Optimal System, Invariant Solutions and Conservation Laws of the Dispersionless B type Kadomtsev-Petviashvili Equation

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Abstract

In this paper, we consider the dispersionless B type Kadomtsev-Petviashvili (dBKP) equation through quasi classical limit of BKP equation. We investigate the existence of one-parameter point transformations in which the dBKP equation remains invariant by admitting a five-dimensional Lie algebra. For the admitted Lie symmetries, we calculate the one-dimensional optimal system, a necessary analysis to perform the reduction process. Using this, we obtain various closed-form similarity solutions for the dBKP equation. In addition to this, we also derive the associated conservation laws of this equation through Ibragimov's method.

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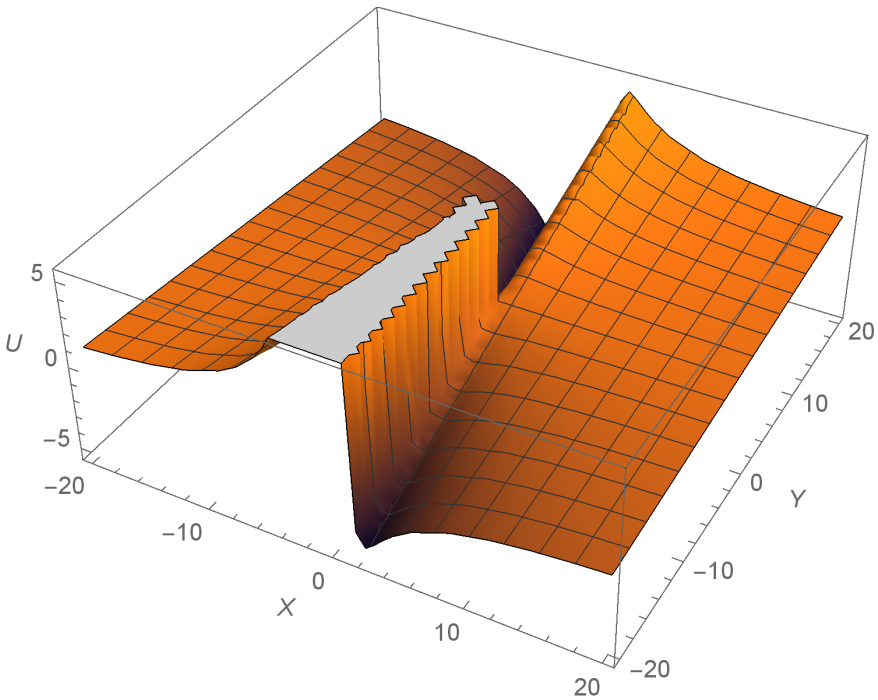

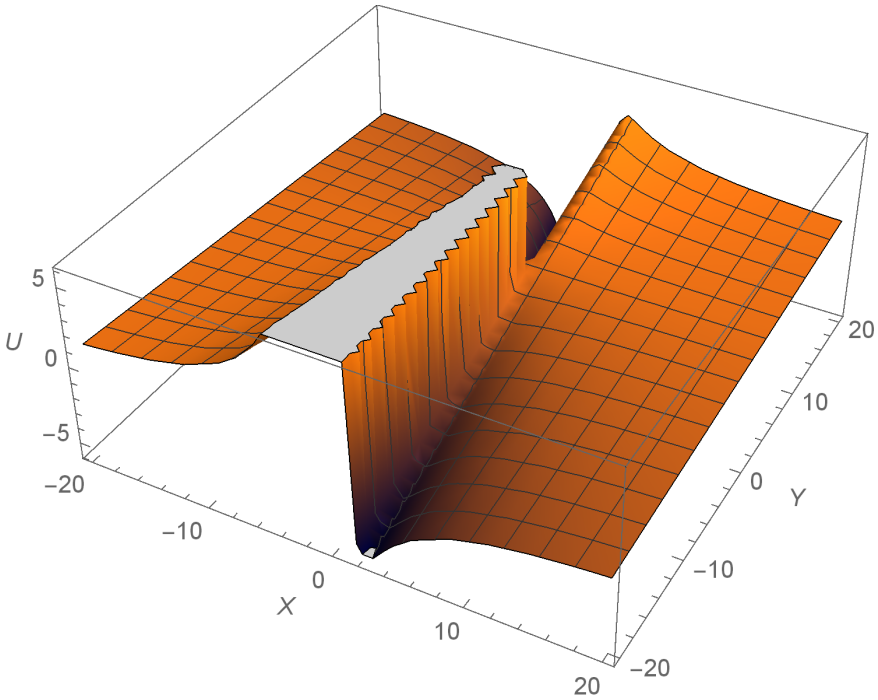

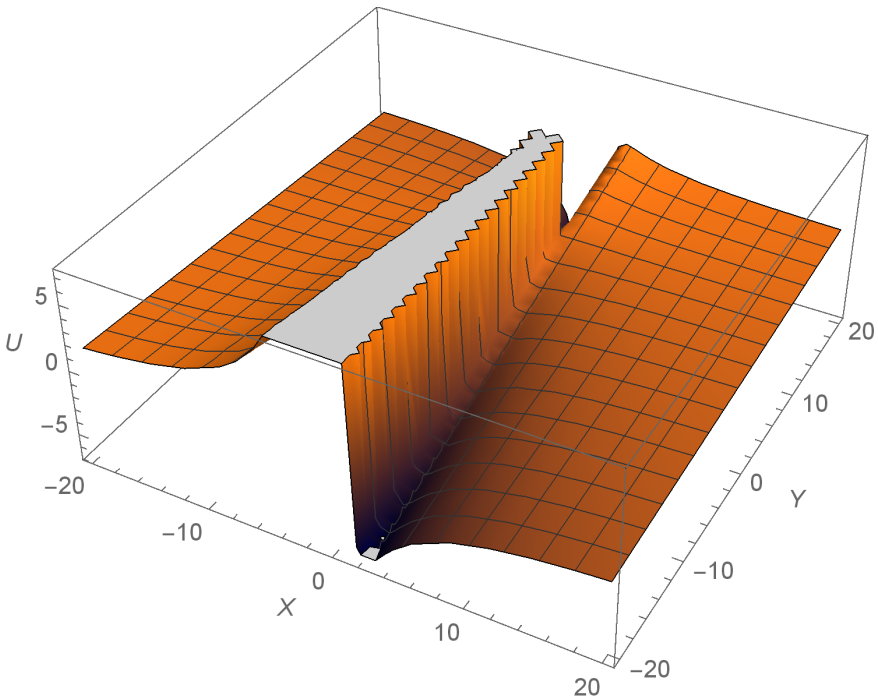

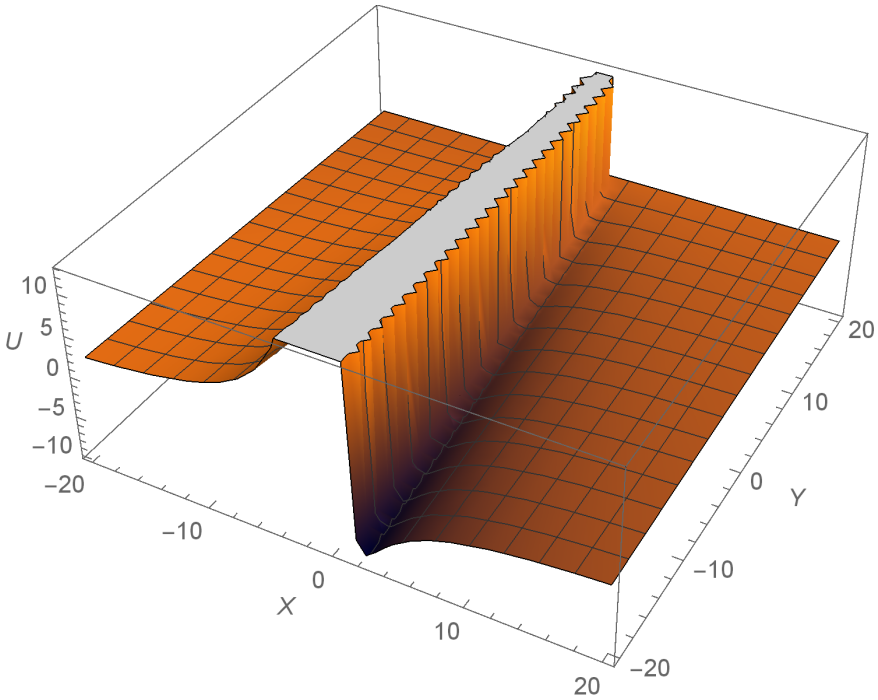

Optimal System, Invariant Solutions and Conservation Laws of the Dispersionless B type Kadomtsev-Petviashvili Equation

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Abstract: In this paper, we consider the dispersionless B type Kadomtsev-Petviashvili (dBKP) equation through quasi classical limit of BKP equation. We investigate the existence of one-parameter point transformations in which the dBKP equation remains invariant by admitting a five-dimensional Lie algebra. For the admitted Lie symmetries, we calculate the one-dimensional optimal system, a necessary analysis to perform the reduction process. Using this, we obtain various closed-form similarity solutions for the dBKP equation. In addition to this, we also derive the associated conservation laws of this equation through Ibragimov's method.

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Key Words and Phrases: Symmetries; Similarity solutions; Lie invariants; dispersionless B type KP(dBKP) equation.

1 Introduction

Dispersionless equations especially dispersionless Kadomtsev-Petviashvili (dKP) hierarchies, a type of non-linear integrable system has been widely studied in the recent past. This equations posses various mathematical richness and physical importance such as the relation with topological field theory, Whitham hierarchy, string theory, two dimensional gravity [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], Orolov functions [5], solutions through hodograph transformations [7, 8], tau function theory and dispersionless analogue of Virasoro constrains [11]. Initially, Lebdev, Manin and Zakhrov [12, 13] investigated this kind of systems through quasi classical limit. Later, Krichever [14] derived the dKP hierarchy through the construction of dispersionless Lax equations. More interestingly, J H Chang et al. [15] studied the correspondence of dKP and dmKP hierarchies through the construction of dispersionless Miura map. Recently, D J Zhang [16] et al. investigated the iso and non isospectral flows of dKP hierarchies by introducing Lax triad approach. Many of the interesting properties are yet to be investigated for

these types of equations such as underlying Lie algebraic structures, similarity reductions and solutions using Lie point symmetries, integrability properties and solutions via discretization, Painlevé analysis, differential-difference analogues etc.

Moreover, dispersionless version of some of the vital integrable equations are still not yet studied in the direction of various mathematical perspectives. In this point of view, we are motivated to consider the dispersionless B type KP(dBKP) equation and study its symmetries and particular solutions. The BKP equation shows significant physical importance in the field of fluid mechanics, plasma physics, optical fibres and solid state physics. Moreover, BKP system posses similar mathematical properties in parallel with KP system such as Lax formulation, soliton, Hirota bilinear equations, tau-function, fermion representation, quasi-periodic solutions etc. This motivates us to consider the dispersionless analogue of BKP equation.

In this piece of work, we consider dBKP equation by taking quasi classical limit on BKP equation. We also obtained various particular solutions of dBKP equation using Lie point symmetries. Lie method is a powerful mathematical tool to derive the exact solutions for nonlinear differential equations [17, 18, 19]. Specifically, Lie point symmetries enable one to find similarity transformations, which is used by researchers to introduce new dependent and independent variables. When applying similarity variable into PDEs, the number of independent variables is reduced, but order of the PDEs remains the same. By applying sequence of reduction on the reduced PDEs we finally end up with ODEs. In the same way, repeatedly implementing the Lie symmetry technique on the reduced ODEs leads to lower order ODEs from which one can compute the particular solutions to the given system. In particular, we construct the one-dimensional optimal system for the obtained Lie group [20] to avoid the same class of solutions. This is achieved by constructing the adjoint representation of the symmetry group [21, 22, 23, 24, 25, 26, 27]. Nowadays researchers are using powerful Computer Algebra Systems (CAS) like Maple and Mathematica (commercial), etc. to do the calculations over the symmetry rapidly. In this work, for the calculation of the symmetries, we use the Mathematica add-on Sym [28, 29, 30, 31].

The plan of the paper is as follows, in Section 2, we derive Lie point symmetries of dBKP and obtain the associated optimal class of vector fields, the similarity solutions of dBKP are presented in Section 3. In section 4, using Ibragimov's method, we obtain the conservation laws of dBKP equation. Finally, in Section 5 we discuss our results and we draw out our conclusions.

2 Lie symmetry classification

Consider the BKP equation as

$$u_t = -\frac{5}{9}\left[\frac{1}{5}u_{xxxxx} + 3uu_{xxx} + 3u_x u_{xx} + 9u^2 u_x - u_{xxy} - 3uu_y - 3u_x\partial^{-1}u_y - \partial^{-1}u_{yy}\right] \tag{2.1}$$

Now, using quasi classical limit i.e. by averaging the independent and field variables as $\epsilon x = X, \epsilon y = Y, \epsilon t = T$ in (2.1) and allow $\epsilon \to 0$, we obtain the dBKP as follows

$$U_T = -5U^2 U_X + \frac{5}{3} U U_Y + \frac{5}{3} U_X \partial^{-1} U_Y + \frac{5}{9} \partial^{-1} U_{YY} \tag{2.2}$$

To perform Lie symmetry analysis, it is convenient to rewrite (2.2) without nonlocal terms. Hence, substituting $U = V_X$ in to equation (2.2), we obtain

$$V_{TX} + 5{V_X}^2 V_{XX} - \frac{5}{3} V_X V_{XY} - \frac{5}{3} V_{XX} V_Y - \frac{5}{9} V_{YY} = 0, \tag{2.3}$$

The symmetry conditions for the latter equation are

$$\left.\begin{aligned} 250a_1(T)\partial_T - (50Va_1{}' + 30XYa_1{}'' + 9Y^3 a_1{}''')\partial_V + (50Xa_1{}' + 45Y^2 a_1{}'')\partial_X + 150Ya_1{}'\partial_Y, \\ -(10Xa_2(T)' + 9Y^2 a_2{}'')\partial_V + 30Ya_2{}'\partial_X + 150a_2\partial_Y, \\ -3Ya_3(T)'\partial_V + 5Ya_3\partial_X, \\ 3V\partial_V + 2X\partial_X + Y\partial_Y, \\ a_4(T)\partial_V. \end{aligned}\right\} \tag{2.4}$$

To reduce the number of arbitrary functions in the Lie point symmetries the equation (2.3) can be rewritten as

$$\left.\begin{aligned} W_X + 5{V_X}^2 V_{XX} - \frac{5}{9} V_{YY} = 0 \\ Z - W - \frac{5}{3} V_X V_Y = 0 \\ Z - V_T = 0 \end{aligned}\right\} \tag{2.5}$$

The system of equation (2.5) provides the following Lie point symmetries

$$\Gamma_1 = \partial_T \tag{2.6a}$$

$$\Gamma_2 = \partial_X \tag{2.6b}$$

$$\Gamma_3 = \partial_Y \tag{2.6c}$$

$$\Gamma_4 = 2V\partial_V + \frac{3}{2} W\partial_W + \frac{3}{2} Z\partial_Z + \frac{1}{2} T\partial_T + \frac{3}{2} X\partial_X + Y\partial_Y \tag{2.6d}$$

$$\Gamma_5 = f(T)\partial_V + f'(T)(\partial_W + \partial_Z) \tag{2.6e}$$

We observe that the presence of arbitrary function $f(T)$ in the symmetry vector Γ_5, indicates there are infinite number of solutions depending only on temporal variable. These kind of solutions do not play any role in determining the exact solution of the equation (2.3). Hence, our study restricted with the vector fields Γ_1 to Γ_4 only. Now, using these vector fields, we construct the commutator table, the associated adjoint representation and one-dimensional optimal system [17, 18, 32, 33, 34, 36, 37, 38, 39, 40, 41] of (2.5), sufficient combination of vector fields (2.6). This allows us to find similarity transformations, reductions and exact solutions of (2.5). The commutator table and adjoint representation are given in table (1) and (2) respectively.

Table 1: Commutator Table

$[\Gamma_I, \Gamma_J]$	Γ_1	Γ_2	Γ_3	Γ_4
Γ_1	0	0	0	$\frac{\Gamma_1}{2}$
Γ_2	0	0	0	$\frac{3\Gamma_2}{2}$
Γ_3	0	0	0	Γ_3
Γ_4	$-\frac{\Gamma_1}{2}$	$-\frac{3\Gamma_2}{2}$	$-\Gamma_3$	0

Table 2: Adjoint representation

$\left[Ad(e^{\epsilon\Gamma_i})\Gamma_j\right]$	Γ_1	Γ_2	Γ_3	Γ_4
Γ_1	Γ_1	Γ_2	Γ_3	$\Gamma_4 - \frac{\epsilon}{2}\Gamma_1$
Γ_2	Γ_1	Γ_2	Γ_3	$\Gamma_4 - \frac{3\epsilon}{2}\Gamma_2$
Γ_3	Γ_1	Γ_2	Γ_3	$\Gamma_4 - \epsilon\Gamma_3$
Γ_4	$e^{\frac{\epsilon}{2}}\Gamma_1$	$e^{\frac{3\epsilon}{2}}\Gamma_2$	$e^{\epsilon}\Gamma_3$	Γ_4

Based on the commutator table (1) and adjoint representation table (2), the one dimensional optimal system are

$$\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_2 + c_1\Gamma_1,\ \Gamma_3 + c_1\Gamma_2,\ \Gamma_3 + c_1\Gamma_1,\ \Gamma_3 + c_2\Gamma_2 + c_1\Gamma_1.$$

3 Similarity solutions

In this section, we compute systematically the invariant solutions for (2.5) corresponding to each optimal system, tabulated in Table (3). In particular, we are motivated in obtaining the invariant solutions with respect to the scaling symmetries of the reduced equations of (2.5) which leads interesting solutions to the dBKP equation (2.2) by back substitution. This is due to the fact that the other symmetries give the solutions of non novelty. Hence, we are forced to make use of the scaling symmetries for the reduction of (2.5) throughout this literature. Moreover, we perform a detailed work for obtaining the similarity solutions of (2.5) using the symmetry $\Gamma_3 + c_1\Gamma_2$ discussed in section 4.1. For the other symmetries in Table (3), we calculate the similarity solutions directly from the reduced equations of (2.5), since it carries the similar computations.

Table 3: Symmetries, Similarity variables and Resultant equations

Optimal System	Similarity variable	Reductions	
Γ_1	$\alpha = X, \beta = Y, Z(T,X,Y) = J(\alpha,\beta)$	(I)	$H_\alpha - \frac{5}{9}G_{\beta\beta} + 5{G_\alpha}^2 G_{\alpha\alpha} = 0$
	$V(T,X,Y) = G(\alpha,\beta), W(T,X,Y) = H(\alpha,\beta)$		$H + \frac{5}{3}G_\alpha G_\beta = 0, J(\alpha,\beta) = 0$
Γ_2	$\alpha = T, \beta = Y, Z(T,X,Y) = J(\alpha,\beta)$	(II)	$G_{\beta\beta} = 0, J(\alpha,\beta) - G_\alpha = 0$
	$V(T,X,Y) = G(\alpha,\beta), W(T,X,Y) = H(\alpha,\beta)$		$J(\alpha,\beta) - H(\alpha,\beta) = 0$
Γ_3	$\alpha = T, \beta = X, Z(T,X,Y) = J(\alpha,\beta)$	(III)	$H_\beta + 5{G_\beta}^2 G_{\beta\beta} = 0, J(\alpha,\beta) - G_\alpha = 0$
	$V(T,X,Y) = G(\alpha,X), W(T,X,Y) = H(\alpha,X)$		$J(\alpha,X) - H(\alpha,X) = 0$
Γ_4	$\alpha = \frac{X}{T^3}, \beta = \frac{Y}{T^3},$	(IV)	$H_\alpha + 5{G_\alpha}^2 G_{\alpha\alpha} - \frac{5}{9}G_{\beta\beta} = 0$
	$V(T,X,Y) = T^4 G(\alpha,\beta), W(T,X,Y) = T^3 H(\alpha,\beta)$		$J(\alpha,\beta) - H(\alpha,\beta) - \frac{5}{3}G_\alpha G_\beta = 0$
	$Z(T,X,Y) = T^3 J(\alpha,\beta)$		$J(\alpha,\beta) - 4G + 2\beta G_\beta + 3\alpha G_\alpha = 0$
$\Gamma_2 + c_1\Gamma_1$	$\alpha = X - c_1 T, \beta = Y,$	(V)	$H_\alpha + 5{G_\alpha}^2 G_{\alpha\alpha} - \frac{5}{9}G_{\beta\beta} = 0$
	$V(T,X,Y) = G(\alpha,\beta), W(T,X,Y) = H(\alpha,\beta)$		$J(\alpha,\beta) - H(\alpha,\beta) - \frac{5}{3}G_\alpha G_\beta = 0$
	$Z(T,X,Y) = J(\alpha,\beta)$		$J(\alpha,\beta) + c_1 G_\alpha = 0$
$\Gamma_3 + c_1\Gamma_2$	$\alpha = T, \beta = Y - c_1 X,$	(VI)	$c_1 H_\beta - 5{c_1}^4 {G_\beta}^2 G_{\beta\beta} + \frac{5}{9}G_{\beta\beta} = 0$
	$V(T,X,Y) = G(\alpha,\beta), W(T,X,Y) = H(\alpha,\beta)$		$J(\alpha,\beta) - H(\alpha,\beta) + \frac{5}{3}c_1 {G_\beta}^2 = 0$
	$Z(T,X,Y) = J(\alpha,\beta)$		$J(\alpha,\beta) - G_\alpha = 0$
$\Gamma_3 + c_1\Gamma_1$	$\alpha = X, \beta = Y - c_1 T,$	(VII)	$H_\alpha + 5{G_\alpha}^2 G_{\alpha\alpha} - \frac{5}{9}G_{\beta\beta} = 0$
	$V(T,X,Y) = G(\alpha,\beta), W(T,X,Y) = H(\alpha,\beta)$		$J(\alpha,\beta) - H(\alpha,\beta) + \frac{5}{3}G_\alpha G_\beta = 0$
	$Z(T,X,Y) = J(\alpha,\beta)$		$J(\alpha,\beta) + c_1 G_\beta = 0$
$\Gamma_3 + c_2\Gamma_2 + c_1\Gamma_1$	$r = Y - c_2 X - c_1 T$	(VIII)	$c_2 H' - 5{c_2}^4 {G'}^2 G'' + \frac{5}{9}G'' = 0$
	$V(T,X,Y) = G(r), W(T,X,Y) = H(r)$		$J(r) - H(r) + \frac{5}{3}c_2 {G'}^2 = 0$
	$Z(T,X,Y) = J(r)$		$J(r) + c_1 G' = 0$

To construct the nontrivial solutions of (2.5), we are intended to consider the reduced equations I, III, IV, VI and VII as listed in Table (3).

3.1 Reduction for (VI)

Here we perform a detailed analysis in tracing the solutions of (2.3) using the symmetry vector field $\Gamma_3 + c_1\Gamma_2$.

The associated characteristic equation of this vector field is given by

$$\frac{dT}{0} = \frac{dX}{c_1} = \frac{dY}{1} = \frac{dV}{0} = \frac{dW}{0} = \frac{dZ}{0}$$

The solution of the above characteristic equation give the transformation as follows

$$\alpha = T, \beta = Y - c_1X, V(T,X,Y) = G(\alpha,\beta),\ W(T,X,Y) = H(\alpha,\beta),\ Z(T,X,Y) = J(\alpha,\beta)$$

Using the above transformations the equation (2.5) can be reduced as

$$c_1H_\beta + \frac{5}{9}G_{\beta\beta} - 5c_1^4G_\beta^2G_{\beta\beta} = 0, \tag{3.1a}$$

$$H - J - \frac{5}{3}c_1G_\beta^2 = 0, \tag{3.1b}$$

$$J - G_\alpha = 0. \tag{3.1c}$$

Rewriting (3.1), we arrive

$$c_1H_\beta + \frac{5}{9}G_{\beta\beta} - 5c_1^4G_\beta^2G_{\beta\beta} = 0, \tag{3.2a}$$

$$H - G_\alpha - \frac{5}{3}c_1G_\beta^2 = 0. \tag{3.2b}$$

The symmetries of equation (3.2) are listed as

$$\Gamma_{61} = \partial_\alpha,\ \Gamma_{62} = \partial_\beta,\ \Gamma_{63} = \partial_G, \tag{3.3}$$

$$\Gamma_{64} = \alpha\partial_\alpha + \beta\partial_\beta + G\partial_G. \tag{3.4}$$

For further reduction, we consider the scaling symmetry (3.4) and the corresponding similarity variables obtained as $r = \frac{\beta}{\alpha}$, $G = \alpha g(r)$ and $H = h(r)$. Using these similarity variables, we get the reduced form of (3.2) is

$$g - h - rg' + \frac{5}{3}c_1g'^2 = 0, \tag{3.5a}$$

$$c_1h' + \frac{5}{9}g'' - 5{c_1}^4g'^2g'' = 0. \tag{3.5b}$$

The solution of the latter system (3.5) is

$$g = I_1 + \frac{135c_1r \pm 2\sqrt{5}(10 - 9c_1r)^{\frac{3}{2}}}{405{c_1}^3}, \tag{3.6a}$$

$$h = I_1 + \frac{225 - 135c_1r - \sqrt{5}(10 - 9c_1r)^{\frac{3}{2}}}{405{c_1}^3}. \tag{3.6b}$$

From (3.6), one can deduce the solution of (2.3) as given by

$$V = T\left(I_1 + \frac{135c_1(\frac{Y-c_1X}{T}) \pm 2\sqrt{5}(10 - 9c_1(\frac{Y-c_1X}{T}))^{\frac{3}{2}}}{405{c_1}^3}\right) \tag{3.7}$$

Then the solution of the original equation (2.2) is given by

$$U = \frac{1}{15{c_1}^2}\left[5 + \left(5\left(10 - 9c_1\left(\frac{Y - c_1X}{T}\right)\right)\right)^{\frac{1}{2}}\right]. \tag{3.8}$$

3.2 Reduction for (IV) :

The similarity variables and resultant equation (IV), which are obtained by applying the symmetry Γ_4, are tabulated in Table (3). One can rewrite the system of (IV) as

$$H_\alpha + 5{G_\alpha}^2 G_{\alpha\alpha} - \frac{5}{9}G_{\beta\beta} = 0, \tag{3.9a}$$

$$H - 4G + 2\beta G_\beta + 3\alpha G_\alpha + \frac{5}{3}G_\alpha G_\beta = 0. \tag{3.9b}$$

The Lie point symmetries of equation (3.9) are given below

$$\Gamma_{41} = \partial_G + 4\partial_H, \tag{3.10}$$

$$\Gamma_{42} = -9\beta\partial_G - 18\beta\partial_H + 5\partial_\alpha, \tag{3.11}$$

$$\Gamma_{43} = 3G\partial_G + 3H\partial_H + \beta\partial_\beta + 2\alpha\partial_\alpha. \tag{3.12}$$

Using the scaling symmetry Γ_{43} with the associated similarity variables $\gamma = \frac{\beta}{\sqrt{\alpha}}$, $H = \alpha^{\frac{3}{2}}h(\gamma)$ and $G = \alpha^{\frac{3}{2}}g(\gamma)$, the reduced system of (3.9) given as

$$\begin{aligned} 1215g^3 + 216h - 135\gamma^3{g'}^3 - 72\gamma h' - 80g'' + 45\gamma^4{g'}^2g'' & \\ +405\gamma g^2(\gamma g'' - 5g') - 135\gamma^2 gg'(2\gamma g'' - 7g') &= 0, \end{aligned} \tag{3.13a}$$

$$6h + \gamma(3 - 5g')g' + 3g(1 + 5g') = 0. \tag{3.13b}$$

Further rewriting the above equation we get a non-solvable second order nonlinear ODE as

$$\begin{aligned} 1215g^3 - 135\gamma^3{g'}^3 + 4(9\gamma^2 - 20)g'' + 405\gamma g^2(\gamma g'' - 5g') - 12\gamma g'(10\gamma g'' + 3) & \\ +15\gamma{g'}^2(3\gamma^3 g'' + 20) + 9g(105\gamma^2{g'}^2 + 20\gamma g'' - 60g' - 30\gamma^3 g'g'') &= 0 \end{aligned} \tag{3.14}$$

3.3 Reduction using Γ_1 :

The associated similarity variables of Γ_1, one can deduce as

$$\alpha = X, \beta = Y, V(T, X, Y) = G(\alpha, \beta),\ W(T, X, Y) = H(\alpha, \beta),\ Z(T, X, Y) = 0.$$

Using the above transformations, (2.5) can be reduced in to the following system of equations which also listed in the Table (3).

$$H_\alpha - \frac{5}{9}G_{\beta\beta} + 5{G_\alpha}^2 G_{\alpha\alpha} = 0, \tag{3.15a}$$

$$H + \frac{5}{3}G_\alpha G_\beta = 0. \tag{3.15b}$$

Further, we find the symmetry vector fields of the system (3.15) as

$$\Gamma_{11} = \partial_\alpha,\ \Gamma_{12} = \partial_\beta,\ \Gamma_{13} = \partial_G, \tag{3.16}$$

$$\Gamma_{14} = \beta\partial_\beta - G\partial_G - 3H\partial_H, \tag{3.17}$$

$$\Gamma_{15} = \alpha\partial_\alpha + 2G\partial_G + 3H\partial_H. \tag{3.18}$$

Here, the symmetries Γ_{14} and Γ_{15} are scaling and leads to a nontrivial solution for the equation (2.2). The reductions of these equations are listed the following table.

Table 4: Symmetries, Similarity variables, Resultant equations and Solutions

Optimal System	Similarity variable	Reductions	
Γ_{14}	$G = -\frac{g(\alpha)}{\beta}$	(IX)	$h = \frac{5}{3}gg'$
	$H = -\frac{h(\alpha)}{\beta^3}$		$h' - \frac{10}{9}g + 5gg'' = 0$
Γ_{15}	$G = \alpha^2 g(\beta)$	(X)	$h + \frac{10}{3}gg' = 0$
	$H = \alpha^3 h(\beta)$		$3h + 40g^3 - \frac{5}{9}g'' = 0$
$\Gamma_{14} - \Gamma_{15}$	$G = \alpha^3 g(r)$	(XI)	$h = -\frac{5}{3}r{g'}^2 - 5gg'$
	$H = \alpha^6 h(r), r = \alpha\beta$		$2430g^3 + 270r^3{g'}^3 + 9rh' - 5g'' + 405rg^2(10g' + rg'') + 45r{g'}^2(r^3g'') + 270gg'(-1 + 7r^2g' + r^3g'') = 0$

Rewriting the system (IX) as

$$5g'^2g'' + \frac{5}{3}gg'' + \frac{5}{3}gg'^2 - \frac{10}{9}g = 0. \tag{3.19}$$

The Lie point symmetries of the equation (3.19) are obtained as ∂_α , $\alpha\partial_\alpha + 2g\partial_g$. For the scaling symmetry $\alpha\partial_\alpha + 2g\partial_g$, we have two set of values for g and h as follows:

$\{g = -\frac{1}{3}\alpha^2, h = \frac{10}{27}\alpha^3\}$, $\{g = \frac{1}{12}\alpha^2, h = \frac{5}{196}\alpha^3\}$. For these two set of g we have obtained solutions

$$U = -\frac{X^3}{9Y}, \tag{3.20a}$$

$$U = -\frac{X^3}{36Y} \tag{3.20b}$$

respectively to the equation (2.2) by using back substitution into series of similarity variables.

Next, we consider the system (X) and rewriting this we get

$$\frac{1}{72}g'' + \frac{1}{4}gg' - g^3 = 0 \tag{3.21}$$

The Lie point symmetries of the equation (3.21) are derived as ∂_β , $\beta\partial_\beta - g\partial_g$. For the scaling symmetry $\beta\partial_\beta - g\partial_g$, we construct the following two sets of solution based on values for g and h: $\{g = -\frac{1}{3\beta}, h = \frac{10}{27\beta^3}\}$ and $\{g = \frac{1}{12\beta}, h = \frac{5}{216\beta^3}\}$ where h can be obtained from the equation (X).

For these two expressions of g, we have solutions

$$U = -\frac{X^3}{9Y}, \text{and} \tag{3.22}$$

$$U = -\frac{X^3}{36Y} \tag{3.23}$$

respectively to the equation (2.2).

Next, rewriting (XI), we get

$$486g^3+54r^3g'^3-g''-6r^2g'g''+81rg^2(10g'+rg'')+g'^2(-30r+9r^4g'')+9g(42r^2g'^2-rg''+6g'(-1+r^3g'')) = 0 \tag{3.24}$$

By performing Lie symmetry analysis for the equation (3.24), we get only one vector field as $g\partial_g - r\partial_r$ and the corresponding similarity variable for (3.24) is obtained as

$$g = \frac{k}{r}, \ k \text{ is a constant.} \tag{3.25}$$

Using (3.25) in (3.24), we get

$$k(1+3k)(12k-1) = 0. \tag{3.26}$$

For the choice of $k = \frac{-1}{3}$, we get the following solution of (3.24)

$$g = \frac{-1}{3r}. \tag{3.27}$$

From (3.27), one can deduce the solution of (2.2) by back substitution method as

$$U = -\frac{2X}{3Y}. \tag{3.28}$$

For all other choices of k, one can obtain similar form as above solution.

3.4 Reductions for (III)

The equation (III) can be rewritten as

$$G_{\alpha\beta} + 5{G_\beta}^2 G_{\beta\beta} = 0 \tag{3.29}$$

The Lie point symmetries of equation (3.29) are derived as

$$\Gamma_{31} = \partial_\alpha,\ \Gamma_{32} = \partial_\beta,\ \Gamma_{33} = f(\alpha)\partial_G, \tag{3.30}$$

$$\Gamma_{34} = 3\alpha\partial_\alpha + \beta\partial_\beta, \tag{3.31}$$

$$\Gamma_{35} = -2\alpha\partial_\alpha + G\partial_G, \tag{3.32}$$

Case 1. For further reduction, take the scaling symmetry $3\alpha\partial_\alpha + \beta\partial_\beta$. The corresponding similarity variables are $r = \frac{\beta}{\alpha^{\frac{1}{3}}}$, $G = \alpha g(r)$. Therefore the equation (3.29) reduced to the following equation

$$g' + rg'' = 15{g'}^2 g'' \tag{3.33}$$

After an integration, the equation (3.33) becomes

$$rg' = 5{g'}^3 + I_1 \tag{3.34}$$

After solving the equation (3.34), one can easily find the solution of the equation (2.2).

Case 2. If we take the scaling symmetry $-2\alpha\partial_\alpha + G\partial_G$ then the corresponding similarity variable is $G = \frac{g(\beta)}{\sqrt{\alpha}}$. Therefore, the equation (3.29) reduced to the following equation

$$g' = 10g'' \tag{3.35}$$

The solution of the equation (3.35) becomes

$$g = \pm\frac{10}{3}\left(\frac{\beta}{5} + 2I_1\right)^{\frac{3}{2}} + I_2. \tag{3.36}$$

From the equation (3.36), one can find the solution of the equation (2.2) as

$$U = \frac{\pm\left(\frac{X}{5} + 2I_1\right)^{\frac{1}{2}}}{\sqrt{T}}. \tag{3.37}$$

3.5 Reduction for (VII)

By means of $J(\alpha,\beta) = -c_1 G_\beta$ the equation (VII) can be rewritten as

$$H + c_1 G_\beta + \frac{5}{3} G_\beta G_\alpha = 0 \tag{3.38}$$

$$H_\alpha - \frac{5}{9} G_{\beta\beta} + 5{G_\alpha}^2 G_{\alpha\alpha} = 0 \tag{3.39}$$

The Lie point symmetries of equation (3.38) are

$$\Gamma_{71} = \partial_\alpha,\ \Gamma_{72} = \partial_\beta,\ \Gamma_{73} = \partial_G, \Gamma_{74} = \alpha\partial_\alpha + \beta\partial_\beta + G\partial_G.$$

For further reduction, take the scaling symmetry $\alpha\partial_\alpha + \beta\partial_\beta + G\partial_G$. The corresponding similarity variables are $r = \frac{\beta}{\alpha},\ G = \alpha g(r)$ and $H = h(r)$. Thus, we obtain the reduced system as

$$h + c_1 g' + \frac{5}{3} g g' - \frac{5}{3} r g'^2 = 0 \tag{3.40a}$$

$$r h' + \frac{5}{9} g'' - 5 r^2 g^2 g'' + 10 r^3 g g' g'' - 5 r^4 g'^2 g'' = 0 \tag{3.40b}$$

Solving the above system, we get

$$g = I_1 + I_2 r \tag{3.41}$$

$$h = \frac{1}{3}(5 r {I_2}^2 - 5 I_2 (I_1 + I_2 r) - 3 c_1 I_2) \tag{3.42}$$

By successive back substitutions, we obtain solution of (2.2).

$$U = I_1 + I_2 \left(\frac{Y - c_1 T}{X}\right)\left(1 - \frac{1}{X}\right). \tag{3.43}$$

We illustrate this solution behaviour graphically at the singular point $X = 0$ by means of the following graphs Fig.(3.5.1), Fig.(3.5.2), Fig.(3.5.3) and Fig.(3.5.4) at different time snapshots. Here, we observe that the wave profile moves along positive Y axis as the time variable T evolves.

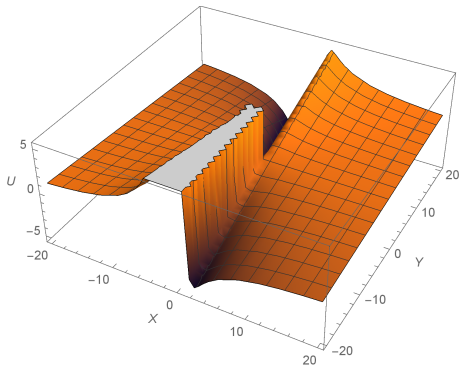

Fig.(3.5.1)

$\mathrm{I}_1 = 0, I_2 = 1, c_1 = 1, T = 0$

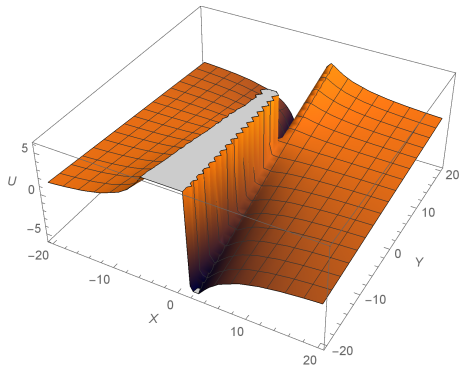

Fig.(3.5.2)

$\mathrm{I}_1 = 0, I_2 = 1, c_1 = 1, T = 5$

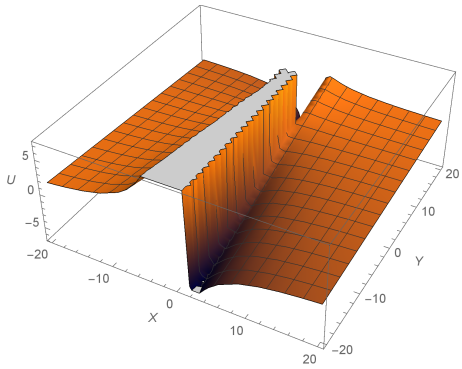

Fig.(3.5.3)

$\mathrm{I}_1 = 0, I_2 = 1, c_1 = 1, T = 10$

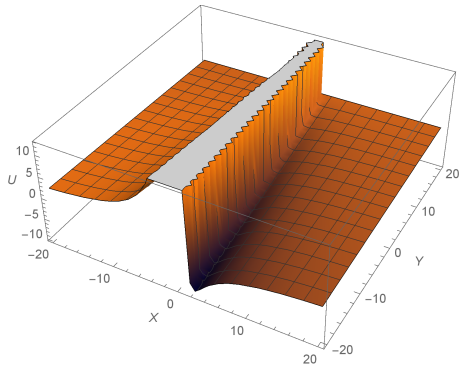

Fig.(3.5.4)

$\mathrm{I}_1 = 0, I_2 = 1, c_1 = 1, T = 20$

4 Conservation laws

It is understood that integrable PDEs admit conservation laws as one of their essential requirements. E. Noether [42] established that the correspondence between symmetries of differential equations and conservation laws. However to obtain this one need to identify the associated Euler-Lagrange equations of the differential equations. This approach forces us to consider a restricted types of specific differential equations only. Recently N.H. Ibragimov's theory [43] enables to derive the conservation laws for broader class of differential equations. This method motivated us to derive the conserved densities of dBKP equation. Using the following Ibragimov's theorem we derive the conserved densities of dBKP equation.

Theorem 1. *Any infinitesimal symmetry*

$$\Gamma = \xi^T \partial_T + \xi^X \partial_X + \xi^Y \partial_Y + \eta^1 \partial_V + \eta^2 \partial_W + \eta^3 \partial_Z$$

of equation (2.5) *leads to a conservation law* $D_i(C^i) = 0$ *constructed by the formula*

$$C^i = \xi^i L + G^\alpha \left[\left(\frac{\partial L}{\partial {u_i}^\alpha} \right) - D_j \left(\frac{\partial L}{\partial {u_{ij}}^\alpha} \right) + D_j D_k \left(\frac{\partial L}{\partial {u_{ijk}}^\alpha} \right) - \dots \right]$$
$$+ D_j(G^\alpha) \left[\left(\frac{\partial L}{\partial {u_{ij}}^\alpha} \right) - D_k \left(\frac{\partial L}{\partial {u_{ijk}}^\alpha} \right) + \dots \right] + D_j D_k(G^\alpha) \left[\left(\frac{\partial L}{\partial {u_{ijk}}^\alpha} \right) - \dots \right], \tag{4.1}$$

where $G^\alpha = \eta^\alpha - \xi^j {u_j}^\alpha$

A local conservation law for the dBKP Equation (2.5) is a continuity equation

$$D_T(C^1) + D_X(C^2) + D_Y(C^3) = 0. \tag{4.2}$$

The formal Lagrangian of the system of equation (2.5) is written as

$$L = \phi_1 (W_X + 5\, {V_X}^2\, V_{XX} - \frac{5}{9} V_{YY}) + \phi_2 (Z - W - \frac{5}{3} V_X\, V_Y) + \phi_3 (Z - V_T) \tag{4.3}$$

By taking variational derivative on (4.3) we obtain the following adjoint equations as follows:

$$\begin{aligned} {F_1}^* &= \frac{\delta L}{\delta V} = \phi_{3,T} + \frac{10}{3}\phi_2 V_{XY} + \frac{5}{3}\phi_{2X} V_Y + \frac{5}{3}\phi_{2,Y} V_X + 5\phi_{1,XX} {V_X}^2 + 10\phi_{1,X} V_X V_{XX} && (4.4) \\ {F_2}^* &= \frac{\delta L}{\delta W} = -\phi_{1,X} - \phi_2 && (4.5) \\ {F_3}^* &= \frac{\delta L}{\delta Z} = \phi_2 + \phi_3 && (4.6) \end{aligned}$$

where ϕ_1, ϕ_2 and ϕ_3 can be determined by ${F_i}^* = 0$ for $i = 1, 2, 3$. This gives an over determined system of ϕ_1, ϕ_2 and ϕ_3 and solving them we get $\phi_1 = f(t)y,\ \phi_2 = \phi_3 = 0$. Here, the arbitrary function $f(t)$ confirms the existence of infinitely many conserved densities.

Case (i): For the infinitesimal symmetry ∂_T. The corresponding extended operator are given as follows $\xi^1 = 1,\ \xi^2 = \ \xi^3\ = \eta^1\ = \eta^2\ = \eta^3 = 0$, $G^1 = -V_T,\ G^2 = -W_T$ and $G^3 = -Z_T$. The equation (4.1) gives the following bellow conserved vectors

$$\begin{aligned} C^1 &= L && (4.7) \\ C^2 &= 5V_T {V_X}^2 \phi_{1X} - 5V_{TX} {V_X}^2 \phi_1 - W_T \phi_1 \\ C^3 &= \frac{5}{9} V_{TY} \phi_1 - \frac{5}{9} V_T \phi_{1Y} && (4.8) \end{aligned}$$

Case (ii): For the infinitesimal symmetry ∂_X. The corresponding extended operator are given as follows $\xi^2 = 1,\ \xi^1 = \ \xi^3\ = \eta^1\ = \eta^2\ = \eta^3 = 0$, $G^1 = -V_X,\ G^2 = -W_X$ and $G^3 = -Z_X$. The equation (4.1) gives the following bellow conserved vectors

$$\begin{aligned} C^1 &= 0 && (4.9) \\ C^2 &= L + 5\, {V_X}^3 \phi_{1X} - 5 V_{XX}\, {V_X}^2 \phi_1 - W_X \phi_1 \\ C^3 &= \frac{5}{9} V_{XY} \phi_1 - \frac{5}{9} V_X \phi_{1Y} && (4.10) \end{aligned}$$

Case (iii):Here, we consider the infinitesimal symmetry ∂_Y. The corresponding extended operator are given as follows $\xi^3 = 1,\ \xi^1 = \ \xi^2\ = \eta^1\ = \eta^2\ = \eta^3 = 0$, $G^1 = -V_Y,\ G^2 = -W_Y$ and $G^3 = -Z_Y$. The equation (4.1) gives the following bellow conserved vectors

$$\begin{aligned} C^1 &= 0 \\ C^2 &= 5V_Y {V_X}^2 \phi_{1X} - 5V_{YX} {V_X}^2 \phi_1 - W_Y \phi_1 \\ C^3 &= L + \frac{5}{9} V_{YY} \phi_1 - \frac{5}{9} V_Y \phi_{1Y} \end{aligned} \tag{4.11}$$

Case (iv): Next, we consider the infinitesimal symmetry $2V\partial_V + \frac{3}{2}W\partial_W + \frac{3}{2}Z\partial_Z + \frac{1}{2}T\partial_T + \frac{3}{2}X\partial_X + Y\partial_Y$. The corresponding extended operator are given as follows $\xi^1 = \frac{1}{2}T,\ \xi^2 = \frac{3}{2}X,\ \xi^3\ = Y,\ \eta^1 = 2V,\ \eta^2 = \frac{3}{2}W,\ \eta^3 = \frac{3}{2}Z$, $G^1 = 2V - \left(\frac{1}{2}V_T + \frac{3}{2}V_X + YV_Y\right),\ G^2 = \frac{3}{2}W - \left(\frac{1}{2}W_T + \frac{3}{2}W_X + YW_Y\right)$ and $G^3 = \frac{3}{2}Z - \left(\frac{1}{2}Z_T + \frac{3}{2}Z_X + YZ_Y\right)$. The equation (4.1) gives the following bellow conserved vectors

$$\begin{aligned} C^1 &= 0 \\ C^2 &= -5\left(2V - \left(\frac{1}{2}TV_T + \frac{3}{2}XV_X + YV_Y\right)\right){V_X}^2\phi_{1X} + \left(\frac{3}{2}W - \left(\frac{1}{2}W_T + \frac{3}{2}W_X + YW_Y\right)\right)\phi_1 \\ &\quad +5{V_X}^2\left(2V_X - \frac{1}{2}TV_{TX} - \frac{3}{2}V_X - \frac{3}{2}XV_{XX} - YV_{XY}\right)\phi_1 \\ C^3 &= -\frac{5}{9}\left(2V_Y - \frac{1}{2}TV_{TY} - \frac{3}{2}XV_{XY} - YV_{YY} - V_Y\right)\phi_1 + \frac{5}{9}\left(2V - \left(\frac{1}{2}V_T + \frac{3}{2}V_X + YV_Y\right)\right)\phi_{1Y} \end{aligned}$$

Case (v): Finally, for the infinitesimal symmetry $h(t)\partial_V + h'(t)\partial_W + h'(t)\partial_Z$. The corresponding extended operator are given as follows $\xi^1 = \ \xi^2 = \ \xi^3\ = 0,\ \eta^1 = h(t),\ \eta^2 = \ \eta^3 = h'(t)$, $G^1 = h(t),\ G^2 = h'(t)$ and $G^3 = h'(t)$. The equation (4.1) gives the following bellow conserved vectors

$$\begin{aligned} C^1 &= 0 \\ C^2 &= -5h(t){V_X}^2\phi_{1X} + h'(t)\phi_1 \\ C^3 &= \frac{5}{9}h'(t)\phi_{1Y} \end{aligned}$$

5 Conclusions

In this work we performed a detailed analysis of the symmetry properties for the dBKP equation. We found that the dBKP equation is invariant under the action of a five-dimensional Lie group of one-parameter point transformations. The infinitesimal generators of the Lie group are used to determine the one-dimensional optimal system for the dBKP equation.

From the Lie point symmetries we derive the Lie invariants which define the similarity solutions, necessary to reduce and simplify the differential equation. We performed a detailed analysis on the reduction process and we were able to find new similarity solutions for the dBKP equation.

This analysis contributes on the subject of the application of Lie point symmetries on nonlinear differential equations. We found that we were able to construct closed-form solutions. Finally, using point symmetries, we derived the conservation laws of dBKP equations through Ibragimov's method.

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