Sign-changing solutions of critical quasilinear Kirchhoff-Schr\"{o}dinger-Poisson system with logarithmic nonlinearity

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Abstract

In the present paper, we deal with the following Kirchhoff-Schr\"{o}dinger-Poisson system with logarithmic and critical nonlinearity: \begin{equation*} \begin{array}{ll} \left \{ \begin{array}{ll} \ds\B(a+b\int_\Omega|\nabla u|^2\mathrm{d}x \B)\Delta u+V(x)u-\frac{1}{2}u\Delta (u^2)+\phi u=\lambda |u|^{q-2}u\ln|u|^2+|u|^4u, &x\in \Omega, \\ -\Delta \phi=u^2,& x\in \Omega, \\ u=0,& x\in \R^3\setminus\Omega, \end{array} \right . \end{array} \end{equation*} where $\lambda,b>0,a>\frac{1}{4},4

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Abstract

In the present paper, we deal with the following Kirchhoff-Schrödinger-Poisson system with logarithmic and critical nonlinearity:

$$\begin{cases} \Big(a+b\displaystyle\int_{\Omega}|\nabla u|^2\mathrm{d}x\Big)\Delta u+V(x)u-\dfrac{1}{2}u\Delta(u^2)+\phi u=\lambda|u|^{q-2}u\ln|u|^2+|u|^4u, & x\in\Omega,\\ -\Delta\phi=u^2, & x\in\Omega,\\ u=0, & x\in\mathbb{R}^3\setminus\Omega, \end{cases}$$

where $\lambda, b>0, a>\frac{1}{4}, 4<q<6$, Ω is a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary. Combining constraint variational methods and perturbation method, we prove that the above problem has a least energy sign-changing solution u_0 which has precisely two nodal domains. Moreover, we show that the energy of u_0 is strictly larger than two times the ground state energy.

Keywords: Quasilinear Kirchhoff-Schrödinger-Poisson; Critical problem; Logarithmic nonlinearity.

Mathematics Subject Classification: 35A15, 35J60, 47G20.

1 Introduction and main results

In this paper, we consider the existence of a least energy sign-changing solution of the following quasilinear Kirchhoff-Schrödinger-Poisson type systems:

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$$\begin{cases} -(a+b\int_{\Omega}|\nabla u|^2\mathrm{d}x)\Delta u+V(x)u-\dfrac{1}{2}u\Delta(u^2)+\phi u=\lambda|u|^{q-2}u\ln|u|^2+|u|^4u, & x\in\Omega,\\ -\Delta\phi=u^2, & x\in\Omega,\\ u=0, & x\in\mathbb{R}^3\setminus\Omega. \end{cases} \tag{1.1}$$

After the pioneer work of Lions [12], some researchers began to pay attention to the following Kirchhoff Dirichlet problem:

$$\begin{cases} -\Big(a+b\int_{\Omega}|\nabla u|^2\mathrm{d}x\Big)\Delta u=f(x,u), \qquad x\in\Omega,\\ u|_{\partial\Omega}=0. \end{cases} \tag{1.2}$$

Problem (1.2) is related to a model firstly proposed by Kirchhoff [7] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings, which is related to the stationary analogue of the equation:

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2\mathrm{d}x\Big)\Big(\frac{\partial u}{\partial x}\Big)^2=0.$$

Because problem (1.2) has nonlocal term $(\int_{\Omega}|\nabla u|^2\mathrm{d}x)\Delta u$, there are some difficulties in the study of the nonlocal problems by means of variational methods. In recently years, many studied about positive solutions, multiple solutions, bound state solutions, semiclassical state solutions and sign-changing solutions for (1.2) can be found in [1, 4, 16, 18, 26, 22, 25] and the references therein.

By using the constraint variational methods and the quantitative deformation lemma, Wang [5] obtained the existence of at least energy sign-changing solutions for the following Kirchhoff-type equation with critical growth:

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u=|u|^4u+\lambda f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where λ is large enough and f satisfy suitable conditions. Lately, Li and Wang [27] studied ground state sign-changing solutions for Kirchhoff equations with logarithmic nonlinearity:

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=|u|^{p-2}u\ln u^2, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $4<p<2^*$, they used constraint variational method, topological degree theory and some new energy estimate inequalities to prove the existence of ground state solutions and ground state sign-changing solutions. Recently, Liang [20] got a more general result about problem (1.4) with critical growth.

Nevertheless, there are relatively few studies on quasilinear Schrödinger-Poisson system. Illner [19] first studied quasilinear Schrödinger-Poisson system. This quasilinear version of the nonlinear Schrödinger equation arises in several models of different physical phenomena, such as superfluid films, plasma physics, condensed matter theory, etc. (see [[2, 14]]).

By using the methods of perturbation and the Mountain Pass theorem, Feng [23] proved the existence of non-trivial solution to the following quasilinear Schrödinger-Poisson equations:

$$\begin{cases} -\Delta u + V(x)u + \phi u - \frac{1}{2}u\Delta(u^2) = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $V \in C(\mathbb{R}^3, \mathbb{R}^1), \lim_{|x|\to\infty} V(x) = \infty$ and $V(x) \geq m > 0$ for some constant m to overcome the lack of compactness.

Lately, under suitable condition of f, Chen and Tang [11] applied some new analytical techniques and non-Nehari manifold method investigated the existence of ground state sign-changing solutions for the following quasilinear Schrödinger equations with a Kirchhoff-type perturbation:

$$\begin{aligned} &\left(1 + b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 \, \mathrm{d}x\right) \left[-\operatorname{div}\left(g^2(u)\nabla u\right) + g(u)g'(u)|\nabla u|^2\right] \\ &+V(x)u = K(x)f(u) \end{aligned} \tag{1.6}$$

As we know, Figueiredo and Siciliano in [8, 9] paid close attention to two different critical systems with 4-Laplacian operator in $\mathbb{R}^3$ and a bounded domain in $\mathbb{R}^2$, they obtained the existence and asymptotic behavior of nontrivial solutions. Wang [15] investigated nontrivial solutions of quasilinear Schrödinger-Kirchhoff-type equation with radial potentials. Fu and Zhu [6] considered the multiple solutions to a class of generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation. However, there are relatively few achievements on the so called quasilinear Kirchhoff-Schrödinger-Poisson type systems with critical growth, furthermore, few studies have included logarithmic terms about quasilinear problem. It is quite natural to ask: what is going to happen with logarithmic nonlinear terms for the critical quasilinear Kirchhoff Schrödinger-Poisson system? In this paper, we will show that there exists a least energy sign-changing solution.

According to the Lax-Milgram Theorem, for $u \in H$, there is a unique $\phi_u \in D^{1,2}(\Omega)$ that satisfies

$$-\Delta\phi_u = u^2.$$

The function is represented by

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}\mathrm{d}y, \ x \in \mathbb{R}^3.$$

Therefore, system (1.1) has an equivalent form

$$\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) + \phi u = \lambda|u|^{q-2}u\ln|u|^2 + |u|^4 u.$$

So, the functional associated with system (1.1) can be defined by

$$\begin{aligned} I_b^\lambda(u) = \; & \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|_1^4 + \frac{1}{2}\int_\Omega |\nabla u|^2 u^2 \mathrm{d}x + \frac{1}{4}\int_\Omega \phi_u u^2 \mathrm{d}x \\ & + \frac{2\lambda}{q^2}\int_\Omega |u|^q \mathrm{d}x - \frac{\lambda}{q}\int_\Omega |u|^q \ln|u|^2 \mathrm{d}x - \frac{1}{6}\int_\Omega |u|^6 \mathrm{d}x. \end{aligned}$$

It is easy to see that $I_b^\lambda(u) \in \mathcal{C}^1(H, \mathbb{R})$. Moreover, for any $u, \varphi \in H$, we have

$$\begin{aligned} \langle (I_b^\lambda)'(u), \varphi\rangle = \; & \int_\Omega a\nabla u \nabla\varphi \mathrm{d}x + b\int_\Omega |\nabla u|^2 \mathrm{d}x \int_\Omega \nabla u\nabla\varphi \mathrm{d}x + \int_\Omega V(x)uv\mathrm{d}x + \int_\Omega \phi_u u\varphi \mathrm{d}x \\ & + \int_\Omega (|\nabla u|^2 u\varphi + |u|^2\nabla u\nabla\varphi)\mathrm{d}x - \lambda\int_\Omega |u|^{q-2}u\varphi \ln|u|^2 \mathrm{d}x - \int_\Omega |u|^4 u\varphi \mathrm{d}x. \end{aligned}$$

There are some difficulties in applying variational methods directly to the problem (1.1) because of the quasilinear term $\int_\Omega u^2|\nabla u|^2$, it seems impossible to find a suitable space in which the corresponding functional possesses both smoothness and compactness properties. On the other hand, it is difficult to apply the dual approach since problem (1.1) exists nonlocal term, In order to overcome the lack of compactness caused by the critical term, we would employ the method from [21, 24]. In fact, we will use the approximation method by adding a 4-Laplacian operator, i.e. we consider the sign-changing critical point of the perturbed functional:

$$I_{b,\mu}^\lambda(u) = I_b^\lambda(u) + \frac{\mu}{4}\int_\Omega \left(|\nabla u|^4 + u^4\right)\mathrm{d}x, \tag{1.7}$$

where $\mu \in (0, 1]$. Then by using the approximation technique, we get the existence of sign-changing solution of problem (1.1).

We first try to seek a minimizer of energy functional $I_{b,\mu}^\lambda$ over the following constraint:

$$\mathcal{M}_{b,\mu}^\lambda = \{u \in H, u^\pm \neq 0 \;\text{ and }\; \langle (I_{b,\mu}^\lambda)'(u), u^+\rangle = \langle (I_{b,\mu}^\lambda)'(u), u^-\rangle = 0\},$$

and consider a minimization problem of $I_{b,\mu}^\lambda$ on $\mathcal{M}_{b,\mu}^\lambda$.
Here

$$u(x) = u^+(x) + u^-(x), \;\; u^+(x) = \max\{u(x),\ 0\} \;\text{ and }\; u^-(x) = \min\{u(x),\ 0\}.$$

We will prove that the minimizer is a critical point of $I_{b,\mu}^\lambda$ and obtain the convergence property as $\mu \to 0$, thus we get the least energy sign-changing solution of problem (1.1). Since problem (1.1) has nonlocal term and logarithmic nonlinearity, it is difficult to prove $\mathcal{M}_{b,\mu}^\lambda \neq \emptyset$. Inspired by [5], we combine modified Miranda's theorem [3], quantitative lemma, topological degree theory and perturbation method to prove that the minimizer of the constrained problem is also a least sign-changing solution.

Our main results of this paper are as follows:

Theorem 1.1. *There exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, problem (1.1) possess one least energy sign-changing solution u_0 which has precisely two nodal domains.*

Theorem 1.2. *There exists $\lambda^{**} > 0$ such that for all $\lambda > \max\{\lambda^*, \lambda^{**}\}$, $c^* := \inf_{u\in\mathcal{N}_b^\lambda} I_b^\lambda(u) > 0$ is achieved either by a positive or a negative function and $I_b^\lambda(u_0) > 2c^*$, where $\mathcal{N}_b^\lambda = \{u \in H\backslash\{0\} | \langle (I_b^\lambda)'(u), u\rangle = 0\}$ and u_0 is the least energy sign-changing solution obtained in Theorem 1.1.*

2 Framework

In this section, we introduce the variational framework associated with problem (1.1). We first describe the working space. Let $L^p(\Omega)$ be the usual Lebesgue space with the norm $\|u\|_p = \left(\int_\Omega |u|^p \mathrm{d}x\right)^{1/p}$ and $H^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm:

$$\|u\|_{H^1}^2 = \int_\Omega \left(|\nabla u|^2 + u^2\right) \mathrm{d}x.$$

Moreover, we denote the completion of $C_0^\infty(\Omega)$ with respect to the norm:

$$\|u\|_1^2 := \|u\|_{D^{1,2}}^2 = \int_\Omega |\nabla u|^2 \mathrm{d}x.$$

We assume that $V \in C(\Omega, \mathbb{R}^+)$, where

$$H = \left\{u \in H^1(\Omega) : \int_\Omega V(x) u^2 \mathrm{d}x < \infty\right\},$$

which is a Hilbert space endowed with the norm:

$$\|u\|^2 = \int_\Omega (a|\nabla u|^2 + V(x)u^2) \mathrm{d}x.$$

In order to use perturbation method, we will use the space

$$E = W^{1,4}(\Omega) \cap H(\Omega),$$

where $W^{1,4}(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_W := \left(\int_\Omega |\nabla u|^4 + u^4 \mathrm{d}x\right)^{\frac{1}{4}}.$$

Moreover, Hölder inequality implies that

$$\int_\Omega |\nabla u|^2 u^2 \mathrm{d}x \leq \left(\int_\Omega |\nabla u|^4\right)^{\frac{1}{2}} \left(\int_\Omega |u|^4 \mathrm{d}x\right)^{\frac{1}{2}} \leq \|u\|_W^4.$$

So the norm of E is denoted by

$$\|u\|_E = \left(\|u\|_W^2 + \|u\|^2\right)^{\frac{1}{2}}.$$

Since $I_{b,\mu}^\lambda(u) = I_b^\lambda(u) + \frac{\mu}{4}\int_\Omega (|\nabla u|^4 + u^4)\,\mathrm{d}x$, we can easily get $I_{b,\mu}^\lambda \in \mathcal{C}^1(E, \mathbb{R})$ for all $\varphi \in E$ and

$$\begin{aligned}\langle (I_{b,\mu}^\lambda)'(u), \varphi\rangle = \;& \mu \int_\Omega \left(|\nabla u|^2 \nabla u \nabla \varphi + |u|^2 u \varphi\right) \mathrm{d}x + \int_\Omega a \nabla u \nabla \varphi \mathrm{d}x + b\int_\Omega |\nabla u|^2 \mathrm{d}x \int_\Omega \nabla u \nabla \varphi \mathrm{d}x \\ & + \int_\Omega V(x) u\varphi \mathrm{d}x + \int_\Omega \phi_u u \varphi \mathrm{d}x + \int_\Omega (|\nabla u|^2 u\varphi + |u|^2 \nabla u \nabla \varphi) \mathrm{d}x \\ & - \lambda \int_\Omega |u|^{q-2} u \varphi \ln |u|^2 \mathrm{d}x - \int_\Omega |u|^4 u \varphi \mathrm{d}x.\end{aligned}$$

It is noticed that if $u^{\pm} \neq 0$, we have

$$I_{b,\mu}^{\lambda} = I_{b,\mu}^{\lambda}(u^+) + I_{b,\mu}^{\lambda}(u^-) + \frac{b}{2}\|u^+\|_1^2\|u^-\|_1^2 + \frac{1}{4}\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x + \frac{1}{4}\int_\Omega \phi_{u^+}(u^-)^2\mathrm{d}x,$$
$$\langle (I_{b,\mu}^{\lambda})'(u), u^+\rangle = \langle (I_{b,\mu}^{\lambda})'(u^+), u^+\rangle + b\|u^+\|_1^2\|u^-\|_1^2 + \int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x,$$
$$\langle (I_{b,\mu}^{\lambda})'(u), u^-\rangle = \langle (I_{b,\mu}^{\lambda})'(u^-), u^-\rangle + b\|u^+\|_1^2\|u^-\|_1^2 + \int_\Omega \phi_{u^+}(u^-)^2\mathrm{d}x,$$

Our goal in this paper is to seek the least energy sign-changing solutions of problem (1.1).

Now, fixed $u \in E$ with $u^{\pm} \neq 0$, we denote $\psi_u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and mapping $T_u : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ by

$$\psi_u(\alpha,\beta) = I_{b,\mu}^{\lambda}(\alpha u^+ + \beta u^-), \tag{2.1}$$

and

$$T_u(\alpha,\beta) = (\langle (I_{b,\mu}^{\lambda})'(\alpha u^+ + \beta u^-), \alpha u^+\rangle,\ \langle (I_{b,\mu}^{\lambda})'(\alpha u^+ + \beta u^-), \beta u^-\rangle). \tag{2.2}$$

Lemma 2.1. For any $u \in H$, according to [13], we have

(1) there exist $C > 0$ such that

$$\int_\Omega \phi_u u^2 \mathrm{d}x \leq C\|u\|_1^4,\ \forall u \in H;$$

(2) $\phi_u > 0,\ \forall u \in H$;

(3) $\phi_{\tau u} = \tau^2\phi_u,\ \forall \tau > 0$ and $u \in H$;

(4) If $u_n \rightharpoonup u$ in H, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.

3 Some technical lemmas

In this section, we give some sueful lemmas as which are critical to the proof of Theorem 1.1.

Lemma 3.1. For any $u \in E$ with $u^{\pm} \neq 0$, then there exists a unique maximum point pair (α_u, β_u) of the function ψ_u such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_{b,\mu}^{\lambda}$.

Proof. Our proof will be divided into three steps.

Step 1: For any $u \in E$ with $u^{\pm} \neq 0$, in the following, we will prove the existence of α_u and β_u.

From sample computation, we have

$$\lim_{\tau \to 0} \frac{|\tau|^{q-1}\ln|\tau|^2}{|\tau|} = 0 \quad \text{and} \quad \lim_{\tau \to \infty} \frac{|\tau|^{q-1}\ln|\tau|^2}{|\tau|^{r-1}} = 0 \tag{3.1}$$

for all $r \in (q, 6)$. Then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|\tau|^{q-1} \ln |\tau|^2 \leq \varepsilon|\tau| + C_\varepsilon|\tau|^{r-1}. \tag{3.2}$$

Since $4 < q < 6$, it follows from (3.2) and the Sobolev embedding theorem that

$$\begin{aligned}
&\langle (I_{b,\mu}^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle \\
&= \mu\alpha^4\|u^+\|_W^4 + \alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|_1^4 + b\alpha^2\beta^2\|u^+\|_1^2\|u^-\|_1^2 + 2\alpha^2\int_\Omega |\nabla u^+|^2|u^+|^2 \mathrm{d}x \\
&\quad + \alpha^4\int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x + \alpha^2\beta^2\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x - \lambda\int_\Omega |\alpha u^+|^q \ln|\alpha u^+|^2\mathrm{d}x - \alpha^6\int_\Omega |u^+|^6\mathrm{d}x \\
&\geq \alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|_1^4 + b\alpha^2\beta^2\|u^+\|_1^2\|u^-\|_1^2 + 2\alpha^2\int_\Omega |\nabla u^+|^2|u^+|^2\mathrm{d}x \\
&\quad + \alpha^4\int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x + \alpha^2\beta^2\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x - \lambda\alpha^2\varepsilon\int_\Omega |u^+|^2\mathrm{d}x \\
&\quad - \lambda C_\varepsilon\alpha^r\int_\Omega |u^+|^r\mathrm{d}x - \alpha^6\int_\Omega |u^+|^6\mathrm{d}x \\
&\geq \alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|_1^4 - \lambda\alpha^2\varepsilon C_1\|u^+\|^2 - \lambda C_\varepsilon\alpha^r C_2\|u^+\|^r - C_3\alpha^6\|u^+\|^6 \\
&= (1-\lambda\varepsilon C_1)\alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|_1^4 - \lambda C_\varepsilon\alpha^r C_2\|u^+\|^r - C_3\alpha^6\|u^+\|^6,
\end{aligned}$$

where C_1, C_2, C_3 are positive constants. Choosing $\varepsilon > 0$ such that $1 - \lambda\varepsilon C_1 > 0$. Since $4 < r < 6$, we have $\langle (I_{b,\mu}^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle > 0$ for α small enough and all $\beta > 0$.

Similarly, we obtain that $\langle (I_{b,\mu}^\lambda)'(\alpha u^+ + \beta u^-), \beta u^-\rangle > 0$ for β small enough and all $\alpha > 0$.

Therefore, there exists $\alpha_1 > 0$ such that

$$\langle (I_{b,\mu}^\lambda)'(\alpha_1 u^+ + \beta u^-), \alpha_1 u^+\rangle > 0, \qquad \langle (I_{b,\mu}^\lambda)'(\alpha u^+ + \alpha_1 u^-), \alpha_1 u^-\rangle > 0 \tag{3.3}$$

for all $\alpha,\ \beta > 0$.

On the other hand, since $u^+ \neq 0$, there exists a constants $\theta > 0$ such that meas$\{x \in \Omega, u^+ > \theta\}$. Since $q > 4$, we deduce that, for any $M > 1$, there exists $T > 0$ such that $\frac{|\tau|^q \ln|\tau|^2}{\tau^4} > M$ for all $\tau > T$. Therefore, for $\alpha > \frac{T}{\theta}$, we have

$$\lambda\int_\Omega |\alpha u^+|^q \ln|\alpha u^+|^2\mathrm{d}x \geq M\alpha^4\int_{\{u^+>\theta\}} (u^+)^4\mathrm{d}x.$$

We can choose $\alpha = \alpha_2^* > \alpha_1$, if $\beta \in [\alpha_1, \alpha_2^*]$ and α_2^* is large enough, it follows that

$$\begin{aligned}
&\langle (I_{b,\mu}^\lambda)'(\alpha_2^* u^+ + \beta u^-), \alpha_2^* u^+\rangle \\
&\leq \mu(\alpha_2^*)^4\|u^+\|_W^4 + (\alpha_2^*)^2\|u^+\|^2 + b(\alpha_2^*)^4\|u^+\|_1^4 + b(\alpha_2^*)^2\beta^2\|u^+\|_1^2\|u^-\|_1^2 \\
&\quad + 2(\alpha_2^*)^4\int_\Omega |\nabla u^+|^2|u^+|^2\mathrm{d}x + (\alpha_2^*)^4\int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x \\
&\quad + (\alpha_2^*)^2\beta^2\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x - M(\alpha_2^*)^4\int_{\{u^+>\theta\}} (u^+)^4\mathrm{d}x - (\alpha_2^*)^6\int_\Omega |u^+|^6\mathrm{d}x \\
&\leq 0.
\end{aligned}$$

Similarly, we have that

$$\langle (I_{b,\mu}^{\lambda})'(\alpha u^{+}+\alpha_2^{*}u^{-}),\alpha_2^{*}u^{-}\rangle \leq 0.$$

Let $\alpha_2>\alpha_2^{*}$ be large enough, we obtain that, for all $\alpha,\beta\in[\alpha_1,\alpha_2]$, we have

$$\langle (I_{b,\mu}^{\lambda})'(\alpha_2 u^{+}+\beta u^{-}),\alpha_2 u^{+}\rangle < 0, \quad \langle (I_{b,\mu}^{\lambda})'(\alpha u^{+}+\alpha_2 u^{-}),\alpha_2 u^{-}\rangle < 0. \tag{3.4}$$

Combining (3.3) and (3.4) with Miranda's theorem, there exist $(\alpha_u,\beta_u)\in(0,+\infty)\times(0,+\infty)$ such that $T_u(\alpha_u,\beta_u)=(0,0)$, i.e, $\alpha_u u^{+}+\beta_u u^{-}\in\mathcal{M}_{b,\mu}^{\lambda}$.

Step 2: In this step, we prove the uniqueness of the pair (α_u,β_u).

Case 1: $u\in\mathcal{M}_{b,\mu}^{\lambda}$.

If $u\in\mathcal{M}_{b,\mu}^{\lambda}$, we have

$$\begin{aligned}
&\mu\|u^{+}\|_W^4+\|u^{+}\|^2+b\|u^{+}\|_1^4+b\|u^{+}\|_1^2\|u^{-}\|_1^2+2\int_\Omega|\nabla u^{+}|^2|u^{+}|^2\mathrm{d}x\\
&\quad+\int_\Omega\phi_{u^{+}}(u^{+})^2\mathrm{d}x+\int_\Omega\phi_{u^{-}}(u^{+})^2\mathrm{d}x\\
&=\lambda\int_\Omega|u^{+}|^q\ln|u^{+}|^2\mathrm{d}x+\int_\Omega|u^{+}|^6\mathrm{d}x
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
&\mu\|u^{-}\|_W^4+\|u^{-}\|^2+b\|u^{-}\|_1^4+b\|u^{+}\|_1^2\|u^{-}\|_1^2+2\int_\Omega|\nabla u^{-}|^2|u^{-}|^2\mathrm{d}x\\
&\quad+\int_\Omega\phi_{u^{-}}(u^{-})^2\mathrm{d}x+\int_\Omega\phi_{u^{+}}(u^{-})^2\mathrm{d}x\\
&=\lambda\int_\Omega|u^{-}|^q\ln|u^{-}|^2\mathrm{d}x+\int_\Omega|u^{-}|^6\mathrm{d}x.
\end{aligned}\tag{3.6}$$

In the following we show that $(\alpha_u,\beta_u)=(1,1)$.

Let (α_u,β_u) be a pair of numbers such that $\alpha_u u^{+}+\beta_u u^{-}\in\mathcal{M}_{b,\mu}^{\lambda}$ with $0<\alpha_u\leq\beta_u$. Hence, one has that

$$\begin{aligned}
&(\alpha_u)^4\mu\|u^{+}\|_W^4+(\alpha_u)^2\|u^{+}\|^2+b(\alpha_u)^4\|u^{+}\|_1^4+b(\alpha_u)^2(\beta_u)^2\|u^{+}\|_1^2\|u^{-}\|_1^2\\
&\quad+2(\alpha_u)^4\int_\Omega|\nabla u^{+}|^2|u^{+}|^2\mathrm{d}x+(\alpha_u)^4\int_\Omega\phi_{u^{+}}(u^{+})^2\mathrm{d}x+(\alpha_u)^2(\beta_u)^2\int_\Omega\phi_{u^{-}}(u^{+})^2\mathrm{d}x\\
&=\lambda\int_\Omega|\alpha_u u^{+}|^q\ln|\alpha_u u^{+}|^2\mathrm{d}x+\int_\Omega|\alpha_u u^{+}|^6\mathrm{d}x
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
&(\beta_u)^4\mu\|u^{-}\|_W^4+(\beta_u)^2\|u^{-}\|^2+b(\beta_u)^4\|u^{-}\|_1^4+b(\alpha_u)^2(\beta_u)^2\|u^{+}\|_1^2\|u^{-}\|_1^2\\
&\quad+2(\beta_u)^4\int_\Omega|\nabla u^{-}|^2|u^{-}|^2\mathrm{d}x+(\beta_u)^4\int_\Omega\phi_{u^{-}}(u^{-})^2\mathrm{d}x+(\alpha_u)^2(\beta_u)^2\int_\Omega\phi_{u^{+}}(u^{-})^2\mathrm{d}x\\
&=\lambda\int_\Omega|\beta_u u^{-}|^q\ln|\beta_u u^{-}|^2\mathrm{d}x+\int_\Omega|\beta_u u^{-}|^6\mathrm{d}x.
\end{aligned}\tag{3.8}$$

According to $0<\alpha_u\leq\beta_u$ and (3.8), we have that

$$
\begin{aligned}
&\mu\|u^-\|_W^4+\frac{\|u^-\|^2}{{\beta_u}^2}+b\|u^-\|_1^4+b\|u^+\|_1^2\|u^-\|_1^2\\
&+2\int_\Omega|\nabla u^-|^2|u^-|^2\mathrm{d}x+\int_\Omega\phi_{u^-}(u^-)^2\mathrm{d}x+\int_\Omega\phi_{u^+}(u^-)^2\mathrm{d}x\\
&\geq\lambda\int_\Omega\frac{|\beta_u u^-|^q\ln|\beta_u u^-|^2}{(\beta_u)^4}\mathrm{d}x+(\beta_u)^2\int_\Omega|u^-|^6\mathrm{d}x.
\end{aligned}\tag{3.9}
$$

If $\beta_u>1$, by (3.6) and (3.9), one has that

$$
\begin{aligned}
&\Big(\frac{1}{(\beta_u)^2}-1\Big)\|u^-\|^2\\
&\geq\lambda\int_\Omega\Big[\frac{|\beta_0 u^-|^q\ln|\beta_u u^-|^2}{(\beta_u)^4}-|u^-|^q\ln|u^-|^2\Big]\mathrm{d}x+((\beta_u)^2-1)\int_\Omega|u^-|^6\mathrm{d}x.
\end{aligned}
$$

The left side of above inequality is negative, which is a contradiction because the right side is positive. Therefore, we conclude that $0<\alpha_u\leq\beta_u\leq1$.

Similarly, by (3.7) and $0<\alpha_u\leq\beta_u$, we have that

$$
\begin{aligned}
&\Big(\frac{1}{(\alpha_u)^2}-1\Big)\|u^+\|^2\\
&\leq\lambda\int_\Omega\Big[\frac{|\alpha_u u^+|^q\ln|\alpha_u u^+|^2}{(\alpha_u)^4}-|u^+|^q\ln|u^+|^2\Big]\mathrm{d}x+((\alpha_u)^2-1)\int_\Omega|u^+|^6\mathrm{d}x.
\end{aligned}
$$

This fact implies that $\alpha_u\geq1$. Consequently, $\alpha_u=\beta_u=1$.

Case 2: $u\notin\mathcal{M}_{b,\mu}^\lambda$.

Suppose that there exist $(\widetilde{\alpha}_1,\widetilde{\beta}_1)$, $(\widetilde{\alpha}_2,\widetilde{\beta}_2)$ such that

$$
u_1=\widetilde{\alpha}_1u^++\widetilde{\beta}_1u^-\in\mathcal{M}_{b,\mu}^\lambda\quad\text{ and }\quad u_2=\widetilde{\alpha}_2u^++\widetilde{\beta}_2u^-\in\mathcal{M}_{b,\mu}^\lambda.
$$

Hence

$$
u_2=(\frac{\widetilde{\alpha}_2}{\widetilde{\alpha}_1})\widetilde{\alpha}_1u^++(\frac{\widetilde{\beta}_2}{\widetilde{\beta}_1})\widetilde{\beta}_1u^-=(\frac{\widetilde{\alpha}_2}{\widetilde{\alpha}_1})u_1^++(\frac{\widetilde{\beta}_2}{\widetilde{\beta}_1})u_1^-\in\mathcal{M}_{b,\mu}^\lambda.
$$

By $u_1\in\mathcal{M}_{b,\mu}^\lambda$, one has that

$$
\frac{\widetilde{\alpha}_2}{\widetilde{\alpha}_1}=\frac{\widetilde{\beta}_2}{\widetilde{\beta}_1}=1.
$$

Hence, $\widetilde{\alpha}_1=\widetilde{\alpha}_2$, $\widetilde{\beta}_1=\widetilde{\beta}_2$.

Step 3: In this step we will prove that (α_u,β_u) is the unique maximum point of ψ_u on $[0,+\infty)\times[0,+\infty)$.

First, it is easy to see that

$$
2\rho^q-q\rho^q\ln|\rho|^2\leq2\ \text{ for all }\rho\in(0,\infty).\tag{3.10}
$$

Let $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ and $\Omega^- = \{x \in \Omega : u(x) < 0\}, u \in H$ with $u^\pm \neq 0$, we have

$$\int_\Omega |\alpha u^+ + \beta u^-|^q \ln |\alpha u^+ + \beta u^-|^2 \mathrm{d}x = \int_\Omega \Big(|\alpha u^+|^q \ln |\alpha u^+|^2 + |\beta u^-|^q \ln |\beta u^-|^2 \Big) \mathrm{d}x. \quad (3.11)$$

Combining (3.10) and (3.11), we get

$$\begin{aligned}
c \ &= I^\lambda_{b,\mu}(\alpha u^+ + \beta u^-) \\
&= \frac{\mu\alpha^4}{4}\|u^+\|^4_W + \frac{\mu\beta^4}{4}\|u^-\|^4_W + \frac{\alpha^2}{2}\|u^+\|^2 + \frac{\beta^2}{2}\|u^-\|^2 + \frac{\alpha^4}{4}b\|u^+\|^4_1 + \frac{\beta^4}{4}b\|u^-\|^4_1 \\
&\quad + \frac{\alpha^2\beta^2}{2}b\|u^+\|^2_1\|u^-\|^2_1 + \frac{\alpha^4}{2}\int_\Omega |\nabla u^+|^2|u^+|^2\mathrm{d}x + \frac{\beta^4}{2}\int_\Omega |\nabla u^-|^2|u^-|^2\mathrm{d}x \\
&\quad + \frac{\alpha^4}{4}\int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x + \frac{\alpha^2\beta^2}{4}\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x + \frac{\alpha^2\beta^2}{4}\int_\Omega \phi_{u^+}(u^-)^2\mathrm{d}x + \frac{\beta^4}{4}\int_\Omega \phi_{u^-}(u^-)^2\mathrm{d}x \\
&\quad + \frac{\lambda}{q^2}\int_\Omega (2|\alpha u^+|^q - q|\alpha u^+|^q \ln|\alpha u^+|^2)\mathrm{d}x + \frac{\lambda}{q^2}\int_\Omega (2|\beta u^-|^q - q|\beta u^-|^q \ln|\beta u^-|^2)\mathrm{d}x \\
&\quad - \frac{\alpha^6}{6}\int_\Omega |u^+|^6 dx - \frac{\beta^6}{6}\int_\Omega |u^-|^6\mathrm{d}x \\
&\leq \frac{\mu\alpha^4}{4}\|u^+\|^4_W + \frac{\mu\beta^4}{4}\|u^-\|^4_W + \frac{\alpha^2}{2}\|u^+\|^2 + \frac{\beta^2}{2}\|u^-\|^2 + \frac{\alpha^4}{4}b\|u^+\|^4_1 + \frac{\beta^4}{4}b\|u^-\|^4_1 \\
&\quad + \frac{\alpha^2\beta^2}{2}b\|u^+\|^2_1\|u^-\|^2_1 + \frac{\alpha^4}{2}\int_\Omega |\nabla u^+|^2|u^+|^2\mathrm{d}x + \frac{\beta^4}{2}\int_\Omega |\nabla u^-|^2|u^-|^2\mathrm{d}x \\
&\quad + \frac{\alpha^4}{4}\int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x + \frac{\alpha^2\beta^2}{4}\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x + \frac{\alpha^2\beta^2}{4}\int_\Omega \phi_{u^+}(u^-)^2\mathrm{d}x + \frac{\beta^4}{4}\int_\Omega \phi_{u^-}(u^-)^2\mathrm{d}x \\
&\quad + \frac{4\lambda}{q^2}|\Omega| - \frac{\alpha^6}{6}\int_\Omega |u^+|^6 dx - \frac{\beta^6}{6}\int_\Omega |u^-|^6\mathrm{d}x,
\end{aligned}$$

which implies that $\lim_{|(\alpha,\beta)|\to\infty} \psi_u(\alpha,\beta) = -\infty$. So it is sufficient to check that a maximum point can not be achieved on the boundary of $[0,+\infty)\times[0,+\infty)$. By contradiction, we suppose that $(0,\beta_u)$ is a maximum point of $\psi_u(\alpha,\beta)$ with $\beta_u \geq 0$. Then, we have that

$$\begin{aligned}
&\psi_u(\alpha,\beta_u) \\
&= \frac{\mu\alpha^4}{4}\|u^+\|^4_W + \frac{\mu(\beta_u)^4}{4}\|u^-\|^4_W + \frac{\alpha^2}{2}\|u^+\|^2 + \frac{(\beta_u)^2}{2}\|u^-\|^2 + \frac{\alpha^4}{4}b\|u^+\|^4_1 + \frac{(\beta_u)^4}{4}b\|u^-\|^4_1 \\
&\quad + \frac{\alpha^2(\beta_u)^2}{2}b\|u^+\|^2_1\|u^-\|^2_1 + \frac{\alpha^4}{2}\int_\Omega |\nabla u^+|^2|u^+|^2\mathrm{d}x + \frac{(\beta_u)^4}{2}\int_\Omega |\nabla u^-|^2|u^-|^2\mathrm{d}x \\
&\quad + \frac{\alpha^4}{4}\int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x + \frac{(\beta_u)^4}{4}\int_\Omega \phi_{u^-}(u^-)^2\mathrm{d}x + \frac{\alpha^2(\beta_u)^2}{4}\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x \\
&\quad + \frac{\alpha^2(\beta_u)^2}{4}\int_\Omega \phi_{u^+}(u^-)^2\mathrm{d}x + \frac{\lambda}{q^2}\int_\Omega (2|\alpha u^+|^q - q|\alpha u^+|^q \ln|\alpha u^+|^2)\mathrm{d}x \\
&\quad + \frac{\lambda}{q^2}\int_\Omega (2|\beta_u u^-|^q - q|\beta_u u^-|^q \ln|\beta_u u^-|^2)\mathrm{d}x - \frac{\alpha^6}{6}\int_\Omega |u^+|^6\mathrm{d}x - \frac{(\beta_u)^6}{6}\int_\Omega |u^-|^6\mathrm{d}x.
\end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned}(\psi_u(\alpha,\beta_u))'_\alpha \;&= \mu\alpha^3\|u^+\|_W^4 + \alpha\|u^+\|^2 + b\alpha^3\|u^+\|_1^4 + b\alpha\beta_u\|u^+\|_1^2\|u^-\|_1^2 + 2\alpha^3\int_\Omega |\nabla u^+|^2|u^+|^2\mathrm{d}x \\ &\quad + \alpha^3\int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x + \frac{\alpha(\beta_u)^2}{2}\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x + \frac{\alpha(\beta_u)^2}{2}\int_\Omega \phi_{u^+}(u^-)^2\mathrm{d}x \\ &\quad - \alpha^{q-1}\lambda\int_\Omega |u^+|^q\ln|\alpha u^+|^2\mathrm{d}x - \alpha^5\int_\Omega |u^+|^6\mathrm{d}x \\ &> 0\end{aligned}$$

if α is small enough, that is, ψ_u is an increasing function with respect to α if α is small enough. This yields the contradiction. Similarly, ψ_u can not achieve its global maximum point at $(\alpha_u, 0)$ with $\alpha_u > 0$. $\square$

Lemma 3.2. For any $u \in$ with $u^\pm \neq 0$, such that $\langle (I^\lambda_{b,\mu})'(u), u^\pm\rangle \leq 0$, then the unique maximum point of ψ_u in $[0,+\infty)\times[0,+\infty)$ satisfies $0 < \alpha_u, \beta_u \leq 1$.

Proof. If $\alpha_u = 0$ or $\beta_u = 0$, according Lemma 3.1, ψ_u can not achieve maximum. Without loss of generality, let $\alpha_u \geq \beta_u > 0$. On the one hand, by $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}^\lambda_{b,\mu}$, we have

$$\begin{aligned}&(\alpha_u)^4\mu\|u^+\|_W^4 + (\alpha_u)^2\|u^+\|^2 + b(\alpha_u)^4\|u^+\|_1^4 + b(\alpha_u)^2(\beta_u)^2\|u^+\|_1^2\|u^-\|_1^2 \\ &\quad + 2(\alpha_u)^4\int_\Omega |\nabla u^+|^2|u^+|^2\mathrm{d}x + (\alpha_u)^4\int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x + (\alpha_u)^2(\beta_u)^2\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x \\ &= \lambda\int_\Omega |\alpha_u u^+|^q\ln|\alpha_u u^+|^2\mathrm{d}x + (\alpha_u)^6\int_\Omega |u^+|^6\mathrm{d}x.\end{aligned} \tag{3.12}$$

On the other hand, by $\langle (I^\lambda_{b,\mu})'(u), u^\pm\rangle \leq 0$, we obtain

$$\begin{aligned}&\mu\|u^+\|_W^4 + \|u^+\|^2 + b\|u^+\|_1^4 + b\|u^+\|_1^2\|u^-\|_1^2 \\ &\quad + 2\int_\Omega |\nabla u^+|^2|u^+|^2\mathrm{d}x + \int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x + \int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x \\ &\leq \lambda\int_\Omega |u^+|^q\ln|u^+|^2\mathrm{d}x + \int_\Omega |u^+|^6\mathrm{d}x.\end{aligned} \tag{3.13}$$

So, it follows form (3.12) and (3.13), we have

$$\begin{aligned}&(\frac{1}{(\alpha_u)^2} - 1)\|u^+\|^2 \\ &\geq \lambda\int_\Omega [(\alpha_u)^{q-4}|u^+|^q\ln|\alpha_u u^+|^2 - |u^+|^q\ln|u^+|^2]\mathrm{d}x + ((\alpha_u)^2 - 1)\int_\Omega |u^+|^6\mathrm{d}x.\end{aligned} \tag{3.14}$$

Since $q > 4$, we conclude that $0 < \beta_u < \alpha_u \leq 1$, so $0 < \alpha_u, \beta_u \leq 1$. $\square$

Lemma 3.3. Let $c^\lambda_{b,\mu} = \inf_{u\in\mathcal{M}^\lambda_b} I^\lambda_{b,\mu}(u)$, then we get $\lim_{\lambda\to\infty} c^\lambda_{b,\mu} = 0$.

Proof. For any $u \in \mathcal{M}_{b,\mu}^{\lambda}$,

$$\begin{aligned}
&\mu\|u^{\pm}\|_W^4 + \|u^{\pm}\|^2 + b\|u^{\pm}\|_1^4 + b\|u^{+}\|_1^2\|u^{-}\|_1^2 \\
&\quad + 2\int_{\Omega} |\nabla u^{\pm}|^2 |u^{\pm}|^2 \mathrm{d}x + \int_{\Omega} \phi_{u^{\pm}} (u^{\pm})^2 \mathrm{d}x + \int_{\Omega} \phi_{u^{\mp}} (u^{\pm})^2 \mathrm{d}x \\
&= \lambda \int_{\Omega} |u^{\pm}|^q \ln |u^{\pm}|^2 \mathrm{d}x + \int_{\Omega} |u^{\pm}|^6 \mathrm{d}x.
\end{aligned}$$

Then by (3.2) and the Sobolev inequalities, we get

$$\|u^{\pm}\|^2 \leq \lambda \int_{\Omega} |u^{\pm}|^q \ln |u^{\pm}|^2 \mathrm{d}x + \int_{\Omega} |u^{\pm}|^6 \mathrm{d}x \leq \lambda\varepsilon C_1 \|u^{\pm}\|^2 + \lambda C_{\varepsilon} C_2 \|u^{\pm}\|^r + C_3 \|u^{\pm}\|^6.$$

Thus

$$(1 - \lambda\varepsilon C_1)\|u^{\pm}\|^2 \leq C_2\|u^{\pm}\|^r + C_3\|u^{\pm}\|^6.$$

Choosing ε small enough such that $1 - \lambda\varepsilon C_1 > 0$, since $r > 4$, there exists $\rho > 0$ such that

$$\|u^{\pm}\|^2 \geq \rho \ \text{ for all } u \in \mathcal{M}_{b,\mu}^{\lambda}. \tag{3.15}$$

Thanks to $u \in \mathcal{M}_{b,\mu}^{\lambda}$, we have $\langle (I_{b,\mu}^{\lambda})'(u), u\rangle = 0$. Then,

$$\begin{aligned}
I_{b,\mu}^{\lambda}(u) &= I_{b,\mu}^{\lambda}(u) - \frac{1}{q}\langle (I_{b,\mu}^{\lambda})'(u), u\rangle \\
&= \mu(\frac{1}{4} - \frac{1}{q})\|u\|_W^4 + (\frac{1}{2} - \frac{1}{q})\|u\|^2 + (\frac{1}{4} - \frac{1}{q})b\|u\|_1^4 + 2(\frac{1}{4} - \frac{1}{q})\int_{\Omega} |\nabla u|^2 |u|^2 \mathrm{d}x \\
&\quad + (\frac{1}{4} - \frac{1}{q})\int_{\Omega} \phi_u u^2 \mathrm{d}x + \frac{2\lambda}{q^2}\int_{\Omega} |u|^q \mathrm{d}x + (\frac{1}{q} - \frac{1}{6})\int_{\Omega} |u|^6 \mathrm{d}x \\
&\geq (\frac{1}{2} - \frac{1}{q})\|u\|^2,
\end{aligned}$$

thus $I_{b,\mu}^{\lambda}(u)$ is bounded below on $\mathcal{M}_{b,\mu}^{\lambda}$.

For any $u \in E$ with $u^{\pm} \neq 0$, by using Lemma 3.1, for each $\lambda > 0$, there exist $\{\alpha_{\lambda}, \beta_{\lambda}\}$ such that $\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-} \in \mathcal{M}_{b,\mu}^{\lambda}$, we have

$$\begin{aligned}
0 &\leq c_b^{\lambda} = \inf I_{b,\mu}^{\lambda}(u) \leq I_{b,\mu}^{\lambda}(\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-}) \\
&\leq \frac{\mu}{4}\|\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-}\|_W^4 + \frac{1}{2}\|\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-}\|^2 + \frac{b}{4}\|\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-}\|_1^4 + \frac{2\lambda}{q^2}\int_{\Omega} |\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-}|^q \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\Omega} |\nabla(\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-})|^2 |\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-}|^2 \mathrm{d}x + \frac{1}{4}\int_{\Omega} \phi_{\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-}} (\alpha_{\lambda}u^{+} + \beta_{\lambda}u^{-})^2 \mathrm{d}x \\
&\leq \frac{\mu}{2}(\alpha_{\lambda})^4\|u^{+}\|_W^4 + \frac{\mu}{2}(\beta_{\lambda})^4\|u^{-}\|_W^4 + (\alpha_{\lambda})^2\|u^{+}\|^2 + (\beta_{\lambda})^2\|u^{-}\|^2 + 2b(\alpha_{\lambda})^4\|u^{+}\|_1^4 \\
&\quad + 2b(\beta_{\lambda})^4\|u^{-}\|_1^4 + (\alpha_{\lambda})^4\int_{\Omega} |\nabla u^{+}|^2 |u^{+}|^2 \mathrm{d}x + (\beta_{\lambda})^4\int_{\Omega} |\nabla u^{-}|^2 |u^{-}|^2 \mathrm{d}x \\
&\quad + 2C(\alpha_{\lambda})^4\|u^{+}\|_1^4 + 2C(\beta_{\lambda})^4\|u^{-}\|_1^4 + \frac{2\lambda}{q^2}\int_{\Omega} |\alpha_{\lambda}u^{+}|^q \mathrm{d}x + \frac{2\lambda}{q^2}\int_{\Omega} |\beta_{\lambda}u^{-}|^q \mathrm{d}x.
\end{aligned}$$

Next we will prove that $\alpha_\lambda \to 0$ and $\beta_\lambda \to 0$ as $\lambda \to \infty$.

Let $G_u = \{(\alpha_\lambda, \beta_\lambda) \in [0, +\infty) \times [0, +\infty) : T_u(\alpha_\lambda, \beta_\lambda) = (0, 0), \lambda > 0\}$, we have

$$
\begin{aligned}
&(\alpha_\lambda)^6 \int_\Omega |u^+|^6 \mathrm{d}x + (\beta_\lambda)^6 \int_\Omega |u^-|^6 \mathrm{d}x \\
&\quad + \lambda(\alpha_\lambda)^q \int_\Omega |u^+|^q \ln |\alpha_\lambda u^+|^2 \mathrm{d}x + \lambda(\beta_\lambda)^q \int_\Omega |u^-|^q \ln |\beta_\lambda u^-|^2 \mathrm{d}x \\
&= \mu\|\alpha_\lambda u^+ + \beta_\lambda u^-\|_W^4 + \|\alpha_\lambda u^+ + \beta_\lambda u^-\|^2 + b\|\alpha_\lambda u^+ + \beta_\lambda u^-\|_1^4 + 2(\alpha_\lambda)^4 \int_\Omega |\nabla u^+|^2 |u^+|^2 \mathrm{d}x \\
&\quad + 2(\beta_\lambda)^4 \int_\Omega |\nabla u^-|^2 |u^-|^2 \mathrm{d}x + \int_\Omega \phi_{\alpha_\lambda u^+ + \beta_\lambda u^-} (\alpha_\lambda u^+ + \beta_\lambda u^-)^2 \mathrm{d}x \\
&\le 2\mu(\alpha_\lambda)^4 \|u^+\|_W^4 + 2\mu(\beta_\lambda)^4 \|u^-\|_W^4 + 2(\alpha_\lambda)^2 \|u^+\|^2 \\
&\quad + 2(\beta_\lambda)^2 \|u^-\|^2 + 4b(\alpha_\lambda)^4 \|u^+\|_1^4 + 4b(\beta_\lambda)^4 \|u^-\|_1^4 \\
&\quad + 2(\alpha_\lambda)^4 \int_\Omega |\nabla u^+|^2 |u^+|^2 \mathrm{d}x + 2(\beta_\lambda)^4 \int_\Omega |\nabla u^-|^2 |u^-|^2 \mathrm{d}x \\
&\quad + 2C(\alpha_\lambda)^4 \|u^+\|_1^4 + 2C(\beta_\lambda)^4 \|u^-\|_1^4.
\end{aligned}
$$

Hence T_u is bounded. Let $\{\lambda_n\} \subset (0, \infty)$ be such that $\lambda_n \to \infty$ as $n \to \infty$. Then there exist α_0 and β_0 such that$(\alpha_{\lambda_n}, \beta_{\lambda_n}) \to (\alpha_0, \beta_0)$ as $n \to \infty$.

Now, we claim $\alpha_0 = \beta_0 = 0$. Suppose, by contradiction, if $\alpha_0 > 0$ or $\beta_0 > 0$ by $\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^- \in \mathcal{M}_{b,\mu}^{\lambda_n}$, for any $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\mu\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|_W^4 + \|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|^2 + b\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|_1^4 \\
&\quad + \int_\Omega |\nabla(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)|^2 |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^2 \mathrm{d}x + \int_\Omega \phi_{\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-} (\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)^2 \mathrm{d}x. \\
&= \lambda_n \int_\Omega |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^q \ln |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^2 \mathrm{d}x + \int_\Omega |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^6 \mathrm{d}x.
\end{aligned}
\tag{3.16}
$$

Thanks to $\alpha_{\lambda_n} u^+ \to \alpha_0 u^+$ and $\beta_{\lambda_n} u^- \to \beta_0 u^-$ in E, (3.2) and the Lebesgue dominated convergence theorem, we have

$$
\int_\Omega |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^q \ln |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^2 \mathrm{d}x \to \int_\Omega |\alpha_0 u^+ + \beta_0 u^-|^q \ln |\alpha_0 u^+ + \beta_0 u^-|^2 \mathrm{d}x > 0
$$

as $n \to \infty$. It follows from $\lambda_n \to \infty$ and the boundness of $\{\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\}$ in E that we have a contradiction with equality (3.16). Hence, $\alpha_0 = \beta_0 = 0$, we conclude that $\lim_{\lambda \to \infty} c_{b,\mu}^\lambda = 0$. □

Lemma 3.4. There exists $\lambda^* > 0$ such that for all $\lambda \ge \lambda^*$, the infimum $c_{b,\mu}^\lambda$ is achieved.

Proof. By the definition of $c_{b,\mu}^\lambda = \inf_{u \in \mathcal{M}_{b,\mu}^\lambda} I_{b,\mu}^\lambda(u)$, there exists a sequence $\{u_n\} \subset \mathcal{M}_{b,\mu}^\lambda$ such that

$$
\lim_{\lambda \to \infty} I_{b,\mu}^\lambda(u_n) = c_{b,\mu}^\lambda.
$$

Obviously, $\{u_n\}$ is bounded in E. Then, up to subsequence, still denoted by $\{u_n\}$, there exists $u \in E$ such that $u_n \rightharpoonup u$. Since the embedding $E \hookrightarrow L^p(\Omega)$ is compact, for all $p \in (2,6)$, we have

$$u_n \to u \ \text{ in } \ L^p(\Omega) \text{ and } u_n \to u \quad \text{ a.e. } x \in \Omega.$$

Hence

$$\begin{aligned} &{u_n}^{\pm} \rightharpoonup u^{\pm} \ \text{ in } E, \\ &{u_n}^{\pm} \to u^{\pm} \ \text{ in } L^p(\Omega), \\ &{u_n}^{\pm} \to u^{\pm} \ \text{ in a.e. } x \in \Omega. \end{aligned}$$

By Lemma 3.1, we have

$$I_{b,\mu}^{\lambda}(\alpha u_n^+ + \beta u_n^-) \le I_{b,\mu}^{\lambda}(u_n)$$

for all $\alpha, \beta \ge 0$. On the one hand, the Vitali convergence theorem yields that

$$\lim_{n\to\infty} \int_\Omega |u_n|^q \ln |u_n|^2 \mathrm{d}x \to \int_\Omega |u|^q \ln |u|^2 \mathrm{d}x. \tag{3.17}$$

Then, by (3.17), Brézis-Lieb lemma and the weak semicontinuity of norm, we get

$$\begin{aligned}
&\liminf_{n\to\infty} I_{b,\mu}^{\lambda}(\alpha {u_n}^+ + \beta {u_n}^-) \\
&\ge \mu \frac{\alpha^4}{4} \lim_{n\to\infty} \left(\left\| u_n^+ - u^+ \right\|_W^4 + \left\| u^+ \right\|_W^4 \right) + \mu \frac{\beta^4}{4} \lim_{n\to\infty} \left(\left\| u_n^- - u^- \right\|_W^4 + \left\| u^- \right\|_W^4 \right) \\
&\quad + \frac{\alpha^2}{2} \lim_{n\to\infty} (\|u_n^+ - u^+\|^2 + \|u^+\|^2) + \frac{\beta^2}{2} (\|u_n^- - u^-\|^2 + \|u^-\|^2) \\
&\quad + \frac{b\alpha^4}{4} \Big([\lim_{n\to\infty} \|u_n^+ - u^+\|_1^2 + \|u^+\|_1^2] \Big)^2 + \frac{b\beta^4}{4} \Big([\lim_{n\to\infty} \|u_n^- - u^-\|_1^2 + \|u^-\|_1^2] \Big)^2 \\
&\quad + \frac{\alpha^4}{2} \liminf_{n\to\infty} \int_\Omega \left|\nabla u_n^+\right|^2 \left|u_n^+\right|^2 \mathrm{d}x + \frac{\beta^4}{2} \liminf_{n\to\infty} \int_\Omega \left|\nabla u_n^-\right|^2 \left|u_n^-\right|^2 \mathrm{d}x + \frac{\alpha^4}{4} \lim_{n\to\infty} \int_\Omega \phi_{u_n^+} (u_n^+)^2 \mathrm{d}x \\
&\quad + \frac{\alpha^2\beta^2}{4} \lim_{n\to\infty} \int_\Omega \phi_{u_n^-} (u_n^+)^2 \mathrm{d}x + \frac{\alpha^2\beta^2}{4} \lim_{n\to\infty} \int_\Omega \phi_{u_n^+} (u_n^-)^2 \mathrm{d}x + \frac{\beta^4}{4} \lim_{n\to\infty} \int_\Omega \phi_{u_n^-} (u_n^-)^2 \mathrm{d}x \\
&\quad - \frac{\alpha^6}{6} \Big(\lim_{n\to\infty} \int_\Omega |u_n^+ - u^+|^6 \mathrm{d}x + \int_\Omega |u^+|^6 \mathrm{d}x \Big) - \frac{\beta^6}{6} \Big(\lim_{n\to\infty} \int_\Omega |u_n^- - u^-|^6 \mathrm{d}x + \int_\Omega |u^-|^6 \mathrm{d}x \Big) \\
&\quad + \frac{2\lambda}{q^2} \int_\Omega |\alpha u_n^+ + \beta u_n^-|^q \mathrm{d}x - \frac{\lambda}{q} \int_\Omega |\alpha u_n^+ + \beta u_n^-|^q \ln |\alpha u_n^+ + \beta u_n^-|^2 \mathrm{d}x \\
&\ge I_{b,\mu}^{\lambda}(\alpha u^+ + \beta u^-) + \frac{\mu\alpha^4}{4} A_1 + \frac{\mu\beta^4}{4} A_2 + \frac{\alpha^2}{2} A_3 + \frac{b\alpha^4}{2} A_5 \|u^+\|_1^2 + \frac{b\alpha^4}{4} A_5^2 - \frac{\alpha^6}{6} B_1 \\
&\quad + \frac{\beta^2}{2} A_4 + \frac{\beta^4}{2} A_6^2 \|u^-\|_1^2 + \frac{b\beta^4}{4} A_6^2 - \frac{\beta^6}{6} B_2,
\end{aligned}$$

where

$$A_1 = \lim_{n\to\infty} \|u_n^+ - u^+\|_W^4, A_2 = \lim_{n\to\infty} \|u_n^- - u^-\|_W^4, A_3 = \lim_{n\to\infty} \|u_n^+ - u^+\|^2,\ A_4 = \lim_{n\to\infty} \|u_n^- - u^-\|^2,$$

$$A_5 = \lim_{n\to\infty} \|u_n^+ - u^+\|_1^2,\ A_6 = \lim_{n\to\infty} \|u_n^- - u^-\|_1^2,\ \ B_1 = \lim_{n\to\infty} |u_n^+ - u^+|_6^6,\ \ B_2 = \lim_{n\to\infty} |u_n^- - u^-|_6^6$$

for all $\alpha \geq 0$ and $\beta \geq 0$. So,

$$\begin{aligned} c_{b,\mu}^{\lambda} &\geq I_{b,\mu}^{\lambda}(\alpha u^+ + \beta u^-) + \frac{\mu\alpha^4}{4}A_1 + \frac{\mu\beta^4}{4}A_2 + \frac{\alpha^2}{2}A_3 + \frac{b\alpha^4}{2}A_5\|u^+\|_1^2 + \frac{b\alpha^4}{4}A_5^2 \\ &- \frac{\alpha^6}{6}B_1 + \frac{\beta^2}{2}A_4 + \frac{\beta^4}{2}A_6^2\|u^-\|_1^2 + \frac{b\beta^4}{4}A_6^2 - \frac{\beta^6}{6}B_2. \end{aligned} \tag{3.18}$$

Denote $\sigma := \frac{1}{3}S^{\frac{3}{2}}$, where $S = \inf_{u\in H\setminus\{0\}} \frac{\|u\|^2}{(\int_\Omega |u|^6 \mathrm{d}x)^{\frac{1}{3}}}$. According to Lemma 3.3, there is $\lambda^* > 0$ such that $c_{b,\mu}^{\lambda} < \sigma$ for all $\lambda \geq \lambda^*$.

Step 1: we prove that $u^{\pm} \neq 0$. By contradiction, we suppose $u^+ = 0$ ($u^- = 0$ is similar).

Case 1: $B_1 = 0$. If $A_1 = A_3 = 0$, that is, $u_n^+ \to u^+$ in E. According to (3.15) in Lemma 3.3, we obtain $\|u^+\| > 0$, which contradicts $u^+ = 0$. If $A_1 > 0,\ A_3 > 0$, by (3.18), we get $\frac{\alpha^2}{2}A_3 < c_{b,\mu}^{\lambda}$ for all $\alpha \geq 0$, which is a contradiction.

Case 2: $B_1 > 0$. According to definition of S, we obtain $\sigma := \frac{1}{3}S^{\frac{3}{2}} \leq \frac{1}{3}(\frac{A_3}{(B_1)^{\frac{1}{3}}})^{\frac{3}{2}}$, by direct calculation, we obtain

$$\frac{1}{3}\Big(\frac{A_3}{(B_1)^{\frac{1}{3}}}\Big)^{\frac{3}{2}} = \max_{\alpha\geq 0}\{\frac{\alpha^2}{2}A_3 - \frac{\alpha^6}{6}B_1\} \leq \max_{\alpha\geq 0}\{\frac{\mu\alpha^4}{4}A_1 + \frac{\alpha^2}{2}A_3 + \frac{b\alpha^4}{2}A_5\|u^+\|_1^2 + \frac{b\alpha^4}{4}A_5^2 - \frac{\alpha^6}{6}B_1\}.$$

Since $c_{b,\mu}^{\lambda} \to 0$ as $\lambda \to \infty$, there exists $\lambda^* > 0, C > 0$ such that for all $\lambda > \lambda^*, c_{b,\mu}^{\lambda} \leq C$. Then, without loss of generality, we can assume $c_{b,\mu}^{\lambda} < \sigma$, choose $\beta = 0$, it follows form (3.18)

$$\sigma \leq \max_{\alpha\geq 0}\{\frac{\alpha^2}{2}A_3 - \frac{\alpha^6}{6}B_1\} \leq \max_{\alpha\geq 0}\{\frac{\mu\alpha^4}{4}A_1 + \frac{\alpha^2}{2}A_3 + \frac{b\alpha^4}{2}A_5\|u^+\|_1^2 + \frac{b\alpha^4}{4}A_5^2 - \frac{\alpha^6}{6}B_1\} < \sigma.$$

It is a contradiction, we have that $u^+ \neq 0$. Similarly, we obtain $u^- \neq 0$.

Step 2: we prove that $B_1 = 0,\ B_2 = 0$. We just prove $B_1 = 0$. By contradiction, we suppose that $B_1 > 0$.

Case 1: $B_2 > 0$. Let $\widehat{\alpha}_1$ and $\widehat{\beta}_1$ satisfy

$$\begin{aligned} &\frac{\mu(\widehat{\alpha}_1)^4}{4}A_1 + \frac{(\widehat{\alpha}_1)^2}{2}A_3 + \frac{b(\widehat{\alpha}_1)^4}{2}A_5\|u^+\|_1^2 + \frac{b(\widehat{\alpha}_1)^4}{4}A_5^2 - \frac{(\widehat{\alpha}_1)^6}{6}B_1 \\ &= \max_{\alpha\geq 0}\{\frac{\mu\alpha^4}{4}A_1 + \frac{\alpha^2}{2}A_3 + \frac{b\alpha^4}{2}A_5\|u^+\|_1^2 + \frac{b\alpha^4}{4}A_5^2 - \frac{\alpha^6}{6}B_1\}, \end{aligned}$$

and

$$\begin{aligned} &\frac{\mu(\widehat{\beta}_1)^4}{4}A_2 + \frac{(\widehat{\beta}_1)^2}{2}A_4 + \frac{b(\widehat{\beta}_1)^4}{2}A_6\|u^-\|_1^2 + \frac{b(\widehat{\beta}_1)^4}{4}A_6^2 - \frac{(\widehat{\beta}_1)^6}{6}B_2 \\ &= \max_{\beta\geq 0}\{\frac{\mu\beta^4}{4}A_2 + \frac{\beta^2}{2}A_4 + \frac{b\beta^4}{2}A_6\|u^-\|_1^2 + \frac{b\beta^4}{4}A_6^2 - \frac{\beta^6}{6}B_2\}. \end{aligned}$$

According to $[0,\widehat{\alpha}_1] \times [0,\widehat{\beta}_1]$ is compact, there exist $(\alpha_u,\beta_u) \in [0,\widehat{\alpha}_1] \times [0,\widehat{\beta}_1]$ such that $\psi_u(\alpha_u,\beta_u) = \max_{(\alpha,\beta)\in[0,\widehat{\alpha}_1]\times[0,\widehat{\beta}_1]} \psi_u(\alpha,\beta)$.

In the following, we will prove $(\alpha_u,\beta_u) \in (0,\widehat{\alpha}_1) \times (0,\widehat{\beta}_1)$. Obvious, if β small enough , we have

$$\psi_u(\alpha,0) < I_{b,\mu}^{\lambda}(\alpha u^+) + I_{b,\mu}^{\lambda}(\beta u^-) \leq I_{b,\mu}^{\lambda}(\alpha u^+ + \beta u^-) = \psi_u(\alpha,\beta),\ \forall\ \alpha \in [0,\widehat{\alpha}_1]$$

Hence, there exists $\beta_0 \in [0, \widehat{\beta}_1]$ such that $\psi_u(\alpha, 0) < \psi_u(\alpha, \beta_0)$ for all $\alpha \in [0, \widehat{\alpha}_1]$. That is, $(\alpha_u, \beta_u) \notin [0, \widehat{\alpha}_1] \times \{0\}$. By similar discussion, we conclude that $(\alpha_u, \beta_u) \notin \{0\} \times [0, \widehat{\beta}_1]$. Obviously, we get

$$\frac{\mu\alpha^4}{4}A_1 + \frac{\alpha^2}{2}A_3 + \frac{b\alpha^4}{2}A_5\|u^+\|_1^2 + \frac{b\alpha^4}{4}A_5^2 - \frac{\alpha^6}{6}B_1 > 0, \ \alpha \in (0, \widehat{\alpha}_1], \tag{3.19}$$

$$\frac{\mu\beta^4}{4}A_2 + \frac{\beta^2}{2}A_4 + \frac{b\beta^4}{2}A_6\|u^-\|_1^2 + \frac{b\beta^4}{4}A_6^2 - \frac{\beta^6}{6}B_2 > 0, \ \beta \in (0, \widehat{\beta}_1]. \tag{3.20}$$

Then, for all $\alpha \in (0, \widehat{\alpha}_1]$ and $\beta \in (0, \widehat{\beta}_1]$, we get

$$\begin{aligned}\sigma \leq\ & \frac{\mu(\widehat{\alpha}_1)^4}{4}A_1 + \frac{(\widehat{\alpha}_1)^2}{2}A_3 + \frac{b(\widehat{\alpha}_1)^4}{2}A_5\|u^+\|_1^2 + \frac{b(\widehat{\alpha}_1)^4}{4}A_5^2 - \frac{(\widehat{\alpha}_1)^6}{6}B_1 \\ & + \frac{\mu\beta^4}{4}A_2 + \frac{\beta^2}{2}A_4 + \frac{b\beta^4}{2}A_6\|u^-\|_1^2 + \frac{b\beta^4}{4}A_6^2 - \frac{\beta^6}{6}B_2\end{aligned}$$

and

$$\begin{aligned}\sigma \leq\ & \frac{\mu\alpha^4}{4}A_1 + \frac{\alpha^2}{2}A_3 + \frac{b\alpha^4}{2}A_5^2\|u^+\|_1^2 + \frac{b\alpha^4}{4}A_5^4 - \frac{\alpha^6}{6}B_1 \\ & + \frac{\mu(\widehat{\beta}_1)^4}{4}A_2 + \frac{\widehat{\beta}_1^2}{2}A_4 + \frac{b(\widehat{\beta}_1)^4}{2}A_6\|u^-\|_1^2 + \frac{b(\widehat{\beta}_1)^4}{4}A_6^2 - \frac{(\widehat{\beta}_1)^6}{6}B_2.\end{aligned}$$

Together with (3.18), we obtain $\psi_u(\alpha, \widehat{\beta}_1) \leq 0$, $\psi_u(\widehat{\alpha}_1, \beta) \leq 0$ for all $\alpha \in [0, \widehat{\alpha}_1]$ and $\beta \in [0, \widehat{\beta}_1]$. That is, $(\alpha_u, \beta_u) \notin [0, \widehat{\alpha}_1] \times \{\widehat{\beta}_1\}$ and $(\alpha_u, \beta_u) \notin \{0, \widehat{\alpha}_1\} \times [0, \widehat{\beta}_1]$.

In conclusion, we get $(\alpha_u, \beta_u) \in (0, \widehat{\alpha}_1) \times (0, \widehat{\beta}_1)$. Hence, $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_{b,\mu}^\lambda$. So, combining (3.18), (3.19) with (3.20), we have that

$$\begin{aligned}c_{b,\mu}^\lambda \ & \geq I_{b,\mu}^\lambda(\alpha_u u^+ + \beta_u u^-) + \frac{\mu(\alpha_u)^4}{4}A_1 + \frac{(\alpha_u)^2}{2}A_3 + \frac{b(\alpha_u)^4}{2}A_5\|u^+\|_1^2 + \frac{b(\alpha_u)^4}{4}A_5^2 \\ & - \frac{{\alpha_u}^6}{6}B_1 + \frac{\mu(\beta_u)^4}{4}A_2 + \frac{{\beta_u}^2}{2}A_4 + \frac{b(\beta_u)^4}{2}A_6\|u^-\|_1^2 + \frac{b(\beta_u)^4}{4}A_6^2 - \frac{(\beta_u)^6}{6}B_2 \\ & > I_{b,\mu}^\lambda(\alpha_u u^+ + \beta_u u^-) \geq c_{b,\mu}^\lambda.\end{aligned}$$

Therefore, we have a contradiction.

Case 2: $B_2 = 0$. In this case, we can maximize in $(0, \widehat{\alpha}_1) \times (0, \infty)$. Indeed, it is possible to show that there exists $\beta_0 \in [0, \infty]$ such that $I_{b,\mu}^\lambda(\alpha u^+ + \beta u^-) < 0$ for all $(\alpha, \beta) \in [0, \widehat{\alpha}_1] \times [\beta_0, \infty)$. Hence, there exists $(\alpha_u, \beta_u) \in [0, \widehat{\alpha}_1] \times [0, \infty)$ that satisfies

$$\psi_u(\alpha_u, \beta_u) = \max_{\alpha \in [0, \widehat{\alpha}_1] \times [0, \infty)} \psi_u(\alpha, \beta).$$

Following, we prove that $(\alpha_u, \beta_u) \in [0, \widehat{\alpha}_1] \times [0, \infty)$.

Since $\psi_u(\alpha, 0) \leq \psi_u(\alpha, \beta)$ for $\alpha \in [0, \widehat{\alpha}_1]$ and β is small enough, we have $(\alpha_u, \beta_u) \notin \{0\} \times [0, \infty)$. On the other hand, for all $\beta \in [0, \infty)$, it is obvious that

$$\begin{aligned}\sigma \leq\ & \frac{\mu(\widehat{\alpha}_1)^4}{4}A_1 + \frac{(\widehat{\alpha}_1)^2}{2}A_3 + \frac{b(\widehat{\alpha}_1)^4}{2}A_5\|u^+\|_1^2 + \frac{b(\widehat{\alpha}_1)^4}{4}A_5^2 - \frac{(\widehat{\alpha}_1)^6}{6}B_1 \\ & + \frac{\mu\beta^4}{4}A_2 + \frac{\beta^2}{2}A_4 + \frac{b\beta^4}{2}A_6\|u^-\|_1^2 + \frac{b\beta^4}{4}A_6^2.\end{aligned}$$

Hence, we have that $\psi_u(\widehat{\alpha}_1,\beta)\le 0$ for all $\beta\in[0,\infty)$. Thus, $(\alpha_u,\beta_u)\notin\{\widehat{\alpha}_1\}\times[0,\infty)$. And so $(\alpha_u,\beta_u)\in[0,\widehat{\alpha}_1]\times[0,\infty)$. That is, $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}^\lambda_{b,\mu}$, therefore, according to (3.19), we have that

$$\begin{aligned}
c^\lambda_{b,\mu} &\ge I^\lambda_{b,\mu}(\alpha_u u^+ + \beta_u u^-) + \frac{\mu(\alpha_u)^4}{4}A_1 + \frac{(\alpha_u)^2}{2}A_3 + \frac{b(\alpha_u)^4}{2}A_5\|u^+\|_1^2 + \frac{b(\alpha_u)^4}{4}A_5^2 \\
&\quad - \frac{(\alpha_u)^6}{6}B_1 + \frac{\mu(\beta_u)^4}{4}A_2 + \frac{(\beta_u)^2}{2}A_4 + \frac{b(\beta_u)^4}{2}A_6\|u^-\|_1^2 + \frac{b(\beta_u)^4}{4}A_6^2 \\
&> I^\lambda_{b,\mu}(\alpha_u u^+ + \beta_u u^-) \ge c^\lambda_{b,\mu},
\end{aligned}$$

which is a contradiction.

Therefore, from above discussion, we have that $B_1 = B_2 = 0$.

Lastly, we prove that $c^\lambda_{b,\mu}$ is achieved.

Since $u^\pm \neq 0$, by Lemma 3.1, there exist $\alpha_u,\beta_u>0$ such that

$$\widetilde{u} = \alpha_u u^+ + \beta_u u^- \in \mathcal{M}^\lambda_{b,\mu}.$$

Furthermore, the norm in E is lower semicontinuous, it is easy to see that

$$\langle (I^\lambda_{b,\mu})'(u), u^\pm\rangle \le 0.$$

By Lemma 3.2, we obtain $\alpha_u,\beta_u\le 1$.
From $u_n\in\mathcal{M}^\lambda_{b,\mu}$, according to Lemma 3.1, we get

$$I^\lambda_{b,\mu}(\alpha_u u_n^+ + \beta_u u_n^-) \le I^\lambda_{b,\mu}(u_n^+ + u_n^-) = I^\lambda_{b,\mu}(u_n).$$

Thanks to $B_1 = B_2 = 0$, we obtain

$$\begin{aligned}
c^\lambda_{b,\mu} &\le I^\lambda_{b,\mu}(\widetilde{u}) - \frac{1}{q}\langle (I^\lambda_{b,\mu})'(\widetilde{u}),\widetilde{u}\rangle \\
&= (\frac14-\frac1q)\|\widetilde{u}\|_W^4 + (\frac12-\frac1q)\|\widetilde{u}\|^2 + (\frac14-\frac1q)\|\widetilde{u}\|_1^4 + (\frac12-\frac1q)\int_\Omega |\nabla\widetilde{u}|^2|\widetilde{u}|^2\,\mathrm{d}x \\
&\quad + (\frac14-\frac1q)\int_\Omega \phi_{\widetilde{u}}\widetilde{u}^2\mathrm{d}x + \frac{2\lambda}{q^2}\int_\Omega|\widetilde{u}|^q\mathrm{d}x + (\frac1q-\frac16)\int_\Omega|\widetilde{u}|^6\mathrm{d}x \\
&= (\frac14-\frac1q)\|\alpha_u u^+ + \beta_u u^-\|_W^4 + (\frac12-\frac1q)\|\alpha_u u^+ + \beta_u u^-\|^2 + (\frac14-\frac1q)\|\alpha_u u^+ + \beta_u u^-\|_1^4 \\
&\quad + (\frac14-\frac1q)(\alpha_u)^4\int_\Omega \phi_{u^+}(u^+)^2\mathrm{d}x + (\frac14-\frac1q)(\alpha_u)^2(\beta_u)^2\int_\Omega \phi_{u^-}(u^+)^2\mathrm{d}x \\
&\quad + (\frac14-\frac1q)(\alpha_u)^2(\beta_u)^2\int_\Omega \phi_{u^+}(u^-)^2\mathrm{d}x + (\frac14-\frac1q)(\beta_u)^4\int_\Omega \phi_{u^-}(u^-)^2\mathrm{d}x \\
&\quad + \frac{2\lambda}{q^2}\Big[\int_\Omega|\alpha_u u^+|^q\mathrm{d}x + \int_\Omega|\beta_u u^-|^q\mathrm{d}x\Big] + (\frac1q-\frac16)\Big[\int_\Omega|\alpha_u u^+|^6\mathrm{d}x + \int_\Omega|\beta_u u^-|^6\mathrm{d}x\Big] \\
&\le (\frac14-\frac1q)\|u\|_W^4 + (\frac12-\frac1q)\|u\|^2 + (\frac14-\frac1q)\|u\|_1^4 + (\frac12-\frac1q)\int_\Omega |\nabla u|^2|u|^2\,\mathrm{d}x \\
&\quad + (\frac14-\frac1q)\int_\Omega \phi_u^t u^2\mathrm{d}x + \frac{2\lambda}{q^2}\int_\Omega|u|^q\mathrm{d}x + (\frac1q-\frac16)\int_\Omega|u|^6\mathrm{d}x \\
&\le \liminf_{n\to\infty}\Big[I^\lambda_{b,\mu}(u_n) - \frac1q\langle (I^\lambda_{b,\mu})'(u_n),u_n\rangle\Big].
\end{aligned}$$

Therefore, $\alpha_u = \beta_u = 1$ and $c^\lambda_{b,\mu}$ is achieved by $u_{b,\mu} = u^+ + u^- \in \mathcal{M}^\lambda_{b,\mu}$. $\square$

4 Proof of Theorems

In order to obtain a sign-changing solution of problem (1.1), we firstly prove that $u_{b,\mu}$ is a sign-changing critical point of $I^\lambda_{b,\mu}$.

Lemma 4.1. *Fixed $\mu \in (0, 1]$, if $u_{b,\mu} \in \mathcal{M}^\lambda_{b,\mu}$ and $I^\lambda_{b,\mu}(u_{b,\mu}) = c^\lambda_{b,\mu}$, then $u_{b,\mu}$ is a sign-changing critical point of $I^\lambda_{b,\mu}$. Moreover, $u_{b,\mu}$ has exactly two nodal domains.*

Proof. Since $u_{b,\mu} \in \mathcal{M}^\lambda_{b,\mu}$, for $(\alpha, \beta) \in (\mathbb{R}^+ \times \mathbb{R}^+)\backslash(1, 1)$, we have

$$I^\lambda_{b,\mu}(\alpha u^+_{b,\mu} + \beta u^-_{b,\mu}) < I^\lambda_{b,\mu}(u^+_{b,\mu} + u^-_{b,\mu}) = c^\lambda_{b,\mu}. \tag{4.1}$$

Arguing by contradiction, we assume that $(I^\lambda_{b,\mu})'(u_{b,\mu}) \neq 0$, then there exist $\delta > 0$ and $\tau > 0$ such that

$$\|(I^\lambda_{b,\mu})'(v)\| > \tau \ \text{ for all } \|v - u_{b,\mu}\| \geq 3\delta.$$

Choose $\tau \in (0, \min\{\frac{1}{2}, \frac{\delta}{\sqrt{2}\|u_{b,\mu}\|}\})$, let

$$D = (1 - \tau, 1 + \tau) \times (1 - \tau, 1 + \tau)$$

and

$$g(\alpha, \beta) = \alpha u^+_{b,\mu} + \beta u^-_{b,\mu} \ \text{ for all } (\alpha, \beta) \in D.$$

According to (4.1), we have that

$$c_\lambda := \max_{\partial\Omega}(I^\lambda_{b,\mu}) \circ g < c^\lambda_{b,\mu}. \tag{4.2}$$

Let $\varepsilon := \min\{c^\lambda_{b,\mu} - c_\lambda, \frac{\tau\sigma}{8}\}$ and $S_\delta = B(u_{b,\mu}, \delta)$, according to Lemma 2.3 in [11], there exists a deformation $\eta \in C([0, 1] \times D, D)$ such that

(a) $\eta(1, v) = v$ *if* $v \notin (I^\lambda_{b,\mu})^{-1}([c^\lambda_{b,\mu} - 2\varepsilon, c^\lambda_{b,\mu} + 2\varepsilon] \cap S_{2\delta})$,

(b) $\eta(1, (I^\lambda_{b,\mu})^{c^\lambda_{b,\mu}+\varepsilon} \cap S_{2\delta}) \subset (I^\lambda_{b,\mu})^{c^\lambda_{b,\mu}-\varepsilon}$,

(c) $I^\lambda_{b,\mu}(\eta(1, v)) \leq I^\lambda_{b,\mu}(v)$ *for all* $v \in H$.

First, from (b) and Lemma 3.1, it is easy to see that

$$I^\lambda_{b,\mu}(g(\alpha, \beta)) < c^\lambda_{b,\mu} < c^\lambda_{b,\mu} + \varepsilon.$$

That is,

$$g(\alpha, \beta) \in (I^\lambda_{b,\mu})^{c^\lambda_{b,\mu}+\varepsilon}.$$

On the other hand, we have

$$\begin{aligned} \|g(\alpha, \beta) - u_{b,\mu}\|^2 &= \|(\alpha - 1)u^+_{b,\mu} + (\beta - 1)u^-_{b,\mu}\|^2 \\ &\leq 2[(\alpha - 1)^2\|u^+_{b,\mu}\|^2 + (\beta - 1)^2\|u^-_{b,\mu}\|^2] \\ &\leq 2\tau^2\|u_{b,\mu}\|^2 \leq \delta^2. \end{aligned}$$

Hence, by (b), we have

$$I_{b,\mu}^{\lambda}(\eta(1,g(\alpha,\beta)) \leq c_{b,\mu}^{\lambda} - \varepsilon < c_{b,\mu}^{\lambda}. \tag{4.3}$$

Next, we prove that $\eta(1,g(D)) \cap \mathcal{M}_{b,\mu}^{\lambda} \neq \emptyset$, which contradicts the definition of $c_{b,\mu}^{\lambda}$. Let $\gamma(\alpha,\beta) := \eta(1,g(\alpha,\beta))$ and

$$\begin{aligned}
\Psi_0(\alpha,\beta): &= (\langle (I_{b,\mu}^{\lambda})'(g(\alpha,\beta)), u_{b,\mu}^{+}\rangle, \langle (I_{b,\mu}^{\lambda})'(g(\alpha,\beta)), u_{b,\mu}^{-}\rangle)\\
&= (\langle (I_{b,\mu}^{\lambda})'(\alpha u_{b,\mu}^{+} + \beta u_{b,\mu}^{-}, u_{b,\mu}^{+}\rangle, \langle (I_{b,\mu}^{\lambda})'(\alpha u_{b,\mu}^{+} + \beta u_{b,\mu}^{-})), u_{b,\mu}^{-}\rangle)\\
&= (\varphi_u^1(\alpha,\beta), \varphi_u^2(\alpha,\beta))
\end{aligned}$$

and

$$\Psi_1(\alpha,\beta): \ = \Big(\frac{1}{\alpha}\langle (I_{b,\mu}^{\lambda})'(\gamma(\alpha,\beta)), (\gamma(\alpha,\beta))^{+}\rangle, \frac{1}{\beta}\langle (I_{b,\mu}^{\lambda})'(\gamma(\alpha,\beta)), (\gamma(\alpha,\beta))^{-}\rangle\Big).$$

Since $u_{b,\mu} \in \mathcal{M}_{b,\mu}^{\lambda}$, by the direct calculation, we have

$$\begin{aligned}
\frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \alpha}\Big|_{(1,1)} &= 3\|u_{b,\mu}^{+}\|_W^4 + \|u_{b,\mu}^{+}\|^2 + 3b\|u_{b,\mu}^{+}\|_1^4 + b\|u_{b,\mu}^{+}\|_1^2\|u_{b,\mu}^{-}\|_1^2\\
&\quad + 3\int_\Omega \phi_{u_{b,\mu}^{+}}(u_{b,\mu}^{+})^2 \mathrm{d}x + \int_\Omega \phi_{u_{b,\mu}^{-}}(u_{b,\mu}^{+})^2 \mathrm{d}x\\
&\quad - \lambda(q-1)\int_\Omega |u_{b,\mu}^{+}|^q \ln|u_{b,\mu}^{+}|^2 \mathrm{d}x - 2\lambda\int_\Omega |u_{b,\mu}^{+}|^q \mathrm{d}x - 5\int_\Omega |u_{b,\mu}^{+}|^6 \mathrm{d}x\\
&= (4-q)\|u_{b,\mu}^{+}\|_W^4 + (2-q)\|u_{b,\mu}^{+}\|^2 + b(4-q)\|u_{b,\mu}^{+}\|_1^4 + (2-q)b\|u_{b,\mu}^{+}\|_1^2\|u_{b,\mu}{}^{-}\|_1^2\\
&\quad + (2-q)\int_\Omega \phi_{u_{b,\mu}^{-}}(u_{b,\mu}^{+})^2 \mathrm{d}x + (4-q)\int_\Omega \phi_{u_{b,\mu}^{+}}(u_{b,\mu}^{+})^2 \mathrm{d}x\\
&\quad - 2\lambda\int_\Omega |u_{b,\mu}^{+}|^q \mathrm{d}x - (6-q)\int_\Omega |u_{b,\mu}^{+}|^6 \mathrm{d}x,
\end{aligned}$$

and

$$\frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \beta}\Big|_{(1,1)} = 2b\|u_{b,\mu}^{+}\|_1^2\|u_{b,\mu}^{-}\|_1^2 + 2\int_\Omega \phi_{u_{b,\mu}^{-}}(u_{b,\mu}^{+})^2 \mathrm{d}x.$$

Similarly,

$$\begin{aligned}
\frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \beta}\Big|_{(1,1)} &= 3\|u_{b,\mu}^{-}\|_W^4 + \|u_{b,\mu}^{-}\|^2 + 3b\|u_{b,\mu}^{-}\|_1^4 + b\|u_{b,\mu}^{+}\|_1^2\|u_{b,\mu}^{-}\|_1^2\\
&\quad + 3\int_\Omega \phi_{u_{b,\mu}^{-}}(u_{b,\mu}^{-})^2 \mathrm{d}x + \int_\Omega \phi_{u_{b,\mu}^{+}}(u_{b,\mu}^{-})^2 \mathrm{d}x\\
&\quad - \lambda(q-1)\int_\Omega |u_{b,\mu}{}^{-}|^q \ln|u_{b,\mu}^{-}|^2 \mathrm{d}x - 2\lambda\int_\Omega |u_{b,\mu}^{-}|^q \mathrm{d}x - 5\int_\Omega |u_{b,\mu}^{-}|^6 \mathrm{d}x\\
&= (4-q)\|u_{b,\mu}^{-}\|_W^4 + (2-q)\|u_{b,\mu}^{-}\|^2 + b(4-q)\|u_{b,\mu}^{-}\|_1^4 + (2-q)b\|u_{b,\mu}^{+}\|_1^2\|u_{b,\mu}^{-}\|_1^2\\
&\quad + (2-q)\int_\Omega \phi_{u_{b,\mu}^{+}}(u_{b,\mu}^{-})^2 \mathrm{d}x + (4-q)\int_\Omega \phi_{u_{b,\mu}^{-}}(u_{b,\mu}^{-})^2 \mathrm{d}x\\
&\quad - 2\lambda\int_\Omega |u_{b,\mu}{}^{-}|^q \mathrm{d}x - (6-q)\int_\Omega |u_{b,\mu}^{-}|^6 \mathrm{d}x,
\end{aligned}$$

and

$$\frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \alpha}\Big|_{(1,1)} = 2b\|u_{b,\mu}^+\|_1^2\|u_{b,\mu}{}^-\|_1^2 + 2\int_\Omega \phi_{u_{b,\mu}^+}(u_{b,\mu}^-)^2 \mathrm{d}x.$$

Let

$$M = \begin{bmatrix} \frac{\varphi_u^1(\alpha,\beta)}{\partial\alpha}\big|_{(1,1)} & \frac{\varphi_u^2(\alpha,\beta)}{\partial\alpha}\big|_{(1,1)} \\ \frac{\varphi_u^1(\alpha,\beta)}{\partial\beta}\big|_{(1,1)} & \frac{\varphi_u^2(\alpha,\beta)}{\partial\beta}\big|_{(1,1)} \end{bmatrix}.$$

Since $q > 4$, then,

$$\det M = \frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \alpha}\Big|_{(1,1)} \times \frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \beta}\Big|_{(1,1)} - \frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \beta}\Big|_{(1,1)} \times \frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \alpha}\Big|_{(1,1)} > 0.$$

Since $\Psi_0(\alpha,\beta)$ is a C^1 function and (1,1) is the unique isolated zero point of Ψ_0, by using the degree theory, we deduce that $\deg(\Psi_0, D, 0) = 1$.
Hence, combing (4.3) and (a), we obtain

$$g(\alpha,\beta) = \gamma(\alpha,\beta) \text{ on } \partial D.$$

Consequently, we obtain $\deg(\Psi_1, D, 0) = 1$. Therefore, $\Psi_1(\alpha_0,\beta_0) = 0$ for some $(\alpha_0,\beta_0) \in D$ so that

$$\eta(1, g(\alpha_0,\beta_0)) = \gamma(\alpha_0,\beta_0) \in \mathcal{M}_{b,\mu}^\lambda,$$

which is contradicted to (4.3).

Finally, we prove that $u_{b,\mu}$ has exactly two nodal domains. To this end, we assume by contradiction that

$$u_{b,\mu} = u_1 + u_2 + u_3$$

with

$$u_i \neq 0, u_1 > 0, u_2 < 0 \text{ and } suppt(u_i) \cap suppt(u_j) = \emptyset,\ for\ i \neq j,\ i,j = 1,2,3$$

and

$$\langle (I_{b,\mu}^\lambda)'(u_{b,\mu}), u_i\rangle = 0, for\ i = 1,2,3.$$

Setting $v := u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2, i.e..\ v^\pm \neq 0$. Then, there exist a unique pair of (α_v, β_v) positive numbers such that

$$\alpha_v u_1 + \beta_v u_2 \in \mathcal{M}_{b,\mu}^\lambda.$$

Hence

$$I_{b,\mu}^\lambda(\alpha_v u_1 + \beta_v u_2) \geq c_{b,\mu}^\lambda.$$

Moreover, using the fact $\langle (I_{b,\mu}^\lambda)'(u_{b,\mu}), u_i\rangle = 0$, we obtain

$$\langle (I_{b,\mu}^\lambda)'(v), v^\pm\rangle = -b\|v^\pm\|_1^2\|u_3\|_1^2 - \int_\Omega \phi_{u_3}(v^\pm)^2 \mathrm{d}x \leq 0.$$

From Lemma 3.2, we get

$$(\alpha_v, \beta_v) \in (0,1] \times (0,1].$$

On the other hand, we obatin

$$
\begin{aligned}
0 &= \frac{1}{4}\langle (I_{b,\mu}^{\lambda})'(u_{b,\mu}), u_3\rangle \\
&= \frac{\mu}{4}\|u_3\|_W^4 + \frac{1}{4}\|u_3\|^2 + \frac{b}{4}\|u_1\|_1^2\|u_3\|_1^2 + \frac{b}{4}\|u_2\|_1^2\|u_3\|_1^2 + \int_\Omega |\nabla u_3|^2|u_3|^2\mathrm{d}x \\
&\quad + \frac{b}{4}\|u_3\|_1^4 + \frac{1}{4}\int_\Omega \phi_{u_b}(u_3)^2\mathrm{d}x - \frac{1}{4}\int_\Omega |u_3|^6\mathrm{d}x - \frac{\lambda}{4}\int_\Omega |u_3|^q\ln|u_3|^2\mathrm{d}x \\
&\le I_{b,\mu}^{\lambda}(u_3) + \frac{b}{4}\|u_1\|_1^2\|u_3\|_1^2 + \frac{b}{4}\|u_2\|_1^2\|u_3\|_1^2 + \frac{1}{4}\int_\Omega \phi_{u_1}(u_3)^2\mathrm{d}x + \frac{1}{4}\int_\Omega \phi_{u_2}(u_3)^2\mathrm{d}x.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
c_{b,\mu}^{\lambda} &\le I_{b,\mu}^{\lambda}(\alpha_v u_1 + \beta_v u_2) \\
&= I_{b,\mu}^{\lambda}(\alpha_v u_1 + \beta_v u_2) - \frac{1}{4}\langle (I_{b,\mu}^{\lambda})'(\alpha_v u_1 + \beta_v u_2), \alpha_v u_1 + \beta_v u_2\rangle \\
&= \frac{1}{4}(\|\alpha_v u_1\|^2 + \|\beta_v u_2\|^2) + \frac{\lambda}{q^2}\Big[\int_\Omega |\alpha_v u_1|^q\mathrm{d}x + \int_\Omega |\beta_v u_2|^q\mathrm{d}x\Big] \\
&\quad + (\frac{1}{4} - \frac{1}{q})\lambda\Big[\int_\Omega |\alpha_v u_1|^q\ln|\alpha_v u_1|^2\mathrm{d}x + \int_\Omega |\beta_v u_2|^q\ln|\beta_v u_2|^2\mathrm{d}x\Big] \\
&\quad + \frac{1}{12}\Big[\int_\Omega |\alpha_v u_1|^6\mathrm{d}x + \int_\Omega |\beta_v u_2|^6\mathrm{d}x\Big] \\
&\le \frac{1}{4}(\|u_1\|^2 + \|u_2\|^2) + \frac{\lambda}{q^2}\Big[\int_\Omega |u_1|^q\mathrm{d}x + \int_\Omega |u_2|^q\mathrm{d}x\Big] \\
&\quad + (\frac{1}{4} - \frac{1}{q})\lambda\Big[\int_\Omega |u_1|^q\ln|u_1|^2\mathrm{d}x + \int_\Omega |u_2|^q\ln|u_2|^2\mathrm{d}x\Big] \\
&\quad + \frac{1}{12}\Big[\int_\Omega |u_1|^6\mathrm{d}x + \int_\Omega |u_2|^6\mathrm{d}x\Big] \\
&= I_{b,\mu}^{\lambda}(u_1 + u_2) - \frac{1}{4}\langle (I_{b,\mu}^{\lambda})'(u_1 + u_2), u_1 + u_2\rangle \\
&\le I_{b,\mu}^{\lambda}(u_1) + I_{b,\mu}^{\lambda}(u_2) + I_{b,\mu}^{\lambda}(u_3) + \frac{b}{4}(\|u_2\|_1^2 + \|u_3\|_1^2)\|u_1\|_1^2 \\
&\quad + \frac{b}{4}(\|u_1\|_1^2 + \|u_3\|_1^2)\|u_2\|_1^2 + \frac{b}{4}(\|u_1\|_1^2 + \|u_2\|_1^2)\|u_3\|_1^2 \\
&\quad + \frac{1}{4}\int_\Omega \phi_{u_1}(u_3)^2\mathrm{d}x + \frac{1}{4}\int_\Omega \phi_{u_2}(u_3)^2\mathrm{d}x + \frac{1}{4}\int_\Omega \phi_{u_1}(u_2)^2\mathrm{d}x \\
&\quad + \frac{1}{4}\int_\Omega \phi_{u_3}(u_2)^2\mathrm{d}x + \frac{1}{4}\int_\Omega \phi_{u_2}(u_1)^2\mathrm{d}x + \frac{1}{4}\int_\Omega \phi_{u_3}(u_1)^2\mathrm{d}x \\
&= I_{b,\mu}^{\lambda}(u) = c_{b,\mu}^{\lambda},
\end{aligned}
$$

which is a contradiction, that is, $u_3 = 0$ and $u_{b,\mu}$ has exactly two nodal domains. □

Lemma 4.2. Let $\mu_n \to 0$ and $\{u_{\mu_n}\} \subset E$ be a sequence of critical points of I_{b,μ_n}^{λ} satisfying $(I_{b,\mu_n}^{\lambda})'(u_{\mu_n}) = 0$ and $I_{b,\mu_n}^{\lambda}(u_{\mu_n}) \le C$ for some C independent of n. Then up to a subsequence $u_{\mu_n} \to u_0$ in E as $n \to \infty$ and u_0 is a critical point of I_b^{λ}.

Proof. We prove the lemma in three parts:

Claim 1. $\{u_{\mu_n}\}$ is bounded in E.

$$\begin{aligned}
C &\geq I_{b,\mu}^{\lambda}(u_{\mu_n}) - \frac{1}{q}\langle (I_{b,\mu}^{\lambda})'(u_{\mu_n}), u_{\mu_n}\rangle \\
&= \mu(\frac{1}{4}-\frac{1}{q})\|u_{\mu_n}\|_W^4 + (\frac{1}{2}-\frac{1}{q})\|u_{\mu_n}\|^2 + (\frac{1}{4}-\frac{1}{q})b\|u_{\mu_n}\|_1^4 + 2(\frac{1}{4}-\frac{1}{q})\int_\Omega |\nabla u_{\mu_n}|^2|u_{\mu_n}|^2\mathrm{d}x \\
&\quad + (\frac{1}{4}-\frac{1}{q})\int_\Omega \phi_{u_{\mu_n}} u_{\mu_n}^2\mathrm{d}x + \frac{2\lambda}{q^2}\int_\Omega |u_{\mu_n}|^q\mathrm{d}x + (\frac{1}{q}-\frac{1}{6})\int_\Omega |u_{\mu_n}|^6\mathrm{d}x \\
&\geq \mu(\frac{1}{4}-\frac{1}{q})\|u_{\mu_n}\|_W^4 + (\frac{1}{2}-\frac{1}{q})\|u_{\mu_n}\|^2,
\end{aligned}$$

which implies that $\{u_{\mu_n}\}$ is bounded in E. Then up to a subsequence, we may assume $u_{\mu_n} \rightharpoonup u_0$ in E as $n \to \infty$.

Claim 2. $\{u_{\mu_n}\}$ is bounded in L^∞. We will do this by using the Moser iteration. Recall that $\{u_{\mu_n}\}$ satisfies the equation $\left\langle (I_{b,\mu_n}^{\lambda})'(u_{\mu_n}), \varphi\right\rangle = 0$, for any $\varphi \in E$, we have

$$\begin{aligned}
&\mu_n \int_\Omega \left(|\nabla u_{\mu_n}|^2 \nabla u_{\mu_n}\nabla\varphi + |u_{\mu_n}|^2 u_{\mu_n}\varphi\right)\mathrm{d}x + \int_\Omega (a+|u_{\mu_n}|^2)\nabla u_{\mu_n}\nabla\varphi + |\nabla u_{\mu_n}|^2 u_{\mu_n}\varphi\mathrm{d}x \\
&\quad + b(\int_\Omega |\nabla u_{\mu_n}|^2\mathrm{d}x)(\int_\Omega \nabla u_{\mu_n}\nabla\varphi\mathrm{d}x) + \int_\Omega V(x)u_{\mu_n}\varphi\mathrm{d}x + \int_\Omega \phi_{u_{\mu_n}} u_{\mu_n}\varphi\mathrm{d}x \\
&= \lambda\int_\Omega |u_{\mu_n}|^{q-2}u_{\mu_n}\varphi\ln|u_{\mu_n}|^2\mathrm{d}x + \int_\Omega |u_{\mu_n}|^4 u_{\mu_n}\varphi\mathrm{d}x.
\end{aligned} \tag{4.4}$$

Now for any $T>0$, we take $\varphi = \left|u_{\mu_n}^T\right|^{2k} u_{\mu_n}$ as test functions in (4.4) with $k \geq k_0$ for some $k_0 > 0$, where $\left|u_{\mu_n}^T\right| = |u_{\mu_n}|$, if $|u_{\mu_n}| \leq T$; $|u_{\mu_n}| = T$; if $|u_{\mu_n}| \geq T$. By (4.4) and the Sobolev embedding, for $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
&\int_{\{T\leq|u_{\mu_n}|\}} \left(a+{u_{\mu_n}}^2\right)|\nabla u_{\mu_n}|^2\left|u_{\mu_n}^T\right|^{2k}\mathrm{d}x + (2k+1)\int_{\{|u_{\mu_n}|\leq T\}}\left(a+{u_{\mu_n}}^2\right)|\nabla u_{\mu_n}|^2\left|u_{\mu_n}^T\right|^{2k}\mathrm{d}x \\
&\quad + \int_\Omega |\nabla u_{\mu_n}|^2{u_{\mu_n}}^2\left|u_{\mu_n}^T\right|^{2k}\mathrm{d}x + \int_\Omega V_\infty {u_{\mu_n}}^2\left|u_{\mu_n}^T\right|^{2k}\mathrm{d}x \\
&\leq \int_\Omega |u_{\mu_n}|^6\left|u_{\mu_n}^T\right|^{2k}\mathrm{d}x + \lambda\int_\Omega |u_{\mu_n}|^q u_{\mu_n}\ln|u_{\mu_n}|^2|u_{\mu_n}^T|^{2k}\mathrm{d}x.
\end{aligned}$$

Choose $0 < \varepsilon < \frac{V_\infty}{2}$, then there exists $C > 0$ such that

$$\begin{aligned}
&\int_\Omega |u_{\mu_n}|^6\left|u_{\mu_n}^T\right|^{2k}\mathrm{d}x + \lambda\int_\Omega |u_{\mu_n}|^q u_{\mu_n}\ln|u_{\mu_n}|^2|u_{\mu_n}^T|^{2k}\mathrm{d}x \\
&\quad \leq \varepsilon\int_\Omega |u_{\mu_n}|^6\left|u_{\mu_n}^T\right|^{2k}\mathrm{d}x + C_\varepsilon\int_\Omega |u_{\mu_n}|^2\left|u_{\mu_n}^T\right|^{2k}\mathrm{d}x.
\end{aligned} \tag{4.5}$$

Hence, It follows from the above estimates and the Sobolev imbedding theorem, let $T \to \infty$, we get

$$S\left(\int_\Omega\left(u_{\mu_n}^2\left|u_{\mu_n}^T\right|^k\right)^6\mathrm{d}x\right)^{\frac{1}{3}}\leq\int_\Omega\left|\nabla\left(u_{\mu_n}^2\left|u_{\mu_n}^T\right|^k\right)\right|^2\mathrm{d}x \tag{4.6}$$
$$\leq C(k+2)^2\int_\Omega|u_{\mu_n}|^2\left(u_{\mu_n}^2\left|u_{\mu_n}^T\right|^k\right)^2\mathrm{d}x.$$

since $|u_{\mu_n}|^2\in L^s(\Omega)$ for some $s\subset(\frac{3}{2},3)$, then we have

$$|(u_{\mu_n}^2\left|u_{\mu_n}^T\right|^k)^2|_{L^3}=\left(\int_\Omega\left(u_{\mu_n}^2\left|u_{\mu_n}^T\right|^k\right)^6\mathrm{d}x\right)^{\frac{1}{3}}\leq C(k+2)^2|u_{\mu_n}|_{L^s}^2|(u_{\mu_n}^2\left|u_{\mu_n}^T\right|^k)^2|_{L^{s'}},$$

where $s'=\frac{s}{s-1}<3$. Assume $u_{\mu_n}\in L^{(4+2k)s'}$. By the above estimate we have $u_{\mu_n}\in L^{(4+2k)3}$ and

$$|u_{\mu_n}|_{L^{(4+2k)3}}\leq C(k+2)^{\frac{1}{k+2}}|u_{\mu_n}|_{L^{(4+2k)s'}}.$$

Hence, Moser's iteration implies $|u_{\mu_n}|_\infty\leq C$.

Claim 3. $u_0\in E\cap L^\infty(\Omega)$ is a solution of problem (1.1). By Claim 1, we may assume $\{u_{\mu_n}\}$ converges to u_0 weakly in E. Taking $\varphi=\psi e^{-u_{\mu_n}}$, where $\psi\in C_0^\infty(\Omega),\psi\geq0$, we have

$$\begin{aligned}
&\mu_n\int_\Omega\left(|\nabla u_{\mu_n}|^2\nabla u_{\mu_n}\left(\nabla\psi e^{-u_{\mu_n}}-\psi e^{-u_{\mu_n}}\nabla u_{\mu_n}\right)+|u_{\mu_n}|^2u_{\mu_n}\psi e^{-u_{\mu_n}}\right)\mathrm{d}x\\
&+b(\int_\Omega|\nabla u_{\mu_n}|^2\mathrm{d}x)(\int_\Omega\nabla u_{\mu_n}\left(\nabla\psi e^{-u_{\mu_n}}-\psi e^{-u_{\mu_n}}\nabla u_{\mu_n}\right)\mathrm{d}x\\
&+\int_\Omega\left(a+|u_{\mu_n}|^2\right)\nabla u_{\mu_n}\left(\nabla\psi e^{-u_{\mu_n}}-\psi e^{-u_{\mu_n}}\nabla u_{\mu_n}\right)\mathrm{d}x+\int_\Omega V(x)u_{\mu_n}\psi e^{-u_{\mu_n}}\mathrm{d}x\\
&+\int_\Omega|\nabla u_{\mu_n}|^2u_{\mu_n}\psi e^{-u_{\mu_n}}\mathrm{d}x+\int_\Omega\phi_{u_{\mu_n}}u_{\mu_n}\psi e^{-u_{\mu_n}}\mathrm{d}x\\
&-\lambda\int_\Omega|u_{\mu_n}|^{q-2}u_{\mu_n}\psi e^{-u_{\mu_n}}\ln|u_{\mu_n}|^2\mathrm{d}x-\int_\Omega|u_{\mu_n}|^4u_{\mu_n}\psi e^{-u_{\mu_n}}\mathrm{d}x\\
\leq&\mu_n\int_\Omega\left(|\nabla u_{\mu_n}|^2\nabla u_{\mu n}\nabla\psi+|u_{\mu_n}|^2u_{\mu_n}\psi\right)e^{-u_{\mu_n}}\mathrm{d}x+\int_\Omega\left(a+|u_{\mu_n}|^2\right)\nabla u_{\mu_n}\nabla\psi e^{-u_{\mu_n}}\mathrm{d}x\\
&+b(\int_\Omega|\nabla u_{\mu_n}|^2\mathrm{d}x)(\int_\Omega\nabla u_{\mu_n}\left(\nabla\psi e^{-u_{\mu_n}}\right)\mathrm{d}x+\int_\Omega V(x)u_{\mu_n}\psi e^{-u_{\mu_n}}\mathrm{d}x\\
&-\int_\Omega\left(a+|u_{\mu_n}|^2-u_{\mu_n}\right)|\nabla u_{\mu_n}|^2\psi e^{-u_{\mu_n}}\mathrm{d}x+\int_\Omega\phi_{u_{\mu_n}}u_{\mu_n}\psi e^{-u_{\mu_n}}\mathrm{d}x\\
&-\lambda\int_\Omega|u_{\mu_n}|^{q-2}u_{\mu_n}\psi e^{-u_{\mu_n}}\ln|u_{\mu_n}|^2\mathrm{d}x-\int_\Omega|u_{\mu_n}|^4u_{\mu_n}\psi e^{-u_{\mu_n}}\mathrm{d}x.
\end{aligned}$$

Since $a>\frac{1}{4}$, then $a+u_{\mu_n}^2-u_{\mu_n}\geq0$ and

$$\begin{aligned}
&\int_\Omega\left(a+|u_{\mu_n}|^2-u_{\mu_n}\right)|\nabla u_{\mu_n}|^2\psi e^{-u_{\mu_n}}\mathrm{d}x\\
=&\int_\Omega\left(a+|u_{\mu_n}|^2-u_{\mu_n}\right)\left(|\nabla\left(u_{\mu_n}-u_0\right)|^2+2\nabla u_{\mu_n}\nabla u_0-|\nabla u_0|^2\right)\psi e^{-u_{\mu_n}}\mathrm{d}x\\
\geq&\int_\Omega\left(a+|u_{\mu_n}|^2-u_{\mu_n}\right)\left(2\nabla u_{\mu_n}\nabla u_0-|\nabla u_0|^2\right)\psi e^{-u_{\mu_n}}\mathrm{d}x.
\end{aligned}$$

Let $v \geq 0, v \in C_0^\infty(\Omega)$. We choose a sequence of nonnegative functions $\{\psi_n\} \subset C_0^\infty(\Omega)$ such that $\psi_n \to ve^{u_0}$ in $E, \psi_n(x) \to ve^{u_0}(x)$ a.e. $x \in \Omega$ and $\{\psi_n\}$ is uniformly bounded in $L^\infty(\Omega)$. By approximations, we may obtain for all $v \geq 0,\ v \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}0 \leq &\int_\Omega a\nabla u_0 \nabla v + V(x)u_0 v \mathrm{d}x + b\int_\Omega |\nabla u_0|^2 \nabla u_0 \nabla v \mathrm{d}x + \int_\Omega |\nabla u_0|^2 u_0 v + |u_0|^2 \nabla u_0 \nabla v \mathrm{d}x \\ &+ \int_\Omega \phi_{u_0} u_0 v \mathrm{d}x - \lambda \int_\Omega |u_0|^{q-2} v \ln |u_0|^2 \mathrm{d}x - \int_\Omega |u_0|^4 u_0 v \mathrm{d}x.\end{aligned}$$

Take $\varphi = \psi e^{u_{\mu_n}}$ in (4.4),

$$\begin{aligned}0 \geq &\int_\Omega a\nabla u_0 \nabla v + V(x)u_0 v \mathrm{d}x + b\int_\Omega |\nabla u_0|^2 \nabla u_0 \nabla v \mathrm{d}x + \int_\Omega |\nabla u_0|^2 u_0 v + |u_0|^2 \nabla u_0 \nabla v \mathrm{d}x \\ &+ \int_\Omega \phi_{u_0} u_0 v \mathrm{d}x - \lambda \int_\Omega |u_0|^{q-2} v \ln |u_0|^2 \mathrm{d}x - \int_\Omega |u_0|^4 u_0 v \mathrm{d}x.\end{aligned}$$

Thus we have for all $v \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}&\int_\Omega a\nabla u_0 \nabla v + V(x)u_0 v \mathrm{d}x + b\int_\Omega |\nabla u_0|^2 \nabla u_0 \nabla v \mathrm{d}x \\ &+ \int_\Omega |\nabla u_0|^2 u_0 v + |u_0|^2 \nabla u_0 \nabla v \mathrm{d}x + \int_\Omega \phi_{u_0} u_0 v \mathrm{d}x \\ &= \lambda \int_\Omega |u_0|^{q-2} v \ln |u_0|^2 \mathrm{d}x + \int_\Omega |u_0|^4 u_0 v \mathrm{d}x.\end{aligned}$$

Therefore, u_0 is a critical point of I_b^λ. The proof is completed. □

Proof of Theorem 1.1. By Lemma 3.4, we choose a sequence $\mu_n \to 0$, there exists $\{u_{\mu_n}\} \subset E$ satisfying $I_{b,\mu_n}^\lambda(u_{\mu_n}) = c_{b,\mu_n}^\lambda$ and $(I_{b,\mu_n}^\lambda)'(u_{\mu_n}) = 0$.

Claim 1. Problem (1.1) possesses one sign-changing solution u_0.

Assume $\varphi \in C_0^\infty(\Omega))$ with $\varphi^\pm \neq 0$, there is a pair of positive numbers (α_0, β_0) independent of n such that

$$\langle I_{b,\mu_n}^\lambda(\alpha_0\varphi^+ + \beta_0\varphi^-), \alpha_0\varphi^+\rangle \leq \langle I_{b,1}^\lambda(\alpha_0\varphi^+ + \beta_0\varphi^-), \alpha_0\varphi^+\rangle < 0,$$

and

$$\langle I_{b,\mu_n}^\lambda(\alpha_0\varphi^+ + \beta_0\varphi^-), \beta_0\varphi^-\rangle \leq \langle I_{b,1}^\lambda(\alpha_0\varphi^+ + \beta_0\varphi^-), \beta_0\varphi^-\rangle < 0.$$

Let $\varphi_1 = \alpha_0\varphi^+ + \beta_0\varphi^-$, according to Lemma 3.1 that there exists a unique pair of positive numbers $(\alpha_n, \beta_n) \subset (0,1] \times (0,1]$ such that $\alpha_n\varphi_1^+ + \beta_n\varphi_1^- \in \mathcal{M}_{b,\mu_n}^\lambda$, we have that

$$
\begin{aligned}
c_{b,\mu_n}^{\lambda} \leq & I_{b,\mu_n}^{\lambda}\left(\alpha_n\varphi_1^{+}+\beta_n\varphi_1^{-}\right)-\frac{1}{4}\left\langle (I_{b,\mu_n}^{\lambda})'\left(\alpha_n\varphi_1^{+}+\beta_n\varphi_1^{-}\right),\alpha_n\varphi_1^{+}+\beta_n\varphi_1^{-}\right\rangle \\
= & \frac{1}{4}\left((\alpha_n)^2\left\|\varphi_1^{+}\right\|^2+(\beta_n)^2\left\|\varphi_1^{-}\right\|^2\right)+\frac{1}{12}\int_\Omega\left((\alpha_n)^6\left|\varphi_1^{+}\right|^6+(\beta_n)^6\left|\varphi_1^{-}\right|^6\right)\mathrm{d}x \\
& +\frac{\lambda}{q^2}\int_\Omega|\alpha_n\varphi_1^{+}|^q\mathrm{d}x+\frac{\lambda}{q^2}\int_\Omega|\beta_n\varphi_1^{-}|^q\mathrm{d}x+(\frac{1}{4}-\frac{1}{q})\int_\Omega\left(|\alpha_n\varphi_1^{+}|^q\ln|\alpha_n\varphi_1^{+}|^2\right)\mathrm{d}x \\
& +(\frac{1}{4}-\frac{1}{q})\int_\Omega\left(|\beta_n\varphi_1^{-}|^q\ln|\beta_n\varphi_1^{-}|^2\right)\mathrm{d}x \\
\leq & \frac{1}{4}\left(\left\|\varphi_1^{+}\right\|^2+\left\|\varphi_1^{-}\right\|^2\right)+\frac{1}{12}\int_\Omega\left(\left|\varphi_1^{+}\right|^6+\left|\varphi_1^{-}\right|^6\right)\mathrm{d}x+\frac{\lambda}{q^2}\int_\Omega(|\varphi_1^{+}|^q+|\varphi_1^{-}|^q)\mathrm{d}x \\
& +\int_\Omega\left((\frac{1}{4}-\frac{1}{q})|\ \varphi_1^{+}|^q\ln|\varphi_1^{+}|^2\right)\mathrm{d}x+\int_\Omega\left((\frac{1}{4}-\frac{1}{q})|\varphi_1^{-}|^q\ln|\varphi_1^{-}|^2\right)\mathrm{d}x \\
= & I_{b,1}^{\lambda}\left(\varphi_1\right)-\frac{1}{4}\left\langle (I_{b,1}^{\lambda})'\left(\varphi_1\right),\varphi_1\right\rangle,
\end{aligned}
$$

Therefore, $\left\{c_{b,\mu_n}^{\lambda}\right\}$ is bounded, according to Lemma 4.2, there exists a critical point u_0 of I_b^{λ} such that $u_{\mu_n}\to u_0$ in E.

Now we prove $u_0^{\pm}\neq 0$. Since $u_{\mu_n}\in\mathcal{M}_{b,\mu_n}^{\lambda}$, we have that

$$
\begin{aligned}
& \mu_n\left\|u_{\mu_n}^{\pm}\right\|_W^4+\left\|u_{\mu_n}^{\pm}\right\|^2+b\|u_{\mu_n}^{\pm}\|_1^4+2\int_\Omega\left|\nabla u_{\mu_n}^{\pm}\right|^2\left|u_{\mu_n}^{\pm}\right|^2\mathrm{d}x \\
& +\int_\Omega\phi_{u_{\mu_n}^{\pm}}\left|u_{\mu_n}^{\pm}\right|^2\mathrm{d}x+\int_\Omega\phi_{u_{\mu_n}^{\mp}}\left|u_{\mu_n}^{\pm}\right|^2\mathrm{d}x \\
= & \int_\Omega\left|u_{\mu_n}^{\pm}\right|^6\mathrm{d}x+\lambda\int_\Omega\left|u_{\mu_n}^{\pm}\right|^q\ln|u_{\mu_n}^{\pm}|^2\mathrm{d}x.
\end{aligned}
\tag{4.7}
$$

So, for $\mu_n\to 0$ and (3.2), we have that

$$
\begin{aligned}
\|u_0{}^{\pm}\|^2 \leq & \int_\Omega|u_0^{\pm}|^6\mathrm{d}x+\lambda\int_\Omega|u_0^{\pm}|^q\ln|u_0^{\pm}|^2\mathrm{d}x \\
\leq & C_8\|u_0^{\pm}\|^6+\lambda\varepsilon C_9\|u_0^{\pm}\|^2+\lambda C_\varepsilon C_{10}\|u_0^{\pm}\|^r.
\end{aligned}
$$

Thus, we get

$$
(1-\lambda\varepsilon C_9)\left\|u_0^{\pm}\right\|^2\leq C_8\left\|u_0^{\pm}\right\|^6+\lambda C_\varepsilon C_{10}\left\|u_0^{\pm}\right\|^r.
$$

Choosing ε small enough such that $(1-\mu\varepsilon C_9)>0$, since $r>4$, there exists u_0 such that

$$
\|u_0{}^{\pm}\|^2\geq\rho>0.
$$

Therefore, $u_0{}^{\pm}\neq 0$. Then we obtain that u_0 is a sign-changing solution of (1.1).

Claim 2. u_0 has also exactly two nodal domains.

Since u_0 is a nonzero critical point of I_b^{λ}, we have that

$$
\begin{aligned}
& a\int_\Omega|\nabla u_0|^2+V(x)\left|u_0\right|^2\mathrm{d}x+b\Big(\int_\Omega|\nabla u_0|^2\mathrm{d}x\Big)^2+2\int_\Omega|\nabla u_0|^2\left|u_0\right|^2\mathrm{d}x \\
& +\int_\Omega\phi_{u_0}\left|u_0\right|^2\mathrm{d}x-\lambda\int_\Omega|u_0|^q\ln|u_0|^2\mathrm{d}x-\int_\Omega\left|u_0\right|^6\mathrm{d}x=0.
\end{aligned}
\tag{4.8}
$$

On the other hand, $\langle (I_{b,\mu_n}^{\lambda})'(u_{\mu_n}), u_{\mu_n}\rangle = 0$ implies that

$$\begin{aligned}&\mu_n\int_\Omega\left(|\nabla u_{\mu_n}|^4+|u_{\mu_n}|^4\right)\mathrm{d}x+a\int_\Omega|\nabla u_{\mu_n}|^2+V(x)|u_{\mu_n}|^2\,\mathrm{d}x+b\Big(\int_\Omega|\nabla u_{\mu_n}|^2\,\mathrm{d}x\Big)^2\\&+2\int_\Omega|\nabla u_{\mu_n}|^2|u_{\mu_n}|^2\,\mathrm{d}x+\int_\Omega\phi_{u_{\mu_n}}|u_{\mu_n}|^2\,\mathrm{d}x-\lambda\int_\Omega|u_{\mu_n}|^q\ln|u_{\mu_n}|^2\mathrm{d}x-\int_\Omega|u_{\mu_n}|^6\,\mathrm{d}x=0.\end{aligned}$$

According to (3.2), we have that

$$\lim_{n\to\infty}\lambda\int_\Omega(|u_{\mu_n}|^q)\ln|u_{\mu_n}|^2\mathrm{d}x\to\lambda\int_\Omega(|u_0|^q)\ln|u_0|^2\mathrm{d}x. \tag{4.9}$$

Moreover, according to the proof in Lemma 3.4 that $B_1=B_2=0$, we have

$$\lim_{n\to\infty}\int_\Omega|u_{\mu_n}|^6\,\mathrm{d}x=\int_\Omega|u_0|^6\,\mathrm{d}x=0. \tag{4.10}$$

Then, combining (4.7)-(4.10) and using Fatou's lemma and weak semicontinuity of norm, up to a subsequence, we get that

$$\begin{aligned}&\int_\Omega a|\nabla u_0|^2+V(x)|u_0|^2\,\mathrm{d}x+b\Big(\int_\Omega|\nabla u_0|^2\,\mathrm{d}x\Big)^2\\&\le\lim_{n\to\infty}\int_\Omega\left(a|\nabla u_{\mu_n}|^2+V(x)|u_{\mu_n}|^2\right)\mathrm{d}x+b\Big(\int_\Omega|\nabla u_{\mu_n}|^2\,\mathrm{d}x\Big)^2\\&=\lim_{n\to\infty}\left(\lambda\int_\Omega|u_{\mu_n}|^q\ln|u_{\mu_n}|^2\mathrm{d}x+\int_\Omega|u_{\mu_n}|^6\,\mathrm{d}x-2\int_\Omega|\nabla u_{\mu_n}|^2|u_{\mu_n}|^2\,\mathrm{d}x-\int_\Omega\phi_{u_{\mu_n}}|u_{\mu_n}|^2\,\mathrm{d}x\right)\\&=\int_\Omega(a|\nabla u_0|^2+V(x)|u_0|^2)\mathrm{d}x+b\Big(\int_\Omega|\nabla u_0|^2\,\mathrm{d}x\Big)^2+2\int_\Omega|\nabla u_0|^2|u_0|^2\,\mathrm{d}x+\int_\Omega\phi_{u_0}|u_0|^2\,\mathrm{d}x\\&-\lim_{n\to\infty}\left(2\int_\Omega|\nabla u_{\mu_n}|^2|u_{\mu_n}|^2\,\mathrm{d}x-\int_\Omega\phi_{u_{\mu_n}}|u_{\mu_n}|^2\,\mathrm{d}x\right)\\&\le\int_\Omega(a|\nabla u_0|^2+V(x)|u_0|^2)\mathrm{d}x+b\Big(\int_\Omega|\nabla u_0|^2\,\mathrm{d}x\Big)^2.\end{aligned}$$

So that $\lim_{n\to\infty}\|u_{\mu_n}\|^2=\|u_0\|^2$. According to Brzis-Lied Lemma [17] that $u_{\mu_n}\to u_0$ strongly in E as $n\to\infty$. That means that u_0 has also exactly two nodal domains.

Claim 3. u_0 is a least-energy sign-changing solution. By Lemma 3.1 it is easy to see that there exists a unique pair $(\alpha_{\mu_n},\beta_{\mu_n})\in(0,\infty)\times(0,\infty)$ such that $\alpha_{\mu_n}u_0^++\beta_{\mu_n}u_0^-\in\mathcal{M}_{b,\mu_n}^{\lambda}$. Then we have

$$\begin{aligned}&\mu_n\alpha_{\mu_n}^4\left\|u_0^+\right\|_W^4+\alpha_{\mu_n}^4\left\|u_0^+\right\|^2+b\alpha_{\mu_n}^2\left\|u_0^+\right\|_1^4+b\alpha_{\mu_n}^2\beta_{\mu_n}^2\left\|u_0^+\right\|_1^2\left\|u_0^-\right\|_1^2\\&+2\alpha_{\mu_n}^4\int_\Omega\left|\nabla u_0^+\right|^2\left|u_0^+\right|^2\mathrm{d}x+\alpha_{\mu_n}^4\int_\Omega\phi_{u_0^+}\left|u_0^+\right|^2\mathrm{d}x+\alpha_{\beta_n}^2\beta_{\mu_n}^2\int_\Omega\phi_{u_0^-}\left|u_0^+\right|^2\mathrm{d}x\\&=\lambda\int_\Omega|\alpha_{\mu_n}u_0^+|^q\ln|\alpha_{\mu_0}u_0|^2\mathrm{d}x+\alpha_{\mu_n}^6\int_\Omega\left|u_0^+\right|^6\mathrm{d}x,\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}&\mu_n\beta_{\mu_n}^4\left\|u_0^-\right\|_W^4+\beta_{\mu_n}^4\left\|u_0^-\right\|^2+b\beta_{\mu_n}^2\left\|u_0^-\right\|_1^4+b\alpha_{\mu_n}^2\beta_{\mu_n}^2\left\|u_0^+\right\|_1^2\left\|u_0^-\right\|_1^2\\&+2\beta_{\mu_n}^4\int_\Omega\left|\nabla u_0^-\right|^2\left|u_0^-\right|^2\mathrm{d}x+\beta_{\mu_n}^4\int_\Omega\phi_{u_0^-}\left|u_0^-\right|^2\mathrm{d}x+\alpha_{\mu_n}^2\beta_{\mu_n}^2\int_\Omega\phi_{u_0^+}\left|u_0^-\right|^2\mathrm{d}x\\&=\lambda\int_\Omega|\beta_{\mu_n}u_0^-|^q\ln|\beta_{\mu_0}u_0^-|^2\mathrm{d}x+\beta{\mu_n}^6\int_\Omega\left|u_0^-\right|^6\mathrm{d}x.\end{aligned}\tag{4.12}$$

According to $\mu_n \to 0$ as $n \to \infty$, $\{\alpha_{\mu_n}\}$ and $\{\beta_{\mu_n}\}$ are bounded. Up to a subsequence, suppose that $\alpha_{\mu_n} \to \alpha_0$ and $\beta_{\mu_n} \to \beta_0$, then it follows from (4.11) and (4.12) that

$$\begin{aligned}
&\alpha_0^2 \left\|u_0^+\right\|^2 + b\alpha_0^2 \left\|u_0^+\right\|_1^4 + b\alpha_0^2\beta_0^2 \left\|u_0^+\right\|_1^2 \left\|u_0^-\right\|_1^2 \\
&+ 2\alpha_0^4 \int_\Omega \left|\nabla u_0^+\right|^2 \left|u_0^+\right|^2 \mathrm{d}x + \alpha_0^4 \int_\Omega \phi_{u_0^+} \left|u_0^+\right|^2 \mathrm{d}x + \alpha_0^2\beta_0^2 \int_\Omega \phi_{u_0^-} \left|u_0^+\right|^2 \mathrm{d}x \\
=&\lambda \int_\Omega |\alpha_0 u_0^+|^q \ln |\alpha_0 u_0^+|^2 \mathrm{d}x + \alpha_0^6 \int_\Omega \left|u_0^+\right|^6 \mathrm{d}x,
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
&\beta_0^2 \left\|u_0^-\right\|^2 + b\beta_0^2 \left\|u_0^-\right\|_1^4 + b\alpha_0^2\beta_0^2 \left\|u_0^+\right\|_1^2 \left\|u_0^-\right\|_1^2 \\
&+ 2\beta_0^4 \int_\Omega \left|\nabla u_0^-\right|^2 \left|u_0^-\right|^2 \mathrm{d}x + \beta_0^4 \int_\Omega \phi_{u_0^-} \left|u_0^-\right|^2 \mathrm{d}x + \alpha_0^2\beta_0^2 \int_\Omega \phi_{u_0^+} \left|u_0^-\right|^2 \mathrm{d}x \\
=&\lambda \int_\Omega \left(|\beta_0 u_0^-|^q\right) \ln |\beta_{\mu_0} u_0^-|^2 \mathrm{d}x + \beta_0^6 \int_\Omega \left|u_0^-\right|^6 \mathrm{d}x.
\end{aligned} \tag{4.14}$$

Because u_0 is a sign-changing solution of problem (1.1), there holds

$$\begin{aligned}
&\left\|u_0^+\right\|^2 + b \left\|u_0^+\right\|_1^4 + b \left\|u_0^+\right\|_1^2 \left\|u_0^-\right\|_1^2 \\
&+ 2 \int_\Omega \left|\nabla u_0^+\right|^2 \left|u_0^+\right|^2 \mathrm{d}x + \int_\Omega \phi_{u_0^+} \left|u_0^+\right|^2 \mathrm{d}x + \int_\Omega \phi_{u_0^-} \left|u_0^+\right|^2 \mathrm{d}x \\
=&\lambda \int_\Omega \left(|u_0^+|^q\right) \ln |u_0^+|^2 \mathrm{d}x + \int_\Omega \left|u_0^+\right|^6 \mathrm{d}x,
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&\left\|u_0^-\right\|^2 + b \left\|u_0^-\right\|_1^4 + b \left\|u_0^+\right\|_1^2 \left\|u_0^-\right\|_1^2 \\
&+ 2 \int_\Omega \left|\nabla u_0^-\right|^2 \left|u_0^-\right|^2 \mathrm{d}x + \int_\Omega \phi_{u_0^-} \left|u_0^-\right|^2 \mathrm{d}x + \int_\Omega \phi_{u_0^+} \left|u_0^-\right|^2 \mathrm{d}x \\
=&\lambda \int_\Omega \left(|u_0^-|^q\right) \ln |u_0^-|^2 \mathrm{d}x + \int_\Omega \left|u_0^-\right|^6 \mathrm{d}x.
\end{aligned} \tag{4.16}$$

Hence, in view of (4.11)-(4.16), we can easily obtain that $(\alpha_0, \beta_0) = (1, 1)$.
According to Fatou's lemma and weak semicontinuity of norm, we get

$$\begin{aligned}
&I_b^\lambda (u_0) - \frac{1}{4} \left\langle (I_b^\lambda)' (u_0), u_0 \right\rangle \\
=&\frac{1}{4} \|u_0\|^2 + \frac{1}{12} |u_0|_6^6 + (\frac{1}{4} - \frac{1}{q})\lambda \Big[\int_\Omega |u_0|^q \ln |u_0|^2 \mathrm{d}x \Big] + \frac{2\lambda}{q^2} \int_\Omega |u_0|^q \mathrm{d}x \\
\le&\frac{1}{4} \|u_{\mu_n}\|^2 + \frac{1}{12} |u_{\mu_n}|_6^6 + (\frac{1}{4} - \frac{1}{q})\lambda \Big[\int_\Omega |u_n|^q \ln |u_n|^2 \mathrm{d}x \Big] + \frac{2\lambda}{q^2} \int_\Omega |u_n|^q \mathrm{d}x \\
\le&\liminf_{n\to\infty} \left[I_{b,\mu_n}^\lambda (u_{\mu_n}) - \frac{1}{4} \left\langle (I_{b,\mu_n}^\lambda)' (u_{\mu_n}), u_{\mu_n} \right\rangle \right] \\
=&\liminf_{n\to\infty} I_{b,\mu_n}^\lambda (u_{\mu_n}) = \lim_{n\to\infty} c_{b,\mu_n}^\lambda = c_{b,0}^\lambda.
\end{aligned}$$

Moreover,

$$c_{b,0}^\lambda = \liminf_{n\to\infty} I_{b,\mu_n}^\lambda (u_{\mu_n}) \le \liminf_{n\to\infty} I_{b,\mu_n}^\lambda (\alpha_{\mu_n} u_0^+ + \beta_{\mu_n} u_0^-) = I_b^\lambda \left(u_0^+ + u_0^-\right) = I_b^\lambda (u_0).$$

So $I_b^\lambda(u_0) = \liminf_{n\to\infty} I_{b,\mu_n}^\lambda (u_{\mu_n}) = c_{b,0}^\lambda$, The proof is completed. □

By Theorem 1.1, we obtain a least-energy sign-changing solution u_0 of problem (1.1). Next, we prove that the energy of u_0 is strictly larger than two times the least energy.

Proof of Theorem 1.2. Similar to the proof of Lemma 3.4, there exists $\lambda^{**} > 0$ such that for all $\lambda \geq \lambda^{**}$ and when $\mu \to 0$, there is $v_\mu \in \mathcal{N}_b^\lambda$ such that $I_{b,\mu}^\lambda (v_\mu) = c^* > 0$. By standard arguments (see Corollary 2.13 in [10]), the critical points of the functional $I_{b,\mu}^\lambda$ on $\mathcal{N}_b^\lambda$ are critical points of $I_{b,\mu}^\lambda$ in E.

For all $\lambda \geq \lambda^*$, according to Theorem 1.1, for each $\mu \to 0$, we know that the problem (1.1) has a least-energy sign-changing solution u_0 which changes sign only once. Let $\lambda_2 = \max\{\lambda^*, \lambda^{**}\}$. Suppose that $u_0 = u^+ + u^-$. As the proof of Lemma 3.1, there exist $\alpha_{u^+}, \beta_{u^-} \in (0,1)$ such that $\alpha_{u^+}u^+ \in \mathcal{N}_b^\lambda, \beta_{u^-}u^- \in \mathcal{N}_b^\lambda$. Therefore, in view of Lemma 3.1, we have that

$$\begin{aligned} 2c^* &\leq \liminf_{\mu\to 0} \left[I_{b,\mu}^\lambda \left(\alpha_{u^+}u^+\right) + I_{b,\mu}^\lambda \left(\beta_{u^-}u^-\right) \right] \\ &\leq \liminf_{\mu\to 0} I_{b,\mu}^\lambda \left(\alpha_{u^+}u^+ + \beta_{u^-}u^-\right) < \liminf_{\mu\to 0} I_{b,\mu}^\lambda \left(u^+ + u^-\right) = I_b^\lambda (u_0) . \end{aligned}$$

which shows that $I_b^\lambda (u_0) > 2c^*$ and $c^* > 0$ cannot be achieved by a sign-changing function in E.

□

References

[1] A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, Trans. Amer. Math. Soc., 348 (1996), 305-330.

[2] A. Nakamura, Damping and Modification of Exciton Solitary Waves, J. Phys. Soc. Japan, 42 (1977) 1824-1835.

[3] C. Miranda, Unosservazione su un teorema di Brouwer, Boll. Un. Mat. Ital., 3 (1940), 5-7.

[4] C. O. Alves, F.J.S.A. Correa, On existence of solutions for a class of problem involving a nonlinear operator, Comm. Appl. Nonlinear Anal., 8 (2001), 43-56.

[5] D. Wang, Least energy sign-changing solutions of Kirchhoff-type equation with critical growth, J.Math. Phys, 61 (2020) 011501.

[6] F. Li, X. Zhu, Multiple solutions to a class of generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation, J. Math. Anal. Appl. , 2016.

[7] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[8] G. M. Figueiredo, G. Siciliano, Quasilinear Schrödinger-Poisson system under an exponential critical nonlinearity: existence and asymptotic of solutions, Arch. Math., 112 (2019) 313-327.

[9] G. M. Figueiredo, G. Siciliano, Existence and asymptotic behaviour of solutions for a quasilinear Schrödinger-Poisson system under a critical nonlinearity, arXiv:1707.05353 (2017).

[10] X. He, W. Zou, Ground states for nonlinear Kirchhoff equations with critical growth, Ann. Mat.Pura. Appl, 193 (2014), 473-500.

[11] J. Chen, X. Tang, Z. Gao, Ground state sign-changing solutions for a class of generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation, J. Fixed Point Theory Appl., 2017.

[12] J. L. Lions. On some questions in boundary value problems of mathematical physics-sciencedirect, North-Holland Mathematics Studies, 30 (1978), 284-346.

[13] K. Teng, Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent, J. Differential Equations, 261 (261) 3061-3106.

[14] Kurihara, Susumu, Large-Amplitude Quasi-Solitons in Superfluid Films, J. Phys. Soc. Japan, 50 (1981) 3262-3267.

[15] L. Wang, On a quasilinear Schrödinger-Kirchhoff-type equation with radial potentials, Nonlinear Anal., 83 (2013) 58-68.

[16] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J.A. Soriano, Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation, Adv. Differential Equations, 6 (2001), 701-730.

[17] M. Willem. Minimax Theorems, Birkh$\ddot{a}$user Basel, (1996).

[18] P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math., 108 (1992), 247-262.

[19] R. Illner, O. Kavian, H. Lange, Stationary Solutions of Quasi-Linear Schrödinger-Poisson Systems, J. Differential Equations, 145 (1998) 1-16.

[20] S. Liang, V. D. Rdulescu, Least-energy nodal solutions of critical Kirchhoff problems with logarithmic nonlinearity, Anal. Math. Phys., 2020, 10(4):45.

[21] T. Bartsch, T. Weth, Three nodal solutions of singularly perturbed elliptic equations on domains without topology, NonLinear Anal., 22 (2005) 259-281.

[22] W. Shuai, Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains, J. Differential Equations, 259 (2015), 1256-1274.

[23] X. Feng, Y. Zhang, Existence of non-trivial solution for a class of modified Schrödinger-Poisson equations via perturbation method, J. Math. Anal. Appl., 42 (2016) 673-684.

[24] X. Liu, J. Liu, Q.Z. Wang, Quasilinear elliptic equations with critical growth via perturbation method, J. Differential Equations, 254 (2013).

[25] X. Tang, B. Cheng, Ground state sign-changing solutions for Kirchhoff type problems in bounded domains, J. Differential Equations, 261 (2016), 2384-2402.

[26] Y. Deng, W. Shuai, Sign-changing solutions for non-local elliptic equations involving the fractional Laplacian, Adv. Differential Equations, 23 (2018), 109-134.

[27] Y. Li, D. Wang, J. Zhang, Sign-changing solutions for a class of p-Laplacian Kirchhoff-type problem with logarithmic nonlinearity, AIMS. Mathematics, 5 (2020) 2100-2112.