Existence of a global attractor governed by differential hemivariational inequalities with parabolic type and its applications

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Abstract

In this paper, we consider an abstract system which consists of a hemivariational inequality of parabolic type combined with a nonlinear differential inclusion (DPHVI, for short) in the framework of Banach spaces. The objective of this paper is fourfold. The first one is to deal with the existence of solutions and the properties of the solution set for parabolic hemivariational inequalities (PHVIs, for short). The second aim is to investigate the existence of mild solutions for DPHVI by means of a fixed point technique. The third target is to study the existence of a global attractor for the m-semiflow governed by DPHVI. Finally, the fourth goal is to illustrate an application of our abstract results.

Existence of a global attractor governed by differential hemivariational inequalities with parabolic type and its applications *

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Abstract

In this paper, we consider an abstract system which consists of a hemivariational inequality of parabolic type combined with a nonlinear differential inclusion (DPHVI, for short) in the framework of Banach spaces. The objective of this paper is fourfold. The first one is to deal with the existence of solutions and the properties of the solution set for parabolic hemivariational inequalities (PHVIs, for short). The second aim is to investigate the existence of mild solutions for DPHVI by means of a fixed point technique. The third target is to study the existence of a global attractor for the m-semiflow governed by DPHVI. Finally, the fourth goal is to illustrate an application of our abstract results.

Key words: *Global attractor; m-semiflow; parabolic hemivariational inequalities; differential inclusions; fixed point theorem; Measure of noncompactness.*

1 Introduction and problem formulation

The goal of this paper is to investigate an abstract system which represents a class of parabolic hemivariational inequalities driven by the nonlinear differential inclusions

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(DPHVIs, for short) in the framework of Banach spaces. Let E be a Banach space and (U, H, U^*) an evolution triple (Gelfand triple) of spaces. Let $A : D(A) \subseteq E \to E$ be the infinitesimal generator of a C_0-semigroup $T(t)$ $(t \geq 0)$ on E and let ∂J denotes the Clarke subdifferential of a locally Lipschitz function $J : H \to \mathbb{R}$. F, B are two set-valued maps and g is a nonlinear map which will be specified in Section 4. With these data, the formulation of our problem is:

$$x'(t) \in Ax(t) + F(t, x(t), u(t)), \qquad a.e.\ t > 0, \tag{1.1}$$

$$u'(t) + B(t, u(t)) + \partial J(u(t)) \ni g(x(t)) \qquad a.e.\ t > 0, \tag{1.2}$$

$$x(0) = x_0, \quad \text{and} \quad u(0) = u_0. \tag{1.3}$$

The study of variational inequalities, which was initially developed to deal with equilibrium problems, is closely related to the convexity of the energy functionals. It should be mentioned that there are growing papers on the mathematical models for mechanical problems with nonconvex and nonsmooth energy functions. The lack of convexity and d-ifferentiability in mechanics and engineering leads to new types of variational formulation called hemivariational inequalities (HVIs, for short) which was first introduced and studied by P.D. Panagiotopoulos in the early 1980's. Such an expression allows one to deal with many practical problems involving nonmonotone and multivalued relations. Consequently, for the description of various mechanical problems which can be formulated as HVIs as well as applications, we refer to [12, 20, 22–24, 26, 28, 29].

It is well known that differential variational inequalities (DVIs, for short) are systems which couple differential or partial differential equations with a time-dependent variational inequality. DVIs were systematically discussed by Pang–Stewart [30] in the framework of Euclidean spaces. From now on, a large number of works have been dedicated to the development of theory associated to DVIs and their applications, we refer the reader to [1, 3, 4, 13–15, 17, 18, 27]. The notion of differential hemivariational inequalities (DHVIs, for short) was firstly introduced by Liu et al. [16]. DHVIs represent an important extension of DVIs, which couple differential or partial differential equations with a hemivariational inequality or a variational-hemivariational inequality. The DHVIs appear in a variety of mechanical problems such as unilateral contact problems in nonlinear elasticity, adhesive and friction effects, nonconvex semipermeability, masonry structures, and delamination in multilayered composites. In the last few years the study of DHVIs has emerged as a new and interesting branch of applied mathematics. For more details we refer to [10, 11, 16].

It is worth mentioning here that, qualitative studies on DPHVIs like our system (1.1)-(1.3) have been less known. Very recently, a version of DPHVIs was studied for the first time in Migórski and Zeng [25] and they established a solvability result by the Rothe method. In comparison with Migórski and Zeng [25], the main novelties of this paper is that we aim at studying behavior of the dynamics generated by the solution set to DPHVI (1.1)-(1.3). In our present paper, by applying the framework developed in [8, 19, 27], we prove the existence of a global attractor for the aforementioned m-semiflow. To the best of our knowledge, there is still little information known for the longtime behavior of solutions to the DPHVIs and this fact is the motivation of the present work.

The rest of the manuscript is structured as follows. In Section 2 we recall some basic definitions and results needed throughout this paper. In Section 3, the existence of solutions and the properties of the solution set for PHVIs is presented. In Section 4, the existence of mild solutions associated to DPHVI (1.1)-(1.3) is obtained by means of a fixed point technique. In Section 5, the existence of a global attractor for the m-semiflow governed by DPHVIs (1.1)-(1.3) is also devoted. Finally, a mathematical model is provided to illustrate our abstract results.

2 Background material

In this section, we review some necessary prerequisites. First, a triple of spaces (U, H, U^*) is called an "evolution triple", if: (a) U is a separable reflexive Banach space; (b) H is a separable Hilbert space; (c) $U \subseteq H \subseteq U^*$, the embedding of U into H is continuous and U is dense in H. We denote this embedding operator by γ.

In the sequel, by $\|\cdot\|_U$ (respectively, $|\cdot|_H$, $\|\cdot\|_{U^*}$) we denote the norm of U (respectively, H, U^*). By $\langle\cdot,\cdot\rangle$, we denote the duality brackets of the pair (U, U^*). And $(\cdot,\cdot)$ is the inner product of H. The two are compatible in the sense that $\langle\cdot,\cdot\rangle_{U\times H} = (\cdot,\cdot)$. Here U^* denotes the dual space of U. We also introduce the following function spaces: $\mathcal{U} = L^2(0, b; U)$, $\mathcal{U}^* = L^2(0, b; U^*)$ and $\mathcal{H} = L^2(0, b; H)$. The pairing of $\mathcal{U}$ and $\mathcal{U}^*$ is denoted by $\langle\langle\cdot,\cdot\rangle\rangle$. Let $L^p(0, b; \mathbb{R})(1 \le p < \infty)$ denote the Banach space of all Lebesgue measurable functions from $[0, b]$ into $\mathbb{R}$ equipped with the norm $\|\varphi\|_{L^p} := (\int_0^b |\varphi(t)|^p dt)^{1/p} < \infty$. By $\mathcal{C}_E = C([0, b]; E)$ we denote the Banach space of continuous functions from $[0, b]$ into E with $\|x\|_{\mathcal{C}_E} = \sup_{t\in[0,b]}\|x(t)\|_E$. Similarly, denote $\mathcal{C}_H = C([0, b]; H)$. $\mathcal{W} = \{u \in \mathcal{U} : u' \in \mathcal{U}^*\}$ (here, the time derivative of u is understood in the sense of vectorial distributions) endowed with the norm $\|u\|_{\mathcal{W}} = \|u\|_{\mathcal{U}} + \|u'\|_{\mathcal{U}^*}$ is a Banach space. It is well known that the space $\mathcal{W}$ is embedded continuously in $\mathcal{C}_H$. Moreover, if U is embedded compactly in H, then so is $\mathcal{W}$ into $\mathcal{H}$. In the remainder of this paper, we will assume that U is embedded compactly in H.

Next, following Clarke [5], we present the generalization of the gradient operator for functionals which are no longer convex, but are locally Lipschitz.

Definition 2.1. *Let $J : U \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of J at $x \in U$ in the direction $v \in U$ is defined by*

$$J^0(x; v) := \limsup_{\lambda\to 0^+,\, \xi\to x} \frac{J(\xi + \lambda v) - J(\xi)}{\lambda}.$$

The generalized gradient of $J : U \to \mathbb{R}$ at $x \in U$ is the subset of U^* given by

$$\partial J(x) := \{\xi \in U^* : \ J^0(x; v) \ge \langle\xi, v\rangle, \ \forall v \in U\}.$$

In the sequel, we proceed with the definition of some classes of operators.

Definition 2.2. *[24, Definition 9] A multivalued operator $B : U \to 2^{U^*}$ is said to be:*

(a) *monotone, if for all (u, u^*), $(v, v^*) \in Gr(B)$, we have $\langle u^* - v^*, u - v\rangle \ge 0$;*

(b) *pseudomonotone, if B has values which are nonempty, bounded, closed, and convex; B is u.s.c. from each finite-dimensional subspace of U to U^* endowed with the weak topology; if $\{u_n\} \subset U$ with $u_n \to u$ weakly in U and $u_n^* \in Bu_n$ is such that*

$$\limsup\langle u_n^*, u_n - u\rangle \le 0,$$

then for $\forall y \in U$, there exists $u^*(y) \in Bu$ such that $\langle u^*(y), u-y\rangle \le \liminf\langle u_n^*, u_n-y\rangle$;

(c) *generalized pseudomonotone, if* $\{u_n\} \subset U$, $\{u_n^*\} \subset U^*$ *with* $u_n^* \in Bu_n$, $u_n \to u$ *weakly in* U, $u_n^* \to u^*$ *weakly in* U^* *and*

$$\limsup \langle u_n^*, u_n - u \rangle \leq 0,$$

we have $u^* \in Bu$ *and* $\langle u_n^*, u_n \rangle \to \langle u^*, u \rangle$.

Next, by $\mathcal{P}(E)$ [$\mathcal{P}_{cl}(E)$, $\mathcal{P}_b(E)$, $\mathcal{P}_{cv}(E)$, $\mathcal{P}_{(w)cp}(E)$], we denote the collections of all nonempty [respectively, closed, bounded, convex, (weakly) compact] subsets of the Banach space E. Now, we list the following definition.

Definition 2.3. *[7, 9] A multimap* $F : E \to \mathcal{P}(U)$ *is said to be:*

(i) *upper semicontinuous (u.s.c., for short), if for every open subset* $O \subset U$ *the set*

$$F^+(O) = \{x \in E : F(x) \subset O\}$$

is open in E*;*

(ii) *closed if its graph*

$$\{(x, y) : x \in E,\ y \in F(x)\}$$

is a closed subset of $E \times U$*;*

(iii) *compact, if its range* $F(E)$ *is relatively compact in* U*, i.e.* $\overline{F(E)}$ *is compact in* U*;*

(iv) *quasicompact, if its restriction to any compact subset* $K \subset E$ *is compact.*

We will use the following Proposition to get our main results.

Proposition 2.4. *[7, 9]. Let* E*,* U *be two metric spaces and* $F : E \to \mathcal{P}_{cp}(U)$ *a closed quasicompact multimap. Then* F *is u.s.c.*

Now, we focus on a few facts about the measure of noncompactness (cf. [9]).

Definition 2.5. *Let* E *be a Banach space. A map* $\beta : \mathcal{P}_b(E) \to \mathbb{R}_+$ *is called a measure of noncompactness (MNC, for short) in* E *if* $\beta(\overline{co}\Omega) = \beta(\Omega)$ *for every* $\Omega \in \mathcal{P}_b(E)$.

In particular, a MNC β *is called:*

(i) *monotone, if* $\Omega_1,\ \Omega_2 \in \mathcal{P}_b(E),\ \Omega_1 \subseteq \Omega_2$ *implies* $\beta(\Omega_1) \leq \beta(\Omega_2)$*;*

(ii) *nonsingular, if* $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ *for every* $a \in E,\ \Omega \in \mathcal{P}_b(E)$*;*

(iii) *invariant with respect to flection through the origin, if* $\beta(-\Omega) = \beta(\Omega)$ *for every* $\Omega \in \mathcal{P}_b(E)$*;*

(iv) *algebraically semiadditive, if* $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$ *for every* $\Omega_1,\ \Omega_2 \in \mathcal{P}_b(E)$*;*

(v) *regular, if* $\beta(\Omega) = 0$ *is equivalent to the relative compactness of* Ω.

An important example of the MNC possessing all of above properties is the Hausdorff MNCχ which can be defined by:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}, \quad \forall \Omega \in \mathcal{P}_b(E).$$

Next, we also recall the concept of χ-norm of a bounded linear operator $\mathcal{T}(\mathcal{T} \in \mathcal{L}(E))$ as follows

$$\|\mathcal{T}\|_\chi = \inf\{\lambda > 0 : \chi(\mathcal{T}(\Omega)) \leq \lambda \cdot \chi(\Omega) \text{ for all bounded set } \Omega \subset E\}.$$

It is clear that $\chi(\mathcal{T}(\Omega)) \leq \|\mathcal{T}\|_\chi \cdot \chi(\Omega)$. Moreover, $\|\mathcal{T}\|_\chi \leq \|\mathcal{T}\|$, where $\|\mathcal{T}\|$ is the operator norm in $\mathcal{L}(E)$ of $\mathcal{T}$. Obviously, $\mathcal{T}$ is a compact operator iff $\|\mathcal{T}\|_\chi = 0$.

We now briefly focus on the following notion. A multimap $F : I \to \mathcal{P}(E)$ is called integrably bounded, if there exists a function $\delta \in L^1(I; \mathbb{R}_+)$ such that

$$\|f(t)\|_E := \sup\{\|f(t)\|_E : f(t) \in F(t)\} \leq \delta(t), \quad \text{for a.e. } t \in I.$$

We have the following statement.

Proposition 2.6. *[9, Proposition 2.5] Let $D \subset L^1(0, b; E)$ such that*

(1) *$\|\varphi(t)\| \leq \sigma(t)$, for all $\varphi \in D$ and for a.e. $t \in [0, b]$,*

(2) *$\chi(D(t)) \leq q(t)$ for a.e. $t \in [0, b]$,*

where $\sigma,\ q \in L^1(0, b; \mathbb{R})$. Then

$$\chi\Big(\int_0^t D(s)ds\Big) \leq 4\int_0^t q(s)ds,$$

here $\int_0^t D(s)ds = \{\int_0^t \varphi(s)ds : \varphi \in D\}$.

We will also use the following definition in this paper.

Definition 2.7. *[9, Definition 2.12] The sequence $\{f_n\}_{n=1}^\infty \subset L^1(0, b; E)$ is said to be semicompact if it is integrably bounded and the set $\{f_n(t)\}_{n=1}^\infty$ is relatively compact in E for a.e. $t \in [0, b]$.*

To get our main results, we make use of the following fixed point theorem, which is a version of Bohnenblust-Karlin fixed point principle for multivalued mappings.

Theorem 2.8. *([2, Theorem 4]). Let E be a Banach space and $D \subset E$ be a nonempty compact convex subset. If the multimap $\mathcal{F} : D \to P(D)$ has closed graph and convex values, then $\mathcal{F}$ has a fixed point.*

We now recall some formulations regarding m-semiflow and global attractors (see [19]). Let Γ be a nontrivial subgroup of the additive group of real numbers $\mathbb{R}$ and $\Gamma_+ = \Gamma \cap [0, \infty)$. In the many applicable situations, Γ_+ can be chosen to be a half-line $\mathbb{R}_+$.

Definition 2.9. *The mapping $G : \Gamma_+ \times E \to \mathcal{P}(E)$ is called an m-semiflow if the following conditions are satisfied*

(i) *$G(0, w) = \{w\}$, for all $w \in E$,*

(ii) $G(t_1+t_2,x)\subset G(t_1,G(t_2,x))$, *for all* $t_1,\ t_2\in\Gamma_+,\ x\in E$, *where* $G(t,D)=\cup_{x\in D}G(t,x),\ D\subset E$.

It is called a strict m*-semiflow if* $G(t_1+t_2,w)=G(t_1,G(t_2,w))$ *for all* $w\in E$ *and* $t_1,\ t_2\in\Gamma_+$. G *is said to be eventually bounded if for each bounded set* $D\subset E$*, there is a number* $\tau(D)>0$ *such that* $\gamma^+_{\tau(D)}(D)$ *is bounded. Here* $\gamma^+_{\tau(D)}(D)$ *is the orbit after time* $\tau(D):\gamma^+_{\tau(D)}(D)=\bigcup_{t\geq\tau(D)}G(t,D)$.

Definition 2.10. *A bounded set* $D_1\subset E$ *is called an absorbing set for* m*-semiflow* G *if for any bounded set* $D\subset E$*, there exists* $\tau=\tau(D)\geq 0$ *such that* $\gamma^+_{\tau(D)}(D)\subset D_1$.

Definition 2.11. *The subset* $\mathcal{A}\subset E$ *is called a global attractors of the* m*-semiflow* G *if it satisfies the following conditions:*

(i) $\mathcal{A}$ *attracts any* $D\in\mathcal{P}_b(E)$*, i.e.* $dist(G(t,D),\mathcal{A})\to 0$ *as* $t\to\infty$*, for all bounded set* $D\subset E$*, where* $dist(\cdot,\cdot)$ *is the Hausdorff semi-distance of two subsets in* E*;*

(ii) $\mathcal{A}$ *is negatively semi-invariant, i.e.* $\mathcal{A}\subset G(t,\mathcal{A}),\ \forall t\in\Gamma_+$.

The following theorem gives a sufficient condition for the existence of a global attractor for m-semiflow G.

Theorem 2.12. *Assume that the* m*-semiflow* G *has the following properties:*

(i) $G(t,\cdot)$ *is u.s.c and has closed values for each* $t\in\Gamma_+$*;*

(ii) G *admits an absorbing set;*

(iii) G *is asymptotically upper semicompact, i.e. if* D *is a bounded set in* E *such that for some* $\tau(D)>0,\ \gamma^+_{\tau(D)}(D)$ *is bounded, any sequence* $\xi_n\in G(t_n,D)$ *with* $t_n\to\infty$ *is precompact in* E.

If G *is eventually bounded, then it possesses a compact global attractor* $\mathcal{A}$ *in* E*. Moreover, if* G *is a strict* m*-semiflow, then* $\mathcal{A}$ *is invariant, that is* $\mathcal{A}=G(t,\mathcal{A})$ *for any* $t\in\Gamma_+$.

3 Parabolic hemivariational inequalities

In this section, we consider the existence of solutions and the properties of the solutions set for parabolic hemivariational inequalities (PHVIs, for short). Before stating and proving the main results of this section, we consider the following hypotheses.

H(B) $B:[0,b]\times U\to 2^{U^*}$ is a multimap such that the following hold:

(1) $t\mapsto B(t,u)$ is graph measurable for all $u\in U$;

(2) $GrB(t,\cdot)$ is sequentially closed in $U_w\times U^*_w$ and $u\mapsto B(t,u)$ is a generalized pseudomonotone multivalued operator with weakly compact and convex values;

(3) there exist $\iota\geq 0$ and $a_1\in L^2(0,b;\mathbb{R}_+)$ such that

$$\|B(t,u)\|_{U^*}=\sup\{\|\beta\|_{U^*},\ \beta\in B(t,u)\}\leq a_1(t)+\iota\|u\|_U,\quad \text{for all } t\in[0,b],\ u\in U;$$

(4) there are $\alpha_B>0$ and $a_2\in L^1(0,b;\mathbb{R}_+)$ such that

$$\langle\beta,u\rangle\geq\alpha_B\|u\|^2_U-a_2(t),\quad \text{for all } t\in[0,b],\ u\in U,\ \beta\in B(t,u).$$

H(J) : The locally Lipschitz functional $J : H \to \mathbb{R}$ is such that

(1) there exists $\kappa_0,\ \kappa_1 \geq 0$ such that

$$\|\partial J(v)\|_H \leq \kappa_0 + \kappa_1 |v|_H \quad \text{for all} \ \ v \in H;$$

(2) there exists $\alpha_J \geq 0$ such that

$$J^0(u; -u) \leq \alpha_J (1 + |u|_H), \qquad \text{for all} \ \ t \in [0, b],\ u \in U.$$

H(g) $g : E \to H$ is a continuous function and there are a constant $\ell \geq 0$ and a function $\xi \in L^2(0, b; \mathbb{R}_+)$ such that

$$\|g(x)(t)\| \leq \xi(t) + \ell \|x\|_E^{\frac{1}{2}}, \ \ \text{for a.e.}\ t \in [0, b], \ \ \text{all} \ \ x \in E.$$

Let $Q : \mathcal{H} \to 2^{\mathcal{C}_H}$ be the set of solutions for PHVIs defined by

$$Q(g) = \{u \in \mathcal{C}_H : u'(t) + B(t, u(t)) + \partial J(u(t)) \ni g(t), \quad a.e.\ t \in [0, b],\ u(0) = u_0\}.$$

It follows from the theory developed in [12, 20] that $Q(g) \neq \emptyset$ for each $g \in \mathcal{H}$. Now, we are concerned with the solutions set $Q(g(x))$ of PHVI (1.2).

Lemma 3.1. *For a given $x \in \mathcal{C}_E$, let* H(B), H(J) *and* H(g) *be satisfied, then there exists a constant $\varrho > 0$ such that*

$$\|u\|_{\mathcal{W}} \leq \varrho.$$

Proof. Let $u(\cdot)$ be a solution to the system (1.2) with the initial condition $u(0) = u_0$. Multiplying (1.2) by $u(t)$ and integrating on I, we have

$$\int_I \langle u'(t), u(t)\rangle dt + \int_I \langle \beta(t), u(t)\rangle dt + \int_I \langle \eta(t), u(t)\rangle dt = \int_I \langle g(x(t)), u(t)\rangle dt. \tag{3.1}$$

with $\beta(t) \in B(t, u(t)),\ \eta(t) \in \partial J(u(t))$ a.e. $t \in I$. It follows from the integration-by-parts formula (see [33, Proposition 23.23(iv)]) that

$$\int_I \langle u'(t), u(t)\rangle dt = \frac{1}{2}(|u(b)|_H^2 - |u(0)|_H^2). \tag{3.2}$$

By H(g), Hölder inequality and Young's inequality with ε (see [23, Lemma 2.6]), we get

$$\begin{aligned} \int_I \langle g(t), u(t)\rangle dt \ &\leq \ \int_I (\xi(t) + \ell \|x(t)\|_E^{\frac{1}{2}})|u(t)|_H ds \\ &\leq \ \frac{\|\gamma\|\|\xi\|_{L^2}^2}{2\varepsilon^2} + \frac{\varepsilon^2\|\gamma\|}{2}\|u\|_{\mathcal{U}}^2 + \ell\|x\|_{C(I;E)}^{\frac{1}{2}}\sqrt{b}\|\gamma\|\|u\|_{\mathcal{U}}. \end{aligned} \tag{3.3}$$

Since $\eta(t) \in \partial J(u(t))$, it follows from H($J$)(2) that

$$-(\eta(t), u(t)) \leq J^0(u(t); -u(t)) \leq \alpha_J(1 + |u(t)|_H).$$

Therefore, we have

$$\int_I \langle \eta(t), u(t)\rangle dt \geq -\int_I \alpha_J (1 + |u(t)|_H) dt \geq -\alpha_J b - \alpha_J \int_I |u(t)|_H dt. \tag{3.4}$$

Combining H(B)(4), (3.2), (3.3) and (3.4) with (3.1), we obtain

$$
\begin{aligned}
&\frac{1}{2}|u(b)|_H^2 - \frac{1}{2}|u(0)|_H^2 + \int_I \left[\alpha_B \|u(t)\|_U^2 - a_2(t)\right] dt \\
\leq\ & \int_I \langle u'(t), u(t)\rangle dt + \int_I \langle \beta(t), u(t)\rangle dt \\
=\ & -\int_I \langle \eta(t), u(t)\rangle dt + \int_I \langle g(x(t)), u(t)\rangle dt \\
\leq\ & \alpha_J b + \frac{\|\gamma\|\|\xi\|_{L^2}^2}{2\varepsilon^2} + \frac{\varepsilon^2\|\gamma\|}{2}\|u\|_{\mathcal{U}}^2 + (\ell\|x\|_{C(I;E)}^{\frac{1}{2}} + \alpha_J)\sqrt{b}\|\gamma\|\|u\|_{\mathcal{U}}.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&\frac{1}{2}|u(b)|_H^2 + \left(\alpha_B - \frac{\varepsilon^2\|\gamma\|}{2}\right)\|u\|_{\mathcal{U}}^2 \\
\leq\ & \frac{1}{2}|u_0|_H^2 + \|a_2\|_{L^2}^2 + \alpha_J b + \frac{\|\gamma\|\|\xi\|_{L^2}^2}{2\varepsilon^2} + (\ell\|x\|_{C(I;E)}^{\frac{1}{2}} + \alpha_J)\sqrt{b}\|\gamma\|\|u\|_{\mathcal{U}}. \qquad (3.5)
\end{aligned}
$$

Choose $\varepsilon > 0$ such that $\alpha_B > \frac{\varepsilon^2\|\gamma\|}{2}$. For such ε, it follows from (3.5) that there is a constant $\varrho_1 > 0$ such that $\|u\|_{\mathcal{U}} \leq \varrho_1$.

Next, from equation (1.2) and the hypotheses H(B)(3), H(J)(2) and H(g), we have

$$
|u'(t)|_{U^*} \leq a_1(t) + \iota\|u(t)\|_U + \kappa_0 + \kappa_1\|\gamma\|\|u(t)\|_U + \xi(t) + \ell\|x(t)\|_E^{\frac{1}{2}}. \qquad (3.6)
$$

By (3.5) and (3.6), it is obvious that $\|u'\|_{\mathcal{U}^*} \leq \varrho_2$ for some constants $\varrho_2 > 0$. Hence there exists a constant $\varrho > 0$ such that $\|u\|_{\mathcal{W}} \leq \varrho$. The proof is complete.

□

Now, let us take

$$
\begin{aligned}
\mathcal{N}_B(u) &= \{\beta \in \mathcal{U}^* : \beta(t) \in B(t, u(t)) \text{ a.e. } t \in [0, b]\}, \quad \text{for all } u \in \mathcal{U}, \\
\mathcal{N}_{\partial J}(u) &= \{\eta \in \mathcal{H} : \eta(t) \in \partial J(u(t)) \text{ a.e. } t \in [0, b]\}, \quad \text{for all } u \in \mathcal{H}.
\end{aligned}
$$

In the sequel, we list the compactness of solutions set $Q(g(x))$ for PHVIs as follows.

Proposition 3.2. *For a given $x \in \mathcal{C}_E$, suppose that hypotheses* H(B), H(J), H(g) *hold. Then $Q(g(x))$ is compact in $\mathcal{C}_H$.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq Q(g(x))$. Then

$$
u_n' + \beta_n + \eta_n = g(x), \quad \beta_n \in \mathcal{N}_B(u_n), \quad \eta_n \in \mathcal{N}_{\partial J}(u_n), \quad \text{for all } n \in \mathbb{N}. \qquad (3.7)
$$

It follows from Lemma 3.1 that $\{u_n\}_{n\geq 1} \subseteq \mathcal{W}$ is bounded. So, by passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in $\mathcal{W}$, $u_n \to u$ in $\mathcal{H}$, $u_n(t) \to u(t)$ in H for all $t \in [0, b]\backslash\Delta$, $m(\Delta) = 0$ (the Lebesgue measure of Δ). The sequence $\{\langle u_n'(t), u_n(t) - u(t)\rangle\}_{n\geq 1}$ is uniformly integrable. Therefore, given $\varepsilon > 0$ we can find $t \in [0, b]\backslash\Delta$ such that

$$
\int_t^b |\langle u_n'(s), u_n(s) - u(s)\rangle| ds < \varepsilon. \qquad (3.8)
$$

Let $\langle\langle \cdot, \cdot\rangle\rangle_t$ denote the duality brackets for the pair $(L^2([0, t]; U), L^2([0, t]; U^*))$ for any $t \in [0, b]$. Using the integration-by-parts formula, we have

$$
\langle\langle u_n', u_n - u\rangle\rangle_t = \frac{1}{2}|u_n(t) - u(t)|_H^2 + \langle\langle u', u_n - u\rangle\rangle_t.
$$

Note that $|u_n(t)-u(t)|_H \to 0$ (since $t \in [0,b]\backslash\Delta$) and $\langle\langle u', u_n-u\rangle\rangle_t \to 0$ (since $u_n \to u$ weakly in $L^2([0,t];U)$). Hence, we have

$$\langle\langle u_n', u_n - u\rangle\rangle_t \to 0, \quad \text{as} \quad n \to \infty. \tag{3.9}$$

We know that

$$\langle\langle u_n', u_n - u\rangle\rangle = \langle\langle u_n', u_n - u\rangle\rangle_t + \int_t^b \langle u_n'(s), u_n(s) - u(s)\rangle ds.$$

By the above equation and (3.9), we have

$$\langle\langle u_n', u_n - u\rangle\rangle \geq \langle\langle u_n', u_n - u\rangle\rangle_t - \varepsilon \ \text{ and } \ \langle\langle u_n', u_n - u\rangle\rangle \geq \langle\langle u_n', u_n - u\rangle\rangle_t + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary and using (3.9), we get

$$\liminf\langle\langle u_n', u_n - u\rangle\rangle \geq 0 \quad \text{and} \quad \limsup\langle\langle u_n', u_n - u\rangle\rangle \geq 0. \tag{3.10}$$

Hence we can infer that

$$\langle\langle u_n', u_n - u\rangle\rangle \to 0. \tag{3.11}$$

From the boundedness of $\{u_n\}_{n\geq 1}$ in $\mathcal{U}$ and condition H(J), we can assume that

$$\eta_n \rightharpoonup \eta \quad \text{in} \ \ \mathcal{H} \ \ \text{with} \ \ \eta \in \mathcal{H}. \tag{3.12}$$

From (3.11), (3.12) and [22, Lemma 11], we know that $\eta \in \mathcal{N}_{\partial J}(u)$.

Clearly, from hypothesis H(B)(3), we may suppose that $\beta_n \rightharpoonup \beta$ in $\mathcal{U}^*$. Hence, it follows from (3.7), (3.10) and (3.12) that $\limsup\langle\langle \beta_n, u_n - u\rangle\rangle \leq 0$, and thus $\beta \in \mathcal{N}_B(u)$.

From the above proof, we obtain

$$u' + \beta + \eta = g(x), \ \ \beta \in \mathcal{N}_B(u), \ \ \eta \in \mathcal{N}_{\partial J}(u), \tag{3.13}$$

which means that $u \in Q(g(x))$.

Subtracting (3.13) from (3.7), then multiplying by $(u_n - u)$ and using the integration-by-parts formula, we obtain, for every $t \in [0,b]$,

$$\frac{1}{2}|u_n(t) - u(t)|_H^2 + \langle\langle \beta_n - \beta, u_n - u\rangle\rangle = \langle\langle \eta_n - \eta, u_n - u\rangle\rangle. \tag{3.14}$$

$$\begin{aligned}
&\Rightarrow \frac{1}{2}|u_n(t) - u(t)|_H^2 \\
&\leq \int_0^b |\langle \beta_n(s) - \beta(s), u_n(s) - u(s)\rangle| ds \\
&+ \int_0^b |\langle \eta_n(s) - \eta(s), u_n(s) - u(s)\rangle| ds.
\end{aligned}$$

Since $\eta_n \rightharpoonup \eta$ in $\mathcal{H}$, $u_n \to u$ in $\mathcal{H}$ and $\beta_n \rightharpoonup \beta$ in $\mathcal{U}^*$, we can conclude that $\|u_n - u\|_{\mathcal{C}_H} \to 0$. The proof is complete. $\square$

Lemma 3.3. *Suppose that the conditions* H(B), H(J) *and* H(g) *hold. In addition, if* B *is monotone and* $\partial J(\cdot)$ *satisfies the relaxed monotonicity, i.e., there exists* $\rho > 0$ *such that*

$$\langle \eta_1 - \eta_2, u_1 - u_2 \rangle \geq -\rho \|u_1 - u_2\|_H^2, \quad \forall u_1,\ u_2 \in H,\ \eta_1 \in \partial J(u_1),\ \eta_2 \in \partial J(u_2),$$

then the set $Q(g(x))$ *is a singleton. Moreover,* $u(\cdot) = Q(g(x))$ *satisfies the following integral equation*

$$|u(t)|_H \leq L + \alpha_J \left(t + \int_0^t |u(s)|_H ds \right) + \frac{\ell}{2\varepsilon^2} \int_0^t \|x(s)\|_E ds,$$

where $L = \frac{1}{2}(|u_0|_H^2 + 1) + \|a_2\|_{L^1} + \frac{\|\xi\|_{L^2}^2}{2\varepsilon^2}$.

Proof. Let $u_i (i = 1, 2)$ be solutions to PHVIs (1.2). Then there exist $\beta_i \in \mathcal{U}^*$, $\eta_i \in \mathcal{H}$ such that

$$u_i(t) + \beta_i(t) + \eta_i(t) = g(x(t)),\ u(0) = u_0,\ \beta_i(t) \in B(t, u_i(t)),\ \eta_i(t) \in \partial J(u_i(t)).$$

Subtracting these two equations, multiplying the result by $u_1(t) - u_2(t)$ and integrating by parts, we obtain, for $t \in [0, b]$,

$$\begin{aligned} &\frac{1}{2}|u_1(t) - u_2(t)|_H^2 + \int_0^t \langle \beta_1(s) - \beta_2(s), u_1(s) - u_2(s) \rangle ds \\ &+ \int_0^t \langle \eta_1(s) - \eta_2(s), u_1(s) - u_2(s) \rangle ds = 0. \end{aligned}$$

Using the hypotheses, we have

$$|u_1(t) - u_2(t)|_H^2 \leq 2\rho \int_0^t |u_1(s) - u_2(s)|_H^2 ds$$

It follows from Gronwall inequality that $u_1(t) = u_2(t)$ on $[0, b]$.

Next, let $u(\cdot)$ be the unique solution to the system (1.2) with the initial condition $u(0) = u_0$. Multiplying (1.2) by $u(s)$ and integrating on $[0, t]$, we have

$$\int_0^t \langle u'(s), u(s) \rangle ds + \int_0^t \langle \beta(s), u(s) \rangle ds + \int_0^t \langle \eta(s), u(s) \rangle ds = \int_0^t \langle g(x(s)), u(s) \rangle ds. \quad (3.15)$$

with $\beta(s) \in B(s, u(s))$, $\eta(s) \in \partial J(u(s))$ a.e. $s \in [0, b]$. It follows from the integration-by-parts formula (see [33, Proposition 23.23(iv)]) that

$$\int_0^t \langle u'(s), u(s) \rangle ds = \frac{1}{2}(|u(t)|_H^2 - |u(0)|_H^2). \quad (3.16)$$

By H(g) and using Young's inequality with ε (see [23, Lemma 2.6]), we obtain

$$\begin{aligned} \int_0^t \langle g(s), u(s) \rangle ds &\leq \int_0^t (\xi(s) + \ell \|x(s)\|_E^{\frac{1}{2}}) |u(s)|_H ds \\ &\leq \frac{\|\xi\|_{L^2}^2}{2\varepsilon^2} + \frac{(\ell+1)\varepsilon^2}{2} \int_0^t |u(s)|_H^2 ds + \frac{\ell}{2\varepsilon^2} \int_0^t \|x(s)\|_E ds. \quad (3.17) \end{aligned}$$

Since $\eta(t) \in \partial J(u(t))$, it follows from H$(J)$(2) that

$$-(\eta(t), u(t)) \leq J^0(u(t); -u(t)) \leq \alpha_J (1 + |u(t)|_H).$$

Therefore, we obtain

$$\int_0^t \langle \eta(s), u(s)\rangle ds \geq -\int_0^t \alpha_J(1+|u(s)|_H)ds \geq -\alpha_J t - \alpha_J \int_0^t |u(s)|_H ds. \tag{3.18}$$

Combining H(B), (3.16), (3.17) and (3.18) with (3.15), we obtain

$$\begin{aligned}
& \frac{1}{2}|u(t)|_H^2 - \frac{1}{2}|u(0)|_H^2 + \int_0^t \left[\frac{\alpha_B}{\|\gamma\|^2}|u(s)|_H^2 - a_2(s)\right] ds \\
\leq\ & \frac{1}{2}|u(t)|_H^2 - \frac{1}{2}|u(0)|_H^2 + \int_0^t \left[\alpha_B\|u(s)\|_U^2 - a_2(s)\right] ds \\
\leq\ & \int_0^t \langle u'(s), u(s)\rangle ds + \int_0^t \langle \beta(s), u(s)\rangle ds \\
=\ & -\int_0^t \langle \eta(s), u(s)\rangle ds + \int_0^t \langle g(x(s)), u(s)\rangle ds \\
\leq\ & \frac{\|\xi\|_{L^2}^2}{2\varepsilon^2} + \alpha_J\left(t + \int_0^t |u(s)|_H ds\right) \\
& \quad + \frac{(\ell+1)\varepsilon^2}{2}\int_0^t |u(s)|_H^2 ds + \frac{\ell}{2\varepsilon^2}\int_0^t \|x(s)\|_E ds.
\end{aligned} \tag{3.19}$$

Choose $\varepsilon > 0$ such that $(\ell+1)\|\gamma\|^2\varepsilon^2 = 2\alpha_B$. For such ε, it follows from (3.19) that

$$\frac{1}{2}|u(t)|_H^2 \leq \frac{1}{2}|u_0|_H^2 + \|a_2\|_{L^1} + \frac{\|\xi\|_{L^2}^2}{2\varepsilon^2} + \alpha_J\left(t + \int_0^t |u(s)|_H ds\right) + \frac{\ell}{2\varepsilon^2}\int_0^t \|x(s)\|_E ds. \tag{3.20}$$

Let $L = \frac{1}{2}(|u_0|_H^2 + 1) + \|a_2\|_{L^1} + \frac{\|\xi\|_{L^2}^2}{2\varepsilon^2}$. From (3.20) and $|u(t)|_H \leq \frac{|u(t)|_H^2 + 1}{2}$, we get

$$|u(t)|_H \leq L + \alpha_J\left(t + \int_0^t |u(s)|_H ds\right)s + \frac{\ell}{2\varepsilon^2}\int_0^t \|x(s)\|_E ds,$$

which concludes the second claim. The proof is complete. □

4 Existence of mild solutions for DPHVIs

The goal of this section is to consider the existence of mild solutions for DPHVIs under some appropriate sufficient conditions by a well known fixed point theorem. To obtain our main results, we suppose that:

<u>H(A)$_1$</u> The closed linear operator A is the infinitesimal generator of a C_0-semigroup $T(t)$ on Banach space E.

<u>H(F)</u> $F : [0, b] \times E \times H \to \mathcal{P}_{cv,cp}(E)$ is such that

(1) for all $(x, u) \in E \times H$, $t \to F(t, x, u)$ is measurable;

(2) for a.e. $t \in [0, b]$, $F(t, \cdot, \cdot)$ has a strongly-weakly closed graph;

(3) there are a function $a \in L^1(0, b; \mathbb{R}_+)$ and constant $c_1,\ c_2 > 0$ such that

$$\|F(t, x, u)\| := \sup\{\|f\|_E :\ f \in F(t, x, u)\} \leq a(t) + c_1\|x\|_E + c_2|u|_H,$$

for a.e. $t \in [0, b]$, all $(x, u) \in E \times H$;

(4) for every bounded subsets $\Omega_1 \subset E,\ \Omega_2 \subset H$, there are two constants $\omega_1,\ \omega_2 > 0$ such that

$$\chi(F(t, \Omega_1, \Omega_2)) \leq \omega_1 \chi_E(\Omega_1) + \omega_2 \chi_H(\Omega_2), \qquad \text{for a.e. } t \in [0, b],$$

where $\chi_E,\ \chi_H$ stand for the Hausdorff MNC in the space E and H, respectively.

First, following the terminology in [10, 11, 14, 15], the solution of DPHVI is understood in the following mild sense.

Definition 4.1. *A pair of functions $(x, u) \in \mathcal{C}_E \times \mathcal{C}_H$ is said to be a mild solution of DPHVI (1.1)-(1.3) if there exists $f \in L^1(0, b; E)$ such that $f(t) \in F(t, x(t), u(t))$ for a.e. $t \in [0, b]$ and*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, \quad \text{a.e. } t \in [0, b].$$

$$u'(t) + B(t, u(t)) + \partial J(u(t)) \ni g(x(t)) \quad \text{a.e. } t \in [0, b], \quad u(0) = u_0, \tag{4.1}$$

Now, let us denote

$$\mathcal{N}_F : \mathcal{C}_E \times \mathcal{C}_H \to \mathcal{P}(L^1(0, b; E)),\ \ \mathcal{N}_F = \{f \in L^1(0, b; E) : f(t) \in F(t, x(t), u(t)),\ \text{a.e. } t \in [0, b]\}.$$

Proposition 4.2. *[23, Lemma 5.3] Under assumption* H(F), *$\mathcal{N}_F$ is well-defined and weakly u.s.c. with weakly compact and convex values.*

In the sequel, we introduce the solution operator for given (x_0, u_0):

$$\Phi : \mathcal{C}_E \times \mathcal{C}_H \to \mathcal{P}(\mathcal{C}_E \times \mathcal{C}_H)$$

$$\Phi(x, u) = \begin{bmatrix} \Upsilon(x, u) = T(\cdot)x_0 + \int_0^t T(t-s)f(s)ds, \quad f \in \mathcal{N}_F(x, u) \\ Q(g(x(\cdot))) \end{bmatrix}. \tag{4.2}$$

Consider the Cauchy operator

$$\Psi : L^1((0, b); E) \to \mathcal{C}_E, \quad \Psi(f)(t) = \int_0^t T(t-s)f(s)ds.$$

Then the solution operator Φ is rewritten by

$$\Phi(x, u) = \begin{bmatrix} \Upsilon(x, u) = T(\cdot)x_0 + \Psi \circ \mathcal{N}_F(x, u) \\ Q(g(x(\cdot))) \end{bmatrix} \tag{4.3}$$

The following proposition is important for obtaining our main results.

Proposition 4.3. *[6, p.150] Let hypotheses* H$(A)_1$ *and* H(F) *hold. If $\Omega \subset L^1(0, b; E)$ is semicompact, then $\Psi(D)$ is relatively compact in $\mathcal{C}_E$. In particular, if sequence $\{f_n\}$ is semicompact and $f_n \to f$ in $L^1(0, b; E)$ then $\Psi(f_n) \to \Psi(f)$ in $\mathcal{C}_E$.*

Now, we are in the position to present the main result concerning the mild solutions set $S(x_0, u_0)$ of DPHVI (1.1)-(1.3).

Theorem 4.4. *If the hypotheses* $\mathrm{H}(A)_1$*,* $\mathrm{H}(F)$ *and the conditions of Lemma 3.3 are satisfied, then the solutions set* $S(x_0,u_0)$ *of DPHVI (1.1)-(1.3) is nonempty.*

Proof. For $\forall(x,u)\in\mathcal{C}_E\times\mathcal{C}_H$, if the multi-valued map Φ admits a fixed point, then DPHVI (1.1)-(1.3) has a mild solution. We now subdivided our proof into three steps.

STEP 1. $\boldsymbol{\Phi}$ has compact convex values.

For $\forall(x,u)\in\mathcal{C}_E\times\mathcal{C}_H$, $\mathcal{N}_F(x,u)$ is a weakly compact set in $L^1(0,b;E)$ due to Proposition 4.2. So it follows from Proposition 4.3 that $\Psi\circ\mathcal{N}_F(x,u)$ is compact in $\mathcal{C}_E$. In addition, because $\mathcal{N}_F(x,u)$ is convex, $\Upsilon(x,u)$ is convex as well. On the other hand $Q(g(x))$ is singleton. That is, the multimap Φ has compact and convex values.

STEP 2. There exists a nonempty compact convex subset $\boldsymbol{\mathcal{D}}\subset\boldsymbol{\mathcal{C}_E}\times\boldsymbol{\mathcal{C}_H}$ such that $\boldsymbol{\Phi(\mathcal{D})}\subset\boldsymbol{\mathcal{D}}$.

First, we prove that there exists a nonempty convex subset $\mathcal{M}_0\subset\mathcal{C}_E\times\mathcal{C}_H$ such that $\Phi(\mathcal{M}_0)\subset\mathcal{M}_0$.

For any $(y,v)\in\Phi(x,u)$, It follows from hypotheses $\mathrm{H}(A)_1$, $\mathrm{H}(F)$ and Lemma 3.3 that

$$\begin{aligned}
\|y(t)\|_E+|v(t)|_H &\le \|T(t)x_0\|_E+\int_0^t\|T(t-s)\|\|f(s)\|_E ds\\
&\quad +L+\alpha_J b+\alpha_J\int_0^t|u(s)|_H ds+\frac{\ell}{2\varepsilon^2}\int_0^t\|x(s)\|_E ds\\
&\le M\|x_0\|_E+M\int_0^t\big[a(s)+c_1\|x(s)\|_E+c_2|u(s)|_H\big]ds\\
&\quad +L+\alpha_J b+\alpha_J\int_0^t|u(s)|_H ds+\frac{\ell}{2\varepsilon^2}\int_0^t\|x(s)\|_E ds\\
&\le M(\|x_0\|_E+\|a\|_{L^1})+L+\alpha_J b\\
&\quad +\left[M(c_1+c_2)+\alpha_J+\frac{\ell}{2\varepsilon^2}\right]\int_0^t\big[\|x(s)\|_E+|u(s)|_H\big]ds.
\end{aligned}$$

Denote

$$\mathcal{M}_0=\{(x,u)\in\mathcal{C}_E\times\mathcal{C}_H:\|x(t)\|_E+|u(t)|_H\le\delta(t),\ \forall t\in[0,b]\},$$

here δ is the unique solution of the integral equation

$$\delta(t)\le M(\|x_0\|_E+\|a\|_{L^1})+L+\alpha_J b+\left[M(c_1+c_2)+\alpha_J+\frac{\ell}{2\varepsilon^2}\right]\int_0^t\delta(s)ds.$$

It is obvious that $\mathcal{M}_0$ is a closed convex set of $\mathcal{C}_E\times\mathcal{C}_H$ and $\Phi(\mathcal{M}_0)\subset\mathcal{M}_0$. Set

$$\mathcal{M}_{k+1}=\overline{\mathrm{co}}\Phi(\mathcal{M}_k),\ k=0,1,2,\cdots$$

where, the notation co stands for the closure of convex hull of a subset in $\mathcal{C}_E\times\mathcal{C}_H$. We see that $\mathcal{M}_k$ is a closed convex set and $\mathcal{M}_{k+1}\subset\mathcal{M}_k$ for all $k\in\mathbb{N}$. Let $\mathcal{M}=\bigcap_{k=0}^{\infty}\mathcal{M}_k$, then $\mathcal{M}$ is a bounded closed convex subset of $\mathcal{C}_E\times\mathcal{C}_H$ and $\Phi(\mathcal{M})\subset\mathcal{M}$.

Next, for each $k\ge 0$, it is easy to know that $\mathcal{N}_F(\mathcal{M}_k)$ is integrably bounded by assumption $\mathrm{H}(F)$. Thus $\mathcal{M}$ is also integrably bounded.

Now, we check that $\mathcal{M}(t)$ is relativelly compact for each $t\ge 0$. By the regularity of the Hausdorff MNC, this will be done if $\mu_k(t)=\chi^*(\mathcal{M}_k(t))\to 0$ as $k\to\infty$, where χ^* is the Hausdorff MNC in $E\times H$ defined by $\chi^*(C,D)=\chi_E(C)+\chi_H(D)$.

Let $\mathcal{M}(t) = \begin{bmatrix} \mathcal{M}^E(t) \\ \mathcal{M}^H(t) \end{bmatrix}$, $\mu_k(t) = \mu_k^E(t) + \mu_k^H(t) = \chi_E(\mathcal{M}_k^E(t)) + \chi_H(\mathcal{M}_k^H(t))$. Because $\mathcal{M}_k^E(t)$ is bounded in E and by H(g), the set $\{g(x) : x \in \mathcal{M}_k^E\}$ is bounded in H. Due to the compactness of Q, we have

$$\mu_k^H(t) = \chi_H(Q(g(\mathcal{M}_k^E(t)))) = 0.$$

Next, it follows from hypothesis H(F)(4) that

$$\begin{aligned} \mu_{k+1}^E(t) &\le \chi_E\left(\int_0^t T(t-s)\mathcal{N}_F(\mathcal{M}_k^E, \mathcal{M}_k^H)(s)ds\right) \\ &\le 4M\int_0^t \chi_E(\mathcal{N}_F(\mathcal{M}_k^E, \mathcal{M}_k^H)(s)ds \\ &\le 4M\int_0^t [\omega_1\chi_E(\mathcal{M}_k^E(s)) + \omega_2\chi_H(\mathcal{M}_k^H(s))]ds \\ &\le 4M\omega_1\int_0^t \chi_E(\mathcal{M}_k^E(s))ds. \end{aligned}$$

Hence, we get

$$\mu_{k+1}^E(t) \le 4M\omega_1\int_0^t \mu_k^E(s)ds.$$

Putting $\mu_\infty(t) = \lim_{k\to\infty} \mu_k(t)$ and passing the limit we get

$$\mu_\infty(t) \le 4M\omega_1\int_0^t \mu_\infty(s)ds.$$

By using the Gronwall inequality, we deduce that $\mu_\infty(t) = 0$ for all $t \in [0, b]$. Hence, $\mathcal{M}(t)$ is relatively compact for all $t \in [0, b]$. By Proposition 4.3, $\Psi(\mathcal{M}_E)$ is a relatively compact in $\mathcal{C}_E$. Then $\Phi(\mathcal{M})$ is a relatively compact subset in $\mathcal{C}_E \times \mathcal{C}_H$.

Now, let us denote

$$D = \overline{\text{co}}\Phi(\mathcal{M}).$$

It is easy to see that D is a nonempty compact convex subset of $\mathcal{C}_E \times \mathcal{C}_H$ and

$$\Phi(D) = \Phi(\overline{\text{co}}\Phi(\mathcal{M})) \subset \Phi(\mathcal{M}) \subset \overline{\text{co}}\Phi(\mathcal{M}) = D,$$

which comes the conclusion.

STEP 3. Φ has a closed graph.

First, we claim that if $\{(x_n, u_n)\} \subset \mathcal{C}_E \times \mathcal{C}_H$ with $x_n \to x$, $u_n \to u$, $f_n \in \mathcal{N}_F(x_n, u_n)$ and $f_n \to f$ weakly in $L^1(0, b; E)$, then $f \in \mathcal{N}_F(x, u)$. In fact, it follows from condition H(F) that $\{f_n(t)\} \subset K(t) := F(t, \{x_n(t), u_n(t)\})$ is a compact set for a.e. $t \in [0, b]$. Hypothesis H(F) implies that $\{f_n\}$ is bounded by an L^1-integrable function. Thus, the sequence $\{f_n\}$ is semi-compact (see Definition 2.7) and by [7, Theorem 3.34], it is weakly compact in $L^1(0, b; E)$. So we can assume that $f_n \to f$ weakly in $L^1(0, b; E)$. According to Mazur's lemma, there exists a sequence $\widetilde{f}_n \in co\{f_i : i \ge n\}$ such that $\widetilde{f}_n \to f$ in $L^1(0, b; E)$ and so $\widetilde{f}_n(t) \to f(t)$ for a.e. $t \in [0, b]$. Assumption H(F) infer that F has compact values and is u.s.c., this means that for $\varepsilon > 0$

$$F(t, x_n(t), u_n(t)) \subset F(t, x(t), u(t)) + B_\varepsilon$$

for all sufficiently large n, here B_ε is the ball in E centered at origin with radius ε. So

$$f_n(t) \subset F(t, x(t), u(t)) + B_\varepsilon, \quad \text{for a.e.} \quad t \in [0, b].$$

Due to the convexity of $F(t, x(t), u(t)) + B_\varepsilon$, we replace $f_n(t)$ by $\widetilde{f_n}(t)$, the last inclusion still holds. Hence, $f \in F(t, x(t), u(t)) + B_\varepsilon$ for a.e. $t \in [0, b]$. Since ε is arbitrary, we get $f \in F(t, x(t), u(t))$ for a.e. $t \in [0, b]$. and so $f \in \mathcal{N}_F(x, u)$.

Next, let $x_n \to x_*$, $u_n \to u_*$ and $(y_n, v_n) \in \Phi(x_n, u_n)$ with $y_n \to y_*$ in $\mathcal{C}_E$, $v_n \to v_*$ in $\mathcal{C}_H$. Then there exists $f_n \in \mathcal{N}_F(x_n, u_n)$ such that $y_n(t) = T(t)x_0 + \int_0^t T(t-s)f_n(s)ds$ and $v_n(t) = Q(g(x_n(t)))$. Because $\mathcal{N}_F$ is weakly u.s.c., by passing to a subsequence if necessary, we may assume that $f_n \rightharpoonup f_*$. By Proposition 4.3 and the compactness of the operator Q we can pass to the limit to get that $y_*(t) = T(t)x_0 + \int_0^t T(t-s)f_*(s)ds$ and $v_*(t) = Q(g(x_*(t)))$.

So we have shown the validity of all the conditions required in Theorem 2.8. Then applying Theorem 2.8 we conclude that Φ has a fixed point $(\hat{x}, \hat{u}) \in \mathcal{C}_E \times \mathcal{C}_H$. Consequently, $(\hat{x}, \hat{u})$ is a solution of DPHVIs (1.1)-(1.3), which implies that $S(x_0, u_0)$ is nonempty. The proof is complete. □

5 Existence of a global attractor

The aim of this section is to present the existence of a grobal attractor for DPHVIs under some appropriate sufficient conditions. In this section, we replace $\mathrm{H}(A)_1$ by the following assumption:

$\underline{\mathrm{H}(A)_2}$ The C_0-semigroup $T(t)$ generated by A is exponentially stable with exponential ς, that is

$$\|T(t)\|_{\mathcal{L}(E)} \le \lambda e^{-\varsigma t}, \quad \forall t > 0, \quad \text{with} \quad \varsigma > 0, \ \lambda \ge 1.$$

Now, let us put $Z = E \times H$. We observe the DPHVIs (1.1)-(1.3) in the universal phase space Z. We are in a position to define the m-semiflow associated with DPHVIs (1.1)-(1.3) as follows $G : \mathbb{R}_+ \times Z \to \mathcal{P}(Z)$, $G(t, x_0, u_0) = \{(x(t), u(t)) : (x(t), u(t))$ is a solution of (1.1)-(1.3), $x(0) = x_0$, $u(0) = u_0\}$. Denote by $\Sigma(x_0, u_0, b)$ the set of all solutions on $[0, b]$ with initial condition (x_0, u_0) and let $\Sigma(x_0, u_0) = \bigcup\limits_{b>0} \Sigma(x_0, u_0, b)$. It is easy to see that $G(t, x_0, u_0) = \{(x(t), u(t)) : (x(\cdot), u(\cdot)) \in \Sigma(x_0, u_0), (x_0, u_0) \in Z\}$. We prove some properties of the solution set, which will be used to get our main result of this section.

Proposition 5.1. *Suppose that $\{(\xi_n, \eta_n)\} \subset Z$ such that $\xi_n \to \xi$ in E and $\eta_n \to \eta$ in H, respectively. Then $\Sigma(\{(\xi_n, \eta_n, b)\}) \subset C([0, b]; E) \times C([0, b]; H)$ is a relatively compact set in $C([0, b]; E) \times C([0, b]; H)$. In particular, $\Sigma(\xi, \eta, b) \subset C([0, b]; E) \times C([0, b]; H)$ is a compact set for each $(\xi, \eta) \in Z$.*

Proof. Let $(x_n, u_n) \in \Sigma(\{(\xi_n, \eta_n, b)\})$. Then we have

$$x_n(t) = T(t)\xi_n + \Psi \circ S_F(x_n, u_n)(t),$$

$$u_n(t) = Q(g(x_n(t))),$$

$$u_n(0) = \eta_n.$$

Now, we prove that $\{x_n\}$ is relatively compact in $C([0,b];E)$ and $\{u_n\}$ is relatively compact in $C([0,b];H)$. By the same estimate as in the proof of Step 2 in Theorem 4.4, we have

$$\begin{aligned}\|x_n(t)\|_E + |u_n(t)|_H &\le M(\|x_0\|_H + \|a\|_{L^1}) + L + \alpha_J b \\ &\quad + \left[M(c_1+c_2) + \alpha_J + \frac{\ell}{2\varepsilon^2}\right]\int_0^t \big[\|x_n(s)\|_E + |u_n(s)|_H\big]ds.\end{aligned}$$

So the Gronwall inequality ensures the boundedness of $\{(x_n,u_n)\}$ in $C([0,b];E)\times C([0,b];H)$.

Next, let $f_n \in \mathcal{N}_F(x_n,u_n)$ such that $x_n = T(\cdot)\xi_n + \Psi(f_n)$. Because $\{(x_n,u_n)\}$ is bounded and by H(F) we see that $\{f_n\}$ is integrably bounded. Using the compactness of Q (see Lemma 3.2), we deduce that $u_n = Q(g(x_n))$ is a relatively compact sequence. Regarding sequence $\{x_n\}$, it follows from hypothesis H$(F)(4)$ that

$$\begin{aligned}\chi_E(\{x_n(t)\}) &\le \chi_E(\{\Psi(f_n)(t)\}) \\ &\le 4\int_0^t \chi_E(\{T(t-s)(f_n)(s)\})ds \\ &\le 4M\int_0^t \chi_E(\{f_n(s)\})ds \\ &\le 4M\int_0^t [\omega_1\chi_E(\{x_n(s)\} + \omega_2\chi_H(\{u_n(s)\}]ds \\ &= 4M\omega_1\int_0^t \chi_E(\{x_n(s)\}ds.\end{aligned}$$

By using the Gronwall inequality, we deduce that $\chi_E(\{x_n(t)\} = 0$. Hence, we know that $\chi_E(\{f_n(t)\}) = 0$, for all $t\in[0,b]$. Thus $\{f_n\}$ is semicompact in $L^1(0,b;E)$ and $\{x_n\}$ is relatively compact in $C([0,b];E)$.

The last statement is testified if we show that the set $\Sigma(\xi,\eta,b)$ is closed in $C([0,b];E)\times C([0,b];H)$. Assume that $(x_n,u_n)\in\Sigma(\xi,\eta,b)$, $x_n\to x^*$ in E and $u_n\to u^*$ in H. By the same arguments as in the proof of step 3 of Theorem 4.4, we get $(x^*,u^*)\in\Sigma(\xi,\eta,b)$. The proof is complete. □

We obtain the following corollary by using Proposition 5.1.

Corollary 5.2. *The multimap G has compact values in $E\times H$.*

We will show that G is a strict m-semiflow generated by DPHVIs (1.1)-(1.3).

Lemma 5.3. *G is a strict m-semiflow, that is, $G(t_1+t_2,x_0,u_0) = G(t_2,G(t_1,x_0,u_0))$.*

Proof. Take $(y,v)\in G(t_1+t_2,x_0,u_0)$ be arbitrary. So $y = x(t_1+t_2)$, $v = u(t_1+t_2)$, where $(x(\cdot),u(\cdot))\in\Sigma(x_0,u_0,b)$, $b\ge t_1+t_2$. It follows from Definition 4.1 that there are selections $f\in\mathcal{N}_F(x,u)$, $\beta(t)\in B(t,u(t))$, $\eta(t)\in\partial J(u(t))$, $t\in[0,b]$ such that

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, \quad t\in[0,b],$$

$$u'(t) + \beta(t) + \eta(t) = g(x(t)), \text{ a.e. } t\in[0,b],$$

$$u(0) = u_0.$$

Now, we define $\hat{x}(t) = x(t+t_1)$, $\hat{u}(t) = u(t+t_1)$, one can obtain

$$\begin{aligned}
&\widehat{x}(t) = T(t)x(t_1) + \int_0^t T(t-s)\widehat{f}(s)ds,\\
&\qquad\qquad \widehat{f}(s) = f(s+t_1),\ \widehat{f}(s) \in \mathcal{N}_F(\widehat{x},\widehat{u})(s+t_1), \quad t \in [0, b-t_1]\\
&\widehat{u}'(t) + \widehat{\beta}(t) + \widehat{\eta}(t) = g(\widehat{x}(t)), \quad \widehat{\beta}(t) = \beta(t+t_1) \in B(t+t_1, u(t+t_1)),\\
&\qquad\qquad\qquad\qquad \widehat{\eta}(t) = \eta(t+t_1) \in \partial J(u(t+t_1)),\\
&\widehat{u}(0) = u(t_1).
\end{aligned}$$

Therefore, we have $(\hat{x}(\cdot), \hat{u}(\cdot)) \in \Sigma(x(t_1), u(t_1))$ and
$(y, v) = (\hat{x}(t_2), \hat{u}(t_2)) \in G(t_2, \hat{x}(t_1), \hat{u}(t_1)) \subset G(t_2, G(t_1, x_0, u_0))$.
Since (y, v) is arbitrary, we obtain $G(t_1+t_2, x_0, u_0) \subset G(t_2, G(t_1, x_0, u_0))$.

Next, we check that $G(t_2, G(t_1, x_0, u_0)) \subset G(t_1+t_2, x_0, u_0)$. In fact, for any $(y, v) \in G(t_2, G(t_1, x_0, u_0))$, we have $y = x_2(t_2)$, $v = u_2(t_2)$, where $(x_2(\cdot), u_2(\cdot)) \in \Sigma(x_2(0), u_2(0), b_2)$, $b_2 \geq t_2$, and $(x_2(0), u_2(0)) \in G(t_1, x_0, u_0)$. Obviously, we have $x_2(0) = x_1(t_1)$, $u_2(0) = u_1(t_1)$ with $(x_1(\cdot), u_1(\cdot)) \in \Sigma(x_0, u_0, b_1)$, $b_1 \geq t_1$. We define

$$x(t) = \begin{cases} x_1(t), & \text{if } 0 \leq t \leq t_1,\\ x_2(t-t_1), & \text{if } t_1 \leq t \leq t_1+b_2. \end{cases}$$

$$u(t) = \begin{cases} u_1(t), & \text{if } 0 \leq t \leq t_1,\\ u_2(t-t_1), & \text{if } t_1 \leq t \leq t_1+b_2. \end{cases}$$

$$f(t) = \begin{cases} f_1(t), & \text{if } 0 \leq t \leq t_1,\\ f_2(t-t_1), & \text{if } t_1 \leq t \leq t_1+b_2. \end{cases}$$

where $f_1(t) \in \mathcal{N}_F(x_1, u_1)(t)$ and $f_2(t) \in \mathcal{N}_F(x_2, u_2)(t)$ are the selections corresponding to x_1, x_2, respectively. Put

$$\beta(t) = \begin{cases} \beta_1(t), & \text{if } 0 \leq t \leq t_1,\\ \beta_2(t-t_1), & \text{if } t_1 \leq t \leq t_1+b_2, \end{cases}$$

$$\eta(t) = \begin{cases} \eta_1(t), & \text{if } 0 \leq t \leq t_1,\\ \eta_2(t-t_1), & \text{if } t_1 \leq t \leq t_1+b_2, \end{cases}$$

where $\beta_1(t) \in B(t, u_1(t))$, $\eta_1(t) \in \partial J(u_1(t))$ and $\beta_2(t) \in B(t, u_2(t))$, $\beta_2(t) \in \partial J(u_2(t))$, respectively. Then we deduce that $(x(\cdot), u(\cdot))$ satisfies

$$\begin{aligned}
&x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq t_1+b_2,\\
&u'(t) + \beta(t) + \eta(t) = g(x(t)), \quad 0 \leq t \leq t_1+b_2,\\
&u(0) = u_0.
\end{aligned}$$

It implies that $(x(\cdot), u(\cdot)) \in \Sigma(x_0, u_0)$ and $(y, v) = (x(t_1+t_2),\ u(t_1+t_2)) \in G(t_1+t_2, x_0, u_0)$. Therefore, $G(t_2, G(t_1, x_0, u_0)) \subset G(t_1+t_2, x_0, u_0)$. The proof is complete. $\square$

We first prove a condensing property of $G(b,\cdot,\cdot)$, which will be used to deduce that the m-semiflow G is asymptotically upper semicompact.

Lemma 5.4. *Assume that* $\mathrm{H}(A)_2$, $\mathrm{H}(F)$ *and the hypotheses of Lemma 3.3 hold for all* $b>0$. *Then, there exists* $b_0>0$ *and a number* $\rho\in[0,1)$ *such that for all* $b\geq b_0$, *we have*

$$\chi^*(G(b,C,D))\leq \rho\chi_E(C),$$

for all bounded set $(C,D)\in Z$, *provided that* $\varsigma>4\lambda\omega_1$.

Proof. Let $(C,D)\in Z$ be a bounded set. Putting $\Gamma=\Sigma(C,D)$, we have

$$\Gamma(t)=\begin{bmatrix}\Gamma_1(t)\\ \Gamma_2(t)\end{bmatrix}\subset\begin{bmatrix}T(t)C+\int_0^t T(t-s)\mathcal{N}_F(\Gamma_1,\Gamma_2)(s)ds\\ Q(g(\Gamma_1(t)))\end{bmatrix}. \tag{5.1}$$

According to the assumption $\mathrm{H}(g)$, we have $g(\Gamma_1(t))$ is bounded in H. By the compactness of Q, we know that $Q(g(\Gamma_1(t)))$ is relatively compact in H.

In the sequel, in terms of $\Gamma_2(t)$, it follows from hypothesis $\mathrm{H}(F)(4)$ that

$$\begin{aligned}\chi_E(\Gamma_1(t)) &\leq \lambda e^{-\varsigma t}\chi_E(C)+4\lambda\int_0^t e^{-\varsigma(t-s)}\chi_E(\mathcal{N}_F(\Gamma_1(s),\Gamma_2(s)))ds\\ &\leq \lambda e^{-\varsigma t}\chi_E(C)+4\lambda\int_0^t e^{-\varsigma(t-s)}[\omega_1\chi_E(\Gamma_1(s))+\omega_2\chi_H(\Gamma_2(s))]ds\\ &\leq \lambda e^{-\varsigma t}\chi_E(C)+4\lambda\omega_1\int_0^t e^{-\varsigma(t-s)}\chi_E(\Gamma_1(s))ds.\end{aligned}$$

Therefore, we get

$$e^{\varsigma t}\chi_E(\Gamma_1(t))\leq \lambda\chi_E(C)+4\lambda\omega_1\int_0^t e^{\varsigma s}\chi_E(\Gamma_1(s))ds.$$

By using the Gronwall inequality, we obtain

$$e^{\varsigma t}\chi_E(\Gamma_1(t))\leq \lambda\chi_E(C)e^{4\lambda\omega_1 t}.$$

$$\chi_E(\Gamma_1(t))\leq \lambda\chi_E(C)e^{-(\varsigma-4\lambda\omega_1)t}.$$

Since $\varsigma>4\lambda\omega_1$, choosing $b_0>\frac{\ln\lambda}{\varsigma-4\lambda\omega_1}$, we obtain the conclusion of the lemma. The proof is complete. □

Lemma 5.5. *Let the assumptions of Lemma 5.4 hold. Then, G is asymptotically upper semicompact.*

Proof. By using Lemma 5.4 and by the statements as [19, Proposition 1], we get the proof of the lemma. □

Lemma 5.6. *Suppose that the hypotheses of Lemma 5.4 hold. Then, for each* $t>0$ *the* m*-semiflow* $G(t,\cdot,\cdot)$ *is u.s.c.*

Proof. Since $G(t,\cdot,\cdot)$ has compact values due to Corollary 5.2. By Proposition 2.4 it remains to show that $G(t,\cdot,\cdot)$ is quasi-compact and has a closed graph.

First, we check that $G(t,\cdot,\cdot)$ is quasi-compact. For this reason, we assume that $K \subset Z$ is a compact set. Let $\{(y_n, v_n)\} \subset G(t, K)$, then one can find a sequence $\{(\xi_n, \eta_n)\} \subset K$ such that $\xi_n \to \xi$ in E, $\eta_n \to \eta$ in H.

Let $(x_n, u_n) \in \Sigma(\xi_n, \eta_n)$ such that $x_n(0) = \xi_n,\ u_n(0) = \eta_n,\ x_n(t) = y_n,\ u_n(t) = v_n$. By Proposition 5.1, $\Sigma(\{(\xi_n, \eta_n, t)\})$ is a relatively compact set in $C([0,t]; E) \times C([0,t]; H)$. Then, there is a subsequence of $\{(x_n, u_n)\}$ (denoted again by $\{(x_n, u_n)\}$ such that $\pi_t(x_n) \to x$ in $C([0,t]; E)$, $\pi_t(u_n) \to u$ in $C([0,t]; H)$, where π_t is the truncation operator to $[0,t]$ acting on $C([0,\infty); E)$ and $C([0,\infty); H)$. Therefore, (y_n, v_n) converges to $(x(t), u(t))$ in Z and $(x(0), u(0)) = (\xi, \eta)$. It implies the quasi-compactness of $G(t,\cdot,\cdot)$.

We now prove that $G(t,\cdot,\cdot)$ has a closed graph. Let $\{(\xi_n, \eta_n)\}$ be a sequence in Z and (ξ_n, η_n) converges to (ξ, η). Choose $(y_n, v_n) \in G(t, \xi_n, \eta_n)$ such that $y_n \to y$ in E, $v_n \to v$ in H. It suffices to show that $(y, v) \in G(t, \xi, \eta)$.

Let $(x_n, u_n) \in \Sigma(\xi_n, \eta_n)$ be such that $x_n(t) = y_n,\ u_n(t) = v_n$. By Proposition 5.1, $\{(x_n, u_n)\}$ has a convergent subsequence (denoted again by $\{(x_n, u_n)\}$. Assume that $\lim x_n = x$ in $C([0,t]; E)$, $\lim u_n = u$ in $C([0,t]; H)$. We obtain that $y = x(t),\ v = u(t)$. Let $f_n \in S_F(x_n, u_n)$ be such that

$$x_n = T(\cdot)\xi_n + \Psi(f_n), \tag{5.2}$$

$$u_n = Q(g(x_n)), \tag{5.3}$$

$$u_n(0) = u_0.$$

Because $\{(x_n, u_n)\}$ is bounded, we obtain that $\{f_n\} \in L^1(0,t,E)$ is integrably bounded, thanks to H(F)(3). Moreover, $\{f_n(r)\} \subset K(r) = F(\overline{\{x_n(r), u_n(r)\}})$, $r \in [0,t]$ is compact. Thus $\{f_n\}$ is a semicompact sequence. By [13, Proposition 4.2.1], we have $\{f_n\}$ weakly converges to f and $\Psi(f_n) \to \Psi(f)$. Then one can pass to the limit in equality (5.3) to get $x = T(\cdot)\xi + \Psi(f)$. Since $\mathcal{N}_F$ is weakly u.s.c., one has $f \in \mathcal{N}_F(x, u)$. Due to the compactness of Q, one can pass to the limit again in equalities (5.4) to obtain $u = Q(g(x)),\ u(0) = \eta$. So $(x, u) \in \pi_t(\xi, \eta)$. The proof is complete. □

Lemma 5.7. *Assume that the conditions of Lemma 5.4 hold. Then the m-semiflow G admits an absorbing set, provided that $\alpha_J = \ell = 0$ and $\varsigma > \max\{c_1, c_2\}$.*

Proof. Let (C, D) be a bounded set in Z. For each $(x(\cdot), u(\cdot)) \in (C, D)$, $x(0) = x_0,\ u(0) = u_0,\ (x_0, u_0) \in (C, D)$, it follows from condition $\mathrm{H}(A)_2$ and H(F)(3) that

$$\begin{aligned}
\|x(t)\|_E &\le \|T(t)x_0\|_E + \int_0^t \|T(t-s)f(s)\|_E ds \\
&\le e^{-\varsigma t}\|x_0\|_E + \int_0^t e^{-\varsigma(t-s)}\big[a(s) + c_1\|x(s)\|_E + c_2|u(s)|_H\big] ds \\
&\le e^{-\varsigma t}\|x_0\|_E + \|a\|_{L^1} + c\int_0^t e^{-\varsigma(t-s)}(\|x(s)\|_E + |u(s)|_H) ds,
\end{aligned}$$

here $c = \max\{c_1, c_2\}$. Since $\alpha_J = \ell = 0$, it follows from Lemma 3.3 that

$$|u(t)|_H \le L,$$

where $L = \frac{1}{2}(|u_0|_H^2 + 1) + \|a_2\|_{L^1} + \frac{\|\xi\|_{L^2}^2}{2\varepsilon^2}$, which is independent of t. From above two inequalities, we obtain

$$e^{\varsigma t}(\|x(t)\|_E + |u(t)|_H) \leq \|x_0\|_E + e^{\varsigma t}(\|a\|_{L^1} + L) + c\int_0^t e^{\varsigma s}(\|x(s)\|_E + |u(s)|_H)ds,$$

Applying the Gronwall inequality, we have

$$\begin{aligned} & e^{\varsigma t}(\|x(t)\|_E + |u(t)|_H) \\ \leq\ & \|x_0\|_E + e^{\varsigma t}(\|a\|_{L^1} + L) + c\int_0^t [\|x_0\|_E + e^{\varsigma s}(\|a\|_{L^1} + L)]e^{c(t-s)}ds \\ \leq\ & \|x_0\|_E + e^{\varsigma t}(\|a\|_{L^1} + L) + \|x_0\|_E(e^{ct} - 1) + \frac{c(\|a\|_{L^1} + L)(e^{\varsigma t} - e^{ct})}{\varsigma - c}, \end{aligned}$$

which is equal to the following inequality

$$\begin{aligned} & \|x(t)\|_E + |u(t)|_H \\ \leq\ & e^{-\varsigma t}\|x_0\|_E + (\|a\|_{L^1} + L) + \|x_0\|_E(e^{-(\varsigma - c)t} - e^{-\varsigma t}) + \frac{c(\|a\|_{L^1} + L)(1 - e^{-(\varsigma - c)t})}{\varsigma - c}, \end{aligned}$$

It follows from the hypothesis $\varsigma > \max\{c_1, c_2\}$ that the last inequality ensures that the ball centered at origin with radius

$$R = \|a\|_{L^1} + L + \frac{c(\|a\|_{L^1} + L)}{\varsigma - c} + 1$$

becomes an absorbing set for the m-semiflow G. The proof is complete. □

Combining Lemma 5.5, 5.6 and 5.7, we arrive at the main result of this section.

Theorem 5.8. *Let the assumptions of Lemma 5.4 hold. Then the m-semiflow G generated by DPHVIs (1.1)-(1.3) admits a compact global attractor provided that $\alpha_J = \ell = 0$, $\varsigma > \max\{4\lambda\omega_1, c_1, c_2\}$.*

6 A dynamic thermoviscoelasticity problem

In recent years, dynamic contact problems with or without thermal effects for viscoelastic bodies have become an active area of investigation in the field of applications. For more details, we refer to [20, 22–24, 26, 28, 29, 32] and the references therein. However, to the best of our knowledge, there is still little information known for the existence of solutions to the system about hemivariational inequalities in dynamic thermoviscoelasticity has studied in a few papers [11, 21, 25]. In this section we introduce and study a mathematical model for which the results of Section 4 and Section 5 can be applied.

Let $\mathbb{S}^d$ be the space of second order symmetric tensors on $\mathbb{R}^d$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$u \cdot v = u_i v_i, \quad \|u\| = (u \cdot u)^{\frac{1}{2}}, \ \forall u = (u_i), \ v = (v_i) \in \mathbb{R}^d,$$

$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\| = (\tau : \tau)^{\frac{1}{2}}, \ \forall \sigma = (\sigma_{ij}), \ \tau = (\tau_{ij}) \in \mathbb{R}^d.$$

For brevity, we suppress the explicit dependence of the quantities on the spatial variable x, or both x and t. By $\nu = (\nu_i)$ the outward unit normal on the boundary, and by $\varepsilon(u) =$

$(\varepsilon_{ij}(u))$ the linearized strain tensor whose components are given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, where $u_{i,j} = \partial u_i/\partial x_j$. For a vector field, we use the notation u_ν and u_τ for the normal and tangential components of u on Γ given by $u_\nu = u \cdot \nu$ and $u_\tau = u - u_\nu \nu$. The normal and tangential components of the stress field σ on the boundary are defined by $\sigma_\nu = (\sigma\nu) \cdot \nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$, respectively.

For the dynamic thermoviscoelasticity problem we study in this section the physical setting can be described as follows. Suppose that a viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz boundary Γ which consists of parts sets $\Gamma_D,\ \Gamma_N,\ \Gamma_C$ and $m(\Gamma_D) > 0$. The body is clamped on Γ_D, the volume forces of density f_0 act in Ω and the surface tractions of density f_N are applied on Γ_N. We also suppose that the body is subjected to a heat source term per unit volume of the domain Ω. To give the classical formulation of our dynamic contact problem, we need to use the notations $\Omega_b = \Omega \times (0, b),\ \Sigma_D = \Gamma_D \times (0, b),\ \Sigma_N = \Gamma_N \times (0, b)$ and $\Sigma_C = \Gamma_C \times (0, b)$. With these data, we consider the following problem.

Problem $\mathcal{P}$. *Find a displacement field $u : \Omega_b \to \mathbb{R}^d$, a stress field $\sigma : \Omega_b \to \mathbb{S}^d$ and a temperature $\theta : \Omega_b \to \mathbb{R}$ such that for all $t \in (0, b)$,*

$$u''(t) = \mathrm{Div}\sigma(t) + f_0(t), \qquad \text{in} \quad \Omega_b, \tag{6.1}$$

$$\sigma(t) = \mathcal{B}(t, \varepsilon u'(t)) + \mathcal{C}(t, \theta(t)), \qquad \text{in} \quad \Omega_b, \tag{6.2}$$

$$\theta'(t) - \Delta\theta = \sum_{i=1}^{m} \xi_i(t)\varphi_i(t, \theta(t), u'(t)), \qquad \text{in} \quad \Omega_b, \tag{6.3}$$

$$u(t) = 0, \qquad \text{on} \quad \Sigma_D, \tag{6.4}$$

$$\sigma(t)\nu = f_N(t), \qquad \text{on} \quad \Sigma_N, \tag{6.5}$$

$$-\sigma_\nu(t) = \partial j_\nu(u'_\nu(t)), \qquad \text{on} \quad \Sigma_C, \tag{6.6}$$

$$\sigma_\tau(t)\nu = 0, \qquad \text{on } \Sigma_C, \tag{6.7}$$

$$\theta(t) = 0, \qquad \text{on} \quad \partial\Omega \times (0, b), \tag{6.8}$$

$$u(0) = u_0,\ u'(0) = \omega_0,\ \theta(0) = \theta_0, \qquad \text{in} \quad \Omega. \tag{6.9}$$

We briefly comment on Problem $\mathcal{P}$. Formulation (6.1) represents the motion equation. Equation (6.2) expresses the constitutive law for viscoelastic materials in which $\mathcal{B}$ is a nonlinear viscosity operator and $\mathcal{C}$ denotes a nonlinear thermal expansion operator. Relation (6.3) is the equation of heat transfer with the thermal conductivity functions ξ_i characterized by m external heat sources whose properties are depending on the state of the system and nonlinear functions φ_i depending on the velocity. We have the clamped boundary condition (6.4) on Γ_D and the surface traction boundary condition (6.5) on Γ_N. Relation (6.6) is the multivalued contact condition with normal damped response in which ∂j_ν denotes the Clarke subdifferential of a given function j_ν. For simplicity, we assume that (6.7) is the frictionless condition and in (6.8), the temperature vanishes on the boundary $\partial\Omega \times (0, b)$. Finally, conditions (6.9) are the initial condition in which u_0 and ω_0 represent the initial displacement and the initial velocity, respectively.

In the study of Problem $\mathcal{P}$, we use the spaces V, Q and Q_∞ defined in [32, Section 3.1]. We also use notation $\|\gamma\|$ for the norm of the trace operator $\gamma : V \to L^2(\Gamma_C;\mathbb{R}^d)$. Moreover, choose $H = L^2(\Omega;\mathbb{R}^d)$. Clearly, $(V; H; V^*)$ forms an evolution triple of spaces. We also introduce the spaces

$$\mathcal{V} = L^2(0,b;V);\ \mathcal{V}^* = L^2(0,b;V^*);\ W = \{w \in \mathcal{V} | w' \in \mathcal{V}^*\},$$

where the time derivative w' is in the sense of vector-valued distributions. Note that, actually, $\mathcal{V}^*$ is the dual of the space $\mathcal{V}$.

We now turn to the analysis of Problem $\mathcal{P}$. To this end, for equation (6.3), we define the operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ as follows

$$A = \Delta, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega). \tag{6.10}$$

It is well know from [31] that A satisfies assumption H$(A)_1$ on the space $E = L^2(\Omega)$. Moreover, the semigroup $T(t)$ generated by A is exponential stable, that is,

$$\|T(t)\|_{\mathcal{L}(E)} \le e^{-\lambda_1 t} \qquad \text{for all } \ t \ge 0, \tag{6.11}$$

where $\lambda_1 = \inf\{\|\nabla\theta\|_E^2 : \|\theta\|_E = 1\}$. This shows that assumption H$(A)_2$ holds, too.

Suppose that the thermal conductivity functions $\xi_i : [0,b] \to \mathbb{R}(i = 1,\cdots,m)$ are measurable and satisfy the feedback condition

$$\xi(t) = (\xi_1(t),\cdots,\xi_m(t)) \in G(z(t,\cdot), r(t,\cdot)),\ t \in [0,b],$$

where $G : E \times V \to \mathbb{R}^m$ is u.s.c. with convex closed values and satisfies:

$$\|G(z,r)\| \le K, \quad \text{for all } \ z \in E,\ r \in V, \quad \text{where } \ K > 0.$$

We suppose that functions $\varphi_i(i = 1,\cdots,m)$ satisfy the following conditions:

(1) $\varphi_i(\cdot,z,r) : [0,b] \to \mathbb{R}$ is measurable for all $z \in \mathbb{R},\ r \in \mathbb{R}^d$;

(2) $|\varphi_i(t,z_1,r_1) - \varphi_i(t,z_2,r_2)| \le k_i|z_1 - z_2| + l_i\|r_1 - r_2\|_{\mathbb{R}^d}$, $\forall\, t \in [0,b],\ z_1,\ z_2 \in \mathbb{R},\ r_1,r_2 \in \mathbb{R}^d$, and $\varphi_i(t,0,0) \equiv 0$.

Then it is easy to see that the map $h : E \times V \times B_K(\mathbb{R}^m) \to E$, where $B_K(\mathbb{R}^m) = \{\zeta \in \mathbb{R}^m : \|\zeta\| \le K\}$, defined by

$$h(t,z,r) = \sum_{i=1}^m \xi_i(t)\varphi_i(t,z(t),r(t))$$

is (k,l)-Lipschitz in (z,r) with

$$k = K\sqrt{\sum_{i=1}^m k_i^2}, \quad l = K\sqrt{\sum_{i=1}^m l_i^2},$$

and compact in (z,r), i.e., the set $h(z,r,B_K(\mathbb{R}^m)$ is relatively compact in (z,r) for each $(z,r) \in E \times V$.

Now, define the multi-valued function

$$F : [0, b] \times E \times V \to \mathcal{P}(E), \quad F(t, z, r)(x) = h(t, z, r, G(z, r)). \tag{6.12}$$

Similar to the work [27], we can check that the multi-valued function F is fulfilled the hypotheses H(F) and equations (6.3) and (6.8) can be reformulated as

$$\theta'(t) \in A\theta(t) + F(t, \theta(t), u'(t)), \quad t \in [0, b].$$

In the sequel, to derive a variational formulation of Problem $\mathcal{P}$, we now consider the following assumptions on the data.

<u>H($\mathcal{B}$)</u> $\mathcal{B} : \Omega_b \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(1) $\mathcal{B}(\cdot, \cdot, \varepsilon)$ is measurable on Ω_b for all $\varepsilon \in \mathbb{S}^d$;

(2) $\mathcal{B}(x, t, \cdot)$ is continuous on $\mathbb{S}^d$ for a.e. $(x, t) \in \Omega_b$;

(3) $\|\mathcal{B}(x, t, \varepsilon)\|_{\mathbb{S}^d} \leq b_0(x, t) + b_1\|\varepsilon\|_{\mathbb{S}^d}$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x, t) \in \Omega_b$ with $b_0 \in L^2(\Omega_b)$ and $b_0,\ b_1 \geq 0$;

(4) $\mathcal{B}(x, t, \varepsilon) : \varepsilon \geq \alpha_{\mathcal{B}}\|\varepsilon\|^2_{\mathbb{S}^d} - b_2(x, t)$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x, t) \in \Omega_b$ with $b_2 \in L^1(\Omega_b)$ and $\alpha_{\mathcal{B}} > 0,\ b_2 \geq 0$;

(5) $(\mathcal{B}(x, t, \varepsilon_1) - \mathcal{B}(x, t, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq 0$ for all $\varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d$, a.e. $(x, t) \in \Omega_b$.

<u>H($\mathcal{C}$)</u> $\mathcal{C} : \Omega_b \times \mathbb{R} \to \mathbb{S}^d$ is such that

(1) $\mathcal{C}(\cdot, \cdot, r)$ is measurable on Ω_b for all $r \in \mathbb{R}$;

(2) $\mathcal{C}(x, t, \cdot)$ is continuous for a.e. $(x, t) \in \Omega_b$;

(3) $\|\mathcal{C}(x, t, r)\|_{\mathbb{S}^d} \leq c_0(x, t) + c_1|r|^{\frac{1}{2}}$ for all $r \in \mathbb{R}$, a.e. $(x, t) \in \Omega_b$ with $c_0 \in L^2(\Omega_b)$ and $c_0,\ c_1 \geq 0$.

<u>H(j_ν)</u> $j_\nu : \Gamma_C\mathbb{R} \to \mathbb{R}$ is such that

(1) $j_\nu(\cdot, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$ and there is $\overline{e} \in L^2(\Gamma_C)$ such that $j_\nu(\cdot, \overline{e}(\cdot)) \in L^2(\Gamma_C)$;

(2) $j_\nu(x, \cdot)$ is locally Lipchitz continuous on $\mathbb{R}$ for a.e. $x \in \Gamma_C$;

(3) there exists $\kappa_0,\ \kappa_1 \geq 0$ such that $|\partial j_\nu(x, r)| \leq \kappa_0 + \kappa_1|r|$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_C$;

(4) $j^0_\nu(x, r; -r) \leq \alpha_{j_\nu}(1 + |r|^2)$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_C$ with $\alpha_{j_\nu} \geq 0$;

(5) $(\zeta_1 - \zeta_2)(r_1 - r_2) \geq -m_{j_\nu}|r_1 - r_2|^2$ for all $r_1,\ r_2 \in \mathbb{R},\ \zeta_1 \in \partial j_\nu(x, r_1),\ \ \zeta_2 \in \partial j_\nu(x, r_2)$, a.e. $x \in \Gamma_C$ with $m_{j_\nu} \geq 0$;

<u>H(0)</u> $f_0 \in L^2(0, b; L^2(\Omega, \mathbb{R}^d)),\ f_N \in L^2(0, b; L^2(\Gamma_N; , \mathbb{R}^d)),\ u_0,\ w_0 \in V,\ \theta_0 \in E.$

We now turn to the variational formulation of Problem $\mathcal{P}$. Let $v \in V$ and $t \in [0, b]$. We use Green's formula, decompose the resulting surface integral on three integrals on $\Gamma_D,\ \Gamma_N$ and Γ_C and then we use the boundary conditions (6.3), (6.4) and equation (6.2) to obtain the following variational formulation of Problem $\mathcal{P}$, in terms of displacement.

(PHVI)$_{\mathcal{P}}$. *Find a displacement field $u : (0, b) \to V$ such that for all $t \in (0, b)$,*

$$\langle u''(t), v - u'(t)\rangle_{V^*\times V} + (\mathcal{B}(t, \varepsilon(u'(t)) + \mathcal{C}(t, \theta(t)), \varepsilon(v - u'(t)))_{\mathcal{H}} \tag{6.13}$$
$$\int_{\Gamma_C} j_\nu^0(u'_\nu(t); v_\nu - u'_\nu(t))d\Gamma \geq \langle h(t), v - u'(t)\rangle_{V^*\times V},$$

where $h \in L^2(0, b; V^*)$ is defined by

$$\langle h(t), v\rangle_{V^*\times V} = \langle f_0(t), v\rangle_H + \langle f_N(t), v\rangle_{L^2(\Gamma_N;\mathbb{R}^d)}, \quad \text{for} \quad v \in V,\ t \in (0, b).$$

Next, define the operator $B : (0, b) \times V \to V^*$, $J : V \to \mathbb{R}$, $g : (0, b) \times V$ by

$$\langle B(t, u), \zeta\rangle_{V^*\times V} = (\mathcal{B}(t, \varepsilon(u), \varepsilon(\zeta))_{\mathcal{H}}, \quad \text{for all} \ \ u,\ \zeta \in V,\ t \in (0, b), \tag{6.14}$$

$$J(u) = \int_{\Gamma_C} j_\nu^0(u_\nu))d\Gamma, \quad \text{for all} \quad u \in V,\ t \in (0, b), \tag{6.15}$$

$$g(t, \theta) = h(t) - \mathcal{C}(t, \theta(t)) \quad \text{for all} \ \ \theta \in E, \ \text{a.e. } t \in (0, b). \tag{6.16}$$

We denote $w = u'$, i.e.,

$$u(t) = \int_0^t w(s)ds + u_0, \quad \text{for all} \ \ t \in [0, b].$$

With the notation and using the definition (6.10) of the operator A, we deduce that Problem $\mathcal{P}$ can be formulated, equivalently, in a form of the following differential variational-hemivariational inequality.

(DPHVI)$_{\mathcal{P}}$. *Find the velocity $w \in \mathcal{W}$ and the temperature $\theta \in \mathcal{C}_E$ such that*

$$w'(t) + B(t, w(t)) + \partial J(w(t)) \ni g(t, \theta(t)), \qquad \text{a.e. } t \in [0, b], \tag{6.17}$$

$$\theta'(t) \in A\theta(t) + F(t, \theta(t), w(t)), \qquad \text{a.e. } \ t \in [0, b], \tag{6.18}$$

$$w(0) = w_0, \quad \theta(0) = \theta_0. \tag{6.19}$$

The existence of mild solution for the system (6.17)–(6.19) is provided by the following.

Theorem 6.1. *Assume that* H($\mathcal{B}$), H($\mathcal{C}$), H(j_ν) *and* H(0) *hold. Then, Problem (6.17)–(6.19) has a mild solution $(x, u) \in \mathcal{C}_E \times \mathcal{C}_V$.*

Proof. The proof of is based on Theorem 4.4. At this end we check in what follows the validity of the conditions of the theorem.

First, note that as already mentioned, the operator A satisfies condition H(A). Moreover, conditions of functions ξ_i, φ_i imply that the mutivalued function F defined by (6.12) satisfy conditions H(F. Next, from condition H($\mathcal{B}$) of the function $\mathcal{B}$, we see that the operator B given by (6.14) satisfies assumption H(B). Moreover, using the standard arguments on subdifferential calculus, the function J defined by (6.15) satisfies the hypothesis H(J). Finally, we note that assumption H($\mathcal{C}$) on the function $\mathcal{C}$ guarantees that the function g defined by (6.16) satisfies condition H(g). Hence, we are now in a position to apply Theorem 4.4 to conclude the proof. □

Now, we also proceed with the following results about the existence of a grobal attractor for problem (6.17)–(6.19).

Theorem 6.2. *Suppose that* $\mathrm{H}(\mathcal{B})$, $\mathrm{H}(\mathcal{C})$, $\mathrm{H}(j_\nu)$ *and* $\mathrm{H}(0)$ *hold. Then the problem (6.17)–(6.19) admits a compact global attractor provided that* $\alpha_{j_\nu} = c_1 = 0$ *and* $\lambda_1 > \max\{4k, l\}$.

Proof. Theorem 6.2 is a direct consequence of Theorem 5.8. So, we can apply Theorem 5.8 and obtain that the system (6.17)–(6.19) admits a compact global attractor. □

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