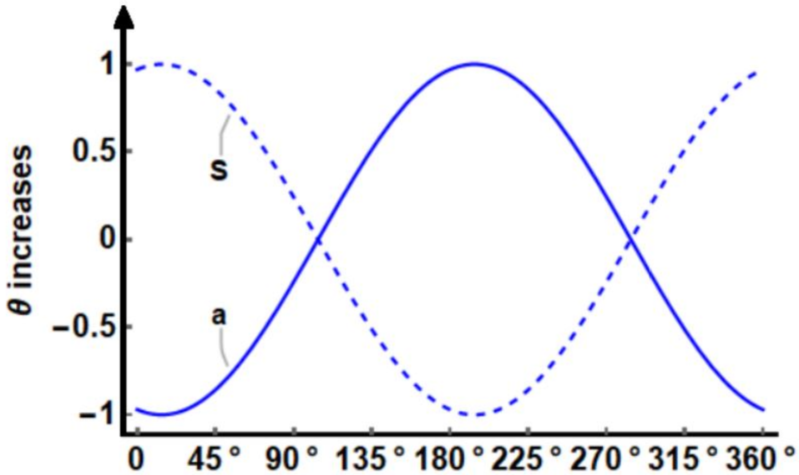

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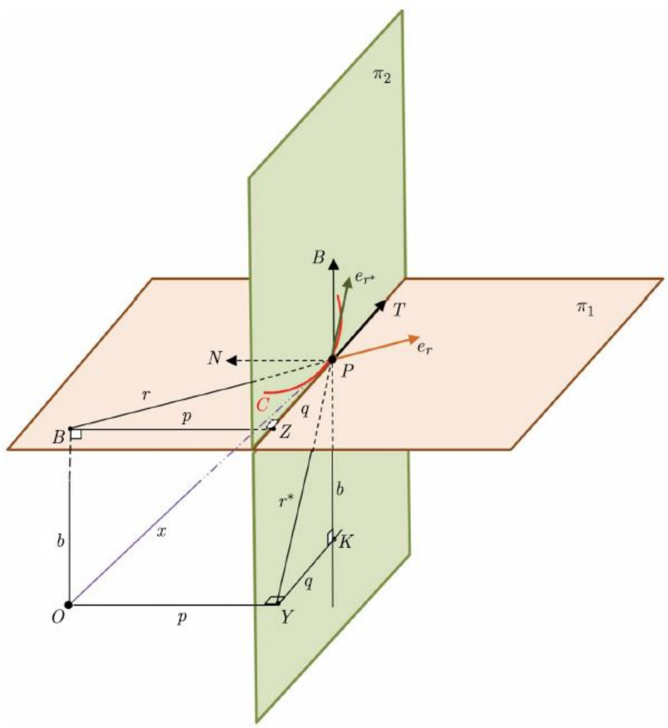

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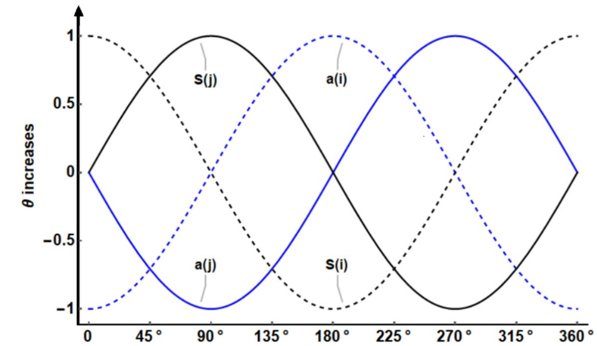
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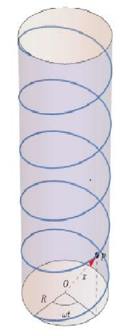

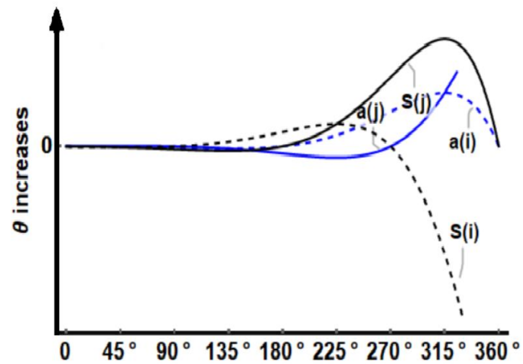
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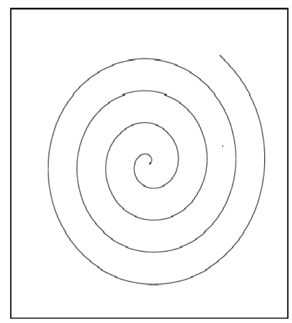

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ARTICLE TYPE

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Summary

The rate of change of jerk or the second derivative of acceleration with respect to time has been called snap or jounce. Snap is an important topic that has many applications in mechanics, acoustics and is used to explain the universe's different phenomena. The main purpose of this paper is to study the snap vector in planar and space motion. For the planar motion, the snap vector is resolved into tangential-normal and radial-transverse components. The oscillation of a simple pendulum and central force proportional to distance are chosen as models for the plane motion to show the several geometric properties of the snap vector. Furthermore, we consider a particle moving in the three-dimensional Euclidean space and resolve its snap vector along the tangential direction, the radial direction in the osculating plane, and the other radial direction in the rectifying plane, respectively. The motion of an electron under a constant magnetic field and the motion of a particle along a logarithmic spiral curve are chosen as models for the three-dimensional motion to show the several geometric properties of the snap vector.

KEYWORDS:
Frenet-Serret apparatus, jerk, snap or jounce, plane and space curves.

1 | INTRODUCTION

In Newtonian physics, the force acting on a particle is obtained with its acceleration through $\mathbf{F} = m\mathbf{a}$. A particle moves in 3-dimensional Euclidean space under the influence of arbitrary forces with an acceleration which is obtained as the derivative of the velocity vector with respect to time. To state the acceleration vector as the sum of tangential and normal components is practical, but when the angular momentum is constant, the sum of these components is more practical. In 1879, the previous study of the acceleration vector was given by Siacci[8]. In 2011, the radial component lies in an instantaneous osculating plane to the trajectory while the tangent component lies along the tangent line vector of the trajectory[1]. In 2020, Siacci theorem studied in Minkowski 3-space[6]. In 2021, a new visualization of fundamental mechanical elements and concepts according to differential geometry is given in[3].

The jerk vector of a particle is the time derivative of the acceleration vector. Jerk is essential when evaluating the destructive effect of motion on a mechanism, and the discomfort caused to passengers in vehicles. The movement of sensitive instruments needs to be kept within specified limits of both acceleration and jerk to avoid damage. From 1978 to 2020, the definition of the jerk and its applications are given in[4],[5],[7],[10] and[9].

The snap or jounce vector of a particle is defined as the time derivative of its jerk vector. But, is the snap vector an important concept to understand? How can we recognize or feel its effect on bodies? what about its manufacturing usages and applications?. Let us discuss the answers to the previous questions.

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In a shaking application, It would be the rate at which the change in acceleration changes, for example, Imagine you are in a roller coaster which just bobs up and down in a sinusoidal fashion, then the high snap would be related to how often you feel the change in this feeling as you go from upwards movement to downwards until your insides scramble (there would also be a varying strength of this feeling throughout the motion due to jerk). In this example, you cannot decouple the high snap from the high jerk, but the question is posed a bit odd. It is like, what are the real-world applications of velocity? things have velocity. So all moving things -same with the snap-, although things would rarely be designed to have jounce since we like our actuation to be governed by simple controls.

On a graph, you can show the action of acceleration on the position as tracing a parabola over time. If you similarly plot snap and acceleration, I would think that snap could be seen in the oscillations upward and downward of the acceleration curve. Jerk is shown with the slope of the curve, and snap is shown with concavity. Thinking about the shape reminded me quite a bit of your roller coaster analogy.

In robotics, I think having a high snap (as well as a high jerk) might cause uncontrolled slipping of moving parts that would cause a loss in precision for what the robot does. This wouldn't matter much if the arm is moving (1 cm) but for high precision robotics like if each movement is like (0.1 mm), it might have an effect.

It is important to think about: it takes time to apply a force to an object in order to accelerate it. The acceleration does not suddenly turn on but instead grows from zero. Therefore, there must be some jerk involved. However, the jerk does not suddenly turn on but also grows from zero. Therefore, there must be some snap involved. However, the snap does not suddenly turn on but also grows from zero. Therefore, there must be some crackle involved. However, etc.

This paper is organized as follows: In section 2, the frame and equations of Frenet-Serret of the curve in 3-dimensional Euclidean space are shown. In section 3, we investigate, for the planar motion, the snap vector that resolved into tangential-normal and radial-transverse components. The oscillation of a simple pendulum and central force proportional to distance are chosen as models for the plane motion to show the several geometric properties of the snap vector. In section 4, we investigate the snap vector in the three-dimensional Euclidean space by considering a particle moving on a curve and resolve its snap vector along the tangential direction, the radial direction in the osculating plane, and the other radial direction in the rectifying plane, respectively. The motion of an electron under a constant magnetic field and the motion of a particle along a logarithmic spiral curve are models chosen for the three-dimensional motion to show the several geometric properties of the snap vector.

2 | PRELIMINARIES

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary curve in 3-dimensional Euclidean space. The curve α which parameterized by arc-length s is said to be a unit speed curve if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle\ ,\ \rangle$ is the standard inner product in 3-dimensional Euclidean space and given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3, \tag{1}$$

for each $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{E}^3$.

Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s), \kappa(s), \tau(s)\}$ be the Frenet-Serret apparatus. $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is a moving frame along a curve α or **TNB** frame, where $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are the tangential, normal and binormal vector fields, receptively. $\kappa(s)$ and $\tau(s)$ are the curvature and torsion of the curve α, respectively. Here $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ are mutually orthonormal vectors satisfying

$$\begin{aligned} \langle \mathbf{T}(s), \mathbf{T}(s) \rangle = \langle \mathbf{N}(s), \mathbf{N}(s) \rangle = \langle \mathbf{B}(s), \mathbf{B}(s) \rangle = 1, \\ \langle \mathbf{T}(s), \mathbf{N}(s) \rangle = \langle \mathbf{T}(s), \mathbf{B}(s) \rangle = \langle \mathbf{N}(s), \mathbf{B}(s) \rangle = 0. \end{aligned} \tag{2}$$

and defined as

$$\mathbf{T}(s) = \alpha'(s), \qquad \mathbf{N}(s) = \frac{\alpha''(s)}{||\alpha''(s)||}, \qquad \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s), \tag{3}$$

where the superposed dash denotes to d/ds.

The Frenet-Serret equations for α are given by

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} \tag{4}$$

Curvature and torsion functions can be calculated from the relations

$$\begin{aligned} \kappa(s) &= \langle \mathbf{T}'(s), \mathbf{N}(s) \rangle = -\langle \mathbf{N}'(s), \mathbf{T}(s) \rangle = ||\mathbf{T}'(s)||, \\ \tau(s) &= \langle \mathbf{N}'(s), \mathbf{B}(s) \rangle = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle. \end{aligned} \tag{5}$$

3 | SNAP IN MOTION ALONG THE PLANE CURVES

In the kinematics of a particle, snap or jounce has a significant effect on the motion of plane curves as on the oscillation of a simple pendulum, the central force that is proportional to distance, Keplerian orbital motion, and in many issues of classical mechanics. In this section, we investigate, For the planar motion, the snap vector that resolved into tangential-normal and radial-transverse components. The oscillation of a simple pendulum and central force proportional to distance are chosen as models for the plane motion to show the several geometric properties of the snap vector

3.1 | Tangential-normal components of the snap

Let C be any differentiable curve described as a function of time t, $\mathbf{R(t)}$ is the position vector from a given fixed origin to any point on the curve. Let the curve C be given in the intrinsic form $\kappa = \kappa(s)$, where s is the arc length of the curve C. Then, the velocity vector $\mathbf{v(t)}$ and the acceleration vector $\mathbf{a(t)}$ can be expressed as follows

$$textbf{v}(t) = \dot{\mathbf{R}} = v\mathbf{T}, \tag{6}$$

$$\mathbf{a(t)} = \dot{\mathbf{v}} = \dot{v}\mathbf{T} + \kappa v^2 \mathbf{N}, \tag{7}$$

where the superposed dot donates d/dt, $v = |\mathbf{v(t)}|$ is called the speed and $\mathbf{T} = \mathbf{T}(t)$, $\mathbf{N} = \mathbf{N}(t)$, and $\kappa = \kappa(s(t))$ are the unit tangent, unit normal, and the curvature of the curve at any time t, respectively. $\dot{v}$ and $\dot{\kappa} v^2$ are the tangential and normal components of the acceleration that called the acceleration along the curve and the centripetal acceleration, respectively.

The jerk vector $\mathbf{J(t)}$ which is the rate of change of the acceleration and the snap vector $\mathbf{S(t)}$ which is the rate of change of the jerk vector can be expressed as follows

$$\mathbf{J(t)} = \dot{\mathbf{a}} = \left(\ddot{v} - \kappa^2 v^3\right)\mathbf{T} + \left(3\kappa v \dot{v} + \dot{\kappa} v^2\right)\mathbf{N}, \tag{8}$$

$$\mathbf{S(t)} = \dot{\mathbf{J}} = \left(\dddot{v} - 3\kappa\dot{\kappa} v^3 - 6\kappa^2 v^2 \dot{v}\right)\mathbf{T} + \left(4\kappa v \ddot{v} + 5\dot{\kappa} v \dot{v} - \kappa^3 v^4 + 3\kappa \dot{v}^2 + \ddot{\kappa} v^2\right)\mathbf{N}. \tag{9}$$

3.2 | Radial-transverse components of the snap

Let r and θ be polar coordinates, where r denotes the radial distance from a fixed origin point to any point on the path and θ is the angle which the radial vector to the point makes with a fixed polar axis. Let $\mathbf{e}_r$ and $\mathbf{e}_\theta$ be the radial vector and perpendicular thereto, respectively, which are mutually orthogonal to each other. Then, on differentiating the radial vector $\mathbf{R} = r\mathbf{e}_r$, the velocity and acceleration vector can be expressed as

$$\mathbf{v(t)} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta, \tag{10}$$

$$\mathbf{a(t)} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e_r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta. \tag{11}$$

by differentiating $\mathbf{a(t)}$ with respect to time t and using the relations $\dot{\mathbf{e}}_\mathbf{r} = \dot{\theta}\mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e_r}$ we calculate the jerk and snap vectors as

$$\mathbf{J(t)} = \left(\dddot{r} - 3\dot{r}\dot{\theta}^2 - 3r\dot{\theta}\ddot{\theta}\right)\mathbf{e_r} + \left(3\ddot{r}\dot{\theta} + 3\dot{r}\ddot{\theta} + r\dddot{\theta} - r\dot{\theta}^3\right)\mathbf{e}_\theta, \tag{12}$$

$$\mathbf{S(t)} = \left(\ddddot{r} - 6\ddot{r}\dot{\theta}^2 - 12\dot{r}\dot{\theta}\ddot{\theta} - 3r\ddot{\theta}^2 - 4r\dot{\theta}\dddot{\theta} + r\dot{\theta}^4\right)\mathbf{e_r} + \left(4\dddot{r}\dot{\theta} - 4\dot{r}\dot{\theta}^3 - 6r\dot{\theta}^2\ddot{\theta} + 6\ddot{r}\ddot{\theta} + 4\dot{r}\dddot{\theta} + r\ddddot{\theta}\right)\mathbf{e}_\theta. \tag{13}$$

3.3 | Planar motion models

In this section, the oscillation of a simple pendulum and central force proportional to distance are chosen as models for the plane motion to show the several geometric properties of the snap vector.

3.3.1 | The oscillation of a simple pendulum

The differential equation of the oscillation of a simple pendulum of length l which subjects only to the gravitational force is given by

$$\ddot{\theta} + \left(\frac{g}{l}\right)\sin\theta = 0, \tag{14}$$

where g is the acceleration of gravity and θ is the deflection angle measured from the equilibrium position. Initially $\theta = \pm\alpha$ and by integrate (14) we get

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\alpha) \tag{15}$$

For simplicity, assume $\alpha = \pm 90$ i.e., the pendulum swings through a lower semicircular arc. Directly from (10)-(13) we get

$$\begin{aligned}
\mathbf{v} &= \left[(2gl)^{1/2}\cos^{1/2}\theta\right]\mathbf{e}_\theta,\\
\mathbf{a} &= -(2g\cos\theta)\mathbf{e_r} - (g\sin\theta)\mathbf{e}_\theta,\\
\mathbf{J} &= \left[3\left(\frac{2g^3}{l}\right)^{1/2}\cos^{1/2}\theta\sin\theta\right]\mathbf{e_r} - \left[3\left(\frac{2g^3}{l}\right)^{1/2}\cos^{3/2}\theta\right]\mathbf{e}_\theta,\\
\mathbf{S} &= \left[\frac{-3g^2}{l}\sin^2\theta + \frac{12g^2}{l}\cos^2\theta\right]\mathbf{e_r} + \left[\frac{15g^2}{l}\sin\theta\cos\theta\right]\mathbf{e}_\theta.
\end{aligned} \tag{16}$$

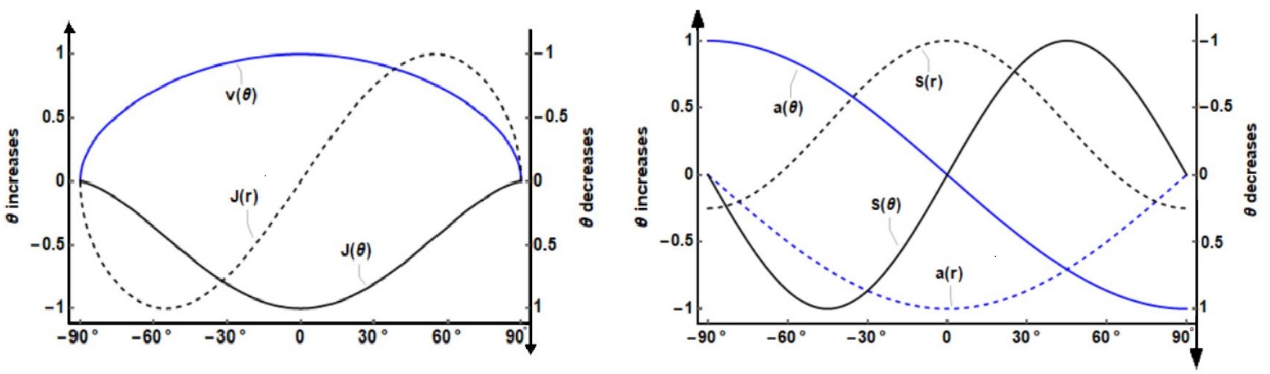

FIGURE 1 Pendulum swinging through ±90. The radial-transverse components of the vectors are plotted separately in normalized form.

According to the FIGURE 1 , it may be noted that:

- Both components of the velocity and jerk vanish at extreme deflection, $\theta = \pm 90$.
- The radial component of jerk changes sign in the midpoint of swing, $\theta = 0$. Furthermore, the transverse component of jerk has only one minimum at $\theta = 0$.
- At $\theta = \pm 90$, the radial component of the acceleration vector and the transverse component of the snap vector vanish.
- Snap could be seen in the oscillations upward and downward of the acceleration curve. Jerk is shown with the slope of the curve, and snap is shown with concavity.

3.3.2 | Central force proportional to distance

The differential equation of the motion of a particle describing a path under the acceleration of a force directed towards a fixed origin and varying as its distance from this point is given by

$$\ddot{\mathbf{R}} + \omega^2\mathbf{R} = 0, \tag{17}$$

where $\mathbf{R(t)}$ is the position vector from a fixed point to the particle, $\omega^2 = k/m$, where k is the proportionality constant in the restoring force $\mathbf{F} = -k\mathbf{R}$, and m is the mass of the particle. The solution of the differential equation (17) is given by

$$\mathbf{R} = \mathbf{R}_0\cos\omega t + \mathbf{v}_0\omega\sin\omega t, \tag{18}$$

where $\mathbf{R}_0$ is the position vector and $\mathbf{v}_0$ is the velocity vector initially at $t = 0$. If $r_0 = |\mathbf{R}_0|$ and $v_0 = |\mathbf{v}_0|$, and let e_R and e_v in the directions of $\mathbf{R}_0$ and $\mathbf{v}_0$, respectively. Then, e_R and e_v span an oblique coordinate system in which the solution $\mathbf{R}$ has the coordinates $\lambda = r_0\ cos\omega t$ and $\mu = v_0\ \omega\ sin\omega t$ satisfying

$$\lambda^2/r_0^2 + \mu^2/v_0^2\omega^2 = 1. \tag{19}$$

This equation says the particle moves in an ellipse which has a conjugate radii of the initial position vector and initial velocity vector, and its center at the center of force. The velocity, acceleration, jerk and snap vectors can be expressed directly from (18) as

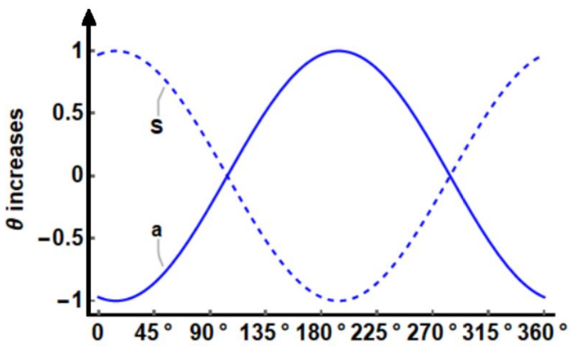

FIGURE 2 The central force proportional to distance. The acceleration and snap vectors are plotted separately in normalized form.

$$\begin{aligned}
\mathbf{v} &= -\mathbf{R}_0\omega \sin \omega t + \mathbf{v}_0\omega^2 \cos \omega t, \\
\mathbf{a} &= -\mathbf{R}_0\omega^2 \cos \omega t - \mathbf{v}_0\omega^3 \sin \omega t = -\omega^2\mathbf{R}, \\
\mathbf{J} &= \mathbf{R}_0\omega^3 \sin \omega t - \mathbf{v}_0\omega^4 \cos \omega t = -\omega^2\mathbf{v}, \\
\mathbf{S} &= \mathbf{R}_0\omega^4 \cos \omega t + \mathbf{v}_0\omega^5 \sin \omega t = -\omega^2\mathbf{a} = \omega^4\mathbf{R}.
\end{aligned} \tag{20}$$

According to the Equation (20) and FIGURE 2 , it may be noted that:

• The velocity vector **v** and the position vector **R** point in two conjugate directions with respect to an ellipse, as **v** can be obtained from **R** by multiplying by ω and replacing ωt by $\omega t + \pi/2$.

• The velocity vector **v** is tangent to the path, as vectors that are drawn along conjugate diameters of an ellipse, given one such vector, the other vector is parallel to the tangent to the ellipse at the endpoint of the given vector (a property of an ellipse).

• The jerk vector **J** is also tangential to the path and directed oppositely of the velocity vector **v**.

• The normal component of the jerk vector **J** vanishes in elliptical orbital motion, as the acceleration vector **a** is directed towards the center.

• The snap vector **S** is directly oppositely of the acceleration vector **a** and takes the same direction of the position vector **R**.

4 | SNAP IN MOTION ALONG THE SPACE CURVES

In the kinematical decomposition for a particle moving along a space curve, discovered by Siacci, the acceleration vector can be expressed as the sum of two special oblique components in the osculating plane, and this concept developed by using an alternative approach in the rectifying plane[4]. Because of the foregoing, we investigate the snap vector in motion along the space curves by using the resolving of the snap vector. In our resolving, one component lies along the tangent line of the path, the second and the third component lie along with the line segments that pass through the particle and the foot of the perpendicular which starts from the origin point of the space to the instantaneous osculating and rectifying planes, respectively. The motion of an electron under a constant magnetic field and the motion of a particle along a logarithmic spiral curve are models chosen for the three-dimensional motion to show the several geometric properties of the snap vector.

Let P be a particle of mass ($m > 0$) moving along a space curve C, parameterized by the arc-length parameter s, in three-dimensional Euclidean space $\mathbb{E}^3$ under the influence of arbitrary forces. Choose an arbitrarily fixed origin O in the space and the position vector of P at time t denoted by $\mathbf{x}$. Let C be the oriented curve traced out by P, and let s be the arc-length parameter of the curve C corresponds to the time t. Then, the unit tangent, velocity, acceleration, jerk and snap vectors of the curve C can be expressed as

$$\mathbf{T} = \frac{d\mathbf{x}}{ds}, \tag{21}$$

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{ds}{dt}\mathbf{T}, \tag{22}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}, \tag{23}$$

$$\mathbf{J} = \frac{d\mathbf{a}}{dt} = \left[\frac{d^3s}{dt^3} - \kappa^2\left(\frac{ds}{dt}\right)^3\right]\mathbf{T} + \left[3\kappa\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^3\right]\mathbf{N} + \left[\kappa\tau\left(\frac{ds}{dt}\right)^3\right]\mathbf{B}, \tag{24}$$

$$\mathbf{S} = \frac{d\mathbf{J}}{dt} = S_T\mathbf{T} + S_N\mathbf{N} + S_B\mathbf{B}, \tag{25}$$

where

$$\begin{aligned}
S_T &= \left[\frac{d^4s}{dt^4} - 3\kappa\left(\frac{ds}{dt}\right)^2\left(2\kappa\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\right)\right],\\
S_N &= \left[4\kappa\frac{ds}{dt}\frac{d^3s}{dt^3} + 3\kappa\left(\frac{d^2s}{dt^2}\right)^2 + 6\frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\frac{d^2s}{dt^2} + \left(\frac{ds}{dt}\right)^4\left(\frac{d^2\kappa}{ds^2} - \kappa^3 - \tau^2\kappa\right)\right],\\
S_B &= \left[6\kappa\tau\left(\frac{ds}{dt}\right)^2\frac{d^2s}{dt^2} + \left(\frac{ds}{dt}\right)^4\left(2\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right)\right].
\end{aligned}$$

According to the FIGURE 3 , let P be a particle moves along the curve C in the 3D Euclidean space curve, and let π_1 and π_2 are the osculating and rectifying planes, respectively. B and Y are the foots of perpendicular line segments that are from the origin O to the osculation plane π_1 and rectifying plane π_2, respectively. e_r and e_{r^*} are the unit vectors in the directions of BP and YP, respectively. BZ and YK are the perpendicular line segments to tangent and binormal axes, respectively. $\mathbf{r}$ and $\mathbf{r^*}$ are the position vectors from the point P to B and Y, respectively. The position vector of the point P on the curve C on the Frenet-Serret basis may be expressed as

$$\mathbf{x} = q\mathbf{T} - p\mathbf{N} + b\mathbf{B}, \tag{26}$$

where

$$q = \langle \mathbf{x}, \mathbf{T}\rangle, \qquad -p = \langle \mathbf{x}, \mathbf{N}\rangle, \qquad b = \langle \mathbf{x}, \mathbf{B}\rangle. \tag{27}$$

On the osculating and rectifying planes π_1 and π_2, respectively, the positions $\mathbf{r}$ and $\mathbf{r^*}$ of the particle P may be expressed as

$$\mathbf{r} = -p\mathbf{N} + q\mathbf{T} \qquad \Rightarrow \qquad r^2 = \langle \mathbf{r}, \mathbf{r}\rangle = p^2 + q^2, \tag{28}$$

$$\mathbf{r^*} = b\mathbf{B} + q\mathbf{T} \qquad \Rightarrow \qquad r^{*2} = \langle \mathbf{r^*}, \mathbf{r^*}\rangle = b^2 + q^2, \tag{29}$$

where r and r^* is the lengths of $\mathbf{r}$ and $\mathbf{r^*}$, respectively. The angular momentum of the particle P about the origin O is obtained by the vector multiplication of the position vector of P and the linear momentum vector of P that defined by

$$\mathbf{H}_O = \mathbf{x} \times m\mathbf{v} = (q\mathbf{T} - p\mathbf{N} + b\mathbf{B}) \times m\frac{ds}{dt}\mathbf{T} = m\frac{ds}{dt}b\mathbf{N} + m\frac{ds}{dt}p\mathbf{B}. \tag{30}$$

Physically, the binormal and normal components of the angular momentum are non-zeros, i.e., p and b never vanish and hence r^2 and r^{*2} are non-zeros. Since $\mathbf{e}_r = \frac{1}{r}\mathbf{r}$, $\mathbf{e}_{\mathbf{r}^*} = \frac{1}{r^*}\mathbf{r^*}$, and by using (28) and (29) we get

$$\mathbf{N} = \frac{1}{p}(-r\mathbf{e}_\mathbf{r} + q\mathbf{T}) \qquad\qquad \mathbf{B} = \frac{1}{b}(r\mathbf{e}_{\mathbf{r}^*} - q\mathbf{T}). \tag{31}$$

Now, we can obtain the jerk and snap vectors by substitute (31) in (24) and (25) as

$$\mathbf{J} = T_t\mathbf{T} + T_r\mathbf{e}_\mathbf{r} + T_{\mathbf{r}^*}\mathbf{e}_{\mathbf{r}^*}, \tag{32}$$

$$\mathbf{S} = S_t\mathbf{T} + S_r\mathbf{e}_\mathbf{r} + S_{r^*}\mathbf{e}_{\mathbf{r}^*}, \tag{33}$$

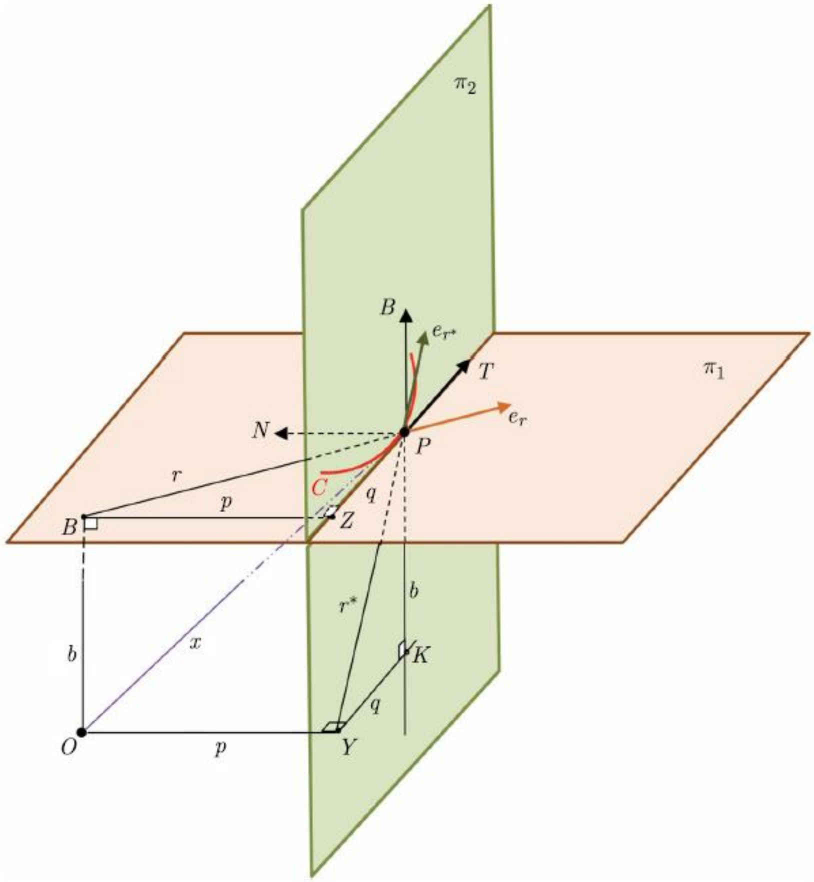

FIGURE 3 The resolution of the snap vector in 3D.

where T_t, T_r, T_{r^*}, S_t, S_r, and S_{r^*} are called tangential, first, second radial components of the jerk, tangential, first and second radial components of the snap, respectively. Their values are given by

$$\begin{aligned}
T_t &= \frac{d^3s}{dt^3} - \kappa^2\left(\frac{ds}{dt}\right)^3 + \frac{q}{p}\left(3\kappa\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{1}{\kappa}\frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^3\right) - \tau\frac{q}{b}\left(\frac{ds}{dt}\right)^3,\\
T_r &= \frac{r}{p}\left(-3\frac{ds}{dt}\frac{d^2s}{dt^2} - \frac{1}{\kappa}\frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^3\right),\\
T_{r^*} &= \tau\frac{r^*}{b}\left(\frac{ds}{dt}\right)^3,\\
S_t &= \frac{d^4s}{dt^4} - 3\kappa\left(\frac{ds}{dt}\right)^2\left(2\kappa\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\right) + \frac{q}{p}\left[4\kappa\frac{ds}{dt}\frac{d^3s}{dt^3} + 3\kappa\left(\frac{d^2s}{dt^2}\right)^2 + 6\frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\frac{d^2s}{dt^2} +\right.\\
&\quad \left.\left(\frac{ds}{dt}\right)^4\left(\frac{d^2\kappa}{ds^2} - \kappa^3 - \tau^2\kappa\right)\right] - \frac{q}{b}\left[6\kappa\tau\left(\frac{ds}{dt}\right)^2\frac{d^2s}{dt^2} + \left(\frac{ds}{dt}\right)^4\left(2\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right)\right],\\
S_r &= -\frac{r}{p}\left[4\kappa\frac{ds}{dt}\frac{d^3s}{dt^3} + 3\kappa\left(\frac{d^2s}{dt^2}\right)^2 + 6\frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\frac{d^2s}{dt^2} + \left(\frac{ds}{dt}\right)^4\left(\frac{d^2\kappa}{ds^2} - \kappa^3 - \tau^2\kappa\right)\right],\\
S_{r^*} &= \frac{r^*}{b}\left[6\kappa\tau\left(\frac{ds}{dt}\right)^2\frac{d^2s}{dt^2} + \left(\frac{ds}{dt}\right)^4\left(2\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right)\right].
\end{aligned}$$

Theorem 1. Let a particle P of mass m moves along a space curve C in three-dimensional Euclidean space, and assume that the normal and binormal components of its angular momentum vector never vanish. In this case, the snap vector of P can be expressed as in equation (33), where S_t, S_r, and S_{r^*} are the tangential, first and second components of the snap vector, respectively. S_t along the tangent line of C. S_r (S_{r^*}) lies along the line that passes through the particle P and the foot of the perpendicular that is from the origin of the space O to the osculating (rectifying) plane.

The proof of the next corollary is easy and similar to the proof of the corollary in[4] so, we omit the proof

Corollary 1. Let the particle P be restricted to a fixed plane and assume the binormal component of its angular momentum never vanishes. hence, the snap vector of the particle P can be expressed as

$$\mathbf{S} = S_t\mathbf{T} + S_r\mathbf{e_r}, \tag{34}$$

where

$$S_t = \frac{d^4s}{dt^4} - 3\kappa\left(\frac{ds}{dt}\right)^2\left(2\kappa\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\right) + \frac{q}{p}\left[4\kappa\frac{ds}{dt}\frac{d^3s}{dt^3} + 3\kappa\left(\frac{d^2s}{dt^2}\right)^2 + 6\frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\frac{d^2s}{dt^2} + \left(\frac{ds}{dt}\right)^4\left(\frac{d^2\kappa}{ds^2} - \kappa^3\right)\right],$$
$$S_r = -\frac{r}{p}\left[4\kappa\frac{ds}{dt}\frac{d^3s}{dt^3} + 3\kappa\left(\frac{d^2s}{dt^2}\right)^2 + 6\frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\frac{d^2s}{dt^2} + \left(\frac{ds}{dt}\right)^4\left(\frac{d^2\kappa}{ds^2} - \kappa^3\right)\right].$$

4.1 | Space motion models

In this section, the motion of an electron under a constant magnetic field and the motion of a particle along a logarithmic spiral curve are models chosen for the three-dimensional motion to show the several geometric properties of the snap vector.

4.1.1 | Motion of an electron along a right-handed circular helix

Consider an electron with electrical charge $-e$ and mass m moves along a right-handed circular helix lying on a cylinder of radius R under the constant magnetic field $(0, 0, B)$ along z-axis. Let α be the angle of helix and determined by $\tan\alpha = R\omega/v_z$ and the helix axis is the z-axis. The position vector of this electron may be expressed as

$$\mathbf{x} = (R\ \cos\omega t, R\ \sin\omega t, v_z t), \tag{35}$$

where $\omega = eB/m$ and R,v_z are positive constants. The velocity, acceleration, jerk ,and snap vectors may be expressed as

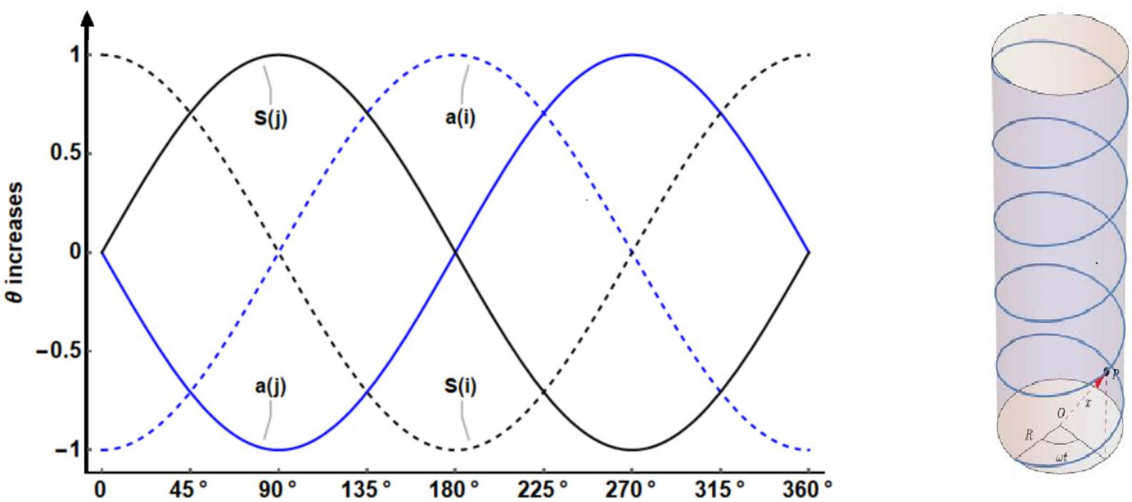

FIGURE 4 An electron P moves along the path of a right-handed helix on a cylinder under a constant magnetic field.

$$\begin{aligned}
\mathbf{v} &= (-R\omega\ \sin\omega t, R\omega\ \cos\omega t, v_z),\\
\mathbf{a} &= (-R\omega^2\ \cos\omega t, -R\omega^2\ \sin\omega t, 0),\\
\mathbf{J} &= (Rw^3\ \sin\omega t, -R\omega^3\ \cos\omega t, 0),\\
\mathbf{S} &= (R\omega^4\ \cos\omega t, R\omega^4\ \sin\omega t, 0) = -\omega^2\mathbf{a}.
\end{aligned} \tag{36}$$

See FIGURE 4 . From the velocity vector, we get

$$dx = -R\omega\ \sin\omega t\ dt, \qquad dy = R\omega\ \cos\omega t\ dt, \qquad dz = v_z\ dt \tag{37}$$

By using the relation $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ we get

$$\frac{ds}{dt} = \sqrt{R^2\omega^2 + v_z^2}, \qquad \frac{d^2 s}{dt^2} = \frac{d^3 s}{dt^3} = \frac{d^4 s}{dt^4} = 0. \tag{38}$$

Let $s = s(t) = \beta t$, where $\beta = \sqrt{R^2\omega^2 + v_z^2}$, be the arc-length of the oriented curve traced out by the electron. Now, we reparametrize the position vector as follows

$$\gamma = \gamma(s) = \Big(R \cos\frac{\omega s}{\beta}, R \sin\frac{\omega s}{\beta}, \frac{v_z s}{\beta}\Big). \tag{39}$$

The Frenet-Serret apparatus are given from (3), (5) and (39) by

$$\begin{aligned}
\mathbf{T}(s) &= \Big(-\sin\alpha\ sin\frac{\omega s}{\beta}, \sin\alpha\ cos\frac{\omega s}{\beta}, \cos\alpha\Big),\\
\mathbf{N}(s) &= \Big(-\cos\frac{\omega s}{\beta}, -\sin\frac{\omega s}{\beta}, 0\Big),\\
\mathbf{B}(s) &= \Big(\cos\alpha\ sin\frac{\omega s}{\beta}, \cos\alpha\ \cos\frac{\omega s}{\beta}, \sin\alpha\Big),\\
\kappa(s) &= \frac{R}{R^2 + v_z^2/\omega^2} = \frac{\sin^2\alpha}{R},\\
\tau(s) &= \frac{v_z/\omega}{R^2 + v_z^2/\omega^2} = \frac{\omega\ \cos^2\alpha}{v_z}.
\end{aligned} \tag{40}$$

Now, we calculate the position vector components of $\mathbf{x}$ from (27), (39) and (40) as

$$q = \frac{s v_z\ \cos\alpha}{\beta} = t v_z\ \cos\alpha, \qquad p = R, \qquad b = \frac{s v_z\ \sin\alpha}{\beta} = t v_z\ \sin\alpha. \tag{41}$$

And the components of the position vector of $\mathbf{r}$ and $\mathbf{r}^*$ from (28) and (29) are

$$r = \sqrt{R^2 + \frac{s^2 v_z^2\ \cos^2\alpha}{\beta^2}} = \sqrt{R^2 + t^2 v_z^2\ \cos^2\alpha}, \qquad r^* = \frac{s v_z}{\beta} = t v_z \tag{42}$$

Consequantly, the components of the sanp vector are given by directly substitution in (33) as

$$S_t = -t v_z \omega^4 \cos\alpha = \frac{-t v_z^2 \omega^4}{\sqrt{R^2\omega^2 + v_z^2}}, \qquad S_r = \omega^4\sqrt{R^2 + t^2 v_z^2 \cos^2\alpha}, \qquad S_{r^*} = 0. \tag{43}$$

By applying theorem (4.1).

4.1.2 | Motion of a particle along a logarithmic spiral curve

Let a particle P moives along a logarithmic spiral curve which can be written in Cartisian coordiantes as

$$\xi(t) = (e^{\omega t}\ \cos\omega t, e^{\omega t}\ \sin\omega t, 0). \tag{44}$$

The position vector of a particle P is given in the MKS unit system as

$$\mathbf{x} = (e^{\omega t}\ \cos\omega t, e^{\omega t}\ \sin\omega t, 0), \tag{45}$$

where ω is the angular frequancy, and t is the time. The velocity, acceleration, jerk, and snap vectors are given by

$$\begin{aligned}
\mathbf{v} &= \omega e^{\omega t}(\cos\omega t - \ \sin\omega t, \sin\omega t + \cos\omega t, 0),\\
\mathbf{a} &= 2\omega^2 e^{\omega t}(-\sin\omega t, \cos\omega t, 0),\\
\mathbf{J} &= 2\omega^3 e^{\omega t}(-\sin\omega t - \cos\omega t, \cos\omega t - \sin\omega t, 0),\\
\mathbf{S} &= -4\omega^4 e^{\omega t}(\cos\omega t, \sin\omega t, 0).
\end{aligned} \tag{46}$$

Notation: In this example, since the motion is planar, we know that $\tau = 0$, and the second radial component of the snap vector vanish. According to FIGURE 5 , in the direction of i, the snap vector could not be seen in the oscillations upward and downward of the acceleration curve which contradicts our predictions in the introduction. So, we predict that another component

of the snap vector will vanish. Let us complete our solution to check that prediction.
From the velocity vector we get

$$dx = \omega e^{\omega t}(\cos\omega t - \sin\omega t)\, dt, \qquad dy = \omega e^{\omega t}(\sin\omega t + \cos\omega t)dt, \qquad dz = 0. \tag{47}$$

By using the relation $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ we get

$$\frac{ds}{dt} = \sqrt{2}\omega e^{\omega t}, \qquad \frac{d^2s}{dt^2} = \sqrt{2}\omega^2 e^{\omega t}, \qquad \frac{d^3s}{dt^3} = \sqrt{2}\omega^3 e^{\omega t}, \qquad \frac{d^4s}{dt^4} = \sqrt{2}\omega^4 e^{\omega t}. \tag{48}$$

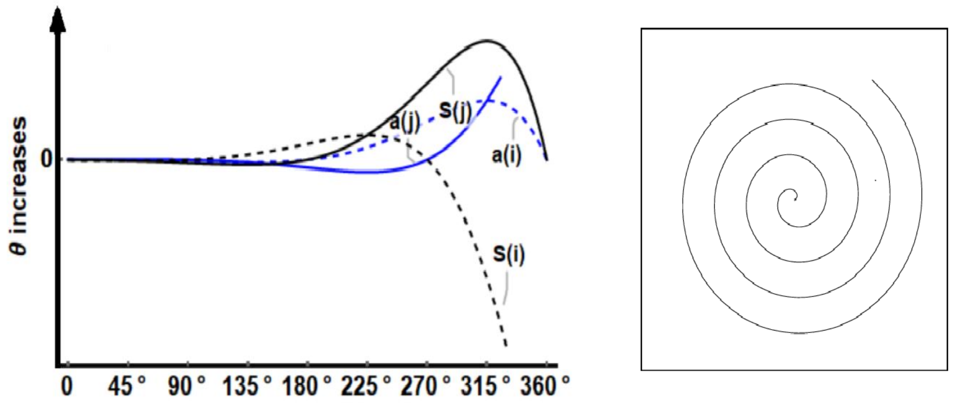

FIGURE 5 The logarithmic spiral, where $\omega = 1$ and $0 < t < 1$ in MKS unit system.

Let $s = s(t) = \sqrt{2}e^{\omega t} - \sqrt{2}$ be the arc-length of the logarithmic spiral curve. Now, we reparametrize the position vector as follows

$$\xi^*(s) = \Big(\frac{s}{\sqrt{2}} + 1\Big)\Big(\cos\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big), \sin\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big), 0\Big). \tag{49}$$

The Frenet-Serret apparatus are given from (3), (5) and (49) by

$$\begin{aligned}
\mathbf{T} &= \frac{1}{\sqrt{2}}\Big(\cos\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big) - \sin\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big), \cos\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big) + \sin\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big), 0\Big),\\
\mathbf{N} &= \frac{1}{\sqrt{2}}\Big(-\cos\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big) - \sin\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big), \cos\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big) - \sin\, ln\Big(\frac{s}{\sqrt{2}} + 1\Big), 0\Big),\\
\mathbf{B} &= (0, 0, 1),\\
\kappa &= \frac{1}{s + \sqrt{2}},\\
\tau &= 0.
\end{aligned} \tag{50}$$

Now, we calculate the position vector components of **x** as

$$q = \frac{s + \sqrt{2}}{2} = p, \qquad b = 0. \tag{51}$$

And the components of the position vector of **r** and $\mathbf{r}^*$ are

$$r = \frac{s + \sqrt{2}}{\sqrt{2}}, \qquad r^* = 0. \tag{52}$$

Consequantly, the components of the sanp vector are given as

$$S_t = 0, \qquad S_r = -4\omega^4 e^{\omega t}, \qquad S_{r^*} = 0. \tag{53}$$

By applying theorem (4.1) ,or simply, by using corollary (4.1) for the planar motion.

Now, let us ask a question, what is the importance of the above notation?.
Things would rarely be designed to have snap since we like our actuation to be governed by simple controls. It will be fine to have zero snap components for the system so, we can check the ability of the appearance of the snap vector by graph before designing our model. The destructive effect of jerk and snap is discussed in[2].

5 | CONCLUSION

In this paper, we investigated, for the planar motion, the snap vector that resolved into tangential-normal and radial-transverse components. The oscillation of a simple pendulum and central force proportional to distance are chosen as models for the plane motion to show the several geometric properties of the snap vector. Furthermore, we investigated the snap vector in the three-dimensional Euclidean space by considering a particle moving on a curve and resolve its snap vector along the tangential direction, the radial direction in the osculating plane, and the other radial direction in the rectifying plane, respectively. The motion of an electron under a constant magnetic field and the motion of a particle along a logarithmic spiral curve are models chosen for the three-dimensional motion to show the several geometric properties of the snap vector.

This studying is more general, efficient and a new contribution to the field. In the future, it may be needed for some special applications in studying dynamical systems, astronomy as harmonic motions, Kepler laws, etc., and in many areas of science.

Author contributions

Hoda K. Elsayied, previewed the paper and was a major contributor in writing that manuscript, Abdelrhman M. Tawfiq, collected the data and perform the calculations and A. Elsharkawy, previewed the calculations and was a major contributor in writing that manuscript.

Financial disclosure

Not applicable.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

SUPPORTING INFORMATION

Availability of data and material

Available on different scientific websites as google scholar, research gate, scopus, etc.

Code availability

TeXstudio (latex) and mathematica program.

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