On some classes of the entire functions

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Abstract

The main target of this article is to prove the products, behaviors and simple zeros for the classes of the entire functions associated with the Weierstrass-Hadamard product and the Taylor series.

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1. Introduction

The theory of the entire functions of different orders and geniuses [1, 2, 3] (see the derailed definitions for them in Section 2) has been developed to propose their classes in order to study the zeros and poles and behaviors for them. One of great interest classes of the entire functions is the Laguerre–Pólya class of the entire functions due to Laguerre [3] and Pólya [4], which deal with the problems for the zeros for the entire functions of the real variable [6, 7]. In this paper we introduce some classes of the entire functions, which are represented by the Weierstrass-Hadamard product [1, 2] and the Taylor series in the theory of the entire functions. Conveniently, let $i = \sqrt{-1}$, $\mathbb{F}^{(n)}(\mathrm{s})$ be the n^{th} derivatives of the entire function $\mathbb{F}(\mathrm{s})$, and $\mathbb{R}$ and $\mathbb{C}$ are the sets of the real and complex numbers.

Definition 1. A entire function of order $\rho = 1$ and genus $\upsilon = 0$, expressed by the Taylor series

$$\mathbb{S}(s) = \mathbb{S}(\varsigma) + \sum_{k=1}^{\infty} \mathbb{S}^{(k)}(\varsigma) \frac{(s-\varsigma)^k}{k!}, \tag{1}$$

is said to be in the class, written $\mathbb{S} \in \mathbb{Y}$, if $\mathbb{S}(s)$ admits a representation of the Weierstrass-Hadamard product

$$\mathbb{S}(s) = \mathbb{S}(0) \prod_{k=1}^{\infty} \left(1 - \frac{s}{\sigma_k}\right), \tag{2}$$

where $\sigma_k = \varsigma_k + i\tau_k \in \mathbb{C}$ for $\varsigma_k \in \mathbb{R}$ and $\tau_k \in \mathbb{R}$, $\sigma_k \neq 0$, $s \in \mathbb{C}$, $\varsigma \in \mathbb{C}$, $\mathbb{S}(0) \neq 0$, $\mathbb{S}(\varsigma) \neq 0$ and $\mathbb{S}^{(k)}(\varsigma) \neq 0$ are the Taylor coefficients for $k \in \mathbb{N}$.

Definition 2. A entire function of order $\rho = 1$ and genus $\upsilon = 1$, expressed by the Taylor series

$$\widehat{\mathbb{S}}(s) = \widehat{\mathbb{S}}(\varsigma) + \sum_{k=1}^{\infty} \widehat{\mathbb{S}}^{(k)}(\varsigma) \frac{(s-\varsigma)^k}{k!}, \tag{3}$$

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is said to be in the class, written $\widehat{\mathbb{S}} \in \mathbb{L}$, if $\widehat{\mathbb{S}}(s)$ admits a representation of the Weierstrass-Hadamard product

$$\widehat{\mathbb{S}}(s) = \widehat{\mathbb{S}}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty} \left(1 - \tfrac{s}{\sigma_k}\right) e^{s/\sigma_k}, \tag{4}$$

where $\sigma_k = \varsigma_k + i\tau_k \in \mathbb{C}$ for $\varsigma_k \in \mathbb{R}$ and $\tau_k \in \mathbb{R}$, $\sigma_k \neq 0$, $s \in \mathbb{C}$, $\varsigma \in \mathbb{C}$, $\mathbb{Q} \in \mathbb{C}$, $\widehat{\mathbb{S}}(0) \neq 0$, $\widehat{\mathbb{S}}(\varsigma) \neq 0$ and $\widehat{\mathbb{S}}^{(k)}(\varsigma) \neq 0$ are the Taylor coefficients for $k \in \mathbb{N}$.

Here, it is expected that the case of $\varsigma_k = \varsigma \equiv \xi$ and $\tau_k \in \mathbb{R}\backslash\{0\}$ for $\xi \in \mathbb{R}\backslash\{0\}$ will produce $\mathbb{S}(s) \in \mathbb{Y}$ and $\widehat{\mathbb{S}} \in \mathbb{L}$, and these yields the subclasses $\widetilde{\mathbb{Y}}$ and $\bar{\mathbb{L}}$ as shown below. In other words, we can define the entire functions as follows:

Definition 3. A entire function of order $\rho = 1$ and genes $\upsilon = 0$, expressed by the Taylor series

$$\widetilde{\mathbb{S}}(s) = \widetilde{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \widetilde{\mathbb{S}}^{(k)}(\xi) \tfrac{(s-\xi)^k}{k!}, \tag{5}$$

is said to be in the class, written $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$, if $\widetilde{\mathbb{S}}(s)$ admits a representation of the Weierstrass-Hadamard product

$$\widetilde{\mathbb{S}}(s) = \widetilde{\mathbb{S}}(0) \prod_{k=1}^{\infty} \left(1 - \tfrac{s}{\sigma_k}\right), \tag{6}$$

where $\sigma_k = \xi + i\tau_k \in \mathbb{C}$ for $\xi \in \mathbb{R}\backslash\{0\}$ and $\tau_k \in \mathbb{R}\backslash\{0\}$, $s \in \mathbb{C}$, $\widetilde{\mathbb{S}}(0) \neq 0$, $\widetilde{\mathbb{S}}(\xi) \neq 0$ and $\widetilde{\mathbb{S}}^{(k)}(\xi) \neq 0$ are the Taylor coefficients for $k \in \mathbb{N}$.

Definition 4. A entire function of order $\rho = 1$ and genes $\upsilon = 1$, expressed by the Taylor series

$$\bar{\mathbb{S}}(s) = \bar{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \bar{\mathbb{S}}^{(k)}(\xi) \tfrac{(s-\xi)^k}{k!}, \tag{7}$$

is said to be in the class, written $\bar{\mathbb{S}} \in \bar{\mathbb{L}}$, if $\bar{\mathbb{S}}(s)$ admits a representation of the Weierstrass-Hadamard product

$$\bar{\mathbb{S}}(s) = \bar{\mathbb{S}}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty} \left(1 - \tfrac{s}{\sigma_k}\right) e^{s/\sigma_k}, \tag{8}$$

where $\sigma_k = \xi + i\tau_k \in \mathbb{C}$ for $\xi \in \mathbb{R}\backslash\{0\}$ and $\tau_k \in \mathbb{R}\backslash\{0\}$, $s \in \mathbb{C}$, $\mathbb{Q} \in \mathbb{C}$, $\bar{\mathbb{S}}(0) \neq 0$, $\bar{\mathbb{S}}(\xi) \neq 0$ and $\bar{\mathbb{S}}^{(k)}(\xi) \neq 0$ are the Taylor coefficients for $k \in \mathbb{N}$.

The purpose of the paper is to study the some classes of the entire functions. In Section 2 we introduce the results in the theory of the entire functions. In Section 3 we consider alternative products for the above classes of the entire functions. In Section 4 we investigate the behaviors for the above classes of the entire functions on the critical line. Finally, we give the simple zeros for some entire functions in Section 5.

2. Preliminary results

In this section we give some results in the theory of the entire functions applied in the present paper.

Definition 5. Let $s \in \mathbb{C}$. The Weierstrass primary factors $\mathbb{H}(s, 0)$ and $\mathbb{H}(s, p)$ are defined as ([3], Lecture 4, p.25)

$$\mathbb{H}(s, 0) = 1 - s \ \ (p = 0)\,, \tag{9}$$

and

$$\mathbb{H}(s, p) = (1 - s) \exp\left(s + \tfrac{1}{2}s^2 + \cdots + \tfrac{1}{p}s^p\right) \ \ (p > 1)\,, \tag{10}$$

where $p \in \mathbb{N}$ is of genus.

Definition 6. Let $\Omega = \{\mu_k\}_{k=1}^{\infty}$ be the set of the sequence of all zeros for the Weierstrass-Hadamard product

$$\widehat{\mathbb{H}}(s) = \prod_{k=1}^{\infty} \mathrm{H}(s, p) \tag{11}$$

such that

$$|\mu_1| < |\mu_2| < |\mu_3| < \cdots < |\mu_k| < |\mu_{k+1}| < \cdots\,, \tag{12}$$

and

$$\lim_{k \to \infty} \mu_k = \infty. \tag{13}$$

We say (11) is a canonical product of genus p ([3], Lecture 4, p.28).

Definition 7. The maximum modulus of $\widehat{\mathbb{H}}(s)$ on a disk of radius v is defined as (See [1], p.1)

$$\mathbb{MV}(v) = \max_{|s|=v} \left|\widehat{\mathbb{H}}(s)\right|. \tag{14}$$

Definition 8. The order β of $\widehat{\mathbb{H}}(s)$ is defined by (See [1], p.8])

$$\beta = \lim_{v \to \infty} \sup \tfrac{\log \log \mathbb{MV}(v)}{\log v}. \tag{15}$$

Definition 9. The exponent of convergence γ (the convergence exponent of its zeros) for $\widehat{\mathbb{H}}(s)$ is defined by (See [1], p.14)

$$\gamma = \inf\left\{\theta \left| |\mu_k|^{-\theta} < \infty\,, \mu_k \in \Omega,\ k \in \mathbb{N}\right.\right\}. \tag{16}$$

It is proved that Theorem 2 in the Levin's book ([3], Theorem 2, p.29) presented the property of the Weierstrass-Hadamard product:

Lemma 1. *Let $\mu_k \in \Omega$ for $k \in \mathbb{N}$ such that*

$$\sum_{k=1}^{\infty} 1/\left|\mu_k\right|^{p+1} < \infty. \tag{17}$$

Then the Weierstrass-Hadamard product

$$\widehat{\mathbb{H}}\left(s\right) = \prod_{k=1}^{\infty} \mathrm{H}\left(x/x_k, p\right), \tag{18}$$

converges uniformly on every compact set $\Re$.

Theorem 2.6.5. in the Boas' book ([1], Theorem 2.6.5., p.19) stated the theorem of Borel as follows:

Lemma 2. *(Theorem of Borel)*

Let $s \in \mathbb{C}$. Then $\widehat{\mathbb{H}}\left(s\right)$ of genus p is an entire function of order equal to the convergence exponent of its zeros.

Lemma 3.

- *Let $\mathbb{S} \in \mathbb{Y}$. Then the exponent of convergence for $\mathrm{S}\left(s\right)$ is $\gamma = 1$ and $\mathbb{S}\left(s\right)$ converges uniformly on every compact set $\aleph$.*
- *Let $\widehat{\mathbb{S}} \in \mathbb{L}$. Then the exponent of convergence for $\widehat{\mathbb{S}}\left(s\right)$ is $\gamma = 1$ and $\widehat{\mathbb{S}}\left(s\right)$ converges uniformly on every compact set $\widehat{\aleph}$.*
- *Let $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$. Then the exponent of convergence for $\widetilde{\mathbb{S}}\left(s\right)$ is $\gamma = 1$ and $\widetilde{\mathbb{S}}\left(s\right)$ converges uniformly on every compact set $\widetilde{\aleph}$.*
- *Let $\bar{\mathbb{S}} \in \bar{\mathbb{L}}$. Then the exponent of convergence for $\bar{\mathbb{S}}\left(s\right)$ is $\gamma = 1$ and $\bar{\mathbb{S}}\left(s\right)$ converges uniformly on every compact set $\bar{\aleph}$.*

Proof. By Lemma 1, Theorem of Borel and $\mathbb{S} \in \mathbb{Y}$, $\widehat{\mathbb{S}} \in \mathbb{L}$, $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$ and $\bar{\mathbb{S}} \in \bar{\mathbb{L}}$, we easily get the required results. □

3. New products for some classes of the entire functions

In this section we investigate new products for some classes of the entire functions.

Now we denote $\sigma_k \in \Omega$ and show the following theorems:

Theorem 1. *Let $\mathbb{S} \in \mathbb{Y}$, $s \in \mathbb{C}$, $\varsigma \in \mathbb{C}$, $\mathbb{Q} \in \mathbb{C}$, $\alpha \in \mathbb{C}\backslash\left\{0\right\}$ and $\alpha \neq \sigma_k$. Then*

$$\mathbb{S}\left(\alpha\right) = \mathbb{S}\left(0\right) e^{\mathbb{Q}\alpha} \prod_{k=1}^{\infty}\left(1 - \tfrac{\alpha}{\sigma_k}\right) e^{\alpha/\sigma_k} \neq 0, \tag{19}$$

$$\mathbb{S}\left(s\right) = \mathbb{S}\left(\alpha\right) e^{\mathbb{Q}\left(s-\alpha\right)} \prod_{k=1}^{\infty}\left(1 - \tfrac{s-\alpha}{\sigma_k-\alpha}\right) e^{\left(s-\alpha\right)/\sigma_k} \tag{20}$$

and

$$\mathbb{S}\left(\varsigma\right) + \sum_{k=1}^{\infty} \mathrm{S}^{(k)}\left(\varsigma\right) \tfrac{\left(s-\varsigma\right)^k}{k!} = \mathbb{S}\left(\alpha\right) e^{\mathbb{Q}\left(s-\alpha\right)} \prod_{k=1}^{\infty}\left(1 - \tfrac{s-\alpha}{\sigma_k-\alpha}\right) e^{\left(s-\alpha\right)/\sigma_k}, \tag{21}$$

where $\sigma_k = \varsigma_k + i\tau_k$ run though all zeros of $\mathbb{S}\left(s\right)$ for $\varsigma_k \in \mathbb{R}\backslash\left\{0\right\}$ and $\tau_k \in \mathbb{R}\backslash\left\{0\right\}$.

Proof. Since $\varsigma \in \mathbb{C}$, $\alpha \in \mathbb{C}\backslash\{0\}$ and $\alpha \neq \sigma_k$, we have $1-\alpha/\sigma_k \neq 0$, $e^{\mathbb{Q}\alpha} \neq 0$, $e^{\alpha/\sigma_k} \neq 0$ and $\mathbb{S}(0) \neq 0$ such that

$$\mathbb{S}(\alpha) = \mathbb{S}(0)\, e^{\mathbb{Q}\alpha} \prod_{k=1}^{\infty}\left(1-\frac{\alpha}{\sigma_k}\right) e^{\alpha/\sigma_k} \neq 0 \tag{22}$$

is valid. Hence, we prove (9).

On the another hand, we show

$$\begin{aligned}
\mathbb{S}(s) &= \mathbb{S}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty}\left(1-\frac{s}{\sigma_k}\right) e^{s/\sigma_k} \\
&= \left(\mathbb{S}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty} e^{s/\sigma_k}\right) \prod_{k=1}^{\infty}\left(1-\frac{s}{\sigma_k}\right) \\
&= \left(\mathbb{S}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty} e^{s/\sigma_k}\right) \prod_{k=1}^{\infty} \frac{\sigma_k - s}{\sigma_k} \\
&= \left(\mathbb{S}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty} e^{s/\sigma_k}\right) \prod_{k=1}^{\infty} \left(\frac{\sigma_k-\alpha}{\sigma_k-\alpha} \cdot \frac{\sigma_k - s}{\sigma_k}\right) \\
&= \left(\mathbb{S}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty} e^{s/\sigma_k}\right) \prod_{k=1}^{\infty} \left(\frac{\sigma_k-\alpha}{\sigma_k} \cdot \frac{\sigma_k - s}{\sigma_k-\alpha}\right) \\
&= \left(\mathbb{S}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty} e^{s/\sigma_k}\right) \cdot \left(\prod_{k=1}^{\infty} \frac{\sigma_k-\alpha}{\sigma_k}\right) \cdot \left(\prod_{k=1}^{\infty} \frac{\sigma_k - s}{\sigma_k-\alpha}\right) \\
&= \left(\mathbb{S}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty} e^{s/\sigma_k}\right) \cdot \prod_{k=1}^{\infty}\left(1-\frac{\alpha}{\sigma_k}\right) \cdot \left[\prod_{k=1}^{\infty} \frac{\sigma_k-\alpha-(s-\alpha)}{\sigma_k-\alpha}\right] \\
&= \left(\mathbb{S}(0)\, e^{\mathbb{Q}s} \prod_{k=1}^{\infty} e^{s/\sigma_k}\right) \cdot \prod_{k=1}^{\infty}\left(1-\frac{\alpha}{\sigma_k}\right) \cdot \prod_{k=1}^{\infty}\left(1-\frac{s-\alpha}{\sigma_k-\alpha}\right) \\
&= \left[\mathbb{S}(0) \prod_{k=1}^{\infty}\left(1-\frac{\alpha}{\sigma_k}\right)\right] \cdot e^{\mathbb{Q}s} \prod_{k=1}^{\infty}\left(1-\frac{s-\alpha}{\sigma_k-\alpha}\right) e^{s/\sigma_k}.
\end{aligned} \tag{23}$$

By (13), we have

$$\begin{aligned}
\mathbb{S}(s) &= \left[\mathbb{S}(0) \prod_{k=1}^{\infty}\left(1-\frac{\alpha}{\sigma_k}\right)\right] \cdot e^{\mathbb{Q}s} \prod_{k=1}^{\infty}\left(1-\frac{s-\alpha}{\sigma_k-\alpha}\right) e^{s/\sigma_k} \\
&= \left[\mathbb{S}(0) \prod_{k=1}^{\infty}\left(1-\frac{\alpha}{\sigma_k}\right)\right] \cdot e^{\mathbb{Q}s} \prod_{k=1}^{\infty}\left(1-\frac{s-\alpha}{\sigma_k-\alpha}\right) e^{s/\sigma_k} \\
&= \left(\mathbb{S}(\alpha)\, e^{-\mathbb{Q}\alpha} \prod_{k=1}^{\infty} e^{-\alpha/\sigma_k}\right) \cdot \left[e^{\mathbb{Q}s} \prod_{k=1}^{\infty}\left(1-\frac{s-\alpha}{\sigma_k-\alpha}\right) e^{s/\sigma_k}\right] \\
&= \mathbb{S}(\alpha)\, e^{\mathbb{Q}(s-\alpha)} \prod_{k=1}^{\infty}\left(1-\frac{s-\alpha}{\sigma_k-\alpha}\right) e^{(s-\alpha)/\sigma_k}
\end{aligned} \tag{24}$$

if

$$\mathbb{S}(0) \prod_{k=1}^{\infty}\left(1-\frac{\alpha}{\sigma_k}\right) = \mathbb{S}(\alpha)\, e^{-\mathbb{Q}\alpha} \prod_{k=1}^{\infty} e^{-\alpha/\sigma_k} \tag{25}$$

is derived from (9).

Combining (1) and (10), we may obtain (11).

This completes the proof of Theorem 1. $\square$

Theorem 2. *Let* $\widehat{\mathbb{S}} \in \mathbb{L}$, $s \in \mathbb{C}$, $\alpha \in \mathbb{C}\backslash\{0\}$ *and* $\alpha \neq \sigma_k$. *Then*

$$\widehat{\mathbb{S}}(\xi) = \widehat{\mathbb{S}}(0) \prod_{k=1}^{\infty} \left(1 - \tfrac{\alpha}{\sigma_k}\right) \neq 0, \tag{26}$$

$$\widehat{\mathbb{S}}(s) = \widehat{\mathbb{S}}(\alpha) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\alpha}{\sigma_k-\alpha}\right) \tag{27}$$

and

$$\widehat{\mathbb{S}}(\varsigma) + \sum_{k=1}^{\infty} \widehat{\mathbb{S}}^{(k)}(\varsigma) \tfrac{(s-\varsigma)^k}{k!} = \widehat{\mathbb{S}}(\alpha) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\alpha}{\sigma_k-\alpha}\right), \tag{28}$$

where $\sigma_k = \varsigma_k + i\tau_k$ *run though all zeros of* $\widehat{\mathrm{S}}(s)$ *for* $\varsigma_k \in \mathbb{R}\backslash\{0\}$ *and* $\tau_k \in \mathbb{R}\backslash\{0\}$.

Proof. By taking $\alpha = \xi$ and

$$e^{\mathbb{Q}(s-\xi)} \prod_{k=1}^{\infty} e^{(s-\xi)/\sigma_k} = 1 \tag{29}$$

in Theorem 1 since $\widehat{\mathbb{S}}(\xi)$ is of genes $\upsilon = 0$, we show (16) and (17), and by $\widehat{\mathbb{S}} \in \mathbb{L}$, we get (18).

Thus, the required results follow. □

Theorem 3. *Let* $\bar{\mathbb{S}} \in \bar{\mathbb{L}}$, $s \in \mathbb{C}$ *and* $\mathbb{Q} \in \mathbb{C}$. *Then*

$$\bar{\mathbb{S}}(\xi) = \bar{\mathbb{S}}(0) e^{\mathbb{Q}\xi} \prod_{k=1}^{\infty} \left(1 - \tfrac{\xi}{\sigma_k}\right) e^{\xi/\sigma_k} \neq 0, \tag{30}$$

$$\bar{\mathbb{S}}(s) = \bar{\mathbb{S}}(\xi) e^{\mathbb{Q}(s-\xi)} \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right) e^{(s-\xi)/\sigma_k} \tag{31}$$

and

$$\bar{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \bar{\mathbb{S}}^{(k)}(\xi) \tfrac{(s-\xi)^k}{k!} = \bar{\mathbb{S}}(\xi) e^{\mathbb{Q}(s-\xi)} \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right) e^{(s-\xi)/\sigma_k}, \tag{32}$$

where $\sigma_k = \xi + i\tau_k$ *run though all zeros of* $\bar{\mathbb{S}}(s)$ *for* $\xi \in \mathbb{R}\backslash\{0\}$ *and* $\tau_k \in \mathrm{R}\backslash\{0\}$.

Proof. By taking $\alpha = \xi$ and $\varsigma = \xi$ in Theorem 1 and considering $\bar{\mathbb{S}} \in \bar{\mathbb{L}}$, we show (30),

$$\begin{aligned} \bar{\mathbb{S}}(s) &= \bar{\mathbb{S}}(\xi) e^{\mathbb{Q}(s-\xi)} \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{\sigma_k-\xi}\right) e^{(s-\xi)/\sigma_k} \\ &= \bar{\mathbb{S}}(\xi) e^{\mathbb{Q}(s-\xi)} \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right) e^{(s-\xi)/\sigma_k} \end{aligned} \tag{33}$$

and

$$\begin{aligned} &\bar{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \bar{\mathbb{S}}^{(k)}(\xi) \tfrac{(s-\xi)^k}{k!} = \bar{\mathbb{S}}(\xi) e^{\mathbb{Q}(s-\xi)} \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{\sigma_k-\xi}\right) e^{(s-\xi)/\sigma_k} \\ &= \bar{\mathbb{S}}(\xi) e^{\mathbb{Q}(s-\xi)} \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right) e^{(s-\xi)/\sigma_k}. \end{aligned} \tag{34}$$

Hence, we finish the proof. □

Theorem 4. *Let* $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$ *and* $s \in \mathbb{C}$. *Then*

$$\widetilde{\mathbb{S}}(\xi) = \widetilde{\mathbb{S}}(0) \prod_{k=1}^{\infty} \left(1 - \tfrac{\xi}{\sigma_k}\right) \neq 0, \tag{35}$$

$$\widetilde{\mathbb{S}}(s) = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right) \tag{36}$$

and

$$\widetilde{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \widetilde{\mathbb{S}}^{(k)}(\xi) \tfrac{(s-\xi)^k}{k!} = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right), \tag{37}$$

where $\sigma_k = \xi + i\tau_k$ *run though all zeros of* $\widetilde{\mathbb{S}}(s)$ *for* $\xi \in \mathbb{R}\backslash\{0\}$ *and* $\tau_k \in \mathrm{R}\backslash\{0\}$.

Proof. By using

$$e^{\mathbb{Q}(s-\xi)} \prod_{k=1}^{\infty} e^{(s-\xi)/\sigma_k} = 1 \tag{38}$$

in Theorem 3 since $\widetilde{\mathbb{S}}(s)$ is of genes $\upsilon = 0$, we show (33) and

$$\widetilde{\mathbb{S}}(s) = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{\sigma_k-\xi}\right) = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right), \tag{39}$$

and by $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$, we reduce to

$$\widetilde{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \widetilde{\mathbb{S}}^{(k)}(\xi) \tfrac{(s-\xi)^k}{k!} = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{\sigma_k-\xi}\right) = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right). \tag{40}$$

Thus, the required results follow. □

Theorem 5. *Let* $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$ *and* $\widetilde{\mathbb{S}}(s) = \widetilde{\mathbb{S}}(2\xi - s)$ *for* $s \in \mathbb{C}$. *Then*

$$\widetilde{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \widetilde{\mathbb{S}}^{(2k)}(\xi) \tfrac{(s-\xi)^{2k}}{(2k)!} = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right), \tag{41}$$

where $\sigma_k = \xi + i\tau_k$ *run though all zeros of* $\widetilde{\mathbb{S}}(s)$ *for* $\xi \in \mathbb{R}\backslash\{0\}$ *and* $\tau_k \in \mathrm{R}\backslash\{0\}$.

Proof. With use of Theorem 4 we deduce

$$\begin{aligned} \widetilde{\mathbb{S}}(2\xi - s) &= \widetilde{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \widetilde{\mathbb{S}}^{(k)}(\xi) \tfrac{[(2\xi-s)-\xi]^k}{k!} \\ &= \widetilde{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \widetilde{\mathbb{S}}^{(k)}(\xi) \tfrac{(\xi-s)^k}{k!} \\ &= \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{\xi-s}{i\tau_k}\right) \end{aligned} \tag{42}$$

and

$$\widetilde{\mathbb{S}}(s) = \widetilde{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \widetilde{\mathbb{S}}^{(k)}(\xi) \tfrac{(s-\xi)^k}{k!} = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right). \tag{43}$$

Since $\widetilde{\mathbb{S}}(s)=\widetilde{\mathbb{S}}(2\xi-s)$, we have

$$\widetilde{\mathbb{S}}(\xi)+\sum_{k=1}^{\infty}\widetilde{\mathbb{S}}^{(k)}(\xi)\frac{(\xi-s)^k}{k!}=\widetilde{\mathbb{S}}(\xi)+\sum_{k=1}^{\infty}\widetilde{\mathbb{S}}^{(k)}(\xi)\frac{(s-\xi)^k}{k!} \tag{44}$$

such that

$$\widetilde{\mathbb{S}}(s)=\widetilde{\mathbb{S}}(\xi)+\sum_{k=1}^{\infty}\widetilde{\mathbb{S}}^{(2k)}(\xi)\frac{(\xi-s)^{2k}}{(2k)!} \tag{45}$$

and

$$\sum_{k=0}^{\infty}\widetilde{\mathbb{S}}^{(2k+1)}(\xi)\frac{(s-\xi)^{2k+1}}{(2k+1)!}=0 \tag{46}$$

if we combine (42) and (43). □

4. Behaviors for some entire functions on the critical line

In this section we consider the behaviors for some entire functions on the critical line $Re(s)=\xi$. We now begin with the following result:

Theorem 6. *Let $\bar{\mathbb{S}}\in\bar{\mathbb{L}}$ such that $\bar{\mathbb{V}}(x)=\bar{\mathbb{S}}(\xi+ix)$, where $x\in\mathbb{R}$. If $\mathbb{Q}\in\mathbb{C}$ and $\bar{\mathbb{V}}(0)\neq 0$, then*

$$\bar{\mathbb{V}}(x)=\bar{\mathbb{V}}(0)\,e^{ix\mathbb{Q}}\prod_{k=1}^{\infty}\left(1-\frac{x}{\tau_k}\right)e^{ix/\sigma_k}, \tag{47}$$

$$\bar{\mathbb{V}}(0)+\sum_{k=1}^{\infty}\bar{\mathbb{V}}^{(k)}(0)\frac{x^k}{k!}=\bar{\mathbb{V}}(0)\,e^{ix\mathbb{Q}}\prod_{k=1}^{\infty}\left(1-\frac{x}{\tau_k}\right)e^{ix/\sigma_k}, \tag{48}$$

$\bar{\mathbb{V}}(x)$ converges uniformly on every real compact set $\bar{\wp}$ and $\tau_k\in\mathbb{R}\backslash\{0\}$ run though all real zeros of $\bar{\mathbb{V}}(x)$.

Proof. Making use of Theorem 4, we suggest

$$\bar{\mathbb{S}}(\xi)=\bar{\mathbb{V}}(0)\neq 0, \tag{49}$$

$$\begin{aligned}\bar{\mathbb{V}}(x)&=\bar{\mathbb{S}}(\xi+ix)=\bar{\mathbb{S}}(\xi)\,e^{\mathbb{Q}[(\xi+ix)-\xi]}\prod_{k=1}^{\infty}\left[1-\frac{(\xi+ix)-\xi}{i\tau_k}\right]e^{[(\xi+ix)-\xi]/\sigma_k}\\&=\bar{\mathbb{V}}(0)\,e^{ix\mathbb{Q}}\prod_{k=1}^{\infty}\left(1-\frac{x}{\tau_k}\right)e^{ix/\sigma_k}\end{aligned} \tag{50}$$

and

$$\begin{aligned}\bar{\mathbb{V}}(x)&=\bar{\mathbb{S}}(\xi+ix)=\bar{\mathbb{S}}(\xi)+\sum_{k=1}^{\infty}\bar{\mathbb{S}}^{(k)}(\xi)\frac{[(\xi+ix)-\xi]^k}{k!}\\&=\bar{\mathbb{V}}(0)+\sum_{k=1}^{\infty}i^k\bar{\mathbb{S}}^{(k)}(\xi)\frac{x^k}{k!}\\&=\bar{\mathbb{V}}(0)+\sum_{k=1}^{\infty}\bar{\mathbb{V}}^{(k)}(0)\frac{x^k}{k!},\end{aligned} \tag{51}$$

where

$$i^k\bar{\mathbb{S}}^{(k)}(\xi) = \bar{\mathbb{V}}^{(k)}(0) \tag{52}$$

and

$$\bar{\mathbb{S}}(\xi)\, e^{\mathbb{Q}[(\xi+ix)-\xi]} \prod_{k=1}^{\infty}\left[1-\frac{(\xi+ix)-\xi}{i\tau_k}\right] e^{[(\xi+ix)-\xi]/\sigma_k} = \bar{\mathbb{V}}(0)\, e^{ix\mathbb{Q}} \prod_{k=1}^{\infty}\left(1-\frac{x}{\tau_k}\right) e^{ix/\sigma_k}. \tag{53}$$

By (51) and (53), $\bar{\mathbb{V}}(x)$ can be rewritten as

$$\bar{\mathbb{V}}(0) + \sum_{k=1}^{\infty} \bar{\mathbb{V}}^{(k)}(0)\,\frac{x^k}{k!} = \bar{\mathbb{V}}(0)\, e^{ix\mathbb{Q}} \prod_{k=1}^{\infty}\left(1-\frac{x}{\tau_k}\right) e^{ix/\sigma_k}. \tag{54}$$

With Lemma 3, we see that $\bar{\mathbb{S}}(s)$ converges uniformly on every compact set $\bar{\aleph}$ and that $\bar{\mathbb{V}}(x)$ converges uniformly on every real compact set $\bar{\wp}$.

Since $\bar{\mathbb{S}} \in \bar{\mathbb{L}}$, $\tau_k \in \mathbb{R}\setminus\{0\}$ run though all real zeros of $\bar{\mathbb{V}}(x)$.

Therefore, the proof of Theorem 6 is complete. □

Theorem 7. *Let $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$ such that $\widetilde{\mathbb{V}}(x) = \widetilde{\mathbb{S}}(\xi + ixn)$, where $x \in \mathbb{R}$. If $\widetilde{\mathbb{V}}(0) \neq 0$, then*

$$\widetilde{\mathbb{V}}(x) = \widetilde{\mathbb{V}}(0) \prod_{k=1}^{\infty}\left(1-\frac{x}{\tau_k}\right), \tag{55}$$

$$\widetilde{\mathbb{V}}(0) + \sum_{k=1}^{\infty} \widetilde{\mathbb{V}}^{(k)}(0)\,\frac{x^k}{k!} = \widetilde{\mathbb{V}}(0) \prod_{k=1}^{\infty}\left(1-\frac{x}{\tau_k}\right), \tag{56}$$

$\widetilde{\mathbb{V}}(x)$ converges uniformly on every real compact set $\widetilde{\wp}$ and $\tau_k \in \mathbb{R}\setminus\{0\}$ run though all real zeros of $\widetilde{\mathbb{V}}(x)$.

Proof. In view of Theorem 4 and $\widetilde{\mathbb{V}}(x) = \widetilde{\mathbb{S}}(\xi + ix)$, we get

$$\widetilde{\mathbb{S}}(\xi) = \widetilde{\mathbb{V}}(0) \neq 0, \tag{57}$$

$$\begin{aligned}\widetilde{\mathbb{V}}(x) &= \widetilde{\mathbb{S}}(\xi+ix) = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty}\left[1-\frac{(\xi+ix)-\xi}{i\tau_k}\right]\\ &= \widetilde{\mathbb{V}}(0) \prod_{k=1}^{\infty}\left[1-\frac{(\xi+ix)-\xi}{i\tau_k}\right]\\ &= \widetilde{\mathbb{V}}(0) \prod_{k=1}^{\infty}\left(1-\frac{x}{\tau_k}\right),\end{aligned} \tag{58}$$

and

$$\begin{aligned}\widetilde{\mathbb{V}}(x) &= \widetilde{\mathbb{S}}(\xi+ix) = \widetilde{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \widetilde{\mathbb{S}}^{(k)}(\xi)\,\frac{[(\xi+ix)-\xi]^k}{k!}\\ &= \widetilde{\mathbb{V}}(0) + \sum_{k=1}^{\infty} i^k\widetilde{\mathbb{S}}^{(k)}(\xi)\,\frac{x^k}{k!}\\ &= \widetilde{\mathbb{V}}(0) + \sum_{k=1}^{\infty} \widetilde{\mathbb{V}}^{(k)}(0)\,\frac{x^k}{k!},\end{aligned} \tag{59}$$

where

$$i^k \widetilde{\mathbb{S}}^{(k)}(\xi) = \widetilde{\mathbb{V}}^{(k)}(0). \tag{60}$$

Hence, from (58) and (59) we arrive at

$$\widetilde{\mathbb{V}}(0) + \sum_{k=1}^{\infty} \widetilde{\mathbb{V}}^{(k)}(0) \tfrac{x^k}{k!} = \widetilde{\mathbb{V}}(0) \prod_{k=1}^{\infty} \left(1 - \tfrac{x}{\tau_k}\right). \tag{61}$$

By Lemma 3, we show $\widetilde{\mathbb{S}}(s)$ converges uniformly on every compact set $\widetilde{\aleph}$, and we obtain $\widetilde{\mathbb{V}}(x)$ converges uniformly on every real compact set $\widetilde{\wp}$.

Since $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$, $\tau_k \in \mathbb{R}\backslash\{0\}$ run over all real zeros of $\widetilde{\mathbb{V}}(x)$.

Therefore, we finish the proof of Theorem 7. □

Remark. Putting $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$ and (56), we suppose that

$$\widetilde{\mathbb{V}}(x) = \widetilde{\mathbb{V}}(-x). \tag{62}$$

Applying (56) and (62), we show

$$\widetilde{\mathbb{V}}(0) + \sum_{k=1}^{\infty} \widetilde{\mathbb{V}}^{(k)}(0) \tfrac{x^k}{k!} = \widetilde{\mathbb{V}}(0) + \sum_{k=1}^{\infty} (-1)^k \widetilde{\mathbb{V}}^{(k)}(0) \tfrac{x^k}{k!}. \tag{63}$$

Then,

$$\sum_{k=1}^{\infty} \widetilde{\mathbb{V}}^{(2k+1)}(0) \tfrac{x^{(2k+1)}}{(2k+1)!} = 0 \tag{64}$$

and

$$\widetilde{\mathbb{V}}(x) = \widetilde{\mathbb{V}}(0) + \sum_{k=1}^{\infty} \widetilde{\mathbb{V}}^{(2k)}(0) \tfrac{x^{2k}}{(2k)!}. \tag{65}$$

By (55) and (62), we get

$$\widetilde{\mathbb{V}}(0) \prod_{k=1}^{\infty} \left(1 - \tfrac{x}{\tau_k}\right) = \widetilde{\mathbb{V}}(0) \prod_{k=1}^{\infty} \left(1 - \tfrac{x^2}{\widehat{\tau}_k^2}\right), \tag{66}$$

where $\widehat{\tau}_k = |\tau_k|$.

From (65) and (66) we see

$$\widetilde{\mathbb{V}}(x) = \widetilde{\mathbb{V}}(0) \prod_{k=1}^{\infty} \left(1 - \tfrac{x^2}{\tau_k^2}\right) = \widetilde{\mathbb{V}}(0) + \sum_{k=1}^{\infty} \widetilde{\mathbb{V}}^{(2k)}(0) \tfrac{x^{2k}}{(2k)!}. \tag{67}$$

By Theorem 7, we find that $\widetilde{\mathbb{V}}(x)$ converges uniformly on every real compact set $\widetilde{\wp}$ and $\tau_k \in \mathbb{R}\backslash\{0\}$ run though all real zeros of $\widetilde{\mathbb{V}}(x)$.

5. Simple zeros for some entire functions

In this section we present two theorems on the simple zeros for some entire functions.

Theorem 8. *Let* $\widetilde{\mathbb{S}} \in \widetilde{\mathbb{Y}}$ *and* $\widetilde{\mathbb{S}}(s) = \widetilde{\mathbb{S}}(2\xi - s)$. *Then all zeros of* $\widetilde{\mathbb{S}}(s)$ *are simple.*

Proof. By Theorem 5, we show

$$\widetilde{\mathbb{S}}(s) = \widetilde{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \widetilde{\mathbb{S}}^{(2k)}(\xi) \tfrac{(s-\xi)^{2k}}{(2k)!} = \widetilde{\mathbb{S}}(\xi) \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right), \tag{68}$$

and all zeros of $\widetilde{\mathbb{S}}(s)$ are $\sigma_k = \xi + i\tau_k$ for $\xi \in \mathbb{R}\setminus\{0\}$ and $\tau_k \in \mathbb{R}\setminus\{0\}$.

Hence, all zeros of $\widetilde{\mathbb{S}}(s)$ are simple, and this completes the proof. □

Theorem 9. *Let* $\bar{\mathbb{S}} \in \bar{\mathbb{L}}$. *Then all zeros of* $\bar{\mathbb{S}}(s)$ *are simple.*

Proof. By using Theorem 5 we get

$$\bar{\mathbb{S}}(\xi) + \sum_{k=1}^{\infty} \bar{\mathbb{S}}^{(k)}(\xi) \tfrac{(s-\xi)^k}{k!} = \bar{\mathbb{S}}(\xi) e^{\mathbb{Q}(s-\xi)} \prod_{k=1}^{\infty} \left(1 - \tfrac{s-\xi}{i\tau_k}\right) e^{(s-\xi)/\sigma_k}, \tag{69}$$

and $\sigma_k = \xi + i\tau_k$ run though all zeros of $\bar{\mathbb{S}}(s)$ for $\xi \in \mathbb{R}\setminus\{0\}$ and $\tau_k \in \mathbb{R}\setminus\{0\}$, which are the required results. □

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