Positive solutions for the critical fractional Kirchhoff-type equations with logarithmic nonlinearity and steep potential well

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Abstract

\begin{abstract} {In this paper, we study a class of critical fractional Kirchhoff-type equations involving logarithmic nonlinearity and steep potential well in $\R^N$ as following: \begin{align*} \renewcommand{\arraystretch}{1.25} \begin{array}{ll} \ds \left \{ \begin{array}{ll} \ds \left(a+b\int_{\R^{N}}|(-\Delta)^\frac{s}{2}u|^2\, dx\right)(-\Delta)^s u+\mu V(x)u=\lambda a(x)u\ln|u|+|u|^{2_{s}^{*}-2}u~~~\text{in}~\mathbb{R}^N, \\ u\in H^s(\R^N), \\ \end{array} \right . \end{array} \end{align*} where $a>0$ is a constant, b is a positive parameter, $s\in(0,1)$ and $N>4s,$ $\mu>0$ is a parameter and $V(x)$ satisfies some assumptions that will be specified later. By applying the Nehari manifold method, we obtain that such equation with sign-changing weight potentials admits at least one positive ground state solution and the associated energy is negative. Moreover, we also explore the asymptotic behavior as $b\to 0$ and $\mu\to\infty,$ respectively.}

Positive solutions for the critical fractional Kirchhoff-type equations with logarithmic nonlinearity and steep potential well*

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Abstract

In this paper, we study a class of critical fractional Kirchhoff-type equations involving logarithmic nonlinearity and steep potential well in $\mathbb{R}^N$ as following:

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\,dx\right)(-\Delta)^s u+\mu V(x)u=\lambda a(x)u\ln|u|+|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^N, \\ u\in H^s(\mathbb{R}^N), \end{cases}$$

where $a>0$ is a constant, b is a positive parameter, $s\in(0,1)$ and $N>4s$, $\mu>0$ is a parameter and $V(x)$ satisfies some assumptions that will be specified later. By applying the Nehari manifold method, we obtain that such equation with sign-changing weight potentials admits at least one positive ground state solution and the associated energy is negative. Moreover, we also explore the asymptotic behavior as $b\to 0$ and $\mu\to\infty$, respectively.

Keywords: Fractional Kirchhoff equations; Positive solutions; Logarithmic nonlinearity; Critical Sobolev exponent; Nehari manifold

MSC Classification: 35A15; 35B09; 35B33; 35R11; 35J60

1 Introduction

The aim of this paper is to discuss the existence of positive solutions for the following critical fractional Kirchhoff-type equations with logarithmic nonlinearity and steep potential well

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\,dx\right)(-\Delta)^s u+\mu V(x)u=\lambda a(x)u\ln|u|+|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^N, \\ u\in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $s\in(0,1),\ \lambda>0,\ N>4s$, $a(x)$ is continuous and bounded weight potentials, $b>0$ small enough and $2_s^*=\frac{2N}{N-2s}$ is the fractional critical Sobolev exponent. Eq. (1.1) is usually called of fractional

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Kirchhoff type, because of the present of fractional operator and Kirchhoff nonlocal term. Denote the fractional Sobolev space $H^s(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with the norm:

$$\|u\|_{H^s} := \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}|^2\, dx\right)^{\frac{1}{2}} + |u|_2.$$

Then $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N), r \in [2, 2_s^*]$ and this embedding is locally compact while $r \in [1, 2_s^*)$ (see [24]).

More precisely, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where $\rho,\ \rho_0,\ h,\ E,\ L$ are constants. This nonlocal model extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Since Lions [19] introduced an abstract framework to Kirchhoff-type equations, the solvability of these nonlocal problems has been well studied in the general dimension by various authors. We refer to D'Ancona and Shibata [11] and D'Ancona and Spagnolo [12] for the global solvability of various classes of Kirchhoff-type problems. We also refer to Carrier [7, 8] who used a more rigorous method to model transverse vibration via the coupled governing equation of planar vibration in order to recover the nonlinear integro partial-differential equation, in which a more general Kirchhoff function was considered. For more details on mathematical theories and its applications of Kirchhoff-type problems, we refer the readers to [1, 12, 17, 21, 22, 29].

Fiscella and Valdinoci [15] studied the following fractional Kirchhoff type equation

$$\begin{cases} M\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \dfrac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u, \quad x\in\Omega, \\ u = 0, \quad x \in \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain and Kirchhoff function M is nondegenerate. By using the mountain pass theorem and the concentration compactness principle, together with a truncation technique, they obtained the existence of non-negative solutions for problem (1.3), see [15, 4, 14] for more physical background involving this subject.

In a recent paper [28], Shuying Tian studied the multiple solutions for a semilinear elliptic equation on a bounded domain with the sign-changing logarithmic nonlinearity. Namely she proved that the following problem

$$\begin{cases} -\Delta u = a(x)u\log|u|, \quad x\in\Omega, \\ u = 0, \quad on\ \partial\Omega \end{cases}$$

has at least two nontrivial solutions provided that $a \in C(\bar\Omega)$ changes sign on Ω, and

$$\max_{x\in\bar\Omega}|a(x)| < 2\pi\exp\left(2 - \frac{4|\Omega|}{ne}\right).$$

Tian's results are quite different from these in the polynomial nonlinearities case, see [2, 6, 31]. Its proof is based on the consideration of the Nehari's manifold associated with the energy function and the using logarithmic Sobolev's inequality. In [27], Truong studied the following fractional p-Laplacian equation with logarithmic nonlinearity

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda a(x)|u|^{p-2}u \ln |u|, \quad x \in \mathbb{R}^N,$$

where $a(x)$ is a sign-changing function. Under some assumptions on V, a and λ, [27] obtained two nontrivial solutions by using Nehari manifold approach. Haining Fan [16] studied the following fractional Schrödinger equations involving logarithmic and critical nonlinearities in $\mathbb{R}^N$

$$(-\Delta)^\alpha + u = \lambda a(x)u \ln |u| + b(x)|u|^{2_\alpha^*-2}u, \quad x \in \mathbb{R}^N.$$

By applying the Nehari manifold and Ljusternik-Schnirelmann category, obtain how the weight potential affects the multiplicity of positive solutions and relationship between the number of positive solutions and the category of some sets related to the weight potential.

It is natural to wonder what the results would be if the fractional Kirchhoff-type equation involving logarithmic nonlinearity and steep potential well in $\mathbb{R}^N$. Motivated by the above discussion, the main goal of this paper is to study the existence of a positive ground state solution for (1.1). To our best knowledge, the critical fractional Kirchhoff-type equations involving logarithmic nonlinearity and steep potential well in $\mathbb{R}^N$ has not been studied yet.

Before stating our main results, we introduce some assumptions on $a(x)$ and $V(x)$:

(A_1) $\lim\limits_{|x|\to\infty} a(x) = 0, x \in \mathbb{R}^N$.

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ *and* $V(x) \geq 0$ *on* $\mathbb{R}^N$.

(V_2) *There is* $c > 0$ *such that* $\mathcal{V}_c := \{x \in \mathbb{R}^N \mid V(x) < c\}$ *is nonempty and has finite measure.*

(V_3) $\Omega = int\ V^{-1}(0)$ *is a nonempty open set with locally Lipschitz boundary and* $\overline{\Omega} = V^{-1}(0)$.

This type of assumptions was first introduced by Bartsch and Wang [5] and they considered a nonlinear Schrödinger equation. In recent years, elliptic equations with steep potential well received much attention of researchers, see e.g.[3, 13, 18, 32, 33]. The hypotheses $(V_1)-(V_3)$ imply that $V(x)$ represents a potential well whose depth is controlled by μ, so $V(x)$ is called a steep potential well if μ is sufficiently large. It is worth mentioning that we do not impose any other hypotheses on the behavior of $V(x)$ for $|x| \to \infty$. We expect to find solutions which are localized near the bottom of the potential $V(x)$.

Theorem 1.1. *Assume that condition* (A_1) *and* $(V_1)-(V_3)$ *hold and* $a(x)$ *is negative or sign-changing. Then there exists* $\Lambda_1 > 0$ *and* $\mu^* > 1$ *such that if* $\lambda \in (0, \Lambda_1)$ *and* $\mu \in (\mu^*, \infty)$, *equation* (1.1) *has a positive ground state solution and the ground energy of* (1.1) *is negative.*

Theorem 1.2. *Assume that condition* (A_1) *and* $(V_1)-(V_3)$ *hold, let* $\varphi_{b,\mu}^+$ *be the positive solution of* (1.1) *obtained by Theorem 1.1. Then there exists* $b_* > 0$ *such that for each* $b \in (0, b_*)$ *fixed,* $\varphi_{b,\mu}^+ \to \varphi_b^+$ *in* $H^s(\mathbb{R}^N)$ *as* $\mu \to \infty$ *up to a subsequence, where* φ_b^+ *is a positive solution of*

$$\begin{cases} \left(a + b\displaystyle\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\, dx\right)(-\Delta)^s u = \lambda a(x)u \ln |u| + |u|^{2_s^*-2}u & \textit{in } \Omega, \\ u\,|_{\partial\Omega} = 0. \end{cases} \tag{1.4}$$

Theorem 1.3. *Assume that condition* (A_1) *and* $(V_1)-(V_3)$ *hold, let* $\varphi^+_{b,\mu}$ *be the positive solution of* (1.1) *obtained by Theorem 1.1. Then there exists* $\mu^*>0$ *such that for each* $\mu\in(\mu^*,\infty)$ *fixed,* $\varphi^+_{b,\mu}\to\varphi^+_\mu$ *in* X *as* $b\to 0$ *up to a subsequence, where* φ^+_μ *is a positive solution of*

$$\begin{cases} a(-\Delta)^s u+\mu V(x)u=\lambda a(x)u\ln|u|+|u|^{2^*_s-2}u \quad in\ \mathbb{R}^N, \\ u\in H^s(\mathbb{R}^N). \end{cases} \tag{1.5}$$

Theorem 1.4. *Assume that condition* (A_1) *and* $(V_1)-(V_3)$ *hold, let* $\varphi^+_{b,\mu}$ *be the positive solution of* (1.1) *obtained by Theorem 1.1. Then* $\varphi^+_{b,\mu}\to\varphi^+$ *in* $H^s(\mathbb{R}^N)$ *as* $b\to 0$ *and* $\mu\to\infty$ *up to a subsequence, where* φ^+ *is a positive solution of*

$$\begin{cases} a(-\Delta)^s u=\lambda a(x)u\ln|u|+|u|^{2^*_s-2}u \quad in\ \Omega, \\ u\mid_{\partial\Omega}=0. \end{cases} \tag{1.6}$$

To achieve our aim, the Nehari manifold is the main tool in this study. First the logarithmic nonlinearity does not satisfy the monotonicity condition or Ambrosetti-Rabinowitz condition and this type of nonlinearity may change sign in $\mathbb{R}^N$, which makes discussions more complicated. Another is the lack of compactness caused by the unbounded domain and the critical nonlinearity. Some concentration compactness results for the fractional Kirchhoff equations seem correct but have not been proved yet and thus cannot be applied directly. All these difficulties prevent us from using the classical variational methods in a standard way, so innovative techniques are highly needed.

This paper is organized as follows. In the forthcoming section we recall some basic definitions, present the variational setting for the problem and study some properties of the corresponding Nehari manifold. In section 3, we present technical lemmas and the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2, Theorem 1.3 and Theorem 1.4.

2 Functional Setting

In this section, we introduce the definition of s-harmonic extension, and present the variational setting for the problem and properties of the corresponding Nehari manifold.

For convenience of our statements, throughout this article we will use the following notations:

- $L^q(\mathbb{R}^N)$, $1\le q\le\infty$, denotes the usual Lebesgue space with the norm $|\cdot|_q$;
- For any $\rho>0$ and $z\in\mathbb{R}^N$, $B_\rho(z)$ denotes the ball of radius ρ centered at z;
- $\mathbb{R}^{N+1}_+=\{(x_1,x_2,\cdots,x_{N+1})\in\mathbb{R}^{N+1}\mid x_{N+1}\ge 0\}$.

To study the corresponding extension problem, we apply an extension method [9] and define the extension function in $H^s(\mathbb{R}^N)$ as follows.

Definition 2.1. *Given a function* $u\in H^s(\mathbb{R}^N)$, *we define the s-harmonic extension* $E_s(u)=\varphi$ *to the problem:*

$$\begin{cases} div(y^{1-2s}\nabla\varphi)=0 & in\ \mathbb{R}^{N+1}_+, \\ \varphi=u, & on\ \mathbb{R}^N\times\{0\}. \end{cases}$$

The extension function $\varphi(x,y)$ *has an explicit expression in term of the Poisson and Riesz kernel, i.e.*

$$\varphi(x,y)=P^s_y*u(x)=\int_{\mathbb{R}^N}P^s_y(x-\xi,y)u(\xi)\,d\xi,$$

where $P_y^s(x,y) = C(N,s)\frac{y^{2s}}{(|x|^2+y^2)^{\frac{N+2s}{2}}}$ *with a constant* $C(N,s)$ *such that* $\int_{\mathbb{R}^N} P_1^s(x)\,dx = 1$ *(see [9]).*

Define the space

$$X^s(\mathbb{R}^{N+1}_+) := \left\{ \varphi(x,y) \in C_0^\infty(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} k_s y^{1-2s}|\nabla\varphi|^2\,dx\,dy + \int_{\mathbb{R}^N} V(x)|\varphi(x,0)|^2\,dx < \infty \right\},$$

equipped with the norm:

$$\|\varphi\| = \left(a\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi|^2\,dx\,dy + \int_{\mathbb{R}^N} V(x)|\varphi(x,0)|^2\,dx \right)^{\frac{1}{2}}.$$

For $\mu > 1$, we also need the following norm

$$\|\varphi\|_X = \left(a\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi|^2\,dx\,dy + \int_{\mathbb{R}^N} \mu V(x)|\varphi(x,0)|^2\,dx \right)^{\frac{1}{2}}.$$

Note that

$$\int_{\mathbb{R}^{N+1}_+} k_s y^{1-2s}|\nabla\varphi|^2\,dx\,dy = [\varphi]_s^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\,dx, \tag{2.1}$$

where $E_s(u) = \varphi$ and k_s is a normal positive constant see [9]. So the function $E_s(\cdot)$ is an isometry between $H^s(\mathbb{R}^N)$ and $X^s(\mathbb{R}^{N+1}_+)$. Then we can rewrite (1.1) as follows:

$$\begin{cases} div(y^{1-2s}\nabla\varphi) = 0, & in\ \mathbb{R}^{N+1}_+, \\ \left(a + b[\varphi]_s^2\right)\left(-k_s\frac{\partial\varphi}{\partial\nu}\right) = -\mu V(x)\varphi + \lambda a(x)\varphi\ln|\varphi| + |\varphi|^{2_s^*-2}\varphi, & on\ \mathbb{R}^N \times \{0\}, \end{cases} \tag{2.2}$$

where

$$-k_s\frac{\partial\varphi}{\partial\nu} = -k_s\lim_{y\to 0^+} y^{1-2s}\frac{\partial\varphi}{\partial y}(x,y) = (-\Delta)^s u(x).$$

For simplicity, we set $k_s = 1$ in follows. If φ is a solution of (2.2), we can get that the trace $u = tr(\varphi) = \varphi(x,0)$ is a solution of (1.1). Conversely, it is also true.

Definition 2.2. *To analyze* (2.2)*, we define the associated energy functional by*

$$\begin{aligned} I_{b,\mu}(\varphi) :&= \frac{1}{2}\|\varphi\|_X^2 + \frac{b}{4}\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi|^2\,dx\,dy\right)^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\ln|\varphi(x,0)|\,dx \\ &+ \frac{\lambda}{4}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*}\,dx, \end{aligned}$$

where $\varphi \in X^s(\mathbb{R}^{N+1}_+)$*. Then* $I_{b,\mu}$ *is Fréchet differentiable and*

$$\begin{aligned} \langle I'_{b,\mu}(\varphi), v\rangle = &\ a\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla\varphi(x,y)\nabla v(x,y)\,dx\,dy + \mu\int_{\mathbb{R}^N} V(x)\varphi(x,0)v(x,0)\,dx \\ &+ b\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla\varphi(x,y)\nabla v(x,y)\,dx\,dy \\ &- \lambda\int_{\mathbb{R}^N} a(x)\varphi(x,0)v(x,0)\ln|\varphi(x,0)|\,dx - \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*-2}\varphi(x,0)v(x,0)\,dx \end{aligned}$$

for any $\varphi \in X^s(\mathbb{R}^{N+1}_+)$. It is notable that finding the weak solution of (2.2) *is equivalent to finding the critical point of the energy functional $I_{b,\mu}$.*

Definition 2.3.

$$\Phi := \{\textit{nontrivial weak solutions of } (2.2)\}.$$

From (2.1) *and Definition 2.1, we define the ground energy of equation* (1.1) *by*

$$d := \inf_{\varphi \in \Phi} I_{b,\mu}(\varphi).$$

If φ is a nontrivial solution of system (2.2) *such that $I_{b,\mu}(\varphi) = d$, we call that $u := \varphi(x,0)$ is a ground state solution of equation* (1.1).

Since $I_{b,\mu}$ is not bounded from below on $X^s(\mathbb{R}^{N+1}_+)$, we consider $I_{b,\mu}$ strictly on the Nehari manifold:

$$\mathcal{N}_\lambda := \{\varphi \in X^s(\mathbb{R}^{N+1}_+) \setminus \{0\} : \langle I'_{b,\mu}(\varphi), \varphi \rangle = 0\}.$$

Then $\varphi \in \mathcal{N}_\lambda$ if and only if

$$\|\varphi\|_X^2 + b[\varphi]_s^4 - \lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)| \, dx - \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*} \, dx = 0. \tag{2.3}$$

We analyze $\mathcal{N}_\lambda$ in terms of the stationary points of fibrering maps [6] that $\phi_\varphi : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$\phi_\varphi(t) := I_{b,\mu}(t\varphi).$$

Then we have

$$\begin{aligned} \phi_\varphi(t) = {} & \frac{t^2}{2}\|\varphi\|_X^2 + \frac{b}{4}t^4 \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \varphi(x,0)|^2 \, dx \, dy \right)^2 - \frac{t^{2_s^*}}{2_s^*} \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*} \, dx \\ & - \lambda \frac{t^2}{2} \left(\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)| \, dx + \ln t \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx \right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \phi'_\varphi(t) = {} & t\|\varphi\|_X^2 + bt^3 \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \varphi(x,0)|^2 \, dx \, dy \right)^2 - t^{2_s^*-1} \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*} \, dx \\ & - \lambda t \left(\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)| \, dx + \ln t \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx \right) \end{aligned}$$

and

$$\begin{aligned} \phi''_\varphi(t) = {} & \|\varphi\|_X^2 + 3bt^2 \left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \varphi(x,y)|^2 \, dx \, dy \right)^2 - (2_s^* - 1)t^{2_s^*-2} \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*} \, dx \\ & - \lambda \left(\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)| \, dx + \ln t \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx + \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx \right). \end{aligned}$$

It is easy to see that $\varphi \in \mathcal{N}_\lambda$ if and only if $\phi'_\varphi(1) = 0$.

We split $\mathcal{N}_\lambda$ into three subsets $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^0$ that correspond to local minima, local maxima, and points of inflection of fibrering maps respectively, i.e.

$$\mathcal{N}_\lambda^+ := \{\varphi \in \mathcal{N}_\lambda : \phi_\varphi''(1) > 0\} = \{t\varphi \in X^s(\mathbb{R}_+^{N+1}) \setminus \{0\} : \phi_\varphi'(t) = 0, \phi_\varphi''(t) > 0\},$$

$$\mathcal{N}_\lambda^- := \{\varphi \in \mathcal{N}_\lambda : \phi_\varphi''(1) < 0\} = \{t\varphi \in X^s(\mathbb{R}_+^{N+1}) \setminus \{0\} : \phi_\varphi'(t) = 0, \phi_\varphi''(t) < 0\},$$

$$\mathcal{N}_\lambda^0 := \{\varphi \in \mathcal{N}_\lambda : \phi_\varphi''(1) = 0\} = \{t\varphi \in X^s(\mathbb{R}_+^{N+1}) \setminus \{0\} : \phi_\varphi'(t) = 0, \phi_\varphi''(t) = 0\}.$$

Note that if $\varphi \in \mathcal{N}_\lambda$, then

$$\phi_\varphi''(1) = 2b\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2 - \lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx - (2_s^*-2)\int_{\mathbb{R}^N}|\varphi(x,0)|^{2_s^*}\,dx.$$

Thus, we get the equivalent expressions:

$$\mathcal{N}_\lambda^+ := \{\varphi \in \mathcal{N}_\lambda : -2b[\varphi]_s^4 + \lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + (2_s^*-2)\int_{\mathbb{R}^N}|\varphi(x,0)|^{2_s^*}\,dx < 0\},$$

$$\mathcal{N}_\lambda^- := \{\varphi \in \mathcal{N}_\lambda : -2b[\varphi]_s^4 + \lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + (2_s^*-2)\int_{\mathbb{R}^N}|\varphi(x,0)|^{2_s^*}\,dx > 0\},$$

$$\mathcal{N}_\lambda^0 := \{\varphi \in \mathcal{N}_\lambda : -2b[\varphi]_s^4 + \lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + (2_s^*-2)\int_{\mathbb{R}^N}|\varphi(x,0)|^{2_s^*}\,dx = 0\}.$$

Let us introduce some properties on the spaces $X^s(\mathbb{R}_+^{N+1})$ and $L^r(\mathbb{R}^N)$.

Proposition 2.1. *[23] The embedding $X^s(\mathbb{R}_+^{N+1}) \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [2, 2_s^*]$ and locally compact for $r \in [1, 2_s^*)$.*

Proposition 2.2. *[23] For every $\varphi \in X^s(\mathbb{R}_+^{N+1})$, there holds*

$$S_s\left(\int_{\mathbb{R}^N}|u(x)|^{\frac{2N}{N-2s}}\right)^{\frac{N-2s}{N}} \le \int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi|^2\,dx\,dy,$$

where $u = tr(\varphi)$. The best constant is given by

$$S_s = \frac{2\pi^s\Gamma(\frac{2-2s}{2})\Gamma(\frac{N+2s}{2})(\Gamma(\frac{N}{2}))^{\frac{2s}{N}}}{\Gamma(s)\Gamma(\frac{N-2s}{2})(\Gamma(N))^{\frac{2s}{N}}},$$

and it is attained when $u = \varphi(x,0)$ takes the form:

$$u_\varepsilon(x) = \frac{C\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2s}{2}}}$$

for an arbitrary $\varepsilon > 0$, $\varphi_\varepsilon = E_s(u_\varepsilon)$ and

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi_\varepsilon|^2\,dx\,dy = \int_{\mathbb{R}^N}|\varphi_\varepsilon(x,0)|^{\frac{2N}{N-2s}}\,dx = S_s^{\frac{N}{2s}}.$$

To handle the logarithmic nonlinearity, we need the following logarithmic Sobolev inequality:

Proposition 2.3. *[10] Let $g \in H^s(\mathbb{R}^N)$ and $\sigma > 0$ be any number. Then*

$$\int_{\mathbb{R}^N} |g|^2 \ln \frac{|g|^2}{|g|_2^2} \, dx \le \frac{\sigma^2}{\pi^s} |(-\Delta)^{\frac{s}{2}} g|_2^2 - \left[N + \frac{N}{s} \ln \sigma + \ln \frac{s\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2s})} \right] |g|_2^2.$$

Remark 2.1. *From* (2.1) *and definition 2.1, we have*

$$\int_{\mathbb{R}^N} |\varphi(x,0)|^2 \ln \frac{|\varphi(x,0)|^2}{|\varphi(x,0)|_2^2} \, dx \le \frac{\sigma^2}{\pi^s} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \varphi(x,0)|^2 \, dx \, dy - \left[N + \frac{N}{s} \ln \sigma + \ln \frac{s\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2s})} \right] |\varphi(x,0)|_2^2$$

for any $\varphi \in X^s(\mathbb{R}^{N+1}_+)$. Furthermore, there holds

$$\begin{aligned}
\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)| \, dx &= \frac{1}{2} \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)|^2 \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln \frac{|\varphi(x,0)|^2}{|\varphi(x,0)|_2^2} \, dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)|_2^2 \, dx \\
&\le \frac{1}{2} |a|_\infty \frac{\sigma^2}{\pi^s} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \varphi(x,0)|^2 \, dx \, dy \\
&\quad + \frac{1}{2} |a|_\infty \left| N + \frac{N}{s} \ln \sigma + \ln \frac{s\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2s})} \right| |\varphi(x,0)|_2^2 \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)|_2^2 \, dx. \qquad (2.5)
\end{aligned}$$

Proposition 2.4. *[16] If φ is a critical point of $I_{b,\mu}$ on $\mathcal{N}_\lambda$ and $\varphi \notin \mathcal{N}_\lambda^0$, then it is a critical point of $I_{b,\mu}$ in $X^s(\mathbb{R}^{N+1}_+)$.*

3 Technical lemmas and proof of Theorem 1.1

In this section, we first give some basic lemmas that will be used in the paper.

Lemma 3.1. *If $\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx \le 0$, then we have either*

$$\|\varphi\|_X \le 1 \qquad (3.1)$$

or

$$\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)| \, dx \le C\|\varphi\|_X^2$$

for some $C > 0$ independent of $\varphi \in X^s(\mathbb{R}^{N+1}_+)$.

Proof. To estimate $\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)| \, dx$, we re-write it as

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln |\varphi(x,0)| \, dx \\
&= \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln \frac{|\varphi(x,0)|}{\|\varphi\|_X} \, dx + \ln \|\varphi\|_X \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx \\
&= I_1 + I_2,
\end{aligned}$$

where $I_1 := \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln \frac{|\varphi(x,0)|}{\|\varphi\|_X}\, dx$ and $I_2 := \ln \|\varphi\|_X \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\, dx$.

If $\|\varphi\|_X \leq 1$, then (3.1) holds. If $\|\varphi\|_X > 1$, due to $\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\, dx \leq 0$ we have $I_2 \leq 0$. This implies

$$\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\, dx \leq I_1. \tag{3.2}$$

We divide I_1 into two parts: $I_1 = I_{11} + I_{12}$, where

$$I_{11} = \int_{\mathbb{R}^N_{a,+}} a(x)|\varphi(x,0)|^2 \ln \frac{|\varphi(x,0)|}{\|\varphi\|_X}\, dx, \quad I_{12} = \int_{\mathbb{R}^N_{a,-}} a(x)|\varphi(x,0)|^2 \ln \frac{|\varphi(x,0)|}{\|\varphi\|_X}\, dx,$$

$$\mathbb{R}^N_{a,+} := \{x \in \mathbb{R}^N : a(x) \geq 0\}, \quad \mathbb{R}^N_{a,-} := \{x \in \mathbb{R}^N : a(x) < 0\}.$$

For all $t > 0$, there exists $\gamma > 0$ such that $\ln t \leq C_\gamma t^\gamma$, it follows from Proposition 2.1 that

$$I_{11} \leq C\|\varphi\|_X^{2-q} \int_{\mathbb{R}^N_{a,+}} a(x)|\varphi(x,0)|^q\, dx \leq C\|\varphi\|_X^2 \tag{3.3}$$

for $2 < q < 2_s^*$ and

$$\begin{aligned} I_{12} &\leq \int_{\Omega_{a,-}} a(x)|\varphi(x,0)|^2 \ln \frac{|\varphi(x,0)|}{\|\varphi\|_X}\, dx \\ &= \int_{\Omega_{a,-}} (-a(x))|\varphi(x,0)|^2 \ln \frac{\|\varphi\|_X}{|\varphi(x,0)|}\, dx \\ &\leq C_{\gamma_0}\|\varphi\|_X^{\gamma_0} \int_{\mathbb{R}^N} |a(x)||\varphi(x,0)|^{2-\gamma_0} dx \\ &\leq C\|\varphi\|_X^2 \end{aligned} \tag{3.4}$$

for $0 < \gamma_0 < 1$, where $\Omega_{a,-} := \{x \in \mathbb{R}^N_{a,-} : |\varphi(x,0)| < \|\varphi\|_X\}$.

As a consequence of (3.2)-(3.4), we obtain

$$\int_{R^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\, dx \leq C\|\varphi\|_X^2,$$

where C is a positive constant independent of $\varphi \in X^s(\mathbb{R}^{N+1}_+)$. □

Lemma 3.2. *There exists $\lambda_2 > 0$ small enough such that if $\lambda \in (0, \lambda_2)$, then the set $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. On the contrary, if $\varphi \in \mathcal{N}_\lambda^0$, then

$$\|\varphi\|_X^2 + b[\varphi]_s^4 - \lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\, dx - \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*}\, dx = 0 \tag{3.5}$$

and

$$\begin{aligned} &\|\varphi\|_X^2 + 3b[\varphi]_s^4 - \lambda \left(\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\, dx + \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\, dx \right) \\ &- (2_s^* - 1) \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*}\, dx = 0. \end{aligned}$$

According to Proposition 2.1 and let $b>0$ small enough

$$\lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx=(2-2_s^*)\int_{\mathbb{R}^N}|\varphi(x,0)|^{2_s^*}\,dx+2b\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2<0. \quad (3.6)$$

In view of (3.6), it follows from Lemma 3.1 that either

$$\|\varphi\|_X\le 1 \quad (3.7)$$

or

$$\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\ln|\varphi(x,0)|\,dx\le C\|\varphi\|_X^2, \quad (3.8)$$

where C is a positive constant independent of $\varphi\in\mathcal{N}_\lambda^0$. If (3.8) holds, for sufficiently small $\lambda>0, b>0$ and exists $\alpha=\frac{|a(x)|_\infty}{2_s^*-2}$, it follows from (3.5)-(3.6) and Proposition 2.1 that

$$\begin{aligned}
0&=\|\varphi\|_X^2+b\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2-\lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\ln|\varphi(x,0)|\,dx-\int_{\mathbb{R}^N}|\varphi(x,0)|^{2_s^*}\,dx\\
&=\|\varphi\|_X^2+\left(b-\frac{2b}{2_s^*-2}\right)\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2-\lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\ln|\varphi(x,0)|\,dx\\
&\quad+\frac{\lambda}{2_s^*-2}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx\\
&\ge\|\varphi\|_X^2(1-\lambda C)-\lambda\alpha\|\varphi\|_X^2+\left(b-\frac{2b}{2_s^*-2}\right)\|\varphi\|_X^4\\
&\ge C\|\varphi\|_X^2.
\end{aligned}$$

Thus, $\|\varphi\|_X=0$, which obviously yields a contradiction to the fact $\varphi\neq 0$. This implies that (3.7) holds.

On the other hand, in view of $\ln t\le t$ for any $t>0$, it follows from (2.4) and Proposition 2.1 that

$$\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\ln|\varphi(x,0)|\,dx\le C(\|\varphi\|_X^2+|\varphi(x,0)|_2^4)\le C(\|\varphi\|_X^2+\|\varphi\|_X^4). \quad (3.9)$$

With the help of (3.5)-(3.6), (3.9) and Proposition 2.1, we obtain

$$\begin{aligned}
\|\varphi\|_X^2&=\lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\ln|\varphi(x,0)|\,dx-\left(b-\frac{2b}{2_s^*-2}\right)\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2\\
&\quad-\frac{\lambda}{2_s^*-2}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx\\
&\le\lambda C(\|\varphi\|_X^2+\|\varphi\|_X^4)+\lambda\alpha\|\varphi\|_X^2,
\end{aligned}$$

which together with (3.7) gives

$$C\le\lambda(1+\|\varphi\|_X^2)+\lambda\alpha\le\lambda(2+\alpha).$$

This contradict with the fact that λ is sufficiently small. $\square$

Lemma 3.3. *There exists $\lambda_3 > 0$ small enough such that if $\lambda \in (0, \lambda_3)$, then $I_{b,\mu}$ is bounded from below on $\mathcal{N}_\lambda$.*

Proof. Let $\varphi \in \mathcal{N}_\lambda^+$. According to the definition of $\mathcal{N}_\lambda^+$, we get

$$\lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx < 0$$

and

$$\int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*}\,dx < \frac{-\lambda}{2_s^*-2}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + \frac{2b}{2_s^*-2}\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2.$$

As discussing for (3.7), for $\lambda > 0$ small enough we can obtain

$$\|\varphi\|_X \le 1. \tag{3.10}$$

Hence, the low bound of $I_{b,\mu}$ restricted on $\mathcal{N}_\lambda^+$ can be attained by Proposition 2.1 and (3.10), i.e.

$$\begin{aligned}
I_{b,\mu}(\varphi) &= I_{b,\mu}(\varphi) - \frac{1}{2}\langle I'_{b,\lambda}(\varphi),\varphi\rangle\\
&= \frac{-b}{4}\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2 + \frac{\lambda}{4}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|\varphi(x,0)|^{2_s^*}\,dx\\
&\ge \frac{\lambda}{4}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx - \frac{b}{4}\|\varphi\|_X^4\\
&\ge -\lambda C\|\varphi\|_X^2 - \frac{b}{4}\|\varphi\|_X^4\\
&\ge -\lambda C - \frac{b}{4}.
\end{aligned} \tag{3.11}$$

For any $\varphi \in \mathcal{N}_\lambda^-$, we have

$$\begin{aligned}
I_{b,\mu}(\varphi) &= I_{b,\mu}(\varphi) - \frac{1}{2}\langle I'_{b,\mu}(\varphi),\varphi\rangle\\
&= \frac{-b}{4}\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2 + \frac{\lambda}{4}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|\varphi(x,0)|^{2_s^*}\,dx.
\end{aligned} \tag{3.12}$$

If $I_{b,\mu}(\varphi) \ge 0$ for all $\varphi \in \mathcal{N}_\lambda^-$, obviously the lower bound of $I_{b,\mu}$ restricted on $\mathcal{N}_\lambda^-$ can be achieved. Otherwise, if there exists $\varphi \in \mathcal{N}_\lambda^-$ such that $I_{b,\mu}(\varphi) < 0$ by (3.12) it follows that

$$\lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx < 0$$

and

$$\int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*}\,dx < \frac{-2_s^*}{2(2_s^*-2)}\lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + \frac{2_s^*}{2(2_s^*-2)}b\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2.$$

As we did for (3.7), there is $\lambda > 0$ small enough such that

$$\|\varphi\|_X \leq 1. \tag{3.13}$$

Similar to (3.11), $I_{b,\mu}$ is bounded from below on $\mathcal{N}_\lambda^-$. $\square$

Lemma 3.4. *For each $\varphi \in X^s(\mathbb{R}_+^{N+1}) \setminus \{0\}$, there exists $\lambda_1 > 0$ small enough such that if $\lambda \in (0, \lambda_1)$, then the following two statements are true.*
(i) If $\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx > 0$, then there exists $t^- := t^-(\varphi) > 0$ such that $t^-\varphi \in \mathcal{N}_\lambda^-$ and $I_{b,\mu}(t^-\varphi) = \max_{t\geq 0} I_{b,\mu}(t\varphi)$.
(ii) If $\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx < 0$, then there exists a unique $0 < t^+ := t^+(\varphi) < t^- := t^-(\varphi) < \infty$ such that $t^+\varphi \in \mathcal{N}_\lambda^+$, $t^-\varphi \in \mathcal{N}_\lambda^-$, $I_{b,\mu}(t\varphi)$ is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on $(t^-, +\infty.)$ Moreover, $I_{b,\mu}(t^+\varphi) = \min_{0\leq t\leq t^-} I_{b,\mu}(t\varphi)$ and $I_{b,\mu}(t^-\varphi) = \max_{t^+\leq t} I_{b,\mu}(t\varphi)$.

Proof. (i) Suppose that $\varphi \in X^s(\mathbb{R}_+^{N+1})\setminus\{0\}$ with $\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx > 0$. Since $2 < 2_s^*$ and $\lim_{t\to 0^+} \ln t = -\infty$, there exists $t_0 > 0$ small enough such that

$$\phi_\varphi(t) > 0 \tag{3.14}$$

for $t \in (0, t_0)$, where $\phi_\varphi(t)$ is defined by (2.4). Moreover, we have

$$\lim_{t\to 0^+} \phi_\varphi(t) = 0 \text{ and } \lim_{t\to +\infty} \phi_\varphi(t) = -\infty. \tag{3.15}$$

From (3.14) with (3.15), there is $t^- := t^-(\varphi) > 0$ such that

$$\phi_\varphi(t^-) = I_{b,\lambda}(t^-\varphi) = \max_{t\geq 0} \phi_\varphi(t) = \max_{t\geq 0} I_{b,\lambda}(t\varphi).$$

This implies $t^-\varphi \in \mathcal{N}_\lambda^-$.

(ii) Suppose that $\varphi \in X^s(\mathbb{R}_+^{N+1}) \setminus \{0\}$ with $\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx < 0$. Note that

$$\begin{aligned}\frac{\phi'_\varphi(t)}{t} &= \|\varphi\|_X^2 + bt^2 \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2 - t^{2_s^*-2}\int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*}\,dx \\ &\quad - \lambda\left(\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\,dx + \ln t\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx\right).\end{aligned}$$

Let $f(t) := \lambda \ln t \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + t^{2_s^*-2}\int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*}\,dx$ Then $t\varphi \in \mathcal{N}_\lambda$ if and only if

$$f(t) = \|\varphi\|_X^2 + bt^2\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2 - \lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\,dx.$$

Since

$$\lim_{t\to 0^+} f(t) = +\infty, \lim_{t\to +\infty} f(t) = +\infty \tag{3.16}$$

and

$$tf'(t) = \lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + (2_s^* - 2)t^{2_s^*-2} \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*}\,dx, \tag{3.17}$$

there exists

$$t_{\min} := \left(\frac{-\lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx}{(2_s^*-2)\int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*}\,dx} \right)^{\frac{1}{2_s^*-2}}$$

such that

$$\begin{aligned} f(t_{\min}) &= \min_{t\geq 0} f(t) \\ &= \frac{\lambda}{2_s^*-2} \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx \ln\left(\frac{-\lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx}{(2_s^*-2)\int_{\mathbb{R}^N} b(x)|\varphi(x,0)|^{2_s^*}\,dx} \right) \\ &\quad - \frac{\lambda}{2_s^*-2} \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx. \end{aligned}$$

Moreover, $f(t)$ is decreasing in $(0, t_{\min})$ and increasing in $(t_{\min}, +\infty)$.

To show that

$$f(t_{\min}) \leq \|\varphi\|_X^2 + bt^2 \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy \right)^2 - \lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\,dx, \tag{3.18}$$

we start with estimating $\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\,dx$. It follows from Lemma **??** that either

$$\|\varphi\|_X \leq 1 \tag{3.19}$$

or

$$\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\,dx \leq C\|\varphi\|_X^2 \tag{3.20}$$

for some $C > 0$ independent of $\varphi \in X^s(\mathbb{R}_+^{N+1})$. Thus we need to consider two cases.

Case 1 Assume that (3.19) holds. On the one hand, it follows from (2.5) and Proposition 2.1 that

$$\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\,dx \leq C\left(\|\varphi\|_X^2 + \|\varphi\|_X^4\right) \leq C\|\varphi\|_X^2$$

for some $C > 0$ independent of φ. So we have

$$\|\varphi\|_X^2 + bt^2 \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy \right)^2 - \lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \ln|\varphi(x,0)|\,dx \geq C\|\varphi\|_X^2 \tag{3.21}$$

for $\lambda > 0$ small enough and some $C > 0$ independent of φ.

On the other hand, in view of the inequality $\ln t \le t$ for $t > 0$, it follows from Proposition 2.1 and (3.19) that

$$\begin{aligned}
f(t_{\min}) &= \frac{\lambda}{2_s^*-2}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx \ln\left(-\lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx\right)\\
&\quad -\frac{\lambda}{2_s^*-2}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx \ln\left((2_s^*-2)\int_{\mathbb{R}^N} b(x)|\varphi(x,0)|^{2_s^*}\,dx\right)\\
&\quad -\frac{\lambda}{2_s^*-2}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx\\
&\le \frac{-\lambda}{2_s^*-2}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx\left[\lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx+(2_s^*-2)\int_{\mathbb{R}^N} b(x)|\varphi(x,0)|^{2_s^*}\,dx+1\right]\\
&\le \lambda C\|\varphi\|_X^2\left[\lambda C\|\varphi\|_X^2+C\|\varphi\|_X^{2_s^*}+1\right]\\
&\le \lambda C\|\varphi\|_X^2. \qquad (3.22)
\end{aligned}$$

for some $C > 0$ independent of φ. As a consequence of (3.21) and (3.22), we see that (3.18) holds for $\lambda > 0$ small enough.

From (3.16)-(3.18), there exists a unique $0 < t^+(\varphi) < t_{\min} < t^-(\varphi) < \infty$ such that

$$f(t^+(\varphi)) = f(t^-(\varphi)) = \|\varphi\|_X^2+bt^2\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2-\lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\ln|\varphi(x,0)|\,dx$$

and

$$t^+(\varphi)\varphi\in\mathcal{N}_\lambda \text{ and } t^-(\varphi)\varphi\in\mathcal{N}_\lambda.$$

Since

$$f'(t^+(\varphi)) < 0 < f'(t^-(\varphi)),$$

it follows from (3.17) that $t^+(\varphi)\varphi\in\mathcal{N}_\lambda^+$ and $t^-(\varphi)\varphi\in\mathcal{N}_\lambda^-$.

Using the fact that

$$f(t)-\|\varphi\|_X^2-bt^2[\varphi]_s^4+\lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\ln|\varphi(x,0)|\,dx = \begin{cases} \ge 0 & 0\le t\le t^+(\varphi),\\ \le 0 & t^+(\varphi)\le t\le t^-(\varphi),\\ \ge 0 & t^-(\varphi)\le t.\end{cases}$$

we obtain

$$\phi'_\varphi = \begin{cases} \le 0 & 0\le t\le t^+(\varphi),\\ \ge 0 & t^+(\varphi)\le t\le t^-(\varphi),\\ \le 0 & t^-(\varphi)\le t.\end{cases}$$

This indicates that $I_{b,\mu}(t\varphi)$ is decreasing on $(0,t^+(\varphi))$, increasing on $(t^+(\varphi),t^-(\varphi))$ and decreasing on $(t^-(\varphi),\infty)$. Moreover, we have

$$I_{b,\mu}(t^+(\varphi)\varphi) = \min_{0\le t\le t^-(\varphi)} I_{b,\mu}(t\varphi) \text{ and } I_{b,\mu}(t^-(\varphi)\varphi) = \max_{t\ge t^+(\varphi)} I_{b,\mu}(t\varphi).$$

Case 2 Assume that (3.20) holds. Then

$$\|\varphi\|_X^2 + bt^2 \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla \varphi(x,y)|^2 \, dx \, dy \right)^2 - \lambda \int_{\mathbb{R}^N} a(x) |\varphi(x,0)|^2 \ln |\varphi(x,0)| \, dx \geq C \|\varphi\|_X^2 \tag{3.23}$$

for $\lambda > 0$ small enough and some $C > 0$ independent of φ.

If $-\lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx \geq (2_s^* - 2) \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*} \, dx - 2b[\varphi]_s^4$ and b small enough, we have

$$\frac{\lambda}{2_s^* - 2} \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx \ln \left(\frac{-\lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx}{(2_s^* - 2) \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*} \, dx} \right) \leq 0.$$

That is,

$$f(t_{\min}) \leq \frac{-\lambda}{2_s^* - 2} \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx \leq \lambda C \|\varphi\|_X^2 \tag{3.24}$$

for some $C > 0$ independent of φ. From (3.23) and (3.24), we arrive at the desired result (3.18) for $\lambda > 0$ small enough.

Due to (3.18), similar to the proof of Case 1, there exist $0 < t^+(\varphi) \leq t_{\min} \leq t^-(\varphi) < \infty$ such that $t^+(\varphi)\varphi \in \mathcal{N}_\lambda^+$ and $t^+(\varphi)\varphi \in \mathcal{N}_\lambda^-$. Furthermore, we can see that $I_{b,\mu}(t\varphi)$ is decreasing on $(0, t^+\varphi)$ increasing on $(t^+\varphi, t^-\varphi)$ and decreasing on $(t^-\varphi, \infty)$. So we have $I_{b,\mu}(t^+\varphi(\varphi)) = \min_{0 \leq t \leq t^-\varphi} I_{b,\mu}(t\varphi)$ and $I_{b,\mu}(t^-\varphi) = \max_{t \geq t^+(\varphi)} I_{b,\mu}(t\varphi)$.

If $-\lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx < (2_s^* - 2) \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*} \, dx - 2b[\varphi]_s^4$, since $2 < 2_s^*$, there exists $t_0 > 0$ such that

$$-\lambda \int_{\mathbb{R}^N} a(x)|t_0\varphi(x,0)|^2 \, dx > (2_s^* - 2) \int_{\mathbb{R}^N} |t_0\varphi(x,0)|^{2_s^*} \, dx - 2b[\varphi]_s^4.$$

Set

$$\varphi_0 = t_0 \varphi.$$

Similarly, we can see that there are $0 < t^+(\varphi_0) < t^-(\varphi_0) < \infty$ such that the desired result in Case 1 holds for some φ_0. Let $t^+(\varphi) = t_0 t^+(\varphi_0)$ and $t^-(\varphi) = t_0 t^-(\varphi_0)$. Consequently, there exist $0 < t^+(\varphi) < t^-(\varphi) < \infty$ such that the result in Case 1 holds for an arbitrary φ. □

In view of Lemmas 3.3 and 3.4, we set

$$\alpha_\lambda^+ := \inf_{\varphi \in \mathcal{N}_\lambda^+} I_{b,\mu}(\varphi) \quad \text{and} \quad \alpha_\lambda^- := \inf_{\varphi \in \mathcal{N}_\lambda^-} I_{b,\mu}(\varphi).$$

Lemma 3.5. *(i)If $a(x)$ is negative or sign-changing, then $\alpha_\lambda^+ < 0$ and $\alpha_\lambda^+ \leq \alpha_\lambda^-$.*
(ii) If $a(x) \geq 0$, then $\mathcal{N}_\lambda^+ = \emptyset$ and $\alpha_\lambda^- > 0$.

Proof. (i) If $a(x)$ is negative or sign-changing, it follows from Lemma 3.4 that $\mathcal{N}_\lambda^+ \neq \emptyset$. Let $\varphi \in \mathcal{N}_\lambda^+$. Then we have

$$\lambda \int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2 \, dx < (2 - 2_s^*) \int_{\mathbb{R}^N} |\varphi(x,0)|^{2_s^*} \, dx + 2b \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla \varphi(x,y)|^2 \, dx \, dy \right)^2 < 0.$$

This, together with $\langle I'_{b,\mu}(\varphi), \varphi\rangle = 0$, leads to

$$\begin{aligned}
I_{b,\mu}(\varphi) &= I_{b,\mu}(\varphi) - \frac{1}{2}\langle I'_{b,\mu}(\varphi), \varphi\rangle \\
&= \frac{-b}{4}\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2 + \frac{\lambda}{4}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + \left(\frac{1}{2} - \frac{1}{2^*_s}\right)\int_{\mathbb{R}^N} |\varphi(x,0)|^{2^*_s}\,dx \\
&< \frac{2-2^*_s}{4}\int_{\mathbb{R}^N} |\varphi(x,0)|^{2^*_s}\,dx + \frac{2^*_s-2}{22^*_s}\int_{\mathbb{R}^N} |\varphi(x,0)|^{2^*_s}\,dx \\
&= \left(\frac{1}{4} - \frac{1}{22^*_s}\right)(2-2^*_s)\int_{\mathbb{R}^N} |\varphi(x,0)|^{2^*_s}\,dx \\
&< 0. \qquad (3.25)
\end{aligned}$$

Thus, we obtain $\alpha^+_\lambda < 0$.

For any $\varphi \in \mathcal{N}^-_\lambda$, if $I_{b,\lambda}(\varphi) \geq 0$, then

$$I_{b,\mu}(\varphi) \geq 0 > \alpha^+_\lambda. \qquad (3.26)$$

If $I_{b,\mu}(\varphi) < 0$, then

$$\begin{aligned}
I_{b,\mu}(\varphi) &= I_{b,\mu}(\varphi) - \frac{1}{2}\langle I'_{b,\mu}(\varphi)\varphi\rangle \\
&= \frac{-b}{4}\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2 + \frac{\lambda}{4}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + \left(\frac{1}{2} - \frac{1}{2^*_s}\right)\int_{\mathbb{R}^N} |\varphi(x,0)|^{2^*_s}\,dx \\
&< 0.
\end{aligned}$$

That is,

$$\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx < 0.$$

With the help of Lemma 3.4 (ii), there exists a unique $t^+(\varphi) < t^-(\varphi) = 1$ such that $t^+(\varphi)\varphi \in \mathcal{N}^+_\lambda$ and

$$I_{b,\mu}(\varphi) \geq I_{b,\mu}(t^+(\varphi)\varphi) \geq \alpha^+_\lambda. \qquad (3.27)$$

Consequently, as a result of (3.26) and (3.27), we obtain

$$\alpha^+_\lambda \leq \alpha^-_\lambda.$$

(ii) If $a(x) \geq 0$, then for any $\varphi \in X^s(\mathbb{R}^{N+1}_+) \setminus \{0\}$ we have

$$\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx \geq 0, \qquad (3.28)$$

which implies $\mathcal{N}^+_\lambda = \emptyset$. Moreover, it follows from Lemma **??** (i) that $\mathcal{N}^-_\lambda \neq \emptyset$.

For any $\varphi \in \mathcal{N}^-_\lambda$, we get

$$\begin{aligned}
I_{b,\mu}(\varphi) &= I_{b,\mu}(\varphi) - \frac{1}{2}\langle I'_{b,\mu}(\varphi), \varphi\rangle \\
&= \frac{-b}{4}\left(\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2 + \frac{\lambda}{4}\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\,dx + \left(\frac{1}{2} - \frac{1}{2^*_s}\right)\int_{\mathbb{R}^N} |\varphi(x,0)|^{2^*_s}\,dx \\
&\geq 0,
\end{aligned}$$

which implies

$$\alpha_\lambda^- \geq 0.$$

We now suppose by contradiction that $\alpha_\lambda^- = 0$. Let $\{\varphi_n\} \subset \mathcal{N}_\lambda^-$ be a sequence such that $I_{b,\lambda}(\varphi_n) \to 0$, as $n \to \infty$. Then we have

$$\begin{aligned}
0 \leftarrow I_{b,\mu}(\varphi_n) &= I_{b,\mu}(\varphi_n) - \frac{1}{2}\langle I'_{b,\mu}(\varphi_n), \varphi_n\rangle \\
&= \frac{-b}{4}\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla \varphi_n(x,y)|^2\,dx\,dy\right)^2 + \frac{\lambda}{4}\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\,dx \\
&\quad + \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N} |\varphi_n(x,0)|^{2_s^*}\,dx \\
&\geq 0, \text{ as } n \to \infty,
\end{aligned}$$

which together with (3.28) and $b > 0$ small enough, we have

$$\lambda \int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\,dx = o_n(1) \quad \text{and} \quad \int_{\mathbb{R}^N} |\varphi_n(x,0)|^{2_s^*}\,dx = o_n(1). \tag{3.29}$$

It follows (3.29) and Proposition 2.1 that

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2 \ln|\varphi_n(x,0)|\,dx \\
&= \int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2 \ln\frac{|\varphi_n(x,0)|}{\|\varphi_n\|_X}\,dx + \ln\|\varphi_n\|_X \int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\,dx \\
&\leq \int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2 \ln\frac{|\varphi_n(x,0)|}{\|\varphi_n\|_X}\,dx + C\|\varphi_n\|_X^2.
\end{aligned} \tag{3.30}$$

Processing as we did for (3.3) and (3.4), we have

$$\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2 \ln\frac{|\varphi_n(x,0)|}{\|\varphi_n\|_X}\,dx \leq C\|\varphi_n\|_X^2.$$

Using this estimate together with (3.30) leads to

$$\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2 \ln|\varphi_n(x,0)|\,dx \leq C\|\varphi_n\|_X^2. \tag{3.31}$$

Taking into account (2.3), (3.29), (3.31) and Proposition 2.1, for sufficiently small $\lambda > 0$ we deduce that

$$\begin{aligned}
0 &= \|\varphi_n\|_X^2 + b\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi_n(x,y)|^2\,dx\,dy\right)^2 - \lambda\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\ln|\varphi_n(x,0)|\,dx - \int_{\mathbb{R}^N}|\varphi_n(x,0)|^{2_s^*}\,dx \\
&= \|\varphi_n\|_X^2 + b\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi_n(x,y)|^2\,dx\,dy\right)^2 - \lambda\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\ln|\varphi_n(x,0)|\,dx + o_n(1) \\
&\geq \|\varphi_n\|_X^2(1 - \lambda C) + o_n(1) \\
&\geq C\|\varphi_n\|_X^2 + o_n(1).
\end{aligned}$$

That is,

$$\|\varphi_n\|_X = o_n(1). \tag{3.32}$$

On the other hand, in view of $\ln t \le t$ for $t > 0$, it follows from (2.5) and Proposition 2.1 that

$$\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2 \ln|\varphi_n(x,0)|\,dx \le C(\|\varphi_n\|_X^2 + |\varphi_n(x,0)|_2^4) \le C(\|\varphi_n\|_X^2 + \|\varphi_n\|_X^4). \tag{3.33}$$

Making use of (2.3), (3.33) and Proposition 2.2, we get

$$\|\varphi\|_X^2 = -b\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi(x,y)|^2\,dx\,dy\right)^2 + \lambda\int_{\mathbb{R}^N} a(x)|\varphi(x,0)|^2\ln|\varphi(x,0)|\,dx + \int_{\mathbb{R}^N}|\varphi(x,0)|^{2_s^*}\,dx.$$

Then

$$\|\varphi\|_X^2 \le \lambda C(\|\varphi_n\|_X^2 + \|\varphi_n\|_X^4) + C\|\varphi_n\|_X^{2_s^*}.$$

That is, $\|\varphi_n\|_X^{2_s^*} + \|\varphi_n\|_X^4 \ge (1-\lambda C)\|\varphi_n\|_X^2 \ge C\|\varphi_n\|_X^2$ for small $\lambda > 0$ and some $C > 0$. Hence, we have $\|\varphi_n\|_X^2 \ge C$ for some $C > 0$ independent of $n \in \mathbb{Z}_+$. Apparently, this yields a contradiction to (3.32). □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Our proof will be divided into two steps.

Step 1 We shall show that there exists $\Lambda_1 > 0$ such that for each $\lambda \in (0,\Lambda_1)$, $I_{b,\mu}$ has a minimizer $\varphi_{b,\mu}^+$ in $\mathcal{N}_\lambda^+$ such that $I_{b,\mu}(\varphi_{b,\mu}^+) = \alpha_\lambda^+$.

Let $\{\varphi_n\}$ be a minimizing sequence $\{\varphi_n\} \subset \mathcal{N}_\lambda^+$, i.e. $\lim_{n\to\infty} I_{b,\mu}(\varphi_n) = \alpha_\lambda^+$. We claim that there is some $C > 0$ such that

$$\|\varphi_n\|_X \le C \quad \text{for all } n \in \mathbb{Z}_+. \tag{3.34}$$

Note that $\{\varphi_n\} \subset \mathcal{N}_\lambda^+$, then

$$\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\,dx < 0$$

and

$$\int_{\mathbb{R}^N}|\varphi_n(x,0)|^{2_s^*}\,dx \le \frac{-\lambda}{2_s^*-2}\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\,dx + \frac{2b}{2_s^*-2}\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi_n(x,y)|^2\,dx\,dy\right)^2.$$

Analogous to the derivation of (3.7), we can see that (3.34) holds for $\lambda > 0$ small enough. Thus there exists a subsequence (still denoted by $\{\varphi_n\}$) and $\varphi_{b,\mu}^+ \in X^s(\mathbb{R}_+^{N+1})$ such that

$$\begin{cases} \varphi_n \rightharpoonup \varphi_{b,\mu}^+ & \text{in } X^s(\mathbb{R}_+^{N+1}), \\ \varphi_n \to \varphi_{b,\mu}^+ & \text{in } L_{loc}^s(\mathbb{R}^N) \ \text{ for } s \in [2, 2_s^*), \\ \varphi_n \to \varphi_{b,\mu}^+ & a.e. \text{on } \mathbb{R}^N. \end{cases} \tag{3.35}$$

To prove that

$$\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\,dx \to \int_{\mathbb{R}^N} a(x)|\varphi^+_{b,\mu}(x,0)|^2\,dx \quad \text{as } n\to\infty. \tag{3.36}$$

For any $\varepsilon>0$, according to condition (H_1), there exists $R>0$ such that $|a(x)|<\varepsilon$ for $|x|\geq R$. It follows from (3.36) and Proposition 2.1 that

$$\left|\int_{\mathbb{R}^N\setminus B_R} a(x)|\varphi_n(x,0)|^2\,dx\right| \leq \varepsilon\|\varphi_n\|_X^2 \leq C\varepsilon, \tag{3.37}$$

$$\left|\int_{\mathbb{R}^N\setminus B_R} a(x)|\varphi^+_{b,\mu}(x,0)|^2\,dx\right| \leq \varepsilon\|\varphi^+_{b,\mu}\|_X^2 \leq C\varepsilon, \tag{3.38}$$

Then, Hölder's inequality and Proposition 2.1 leads to

$$\left|\int_{B_R} a(x)|\varphi_n(x,0)|^2\,dx - \int_{B_R} a(x)|\varphi^+_{b,\mu}(x,0)|^2\,dx\right| \to 0, \text{ as } n\to\infty. \tag{3.39}$$

From (3.37)-(3.39), we get (3.36).

To prove that

$$\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\ln|\varphi_n(x,0)|\,dx \to \int_{\mathbb{R}^N} a(x)|\varphi^+_{b,\mu}(x,0)|^2\ln|\varphi^+_{b,\mu}(x,0)|\,dx, \text{ as } n\to\infty. \tag{3.40}$$

From (3.35) that

$$a(x)|\varphi_n(x,0)|^2\ln|\varphi_n(x,0)| \to a(x)|\varphi^+_{b,\mu}(x,0)|^2\ln|\varphi^+_{b,\mu}(x,0)| \quad a.e.\ x\in\mathbb{R}^N.$$

Note that for any $\beta,\gamma>0$, there exists a constant $C_{\beta,\gamma}>0$ such that

$$|\ln t| \leq C_{\beta,\gamma}(t^\beta + t^{-\gamma}),\ t>0.$$

This gives

$$\left|\int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2\ln|\varphi_n(x,0)|\,dx\right| \leq C\int_{\mathbb{R}^N} |a(x)|\left(|\varphi_n(x,0)|^{2-\sigma} + |\varphi_n(x,0)|^{2+\sigma}\right)\,dx$$

for small $\sigma>0$. By virtue of Proposition 2.1 and Lebesgue's dominated convergence theorem, we obtain (3.40) immediately.

Set $\Psi^n_{b,\mu} = \varphi_n - \varphi^+_{b,\mu}$. It follows from Brezis-Lieb's lemma [30] that

$$\|\Psi^n_{b,\mu}\|_X^2 = \|\varphi_n\|_X^2 - \|\varphi^+_{b,\mu}\|_X^2 + o_n(1), \tag{3.41}$$

$$[\Psi^n_{b,\mu}]_s^4 = [\varphi_n]_s^4 - [\varphi^+_{b,\mu}]_s^4 \tag{3.42}$$

and

$$\int_{\mathbb{R}^N} |\Psi^n_{b,\mu}(x,0)|^{2^*_s}\,dx = \int_{\mathbb{R}^N} |\varphi_n(x,0)|^{2^*_s}\,dx - \int_{\mathbb{R}^N} |\varphi^+_{b,\mu}(x,0)|^{2^*_s}\,dx + o_n(1). \tag{3.43}$$

From (3.36) and (3.40)-(3.43) we deduce that

$$\frac{1}{2}\|\Psi^n_{b,\mu}\|^2_X+\frac{b}{4}[\Psi^n_{b,\mu}]^4_s-\frac{1}{2^*_s}\int_{\mathbb{R}^N}|\Psi^n_{b,\mu}(x,0)|^{2^*_s}\,dx=\alpha^+_\lambda-I_{b,\mu}(\varphi^+_{b,\mu})+o_n(1). \tag{3.44}$$

As we discussed for (3.40), there holds

$$\int_{\mathbb{R}^N}a(x)\varphi_n(x,0)\varphi^+_{b,\mu}(x,0)\ln|\varphi_n(x,0)|\,dx\to\int_{\mathbb{R}^N}a(x)|\varphi^+_{b,\mu}(x,0)|^2\ln|\varphi^+_{b,\mu}(x,0)|\,dx,\ \text{as } n\to\infty. \tag{3.45}$$

Combining (3.35) and (3.45), we have

$$\langle I'_{b,\mu}(\varphi^+_{b,\mu}),\varphi^+_{b,\mu}\rangle=0,\ \textit{i.e.}\ \varphi^+_{b,\mu}\in\mathcal{N}_\lambda\cup\{0\}.$$

Note that

$$(2^*_s-2)\int_{\mathbb{R}^N}|\varphi^+_{b,\mu}(x,0)|^{2^*_s}\,dx\le\liminf_{n\to\infty}(2^*_s-2)\int_{\mathbb{R}^N}|\varphi_n(x,0)|^{2^*_s}\,dx.$$

It follows from (3.36) that

$$\varphi^+_{b,\mu}\in\mathcal{N}^+_\lambda. \tag{3.46}$$

According to Proposition 2.1, it follows (3.35)-(3.36) and (3.40)-(3.45) that

$$o_n(1)=\langle I'_n(\varphi_n),\Psi^n_{b,\mu}\rangle=\langle(I'_{b,\mu}(\varphi_n)-I'_{b,\mu}(\varphi^+_{b,\mu})),\Psi^n_{b,\mu}\rangle=\|\Psi^n_{b,\mu}\|^2_X+b[\Psi^n_{b,\mu}]^4_s-\int_{\mathbb{R}^N}|\Psi^n_{b,\mu}(x,0)|^{2^*_s}\,dx, \tag{3.47}$$

as $n\to\infty$.

We suppose that

$$\|\Psi^n_{b,\mu}\|^2_X+b[\Psi^n_{b,\mu}]^4_s\to l \text{ and } \int_{\mathbb{R}^N}|\Psi^n_{b,\mu}(x,0)|^{2^*_s}\,dx\to l,\ \text{as } n\to\infty$$

for some $l\in[0,+\infty)$.

If $l=0$, we obtain the desired result immediately. If $l>0$, we have $l\ge S_s l^{\frac{2}{2^*_s}}$ by Proposition 2.2 and then

$$l\ge S_s^{\frac{N}{2\alpha}}. \tag{3.48}$$

It follows from (3.44) and (3.46)-(3.48) that

$$\alpha^+_\lambda=I_{b,\mu}(\varphi^+_{b,\mu})+\frac{l}{2}-\frac{l}{2^*_s}\ge\alpha^+_\lambda+\frac{\alpha}{N}l\ge\alpha^+_\lambda+\frac{\alpha}{N}S_s^{\frac{N}{2\alpha}}.$$

This is a contradiction. Hence, the only choice is $l=0$, i.e., $\varphi_n\to\varphi^+_{b,\lambda}$ in $X^s(\mathbb{R}^{N+1}_+)$ as $n\to\infty$.

Step 2 We show that $\varphi^+_{b,\lambda}(x,0)$ is a positive ground state solution of equation (1.1).

Since $\varphi^+_{b,\mu}\in X^s(\mathbb{R}^{N+1}_+)$ is a local minimizer for $\mathcal{N}_\lambda$. Proposition 2.4 tells us that $\varphi^+_{b,\mu}$ is a nontrivial solution of (2.2), and so $\varphi^+_{b,\mu}(x,0)$ is a nontrivial solution of equation (1.1). Note that $I_{b,\mu}(|\varphi^+_{b,\mu}|)=\alpha^+_\lambda$. So we assume $\varphi^+_{b,\mu}(x,0)\ge0$. By virtue of the Maximum Principle for fractional elliptic equations [25], $\varphi^+_{b,\mu}$ is positive. Consequently, $\varphi^+_{b,\mu}(x,0)$ is a positive ground state solution of equation (1.1). □

4 Asymptotic behavior of positive solutions

In this section, we investigate the asymptotic behavior of positive solutions for (1.1) and give the proofs of Theorems 1.2, 1.3 and 1.4.

Proof of Theorem 1.2. We follow the argument in [3] (or see [13, 26, 33]). For any sequence $\mu_n \to \infty$, there exists $b_* > 0$ for each $b \in (0, b_*)$ be fixed, then let $\varphi_n := \varphi^+_{b,\mu_n}$ be the positive solution of (1.1) obtained by Theorem 1.1. It follows from (3.34) that

$$\|\varphi_n\|_X \le C \quad \text{for all } n \in \mathbb{Z}^+. \tag{4.1}$$

Consequently, up to a subsequence, we may assume that

$$\begin{cases} \varphi_n \rightharpoonup \varphi_b^+ & \text{in } X, \\ \varphi_n \to \varphi_b^+ & \text{in } L^s_{loc}(\mathbb{R}^N) \ \text{ for } s \in [2, 2_s^*), \\ \varphi_n \to \varphi_b^+ & a.e. \text{on } \mathbb{R}^N. \end{cases} \tag{4.2}$$

By Fatou's lemma and (4.1), we obtain

$$\int_{\mathbb{R}^N} V(x) {\varphi_b^+}^2 \, dx \le \lim_{n\to\infty} \inf \int_{\mathbb{R}^N} V(x)\varphi_n^2 \, dx \le \lim_{n\to\infty} \inf \frac{\|\varphi_n\|_X^2}{\lambda_n} = 0.$$

Thus, $\varphi_b^+ = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$ and so $\varphi_b^+ \in H^s(\Omega)$ by the condition (V_3).

Our proof will be divided into three steps.

Step 1: We intend to show that $\varphi_n \to \varphi_b^+$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2_s^*$. If that doesn't hold up, applying Lion's vanishing lemma (see e.g. [20, 30]) there exist $\delta,\ r > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\int\limits_{B_r(x_n)} (\varphi_n - \varphi_b^+)^2 \, dx \ge \delta.$$

This implies that $|x_n| \to \infty$ and thus $|B_r(x_n) \cap \mathcal{V}_c| \to 0$. By the Hölder inequality, we can refer that

$$\int\limits_{B_r(x_n)\cap\mathcal{V}_c} (\varphi_n - \varphi_b^+)^2 \, dx \to 0.$$

Thus, we obtain

$$\begin{aligned} \|\varphi_n\|_X^2 &\ge \mu_n(x)c \int\limits_{B_r(x_n)\cap\{V(x)\ge c\}} \varphi_n^2 \, dx = \mu_n(x)c \int\limits_{B_r(x_n)\cap\{V(x)\ge c\}} (\varphi_n - \varphi_b^+)^2 \, dx \\ &= \mu_n(x)c \left(\int\limits_{B_r(x_n)} (\varphi_n - \varphi_b^+)^2 \, dx - \int\limits_{B_r(x_n)\cap\mathcal{V}_c} (\varphi_n - \varphi_b^+)^2 \, dx \right) \\ &\to \infty. \end{aligned}$$

This is in conflict with the fact that $\|\varphi_n\|_X$ is bounded.

Step 2: We prove that $\varphi_n \to \varphi_b^+$ in X. Since

$$\langle I_{b,\lambda}^{'}(\varphi_n), \varphi_n\rangle = \langle I_{b,\lambda}^{'}(\varphi_n), u_b^+\rangle = 0,$$

Therefore

$$\|\varphi_n\|_X^2 + b[\varphi_n]_s^4 = \lambda \int_{\mathbb{R}^N} a(x)|\varphi_n(x,0)|^2 \ln|\varphi_n(x,0)|\,dx + |\varphi_n|_{2_s^*}^{2_s^*} \tag{4.3}$$

and

$$\|u_b^+\|^2 + b[\varphi_n]_s^2[\varphi_b^+]_s^2 = \lambda \int_{\mathbb{R}^N} a(x)|\varphi_b^+|^2 \ln|\varphi_b^+|\,dx + |\varphi_b^+|_{2_s^*}^{2_s^*} + o_n(1). \tag{4.4}$$

Going to subsequence if necessary, let $\|\varphi_n\|_X^2 \to l_1$ and $[\varphi_n]_s^2 \to l_2$. By Fatou's lemma, we can obtain

$$[\varphi_b^+]_s^2 \le \lim_{n\to\infty} \inf[\varphi_n]_s^2 = l_2, \tag{4.5}$$

$$|\varphi_n|_{2_s^*}^{2_s^*} = |\varphi_b^+|_{2_s^*}^{2_s^*} + o_n(1). \tag{4.6}$$

From (4.3)–(4.6), we can infer that

$$l_1 + bl_2^2 = \|\varphi_b^+\|^2 + bl_2[\varphi_b^+]_s^2 \le \|\varphi_b^+\|^2 + bl_2^2.$$

Thus, $l_1 \le \|\varphi_b^+\|^2$. It then follows from the weakly lower semi-continuity of norm that

$$\|\varphi_b^+\|^2 \le \lim_{n\to\infty} \inf\|\varphi_n\|^2 \le \lim_{n\to\infty} \sup\|\varphi_n\|^2 \le \lim_{n\to\infty} \|\varphi_n\|_X^2 = l_1 \le \|\varphi_b^+\|^2. \tag{4.7}$$

Hence, we yield that $\varphi_n \to \varphi_b^+$ in X.

Step 3: We investigate that φ_b^+ is a positive solution of (1.4). Since $\langle I_{b,\mu}^{'}(\varphi_n), v\rangle = 0$, for any $v \in C_0^\infty(\Omega)$. we get

$$\begin{aligned} &a\int_{\mathbb{R}_+^{N+1}} y^{1-2s}\nabla\varphi_b^+(x,y)\nabla v(x,y)\,dx\,dy + \mu\int_{\mathbb{R}^N} V(x)\varphi_b^+(x,0)v(x,0)\,dx \\ &\quad + b\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\varphi_b^+(x,y)|^2\,dx\,dy \int_{\mathbb{R}_+^{N+1}} y^{1-2s}\nabla\varphi_b^+(x,y)\nabla v(x,y)\,dx\,dy \\ &= \lambda\int_{\mathbb{R}^N} a(x)\varphi_b^+(x,0)v(x,0)\ln|\varphi_b^+(x,0)|\,dx + \int_{\mathbb{R}^N} |\varphi_b^+(x,0)|^{2_s^*-2}\varphi_b^+(x,0)v(x,0)\,dx \end{aligned}$$

i.e., φ_b^+ is a nonnegative solution of (1.4) by the density of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. By (4.1) and (4.7), we infer that

$$\|u_b\| = \lim_{n\to\infty} \|u_n\|_X > 0,$$

it shows that $\varphi_b \ne 0$. By the strong maximum principle, $\varphi_b^+ > 0$ in $\mathbb{R}^N$. The proof is finished. □

Proof of Theorem 1.3. There exists $\mu^* > 0$, let $\mu \in (\mu^*, \infty)$ be fixed, then for any sequence $b_n \to 0$, let $\varphi_n := \varphi_{b_n,\mu}^+$ be the positive solution obtained by Theorem 1.1. It follows from (3.34) that

$$\|\varphi_n\|_X \le C \quad \text{for all } n \in \mathbb{N}. \tag{4.8}$$

Going to a subsequence if necessary, we may assume that $\varphi_n \rightharpoonup \varphi_\mu^+$ in X. Since $I'_{b_n,\lambda}(\varphi_n) = 0$. using the argument in the proof of Theorem 1.1 we may infer that $\varphi_n \to \varphi_\mu^+$ in X.

To complete the proof, it suffices to show that φ_μ^+ is a positive solution of (1.5). By the same arguments as in the proof of Theorem 1.1, we can get that $\varphi_\mu^+ > 0$ for all $x \in \mathbb{R}^N$. This completes the proof. $\square$

Proof of Theorem 1.4. The proof of Theorem 1.4 is similar to the proof of Theorem 1.2 and we omit the proof. $\square$

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